Lecture 4: (Feedback Linearization, Computed Torque Control, Robust Control)

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4.1 Background Review

4.1.1 Norms

Definitionally, a norm must satisfy three properties. First, it must be nonnegative, and the norm can be 0 if and only if the vector itself is 0. Second, the norm must scale linearly. Third, the norm must obey the triangle inequality. Mathematically, these conditions can be expressed as the following:

1. $\forall x, \|x\| \geq 0; \|x\| = 0 \iff x = 0$
2. $\forall x, \alpha \in \mathbb{R}, \|\alpha x\| = |\alpha|\|x\|$
3. $\forall x, y, \|x + y\| \geq \|x\| + \|y\|$

Three commonly-used norms are defined below:

- Two-norm: $\|(v)\|_2 = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$
- One-norm: $\|(v)\|_1 = |v_1| + |v_2| + ... + |v_n|$
- ∞-norm: $\|(v)\|_\infty = \max_i |v_i|$

Using the norm, we can define a ball of radius a. Note that by convention, the ball is open, and does not include the boundary at radius a.

$$B_a(x_0) := \{x \in V \mid \|x - x_0\| < a\}$$

4.1.2 Taylor Series & Linearization

Recall that any function can be locally represented as a power series known as the Taylor Series. For x in the neighborhood of a, we can write:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + ...$$

The first derivative term of this series is used to represent the linearization of a function about a setpoint. Note that this extends naturally to higher dimensions as well. The cross terms and higher-order derivatives are collectively referred to as Higher Order Terms, and are omitted from the linearization.

$$\begin{aligned}\dot{x} = F(x,u) &= F(x_{eq},u_{eq}) + \frac{\partial F}{\partial x}(x_{eq},u_{eq})(x - x_{eq}) + \frac{\partial F}{\partial u}(x_{eq},u_{eq})(u - u_{eq}) + \text{Higher Order Terms} \\ &\approx Ax + Bu\end{aligned}$$

4.1.3 Lipschitz Continuity

Lipschitz continuity is a stronger condition than merely continuous, but not as strong as continuously differentiable. Lipschitz continuity requires that the slope at all points on the function is bounded by some finite quantity, which is known as the Lipschitz constant.

4.1.4 Nonlinear Systems

We define three kinds of nonlinear systems. A general nonlinear system has the derivative of state represented by some nonlinear function of both the state and the input. A control-affine nonlinear system is specifically linear with respect to the control u, while nonlinear overall. Finally, an autonomous nonlinear system has no input, as the name would suggest.

- General Nonlinear System: $\dot{x} = F(x,u)$
- Control-Affine Nonlinear System: $\dot{x} = f(x) + g(x)u$
- Autonomous Nonlinear System: $\dot{x} = f(x)$

4.1.5 Equilibrium Point

We define an equilibrium point as a point (x_{eq}, u_{eq}) such that $\dot{x}|_(x_{eq}, u_{eq}) = F(x_{eq}, u_{eq}) = 0$. Supposing we can represent $\dot{x} = Ax$, then the equilibria can only exist in the nullspace of A. For a non-singular A, the origin is the only equilibrium.

For the sake of example, we will analyze the stability of three equilibria for three separate systems. First, consider Σ_1 defined with the following:

$$\begin{aligned}\dot{x_1} &= -x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x_2} &= x_1 - x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

Using a polar coordinate transformation in which we define $r^2 = x_1^2 + x_2^2$ and $\theta = \arctan(\frac{x_2}{x_1})$, we get:

$$\dot{r} = -r(r^2 - 1)$$

$$\dot{\theta} = 1$$

We have reached what is seemingly a contradiction, because $\dot{\theta}$ can never be 0. In fact, the failure was in applying the arctangent operation, because arctangent is undefined when $x_1 = x_2 = 0$. Indeed, this is the only equilibrium point of the system. We also find that $r = 1$ corresponds not to an equilirbium point, but instead a limit cycle of the system.

Still, the polar coordinate transformation is useful to help us identify the stability of the $r = 0$ case; ie, the only equilibrium of the system. We see that $\dot{r} = r - r^3$, and so for small values of r, the derivative is of the same sign as the deviation. As a result, the system will continue to move away from its equilibrium point and this system is not stable.

By contrast, consider Σ_2 defined with the following polar representation:

$$\dot{r} = r(r^2 - 1) = r^3 - r$$

$$\dot{\theta} = 1$$

In this case, the derivative will be in the opposite direction and of a smaller magnitude than a small deviation r, and so the system is stable.

