EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2020**

Lecture 9: Lie Bracket

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9.1 Lecture Outline

1. Continuing the introduction of Lie Bracket
2. Controllability of Lie Algebra
3. Chow's Theorem
4. Frobenius Theorem
5. Two Examples

9.2 Lie Bracket

The intuition of Lie Bracket is the resulting new direction after a sequence of alternating directions. If the the system is defined as:

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 \tag{9.1}$$

Alternating between the following input will result in a new direction, which we defined as $[g_1, g_2]$.

$$\begin{cases} u_1 = 1 \\ u_2 = 0 \end{cases} \quad \begin{cases} u_1 = 0 \\ u_2 = 1 \end{cases} \quad \begin{cases} u_1 = -1 \\ u_2 = 0 \end{cases} \quad \begin{cases} u_1 = 0 \\ u_2 = -1 \end{cases} \tag{9.2}$$

Formally, the Lie Bracket of the direction $g_1(q)$ and $g_2(q)$ is defined as:

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 \tag{9.3}$$

It captures the above sequence of movements. Intuitively, the value of Lie Bracket measures how much two directions or values are commutable. For example:

$$\begin{aligned} g_1(q) &= Aq \\ g_2(q) &= Bq \\ \rightarrow [g_1, g_2] &= B^T Aq - A^T Bq \end{aligned} \tag{9.4}$$

Therefore, the Lie Bracket measures how much matrix A and B can commute.

9.3 Controllability of Lie Algebra

Define Δ as the distribution of the Lie Bracket:

$$\Delta = span\{g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], \cdots\} \tag{9.5}$$

The dimension of the distribution satisfies:

$$dim(\Delta) \leq n \tag{9.6}$$

where n is the order of the system. Δ indicates the directions that the system can go. If $dim(\Delta) = n$, then the system can go anywhere in the space R^n, which is known as completely non-holonomic. We will discuss more about that in the following sections.

Since there are infinite Lie Brackets, we may not be able to calculate them all. A useful theorem is that if $g(q)$ is an analytical function, then after finite steps of computations, $\Delta(q)$ will saturate. This is useful in the sense that we always deal with analytical function in practice, and we know the dimension of Δ in finite time.

9.4 Chow's Theorem and Frobenius theorem

Theorem 1 *If $dim(\Delta(q)) = n$, then the system is completely non-holonomic, and the system is controllable. For any start state q_s and final state q_f, there always exists a sequence of control u that can drive the system from q_s to q_f in finite time T.*

Chow's Theorem shows that if the dimension of $\Delta(q)$ is equal to the order of the system, then system is completely nonholonomic and controllable.

Frobenius theorem, on the other hand, gives us the properties of the system when the dimension of the distribution Δ is less than n.

Definition 1 *For any $g_i(1)$ and $g_j(q) \in \Delta$, and $[g_i, g_j] \in \Delta$, then we say Δ is involutive.*

Theorem 2 *Given a distribution $\Delta(q)$ with dimension $r < n$, and it is involutive. Then $\Delta(q)$ is integrable (holonomic), and there exist $h_1(q), h_2(q), \cdots, h_{n-r}(q)$, such that $dh_i g = 0$, for all $g \in \Delta$.*

The Frobenius theorem can be interpreted as follows. There exists a manifold $M = \{h_i(q) = c_i; i = 1, \cdots, n-r\}$, such that g is in the null space of $\{dh_i; i = 1, \cdots, n-r\}$. dh_i denotes the normal direction of the manifold, and g_i is the tangent space of the manifold.

9.5 Two Examples

Example 1

There is an equation $q_1^2 + q_2^2 + q_3^2 = k$
Take derivative on both sides
$q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 = 0$
The equation can be rewrite as

$$\begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = 0$$

The control system can be written as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -q_2 \\ q_1 \\ 0 \end{bmatrix} \mathrm{u}_1 + \begin{bmatrix} 0 \\ q_3 \\ -q_2 \end{bmatrix} u_2$$

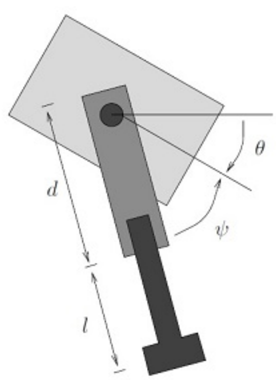
and the Lie bracket is

$$\mathrm{g}_3 = [q_1, \quad q_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -q_2 \\ q_1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ q_3 \\ -q_2 \end{bmatrix} = \begin{bmatrix} q_3 \\ 0 \\ -q_1 \end{bmatrix}$$

$Therefore, we have$

$$\Delta = span\{ \begin{bmatrix} -q_2 \\ q_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ q_3 \\ -q_2 \end{bmatrix}, \begin{bmatrix} q_3 \\ 0 \\ -q_1 \end{bmatrix} \}$$

Example 2



The total angular momentum of the robot is given by

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\psi}) = 0$$

The equation can be rewritten as

$$\begin{bmatrix} \dot{\psi} \\ \dot{l} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -m(l+d)^2/(I + m(l+d)^2) \end{bmatrix} \mathrm{u}_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

$$g_3 = [q_1, \quad q_2] = 0g_1 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2Im(l+d)/(I+m(l+d)^2)^2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -2Im(l+d)/(I+m(l+d)^2)^2 \end{bmatrix}$$

$$\Delta = span\{ \begin{bmatrix} 1 \\ 0 \\ -m(l+d)^2/(I+m(l+d)^2) \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2Im(l+d)/(I+m(l+d)^2)^2 \end{bmatrix} \}$$