EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2020**

Lecture 6: (Nonholonomic Systems)

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6.1 Nonholonomic Systems

6.1.1 Pfaffian Constraints

Given state space $q \in \mathbb{R}^n$, a set of constraints on velocities of the form

$$\omega_i(q)\dot{q} = 0 \qquad\qquad i = 1, ..., k \tag{6.1}$$

with $\omega_i^T(q) \in \mathbb{R}^n$ is referred to as a system of Pfaffian constraints. We will assume that the rows $\omega_i(q)$ are linearly independent at q so that the k constraints are independent.

Can the velocity constraints be converted into state space constraints of the form below?

$$h_i(q) = 0 \qquad\qquad i = 1, ..., k \tag{6.2}$$

In other words, is the state space q constrained to lie in a manifold of dimension $n - k$?

We may be encouraged by the fact that $h(q) = 0 \Longleftrightarrow dh(q)\dot{q} = 0$ for a single constraint. However, the answer to this question is neither easy nor obvious.

6.1.2 Integrability

A single constraint

$$\omega(q)\dot{q} = \sum_{i=1}^{n} w_j(q)\dot{q}_j = 0 \tag{6.3}$$

is said to be **integrable** if there exists a function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\omega(q)\dot{q} = 0 \Longleftrightarrow h(q) = 0 \tag{6.4}$$

That is,

$$\sum_{j=1}^{n} \omega_j(q)\dot{q}_j = 0 \implies \sum_{j=1}^{n} \frac{\partial h}{\partial q_j}\dot{q} = 0 \tag{6.5}$$

This implies that there exists some function $\alpha(q)$ called the integrating factor such that

$$\alpha(q)\omega_j(q) = \frac{\partial h}{\partial q_j}(q) \qquad\qquad j = 1, ..., n \tag{6.6}$$

From the equality of mixed partials of h, i.e.

$$\frac{\partial^2 h}{\partial q_i \partial q_j} = \frac{\partial^2 h}{\partial q_j \partial q_i} \tag{6.7}$$

it follows that

$$\frac{\partial(\alpha\omega_j)}{\partial q_i} = \frac{\partial(\alpha\omega_i)}{\partial q_j} \qquad\qquad i, j = 1, ..., n \tag{6.8}$$

However, this condition relies on finding the integrating factor $\alpha(q)$. This becomes even harder when there are k constraints: You must check not only the integrability of each constraint, but also that of linear combinations of the constraints, i.e. the integrability of

$$\sum_{j=1}^{k} \alpha_i(q)\omega_i(q)\dot{q} \tag{6.9}$$

6.1.3 Definition of Holonomic Systems

A set of Pfaffian constraints $\omega_i(q), i = 1, ..., k$ is said to be **holonomic** if there exist k functions $h_i(q), i = 1, ..., k$ such that

$$\omega_i(q)\dot{q} \Longleftrightarrow h_i(q_j) = c_i \qquad i = 1, ..., k \tag{6.10}$$

That is, the number of constraints on q is precisely k, and thus q lies on a manifold of dimension $(n - k)$.

A set of Pfaffian constraints is said to be **nonholonomic** if there are only $p < k$ functions such that

$$\omega_i(q)\dot{q} \Longleftrightarrow h_i(q_j) = c_i \qquad i = 1, ..., p \tag{6.11}$$

A Pfaffian system is said to be **completely nonholonomic** if $p = 0$, or else **partially nonholonomic** if $0 < p < k$.

For nonholonomic systems, there are fewer than k constraints on the state space q. For completely nonholonomic systems, there are NO constraints on q.

6.1.4 Equivalent Control Systems

Whereas Pfaffian constraints give you the directions that the body coordinates q cannot move, the equivalent control system gives you the directions that they can move.

To obtain the equivalent control system, we first construct the right null space of the constraints, denoted $g_j(q), j = 1, ..., n - k =: m$. That is,

$$\omega_i(q)g_j(q) = 0 \qquad i = 1, ...k \tag{6.12}$$
$$j = 1, ..., n - k \tag{6.13}$$

Then the allowable trajectories satisfying the Pfaffian constraints are the trajectories of the control system

$$\dot{q} = g_1(q)u_1 + ... + g_m(q)u_m \tag{6.14}$$

for suitably chosen inputs $u_1(\cdot), ..., u_m(\cdot), i = 1, ..., m$. This is a *drift free* control system.