Finally, we consider Σ_3:

$$\dot{x_1} = -x_2 - x_1(x_1^2 + x_2^2 - 1)^2$$

$$\dot{x_2} = x_1 + x_2(x_1^2 + x_2^2 - 1)^2$$

$$\dot{r} = r(r^2 - 1)^2$$

$$\dot{\theta} = 1$$

We realize that the linear term dominates at small deviations, and the linear term will be a positive r in the radial derivative. As a result, the system is unstable, similar to the first system.

4.2 Stability

We will now consider stability more formally, focusing on three kinds of stability:

- Stability in the Sense of Lyapunov (SISL)
- Asymptotic Stability (AS)
- Exponential Stability (ES)

4.2.1 Stability in the Sense of Lyapunov

We refer to x_e as stable in the sense of Lyapunov if for any $\epsilon > 0$, there exists some $\delta > 0$ such that $\|x(t, x_0) - x_e\| < \epsilon$ for all x_0 such that $\|x_0 - x_e\| < \delta$. Since this condition is defined around a small neighborhood of the setpoint x_e, SISL is a local condition.

4.2.2 Asymptotic Stability

We can also refer to x_e as asymptotically stable if x_e is stable in the sense of Lyapunov, and there exists some $\eta > 0$ such that for all $\|x_0 - x_e\| < \eta$, $\lim_{t\to\infty} \|x_0 - x_e\| = 0$. If η is infinity, then the function is globally asymptotically stable.

4.2.3 Exponential Stability

A third form of stability not due to Lyapunov is Exponential Stability. We call x_e exponentially stable with rate of convergence α if $x_e \equiv 0$ is stable and there exists an $M, \alpha > 0$ such that $\|x(t)\| \leq Me^{-\alpha(t-t_0)}|x_0|$.

Equilibria that are exponentially stable are referred to as strong attractors, in contrast with weak attractors like those that converge at a rate of $\frac{1}{\sqrt{t}}$.

4.3 Lyapunov's Direct Method

In Lyapunov's Direct Method, we define an energy-like function V that is a Lyapunov function, as well as its derivative. An energy-like function is one that is positive definite. For the set $\mathcal{D}$ around the origin (which we require to be an equilibrium point), we have that:

$$V : \mathcal{D} \to \mathbb{R}$$
$$V > 0, \forall x \in \mathcal{D}, x \neq 0$$

Then, V is SISL if:

$$\dot{V} \leq 0, \forall x \in \mathcal{D}$$

If the inequality is strict for $x \neq 0$, then V is AS:

$$\dot{V} < 0, \forall x \in \mathcal{D}, x \neq 0$$

If V is sufficiently negative for $x \neq 0$, then V is ES with rate of convergence γ:

$$\dot{V} < -\gamma V, \forall x \in \mathcal{D}, x \neq 0$$

In order to make an argument about global stability, it is necessary for the set $\mathcal{D}$ to represent the entire state space of the system, and for V to be radially unbounded: $\|x\| \to \infty \implies \|V(x)\| \to \infty$.

While these conditions remain trivial to apply, the core difficulty is in constructing a Lyapunov function V in the first place. The modern approach involves using the Hamilton-Jacobi equations and solving the partial differential equation, though current research in the area includes attempting to use deep learning and reinforcement learning to approximate a solution.

4.3.1 Example: Simple Pendulum

We can consider the simple, frictionless pendulum in 4.1 by the following equations:

$$\ddot{\theta} = -g\sin(\theta)$$
$$PE = 1 - \cos(\theta)$$
$$KE = \dot{\theta}^2$$

Naturally, the two equilbria that satisfy $\dot{\theta} = 0$ are $\theta = 0$ and $\theta = \pi$. The situation where $\theta = \pi$ (ie, the bob is vertically upwards) is clearly unstable by virtue of a positive energy function. The situation where $\theta = 0$ (ie, the bob is vertically downwards) is stable, but not asymptotically so. This is due to the lack of friction; a small deviation will cause the pendulum to oscillate forever, and there are no dampening effects.

