**EECS C106B - Robotic Manipulation and Interaction** **(Week 3)**

# Discussion #3 Solutions

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## Problem 1 - Lyapunov's Indirect Method: Modified Van Der Pol Oscillator

Consider the following model for an oscillator with nonlinear damping.

$$\ddot{x} + \mu(1 - x^2)\dot{x} + x = 0 \tag{0.1}$$

where $\mu$ is a scalar damping coefficient.

1. By choosing a good set of state variables, write the above model in state space form.
2. Find all equilibria of this system.
3. Linearize the system about the equilibria. Using the indirect method of Lyapunov, comment on the stability of the equilibria for the cases where $\mu > 0$ and $\mu = 0$.

**Solution:**

1. We will use $x_1 = x, x_2 = \dot{x}$ as our system state. We get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - \mu(1 - x_1^2)x_2 \end{bmatrix}$$

in state-space form. The RHS of the above is the required function $f(x)$.

2. We must equate the RHS of the state space representation to zero and solve for $x$.

$$\begin{aligned} x_2 &= 0 \\ -x_1 - \mu(1 - x_1^2)x_2 &= 0 \end{aligned}$$

From the first equation, we immediately get $x_2 = 0$. Substituting this in the bottom equation, we get $x_1 = 0$, implying that the only equilibirum of the system is the origin, $x = (0, 0)^\top$.

3. To linearize the equation near the origin, we need to approximate the system $\dot{x} = f(x)$ as a linear system $\dot{x} = Ax$ near the origin. Using a Taylor expansion near 0, we can write

$$f(x) \approx f(0) + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + \cdots$$

where the $\cdots$ hide higher order terms. Since $f(0) = 0$, we find that the approximation we seek is $f(x) = Ax$ where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

which is the Jacobian of $f$ evaluated at $x = 0$. We first compute the derivative

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + 2\mu x_1 x_2 & -\mu(1 - x_1^2) \end{bmatrix}$$

Substituting $x = (0, 0)^\top$, we get

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix}$$

We can now examine the stability of the equilibrium of the nonlinear system by studying the stability of the linearized system. We can compute the eigenvalues of the linear system by forming the quadratic charecteristic polynomial and finding its roots. We find that the two eigenvalues are

$$\lambda_1 = \frac{-\mu - \sqrt{\mu^2 - 4}}{2}$$
$$\lambda_2 = \frac{-\mu + \sqrt{\mu^2 - 4}}{2}$$

First, let's consider the case where $\mu > 0$. There are two options

(a) $\mu^2 - 4 < 0$. In this case, both eigenvalues are complex, and the real part of both eigenvalues is $-\mu/2$, which is negative.

(b) $\mu^2 - 4 > 0$. In this case, both eigenvalues are *real*. In this case, $\lambda_1 < 0$ clearly, since it is a negative number minus a positive number. We can also conclude that $\lambda_2 < 0$. Note that when the discriminant is positive, the term $\sqrt{\mu^2 - 4}$ has smaller magnitude than $\mu$, and hence $-\mu + \sqrt{\mu^2 - 4} < 0$.

In either case, the eigenvalues lie in the open left-half-plane, and hence the linearized system is exponentially stable. By the indirect method of Lyapunov, we can conclude that the equilibrium of the original nonlinear system is *locally asymptotically stable*.

In the case where $\mu = 0$, note that both eigenvalues of the linearized system lie on the imaginary axis. As such, we cannot conclude anything about the stability of the nonlinear system from the indirect method of Lyapunov alone. However, when $\mu = 0$, the dynamics of the system are in fact the *linear* dynamics of a harmonic oscillator

$$\ddot{x} = -x$$

which is easily seen to be *stable in the sense of Lyapunov*.

