EECS 106B/206B
Robotic Manipulation and Interaction

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Goals of this lecture

Describe some of the characteristics of nonlinear systems and learn about definitions of stability on certain kinds of nonlinear systems

- Stability in the Sense of Lyapunov
- Asymptotic Stability, Exponential Stability

Learn how to prove stability

- Lyapunov's Direct Method
- Lyapunov's Indirect Method

Introduce Controlled Lyapunov Functions

SOME USEFUL MATH

Content draws from Roberto Horowitz's ME 232 Slides, some of Claire Tomlin's EE 221A notes, and Kameshwar Poolla's Linear Algebra Primer

Norms

- Two-norm

$$\|v\|_2 = \sqrt{v_1^2 + v_2^2 \cdots v_n^2}$$

- One-norm

$$\|v\|_1 = |v_1| + |v_2| \cdots |v_n|$$

- Infinity-norm

$$\|v\|_\infty = \max_i |v_i|$$

- Balls of radius n

$$B_a(x_0) := \{x \in V \mid \|x - x_0\| < a\}$$

Def of a Norm:

$$\forall x, \|x\| \geq 0 \ \ and \ \|x\| = 0 \iff x = 0$$

$$\forall x, \alpha \in \mathbb{R}, \|\alpha x\| = |\alpha| \cdot \|x\|$$

$$\forall x, y, \|x + y\| \leq \|x\| + \|y\|$$

Optimization

Linear Program (LP): Convex

$$
\begin{aligned}
&\min c^T x \\
st.\quad & Ax \leq b \\
& Cx = d
\end{aligned}
$$

Quadratic Program (QP):

$$
\begin{aligned}
&\min x^T H x + c^T x \\
st.\quad & Ax \leq b \\
& Cx = d
\end{aligned}
$$

$$
\begin{aligned}
&\min x^T H x + c^T x \\
st.\quad & x^T P_i x + q_i^T x + r_i \leq 0 \\
& Cx = d
\end{aligned}
$$

nlinear Program

$$
\begin{aligned}
&\min f(x) \\
st.\quad & g_i(x) \leq 0 \\
& h_i(x) = d
\end{aligned}
$$

Taylor Series Expansion

Any function can be locally represented by a power series

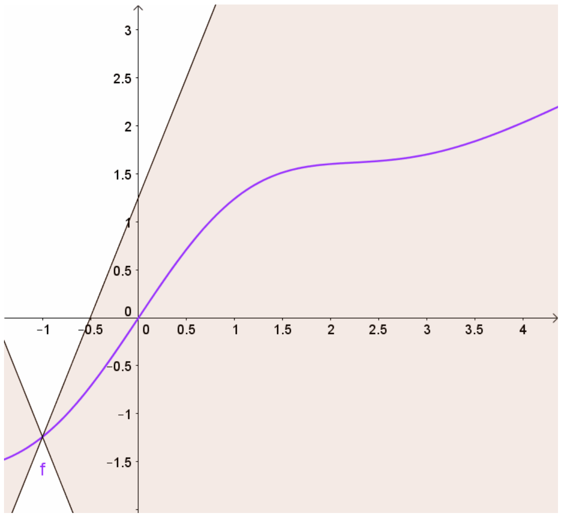
$$f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots,$$

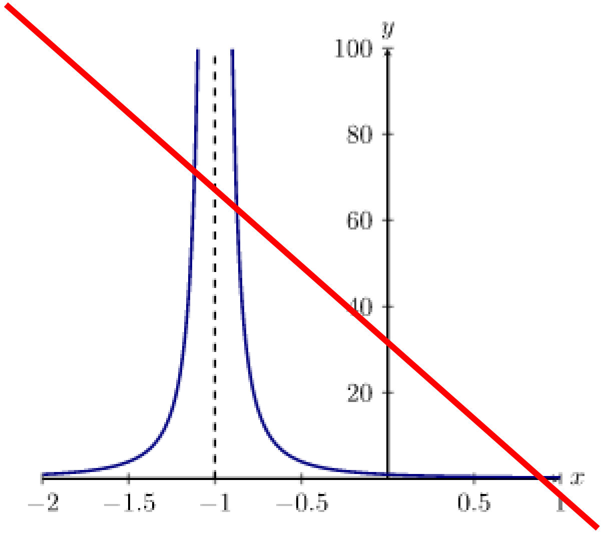
The first term of that power series is called a Linearization