6.1.5 Angular Momentum Constraints on Control

When Raibert's hopper (Fig. 6.1) is in the air, angular momentum is conserved. I is the moment of inertia of the body, and the leg mass m is concentrated at the foot. The formula for angular momentum set to zero is

$$I\dot{\theta} + m(I + d)^2(\dot{\theta} + \psi) = (I + m(I + d)^2)\dot{\theta} + m(I + d)^2\psi = 0 \tag{6.15}$$

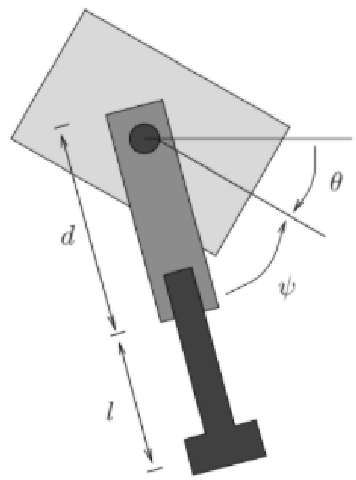

Figure 6.1: Raibert's Hopper

If $q = \psi, I, \theta)^T$, then an equivalent control system for describing it is found by finding a basis for the null space of

$$\omega_1(q) = \begin{bmatrix} m(I+d)^2 & 0 & I + m(I+d)^2 \end{bmatrix} \tag{6.16}$$

An especially convenient one is

$$\omega_1(q) = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(I+d)^2}{I+m(I+d)^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2 \tag{6.17}$$

6.2 Examples

6.2.1 Planar Space Robot

The planar space robot has mass and moment of inertia M, I in the central body. The mass of each arm is m, concentrated at the ends of the arms of length l.

(See MLS page 335 for a detailed derivation of the Lagrangian equations for the Space Robot)

The Lagrangian does not depend on the body angle θ. Hence, the statement of angular momentum conservation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0 = a_{13}(\psi)\dot{\psi}_1 + a_{23}(\psi)\dot{\psi}_2 + a_{33}(\psi)\dot{\psi}_3 \tag{6.18}$$

Setting $q = (\psi_1, \psi_2, \theta)^T$, we get the equivalent control system

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{a_{13}}{a_{33}} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ -\frac{a_{23}}{a_{33}} \end{bmatrix} u_2 \tag{6.19}$$

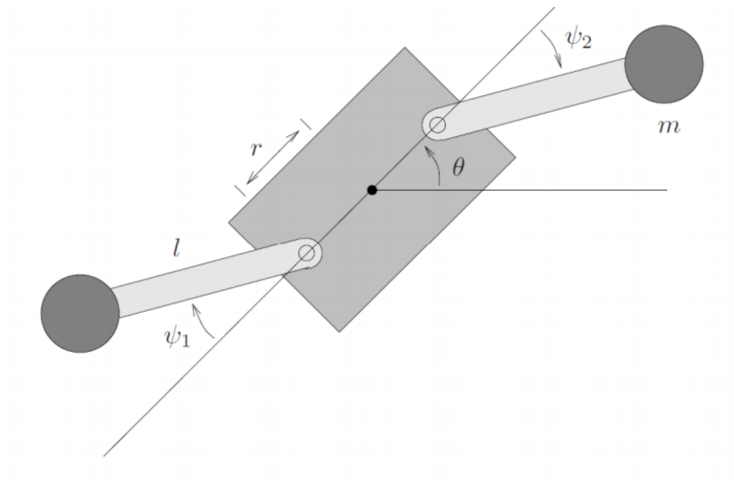

Figure 6.2: Planar Space Robot

6.2.2 Rolling Without Slipping

A second source of nonholonomy is from constraints that arise from discs; wheels which roll without slipping. Coinsider a penny rolling on a surface:

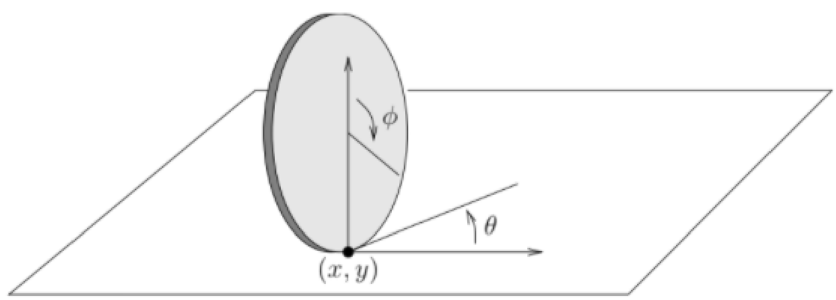

Figure 6.3: Disk Rolling on a Plane

Here, x, y are the location of the contact point on the plane. θ is the angle that the disk makes with the horizontal. ϕ is the angle made by a fixed line on the disk relative to the vertical axis. ρ is the radius of the disk.

If the disk rolls without slipping, then for $q = (x, y, \theta, \phi)^T \in \mathbb{R}^4$ we have

$$\dot{x} - \rho\cos\theta\dot{\phi} = 0 \tag{6.20}$$

$$\dot{y} - \rho\sin\theta\dot{\phi} = 0 \tag{6.21}$$

This may also be written as

$$\begin{bmatrix} 1 & 0 & 0 & -\rho\cos\theta \\ 0 & 1 & 0 & -\rho\sin\theta \end{bmatrix} \dot{q} = 0 \tag{6.22}$$

Thus, there are 2 Pfaffian constraints on $\mathbb{R}^4$. A convenient choice of control system, with $\dot{\theta} = u_1$ and $\dot{\phi} = u_2$ is

$$\dot{q} = \begin{bmatrix} \rho\text{cos}\theta \\ \rho\text{sin}\theta \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 \tag{6.23}$$

This is a two input control system with inputs u_1, u_2.

6.2.3 Front Wheel Drive Car

For the front wheel drive car, the steering angle is ϕ, the angle of the car body is θ, and the position of hte midpoint of the rear axle is x, y. This is sometimes referred to as the kinematic model of the car, and is frequently used in the analysis of self-driving cars and their motion plans.

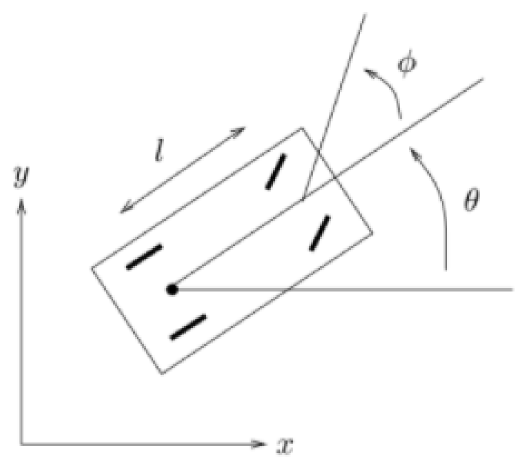

Figure 6.4: Front Wheel Drive Car

The rolling without slipping constraints for the front wheels and back wheels state that their respective velocities perpendicular to their respective directions of travel are 0:

$$\sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - I\text{cos}\phi\dot{\theta} = 0 \tag{6.24}$$
$$\text{sin}\theta\dot{x} - \text{cos}\theta\dot{y} = 0 \tag{6.25}$$

Using the steering velocity as $u_2 = \dot{\phi}$ and $q = (x, y, \theta, \phi)^T \in \mathbb{R}^4$ gives the control system

$$\dot{q} = \begin{bmatrix} \text{cos}\theta \\ \text{sin}\theta \\ \frac{1}{I}\text{tan}\phi \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 \tag{6.26}$$

u_1 is interpreted as the driving input, and u_2 as the steering input.

6.2.4 Car With N Trailers

The figure below shows a car with N trailers attached. The front hitch of each trailer is attached to the center of the rear axle of the previous trailer. The wheels and hitch of the individual trailers are aligned with the body of the trailer.