**Problem 2 - Lyapunov's Direct Method: Unicycle Model Robot**

Consider the following model for a unicycle model robot. The state is $(x, y, \theta)$ which represents the position of the center of the robot relative to some fixed origin along with its current heading. The control inputs are the linear velocity $v$ and the angular velocity $\omega$.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} \tag{0.2}$$

In this problem, we will explore a technique called *point-offset* control for controlling unicycle model robots like the Turtlebot. Consider a point $p$ attached rigidly to the robot at a distance $\delta$ from the center, in front of the robot (see figure 0.7). So, the position of $p$ is given by:

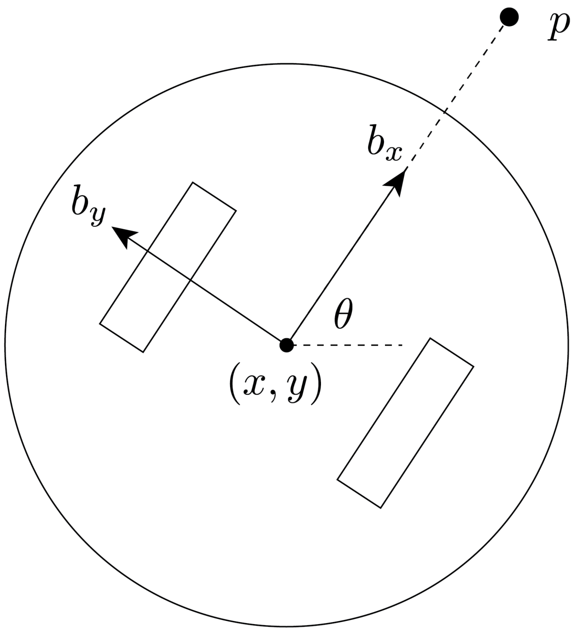
$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} x + \delta\cos\theta \\ y + \delta\sin\theta \end{bmatrix} \tag{0.3}$$

Now consider the problem of driving the turtlebot to some neighbourhood of the origin. Instead of driving the turtlebot directly, we will instead attempt to control the robot so that the point $p$ goes to the origin. Then, the turtlebot will be in a neighbourhood of radius $\delta$ around the origin. In the next few problems, we will develop a control law to drive $p$ to the origin, and prove its stability.

1. Let the body frame axes of the turtlebot be $b_x = (\cos\theta, \sin\theta)^T$ and $b_y = (-\sin\theta, \cos\theta)^T$, as shown in figure 0.7. Show that
$$\dot{p} = v b_x + \delta\omega b_y \tag{0.4}$$

2. Say we apply the following feedback control law on the robot:
$$v = -b_x^T p, \qquad \omega = -\frac{1}{\delta} b_y^T p \tag{0.5}$$
Using the Lyapunov function
$$V = \frac{1}{2} p^T p \tag{0.6}$$
show that the point $p$ converges asymptotically to the origin. Is the stability global?

3. Is it exponentially stable? If so, is the stability global?




(0.7)

**Solution:**

1. By directly differentiating equation 0.3, we get

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \begin{bmatrix} \dot{x} - \delta\dot{\theta}\sin\theta \\ \dot{y} + \delta\dot{\theta}\cos\theta \end{bmatrix} = \begin{bmatrix} v\cos\theta - \delta\omega\sin\theta \\ v\sin\theta + \delta\omega\cos\theta \end{bmatrix} = vb_x + \delta\omega b_y$$

2. Under this control law, we find that

$$\dot{p} = vb_x + \delta\omega b_y = -((b_x^\top p)b_x + (b_y^\top p)b_y)$$

Now note that $b_x^\top p$ and $b_y^\top p$ are simply the projections of $p$ onto the body-frame axes of the robot. So, the above expression in fact states that

$$\dot{p} = -p$$

i.e. the point $p$ has *linear* dynamics under the given control law. Now, we can differentiate the given candidate Lyapunov function (which is easily seen to be positive definite) to get

$$\dot{V} = p^\top \dot{p} = -p^\top p$$

which is negative definite (i.e. $\dot{V} < 0$ whenever $p \neq 0$). Hence the equilibrium $p = 0$ is asymptotically stable under the given control law. Since $V$ satisfies the required conditions globally, and is radially unbounded ($V \to \infty$ as $\|p\| \to \infty$) we can also conclude that the stability is global.

3. By the form of $\dot{V}$, it is clear that $\dot{V} = -V$, and so the stability is also globally exponential.