$$\begin{aligned}\dot{x} &= F(x,u)\\ &= F(x_{eq},u_{eq})+\underbrace{\frac{\partial F}{\partial x}(x_{eq},u_{eq})}_{\equiv A}\cdot(x-x_{eq})+\underbrace{\frac{\partial F}{\partial u}(x_{eq},u_{eq})}_{\equiv B}\cdot(u-u_{eq})+h.o.t.\\ &\approx Ax+Bu,\end{aligned}$$

Lipschitz Continuity

- A stronger condition than continuous
- Not as strong as continuously differentiable
- Intuitively, it means that the function has a bounded slope





Figures from: Taschee, https://en.wikipedia.org/wiki/Lipschitz_continuity#/media/File:Lipschitz_Visualisierung.gif and https://ximera.osu.edu/mooculus/calculus1/asymptotesAsLimits/digInVerticalAsymptotes

INTRO TO LYAPUNOV STABILITY

Content draws from MLS 4.4 and personal notes from Koushil Sreenath’s EE 222

Nonlinear Systems

General Nonlinear System:

$$\dot{x} = F(x, u)$$

Control-Affine Nonlinear System:

$$\dot{x} = f(x) + g(x)u$$

Autonomous Nonlinear System:

$$\dot{x} = f(x)$$

Equilibrium Points

An equilibrium point is a point (x, u) satisfying

$$\dot{x}\Big|_{(x_{eq},u_{eq})} = F(x_{eq}, u_{eq}) = 0$$

What are the equilibrium points of an autonomous linear system?

- The origin
- Lines or hyperplanes corresponding to marginally stable modes

Stability of Equilibria

$$\Sigma_1 : \quad \begin{aligned} \dot{x}_1 &= -x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= x_1 - x_2(x_1^2 + x_2^2 - 1) \end{aligned}$$

$$\Sigma_3 \quad \begin{aligned} \dot{x}_1 &= -x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_2 &= x_1 + x_2(x_1^2 + x_2^2 - 1)^2 \end{aligned}$$

$$\Sigma_1 : \dot{r} = -r(r^2 - 1), \qquad \dot{\theta} = 1,$$

$$\Sigma_2 : \dot{r} = r(r^2 - 1), \qquad \dot{\theta} = 1,$$

$$\Sigma_3 : \dot{r} = r(r^2 - 1)^2, \qquad \dot{\theta} = 1.$$

Is the equilibrium point r = 0 stable for system 1, neglect θ ?

- Yes

Is the equilibrium point r = 0 stable for system 2, neglect θ ?

- No

Is the equilibrium point r = 0 stable for system 3, neglect θ ?

- No

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \\ \theta &= \tan^{-1} \tfrac{x_2}{x_1} \\ r &\geq 0 \end{aligned}$$

Types of Stability

- Stability in the Sense of Lyapunov
- Asymptotic Stability
- Exponential Stability

Stability in the Sense of Lyapunov

x_e *is called* ***stable in the sense of Lyapunov*** *if, for each* $\epsilon > 0$, *there exists some* $\delta > 0$ *such that:*

$$\|x(t, x_0) - x_e\| < \epsilon$$

whenever $\|x_0 - x_e\| < \delta$.

Is SISL a local or global condition?

Asymptotic Stability

x_e *is called* ***asymptotically stable*** *if:*

- *It is stable in the sense of Lyapunov.*
- *There exists some* $\eta > 0$ *such that when* $\|x_0 - x_e\| < \eta$, *we have* $\lim_{t\to\infty} \|x(t, x_0) - x_e\| = 0$.

