EECS C106B - Robotic Manipulation and Interaction **(Week 11)**

Discussion #10

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1 Convex Optimization

A generic *optimization problem* has the following form

$$\begin{aligned}
&\min_x f_0(x) \\
&\text{subject to} \\
&\qquad f_i(x) \leq 0, \quad i = 1, \cdots, k \\
&\qquad h_j(x) = 0, \quad j = 1, \cdots m
\end{aligned}$$

where f_0 is called the *objective function*, f_i are the *inequality constraint functions* and h_j are the *equality constraint functions.* The set $\mathcal{X} = \{x : f_i(x) \leq 0, h_j(x) = 0\}$ is called the *feasible set* of the problem, and it is the set of all x that satisfy the constraints of the problem. Every such point is called a *feasible point.* The optimization problem then seeks to find a feasible point at which the value of the objective function is minimum.

A subset $S \subseteq \mathbb{R}^n$ is called *convex* if for every $x, y \in S$, the line segment connecting x and y is also fully contained in S. Precisely, we must have

$$tx + (1-t)y \in S \quad \text{for all } t \in (0,1)$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* if for every $x, y \in \mathbb{R}^n$, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \text{for all } t \in (0,1)$$

and it is called *strictly convex* if the above inequality is strict. Intuitively, this condition states that the line connecting $(x, f(x))$ and $(y, f(y))$ in $\mathbb{R}^{n+1}$ always lies above the graph of f, which is a way of formalizing the notion that convex functions are "bowl shaped".

An optimization problem is *convex* if the objective function and the inequality constraint functions are all convex, and the equality constraints are affine. Equivalently, a convex optimization problem is one that seeks to minimize a convex function over a convex feasible set. Convex optimization problems are nice, since we are guaranteed that any local optimum is also a global optimum.

For our purposes, it will suffice to note that convex problems are easy to solve and there exists software out there that will solve them for us. We will concern ourselves with two classes of convex optimization problems: linear programs (LP) and quadratic programs (QP). If we can re-write a given optimization problem in one of these forms, then we can easily solve it using convex optimization solvers. Popular choices in Python include `cvxpy` and `casadi`.

1.1 Liner programs

A linear program is one where the objective function is linear, and so are the equality and inequality constraints.

$$\begin{aligned}
&\min_x c^\top x \\
&\text{subject to} \\
&\qquad Ax \leq b \\
&\qquad Dx = e
\end{aligned}$$

where $A \in \mathbb{R}^{k \times n}$ is the stacked matrix of inequality constraints and $D \in \mathbb{R}^{m \times n}$ is the stacked matrix of equality constraints. Here, the inequality $Ax \leq b$ is understood to be element-wise.

1.2 Quadratic programs

A *quadratic function* is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = x^\top Qx + b^\top x + c$$

where $Q \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$. $b^\top x$ is called the *linear term.* When Q is positive semi-definite, f is a convex function (regardless of what b, c are). When Q is positive *definite*, f is strictly convex.

A *quadratic program* is an optimization problem where the objective function is a convex quadratic and the inequality and equality constraints are linear.

$$\begin{aligned} &\min_x x^\top Qx + b^\top x + c \\ &\text{subject to} \\ &\qquad Ax \le b \\ &\qquad Dx = e \end{aligned}$$

Once again, the inequality above is element-wise.

Problem 1

1. Show that the problem of finding the minimum norm solution of a linear equation $Ax = b$ can be written as a quadratic program.

2 Grasping

2.1 Wrenches

A *wrench* is a generalized force acting on a rigid body consists of a linear component (pure force) and an angular component (pure moment) acting at a point. A wrench is a 6D vector F that specifies the force and torques acting at the center of a particular reference frame. It is the inertial analogue of a twist. A wrench is specified as $F = (f_x, f_y, f_z, \tau_x, \tau_y, \tau_z)$, where $f = (f_x, f_y, f_z)$ is the force acting at a point and $\tau = (\tau_x, \tau_y, \tau_z)$ is the *torque vector* about that point.