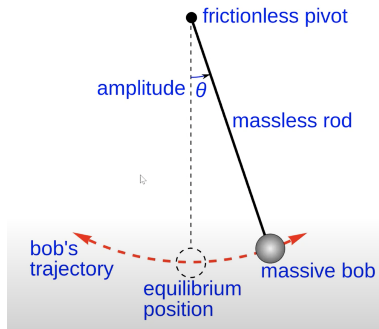

Figure 4.1: A simple pendulum

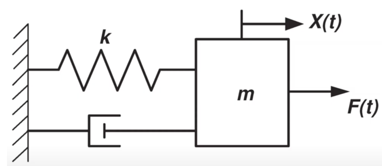

Figure 4.2: A mass-spring damper

4.3.2 Example: Nonlinear Mass-Spring Damper

We can consider another canonical example of the mass-spring damper in 4.2 with nonlinear function $k(x)$ defined by the following equations:

$$\ddot{x} = -b\dot{x} - k(x)$$
$$KE = \frac{1}{2}\dot{x}^2$$
$$PE = \int_0^x k(\alpha)d\alpha$$

In this case, assuming a well-behaved $k(x)$, the equilibrium point is where $x = 0$ (ie, unstretched spring) and we see that it is stable. It is additionally asymptotically stable by meeting the appropriate Lyapunov condition, which matches our intuition about the dampening effects of viscous friction.

4.4 La Salle's Invariance Theorem

States that we can define a set

$$S := \{x \in \mathcal{D} | \dot{V}(x) = 0\}$$

if we have a function $V(x)$ such that

$$V(x) > 0, \forall x \in \mathcal{D}, x \neq 0$$
$$\dot{V}(x) \leq 0, \forall x \in \mathcal{D}$$

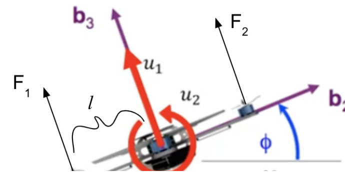

Figure 4.3: A 2D quadcopter model

Our system is asymptotically stable if the only solution to the differential equation that lies entirely within S is $x \equiv 0$.

4.5 Linearization

Given the nonlinear control system with $x \in \Re^n, u \in \Re^{n_i}$

$$\dot{x} = f(x, u)$$

constant ($x_0 \in \Re^n, u_0 \in \Re^{n_i}$) (i subscripts are inputs and 0 subscripts are outputs) is called an equilibrium pair if

$$f(x_0, u_0) = 0$$

The Jacobian linearization is taken by writing the Taylor series of $f(x, u)$ which is

$$\begin{aligned} f(x, u) &= f(x_0, u_0) + \frac{\partial f}{\partial x}(x, u)|_{(x_0, u_0)}(x - x_0) + \frac{\partial f}{\partial u}(x, u)|_{(x_0, u_0)}(u - u_0) + Tayl \\ &= D_1 f(x_0, u_0)(x - x_0) + D_2 f(x_0, u_0)(u - u_0) + Tayl \end{aligned}$$

The linearization about ($x_0 \in \Re^n, u_0 \in \Re^{n_i}$) is given by defining

$$A := D_1 f(x_0, u_0) \in \Re^{n \times n}$$

$$B := D_2 f(x_0, u_0) \in \Re^{n \times n_i}$$

$$\dot{z} = Az + Bu \qquad z := (x - x_0)$$

4.5.1 Linearization of Planar Quadcoptor

Currently, the dynamics equations for the quadcoptor shown in 4.3 are nonlinear:

$$\begin{aligned} \ddot{y} &= \frac{u_1}{m} \sin(\phi) \\ \ddot{z} &= -g + \frac{u_1}{m} \cos(\phi) \\ \ddot{\phi} &= \frac{u_2}{I_{xx}} \end{aligned}$$

with the equilibrium configuration:

$$\mathbf{q}_e = \begin{bmatrix} y_0 \\ z_0 \\ 0 \end{bmatrix}, \mathbf{x}_e = \begin{bmatrix} \mathbf{q}_e \\ \mathbf{0} \end{bmatrix}$$

but we want to convert it to the linearized model:

$$\begin{aligned} \ddot{y} &= -g\phi \\ \ddot{z} &= \frac{u_1}{m} \\ \ddot{\phi} &= \frac{u_2}{I_{xx}} \end{aligned}$$

with the equilibrium hover condition being that $\phi_0 = 0, u_{1,0} = mg, u_{2,0} = 0$. So, we'll do that using Jacobian linearization below:

$$\underset{\overline{x_e}, \overline{u_e}}{\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \end{bmatrix}} = \underset{f(x)}{\begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ -g \\ 0 \end{bmatrix}} + \underset{g_1(x) \qquad\qquad g_2(x)}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin x_3 & 0 \\ \frac{1}{m}\cos x_3 & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with $u_1 = F_1 + F_2$ and $u_2 = (F_2 - F_1)l$.