If η is infinity, the system is **globally asymptotically stable**

Exponential Stability

Definition 2.37 (Exponentially Stable, Rate of Convergence). *The state $x_e \equiv 0$ is called* ***exponentially stable*** *with* ***rate of convergence*** *α if $x_e \equiv 0$ is stable, and $\exists M, \alpha > 0$ such that:*

$$\|x(t)\| \leq Me^{-\alpha(t-t_0)} \cdot |x_0|$$

Lyapunov's Direct Method

We define an "energy like function" V (a Lyapunov function) and consider its derivative

We define a set D around the origin (which we assume is an equilibrium point)

$$
\begin{aligned}
&V : \mathcal{D} \to \mathbb{R} \\
&V > 0, \qquad \forall x \in \mathcal{D}, x \neq 0 \\
&\dot{V} \leq 0, \qquad \forall x \in \mathcal{D} \qquad\qquad \implies \text{SISL} \\
&\dot{V} < 0, \qquad \forall x \in \mathcal{D}, x \neq 0 \implies \text{AS} \\
&\dot{V} < -\gamma V, \forall x \in \mathcal{D}, x \neq 0 \implies \text{ES}
\end{aligned}
$$

Proving Global Stability

In order to prove global stability, we need two additional conditions:

1. The set D must be the entire state space of the system
2. V must be *radially unbounded*, which means that

$$\|x\| \to \infty \implies V(x) \to \infty$$

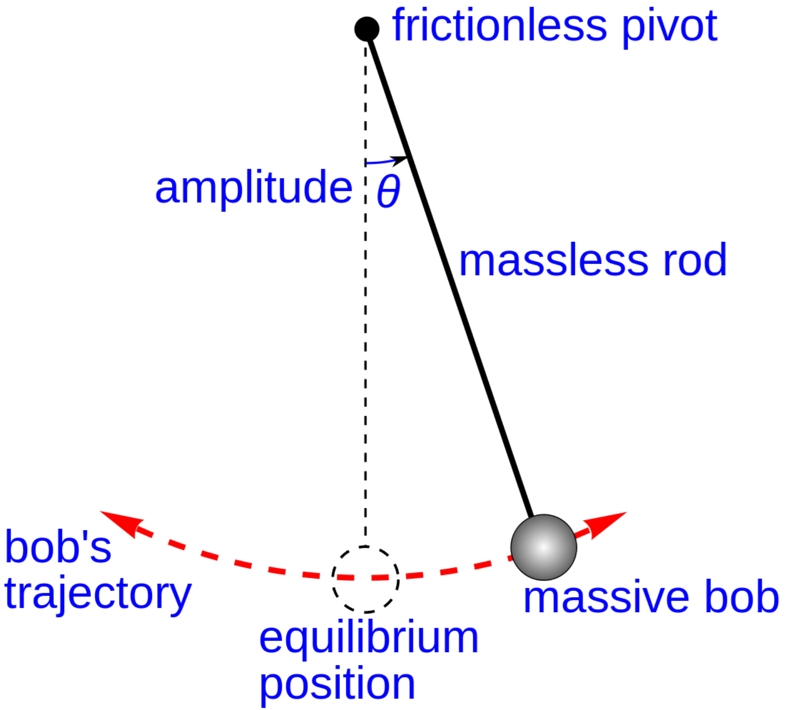
Example: Pendulum

Is a pendulum stable?

$$\ddot{\theta} = -g\sin(\theta)$$

$$PE = 1 - \cos(\theta)$$

$$KE = \dot{\theta}^2$$



Figures from:Chetvorno, https://en.wikipedia.org/wiki/Pendulum#/media/File:Simple_gravity_pendulum.svg

Example: Nonlinear Mass-Spring Damper

Is this pendulum stable?