2.2 Grasp map

We can define a contact as

$$F_{c_i} = B_{c_i} f_{c_i}$$

Where B is the contact basis, or the directions in which the contact can apply force, and f is a vector in that basis. F is the wrench which the contact applies. In our case, we use a soft contact model, which has both lateral and torsional friction components, so the basis is

$$B_{c_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, in the real world, friction is not infinite. For the contact to resist a wrench without slipping, the contact vector must lie within the *friction cone*, which is defined

$$FC_{c_i} = \{f \in \mathbb{R}^4 : \sqrt{f_1^2 + f_2^2} \le \mu f_3, f_3 > 0, |f_4| \le \gamma f_3\}$$

However, we want the wrenches that a contact point can resist in the world frame, not the contact frame. So we define

$$G_i := Ad^T_{g_{oc_i}^{-1}} B_{c_i}$$

A grasp is a set of contacts, so we define the wrenches (in the world frame) a grasp can resist as:

$$F_o = G_1 f_{c_1} + \cdots + G_k f_{c_k} = \begin{bmatrix} G_1 & \cdots & G_k \end{bmatrix} \begin{bmatrix} f_{c_1} \\ \vdots \\ f_{c_k} \end{bmatrix} = Gf$$

The resulting compound matrix G above is called the *grasp map.*

2.3 Force closure

A grasp is in *force closure* when finger forces lying in the friction cones span the space of object wrenches

$$G(FC) = \mathbb{R}^6$$

Essentially, this means that any external wrench applied to the object can be countered by the sum of contact forces (provided the contact forces are high enough).

For a two-contact soft-fingered grasp, we also have the following theorem which makes it very easy to check when a grasp is in force closure. This is theorem 5.7 from MLS.

Theorem. *A spatial grasp with two soft-finger contacts is force-closure if and only if the line connecting the contact point lies inside both friction cones.*

2.4 Discretizing the Friction Cone

Checking that $f \in FC$ can be difficult. Often when evaluatin grasps, we will write down an optimization problem that has $f \in FC$ as a constraint.

$$FC_{c_i} = \begin{cases} \sqrt{f_1^2 + f_2^2} \leq \mu f_3 \\ f_3 > 0 \\ |f_4| \leq \gamma f_3 \end{cases}$$

We can approximate the (conical) friction cone as a pyramid with n vertices. The level sets of the friction cone are circles, but the level sets for this convex approximation are n sided polygons circumscribed by the circle. Thus, the interior of this convexified friction cone is a conservative approximation of the friction cone itself.

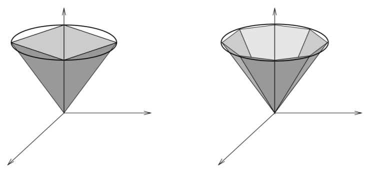

Figure 1: Approximations of the friction cone. From section 5.3 of MLS.

Any point in the interior of this pyramid can be described as a sum of

$$f = \alpha_0 f_0 + \sum_{i=1}^{n} \alpha_i f_i = F\alpha$$

where f_i are the edges of the pyramid and f_0 a straight line in z, and the weights α are all non-negative. Here, we can write a composite matrix F with the f_i vectors as its columns. This lets us more easily characterize any f in the friction cone. We make the approximation that $f \in FC$ if and only if there exists a non-negative vector α such that $f = F\alpha$. Note that here we have also not made explicit how to incorporate the torque component f_4 into our friction cone, but that is an easy extension, and one you will need to implement for your project.

With this approximation, the condition that $f \in FC$ is equivalent to the pair of linear constraints $\{f = F\alpha, \alpha \geq 0\}$ (where this inequality is understood to be element-wise).

Problem 2

Let w be a given wrench. Let a two-contact grasp be given to you with contact grasp maps G_1 and G_2. We wish to find the input force $f \in FC$ with the smallest norm that can resist the wrench w applies at the center of mass of the object being grasped. Using the polyhedral approximation of the friction cone, write this as a quadratic program.

Problem 3

Consider the box grasped by 2 soft-finger contacts shown in figure 2. Find the grasp map. Assume the object is a cube of side-length 2.

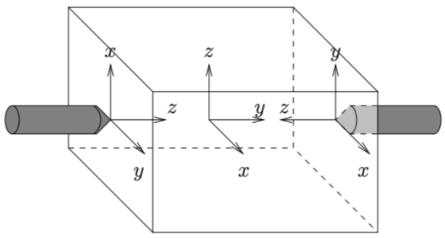

Figure 2: Two finger grasp.