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 = f(x, u) \tag{4.1}$$

In order to find the equilibrium point, we want to set the equation above equal to zero:

$$\begin{aligned} 0 &= \dot{x_1} = \overline{x_4} \\ 0 &= \dot{x_2} = \overline{x_5} \\ 0 &= \dot{x_3} = \overline{x_6} \\ 0 &= \dot{x_4} = -\frac{1}{m}\sin \bar{x_3}\bar{u_1} \\ 0 &= \dot{x_5} = -g + \frac{1}{m}\cos \bar{x_3}\bar{u_1} \\ 0 &= \dot{x_6} = \frac{1}{I}\overline{u_2} \end{aligned}$$

$$\overline{u_2} = 0 \qquad\qquad 0 = \overline{x_4} = \overline{x_5} = \overline{x_6}$$

with no constraints on $\overline{x_1}, \overline{x_2}$. From this, we can get

$$\begin{Bmatrix} 0 = -\frac{1}{m}\sin\bar{x}_3\bar{u}_1 \\ g = \frac{1}{m}\cos\bar{x}_3\bar{u}_1 \end{Bmatrix} \implies \overline{u_1} = g$$

$$\therefore \sin\overline{x_3} = 0 \qquad \cos\overline{x_3} = 1$$

which corresponds to $\overline{x_3}$ being either 0 or π (π isn't physically realizable). Now, we have the equilibrium point

$$\begin{aligned} &\overline{x_1} && \overline{u_1} = mg \\ &\overline{x_2} && \overline{u_2} = 0 \\ &\overline{x_3} = 0 \\ &\overline{x_4} = 0 \\ &\overline{x_5} = 0 \\ &\overline{x_6} = 0 \end{aligned}$$

Now we'll take the Jacobian of equation (4.1)

$$A = D_1 f(x,u) = \frac{\partial}{\partial x}[f(x) + g_1(x)u_1 + g_2(x)u_2]$$

$$= \begin{bmatrix} \mathbf{0}^{3\times3} & \mathbf{I}^{3\times3} \\ \mathbf{0}^{3\times3} & \mathbf{0}^{3\times3} \end{bmatrix} + \begin{bmatrix} & \mathbf{0}^{3\times3} & & \mathbf{0}^{3\times3} \\ 0 & 0 & -\frac{1}{m}\cos\bar{x}_3 & \\ 0 & 0 & -\frac{1}{m}\sin\bar{x}_3 & \mathbf{0}^{3\times3} \\ 0 & 0 & 0 & \end{bmatrix}\overline{u_1} + 0\overline{u_2}$$

$$= \begin{bmatrix} & \mathbf{0}^{3\times3} & & \mathbf{I}^{3\times3} \\ 0 & 0 & -\frac{1}{m}\overline{u_1} & \\ 0 & 0 & 0 & \mathbf{0}^{3\times3} \\ 0 & 0 & 0 & \end{bmatrix} = \begin{bmatrix} & \mathbf{0}^{3\times3} & & \mathbf{I}^{3\times3} \\ 0 & 0 & -g & \\ 0 & 0 & 0 & \mathbf{0}^{3\times3} \\ 0 & 0 & 0 & \end{bmatrix}$$

$$B = D_2 f(x,u) = \frac{\partial}{\partial u}[f(x) + g_1(x)u_1 + g_2(x)u_2]$$

$$= \begin{bmatrix} g_1(\overline{x}) & g_2(\overline{x}) \end{bmatrix} |\overline{x_3} = 0$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{I} \end{bmatrix}$$

Using both the definitions of the A and B matrices, we have

$$\dot{z} = Az + Bu$$

$$= \begin{bmatrix} & \mathbf{0}^{3\times 3} & & \mathbf{I}^{3\times 3} \\ 0 & 0 & -g & \\ 0 & 0 & 0 & \mathbf{0}^{3\times 3} \\ 0 & 0 & 0 & \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{I} \end{bmatrix} u$$

4.5.2 Lyapunov's Indirect Method Stability from Linearization

When you linearize a nonlinear system Σ around an equilibrium point, the linear terms overpower the nonlinear ones around that point. You can use the eigenvalues of the resulting system to check for the stability of the origin of the linear system:

$\mathfrak{Re}(\lambda_i) < 0 \;\; \forall i \in (1, \ldots, n) \implies \Sigma$ is locally ES

$\mathfrak{Re}(\lambda_i) > 0 \;\; \exists i \in (1, \ldots, n) \implies \Sigma$ is unstable

$\mathfrak{Re}(\lambda_i) = 0 \;\; \exists i \in (1, \ldots, n)$ (i.e. they're on the $j\omega$ axis) $\implies$ we cannot make conclusions.

4.6 Quadratic yapunov Functions

The quadratic function $V = x^T P x$ should always be the first Lyapunov function that you try, since it'll work locally for nonlinear systems as long as their linearizations aren't marginally stable.