$$\ddot{x} = -b\dot{x} - k(x)$$

$$KE = \frac{1}{2}\dot{x}^2$$

$$PE = \int_0^x k(\sigma)d\sigma$$

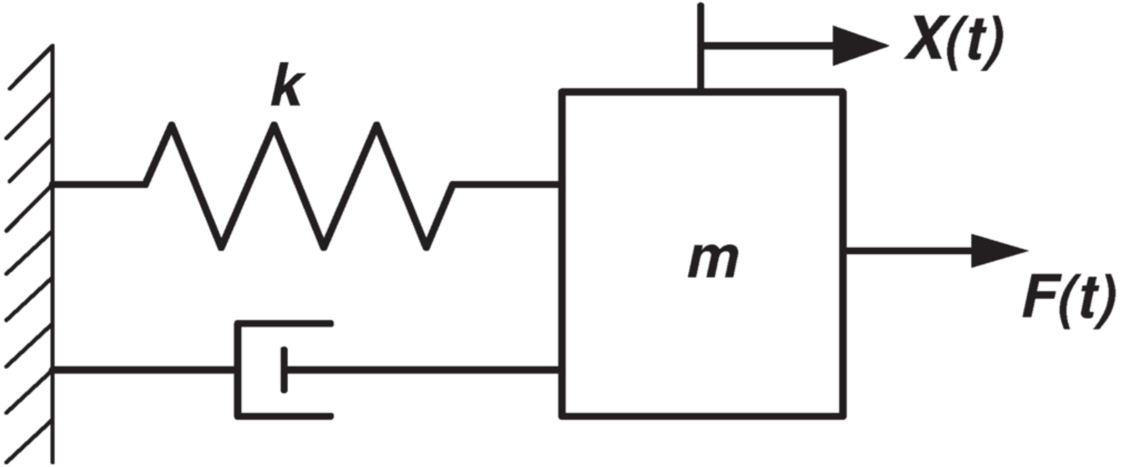


Figure from: Gómez-Aguilar, J.F.; Yépez-Martínez, H.; Calderón-Ramón, C.; Cruz-Orduña, I.; Escobar-Jiménez, R.F.; Olivares-Peregrino, V.H. Modeling of a Mass-Spring-Damper System by Fractional Derivatives with and without a Singular Kernel. *Entropy* **2015**, *17*, 6289-6303.

La Salle’s Invariance Theorem

Assume we have a function V such that

$$V(x) > 0, \forall x \in \mathcal{D}, x \neq 0$$
$$\dot{V}(x) \leq 0, \forall x \in \mathcal{D}$$

We can define a set $S := \{x \in \mathcal{D} | \dot{V}(x) = 0\}$

Suppose the only solution to our system differential equation that lies entirely within S is x ≡ 0. Then our system is asymptotically stable.

Lyapunov’s Indirect Method

Linearize the a nonlinear system Σ about an equilibrium point, and check the stability of the origin of the linear system.

- If all eigenvalues have negative real parts, Σ is *locally* ES
- If any eigenvalue has a positive real part, Σ is unstable
- If any eigenvalue has a zero real part, we cannot make any conclusions

Quadratic Lyapunov Functions

The easiest Lyapunov function to describe is a quadratic Lyapunov function

$$V = x^T P x$$

Since the quadratic Lyapunov function works for all linear systems, it will work locally for all nonlinear systems (as long as their linearizations aren't marginally stable). This should always be the first Lyapunov function that you try.

Lyapunov Stability for Driven Systems

We can define our Lyapunov function the same way, but its derivative will now depend on our input u

$$\dot{V} = L_f V + L_g V u$$

Now if we can prove

$$\forall x \in \mathcal{D}, \exists u \ s.t. \dot{V} \leq 0 \implies \text{SISL}$$

Usually we do this by defining a controller / Lyapunov function pair, and checking if they fulfill the autonomous system criteria.

Controlled Lyapunov Functions

If we have a Lyapunov function that fulfills the above condition, we don't actually need to define a controller. $\dot{V} = L_f V + L_g V u$ is a linear function with respect to u. Thus we can define

$$u = \text{argmin } u^T R u$$
$$s.t. \ \ L_f V(x) + L_g V(x) u \leq -\gamma V(x)$$

If we can prove that such an input exists for every x, we can solve this quadratic program at each time step, thus producing an exponentially stable controller.

Sources

MLS 4.4

EE 222 Notes by Frank Chiu, Valmik Prabhu, David Fridovich-Keil and Sarah Fridovich-Keil. Course taught by Koushil Sreenath