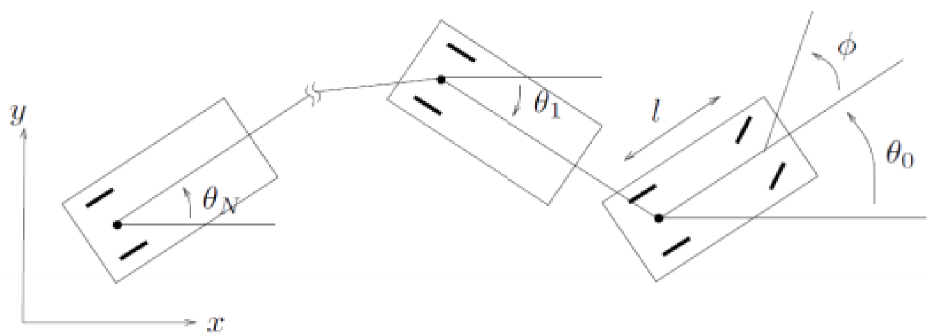

Figure 6.5: Car with N Trailers

Satisfy yourself that $q = (x, y, \phi, \theta_0, ..., \theta_N)^T \in \mathbb{R}^{N+4}$. There are $N + 2$ sets of wheels which roll without slipping to give $N + 2$ Pfaffian constraints (and still just two inputs). (See Exercise 6 in Chapter 7 of MLS).

6.2.5 Firetruck

The figure below shows a kinematic model of a fire truck. There is a driver at the front, as well as a driver at the back. It is not unlike a car with one trailer, except that the rear axle is also steerable.

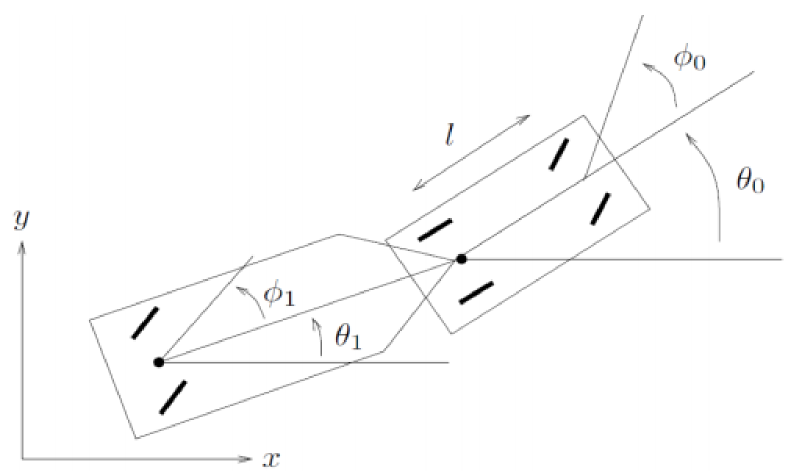

Figure 6.6: Firetruck

How many Pfaffian constraints are there? What is the dimension of q? (See Exercise 7 in Chapter 7 of MLS).

6.3 Controllability

6.3.1 The Lie Bracket

Consider the control system with $q \in \mathbb{R}^n$:

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 \tag{6.27}$$

Clearly, you can move in directions g_1, g_2 at point q, but more directions may exist. Consider the **Lie Bracket Motion**: follow $g_1, g_2, -g_1, -g_2$ respectively, each for ϵ seconds as seen in the figure:

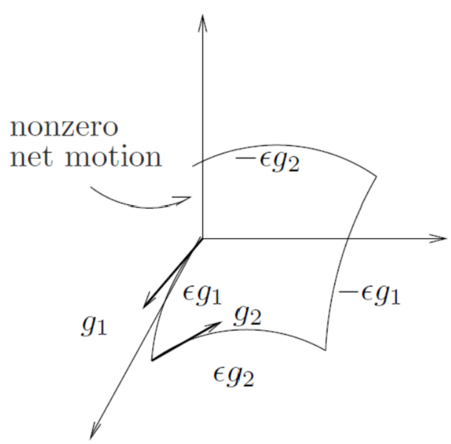

Figure 6.7: The Lie Bracket Motion

A nonzero net motion suggests that g_1, g_2 do not commute. Taylor series expansion yields:

$$q(4\epsilon) = q(0) + \epsilon^2 \left(\frac{\partial g_2}{\partial q} g_1(q_0) - \frac{\partial g_1}{\partial q} g_2(q_0) \right) + O(\epsilon^3) \tag{6.28}$$

The leading term is $O(\epsilon^3)$, and its coefficient measures the extent to which g1, g2 do not commute! This is the **Lie Bracket**:

$$[f, g](q) = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q) \tag{6.29}$$

A **Lie Product** is a nested set of Lie Brackets, e.g.:

$$[[f, g], [f[f, g]]] \tag{6.30}$$

6.3.2 Properties of Lie Brackets

Given vector fields f, g, h on $\mathbb{R}^n$ and smooth functions $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$, we have:

- Skew Symmetry

$$[f, g] = -[g, f] \tag{6.31}$$

- Jacobi Identity

$$[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0 \tag{6.32}$$

- Chain Rule

$$[\alpha f, \beta g] = \alpha\beta[f, g] + \alpha(L_f \beta)g - \beta(L_g \alpha)f \tag{6.33}$$

6.3.3 Distributions and Involutivity

A *distribution* $\triangle \subset \mathbb{R}^n$ assigns a subspace of vector fields at each $q \in \mathbb{R}^n$. Thus:

$$\triangle(q) = \text{span}\{g_1(q), ..., g_m(q)\} \tag{6.34}$$

The distribution is said to be **regular** if the dimension of the subspace $\triangle(q)$ does not vary with q. $\triangle$ is said to be **involutive** if it is closed under the Lie Bracket. That is,

$$\triangle \text{ is involutive} \qquad \Longleftrightarrow \qquad \forall f, g \in \triangle, [f, g] \in \triangle \tag{6.35}$$

6.3.4 The Frobenius Theorem

Frobenius Theorem: *A regular distribution $\triangle$ is integrable if and only if $\triangle$ is involutive.*

A regular distribution $\triangle$ of dimension p is said to be **integrable** if there exist functions $h_1, ..., h_{n-p} : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\forall f \in \triangle L_f h_i = \frac{\partial h_i}{\partial q} f(q) = L_f h_i(q) \equiv 0 \qquad\qquad i = 1, ..., (n-p) \tag{6.36}$$

Then, the manifolds M_c parameterized by $c \in \mathbb{R}^{n-k}$

$$M_c = \{h_1(q) = c_1, ..., h_{n-k}(q) = c_{n-k}\} \tag{6.37}$$

are called the integral manifolds of $\triangle$ of dimension $n - k$.

6.3.5 Frobenius Theorem and Integrability of Pfaffians

Given Pfaffian constraints $\omega_i(q)\dot{q} = 0, i = 1, ..., k$, obtain the equivalent control system:

$$\dot{q} = g_1(q)u_1 + ... + g_{n-k}(q)u_{n-k} \tag{6.38}$$

The distribution $\triangle := \text{span}\{g_1, ..., g_n - k\}$ is not necessarily involutive.

Consider the **involutive closure**, denoted $\bar{\triangle}$, which can be constructed as the set of all possible Lie brackets and Lie products of the vector fields in $\triangle$. The involutive closure is by definition involutive. Let it be regular and have dimension $p \leq (n - k)$.

By Frobenius' Theorem, $\bar{\triangle}$ is integrable. Let functions $h_1, ..., h_{n-p}$ define the integral manifolds of $\bar{\triangle}$. Note that $n - p \leq k$. Then, one of the following is true of the Pfaffian system:

- If $p = n$, the Pfaffian system is **completely nonholonomic**, i.e. there are no integral manifolds.
- If $p < n$, the Pfaffian system is **partially nonholonomic**.
- If $p = k$, the Pfaffian system is **holonomic**.

6.3.6 Reach Set

The reachable states of a nonlinear control system:

$$\dot{q} = g_1(q)u_1 + ... + g_m(q)u_m \tag{6.39}$$

from an initial state $q_0 \in V$ are defined by first defining $\mathcal{R}^V(q_0, t)$ to be the set of all states that you can steer the system to at t seconds starting from q_0 and staying inside V. Then:

$$\mathcal{R}^V(q_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}^V(q_0, t) \tag{6.40}$$

is called the **reach set**.

6.3.7 Chow's Theorem

Chow's theorem relates the reach set with the involutive closure $\bar{\triangle}$ of $\triangle = \text{span}\{g_1, ..., g_m\}$.

Chow's Theorem: *If $\bar{\triangle}(q) = \mathbb{R}^n$ for all q in a neighborhood of q_0, then $\mathcal{R}^V(q_0, \leq T)$ has a non-empty interior.*

In other words, if condition is satisfied, then the set of reachable states from q_0 has bulk/interior (i.e. is of full dimension). The condition $\bar{\triangle}(q) = \mathbb{R}^n$ is referred to as the **controllability rank condition**.