EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2020**

Lecture 4: (Nonlinear Systems and Lyapunov Stability)

Scribes: William Mullen and Gyanendra Tripathi

4.1 Math Review

We begin by covering some useful math before launching into our discussion of Lyapunov stability.

4.1.1 Norms

At its core, the norm is a measure of the size of a vector. In practice, many different norms exist. For this section we will focus on three: the two-norm, the one-norm, and the infinity-norm. We will also discuss different graphical representations for each.

First, we formalize the definition of a norm. A norm must satisfy the following properties:

1. $\forall x, ||x|| \geq 0$ and $||x|| = 0 \iff x = 0$
2. $\forall x, \alpha \in \mathbb{R}, ||\alpha x|| = |a| \cdot ||x||$
3. $\forall x, y, ||x + y|| \leq ||x|| + ||y||$

4.1.1.1 Two-norm

The two-norm is often used and is usually known as the magnitude of a vector:

$$||v||_2 = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$$

4.1.1.2 One-norm

The one-norm is defined as follows:

$$||v||_1 = |v_1| + |v_2| + ... + |v_n|$$

4.1.1.3 Infinity-norm

The infinity-norm is defined as follows:

$$||v||_\infty = \max_i |v_i|$$

4.1.1.4 Balls of Radius n

We can graph each of these norms in a two-dimensional setting. We define a ball of radius n centered at x_0 as follows:

$$B_n(x_0) := x \in V \mid ||x - x_0|| < n$$

We can graph our three different norms for a geometric representation as shown in Fig 1. The outer square denotes the ball for $||x||_\infty = 1$, the inscribed circle is a ball for $||x||_2 = 1$, and the inner diamond is for $||x||_1 = 1$.

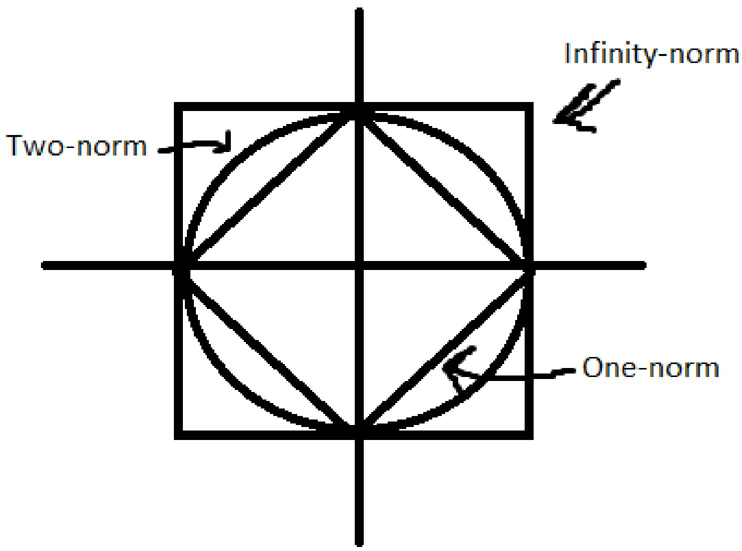

Fig. 1

4.1.2 Optimization

We now briefly discuss four different optimization problems:

1. Linear Program (Convex): $\min c^T x$ st. $Ax \leq b, Cx = d$
2. Quadratic Program (Convex): $\min x^T Hx + c^T x$ st. $Ax \leq b, Cx = d$
3. QCQP (Not convex): $\min x^T Hx + c^T x$ st. $x^T P_i x + q_i^T x + r_i \leq 0, Cx = d$
4. Nonlinear Program (Not Convex): $\min f(x)$ st. $g_i(x) \leq 0, h_i(x) = d$

While this set of notes will not go into detail on each of these scenarios, it is important that we want to have either of the first two situations. Since Linear and Quadratic Programs are convex, we can tell if a solution exists in finite time. In the third and fourth situations we have no guarantees about the solution, making them much less useful.

4.1.3 Taylor Series Expansion

We briefly cover Taylor Series Expansions. Any function can be locally represented by a power series:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... \tag{4.1}$$

The first term of that power series is called a Linearization:

$$\begin{aligned} \dot{x} &= F(x,u) \\ &= F(x_{eq}, u_{eq}) + \underbrace{\frac{\partial F}{\partial x}(x_{eq}, u_{eq})}_{\text{A}} \cdot (x - x_{eq}) + \underbrace{\frac{\partial F}{\partial u}(x_{eq}, u_{eq})}_{\text{B}} \cdot (u - u_{eq}) + h.o.t. \end{aligned} \tag{4.2}$$

$$= Ax + Bu$$

Where $h.o.t.$ stands for higher order terms.

4.1.4 Lipschitz Continuity

Lipschitz continuity is intuitively defined as a function with a bounded slope. See Fig. 2 for an example of a function that satisfies this condition on the left and one that does not on the right. It is a stronger condition than pure continuity but not as strong as continuously differentiable. The functions discussed in this document will all be Lipschitz continuous.

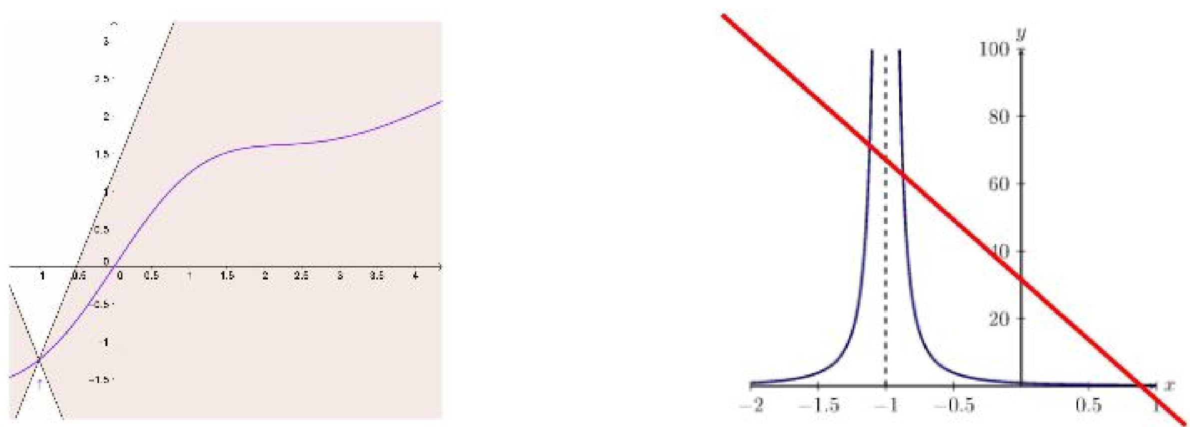

Fig. 2[1]

4.2 Introduction to Lyapunov Stability

We now tackle an introduction to Lyapunov Stability. First, we introduce some basic information about nonlinear systems, then discuss various types of stability before closing with Lyapunov's Direct and Indirect Methods.

[1]Figures from: Taschee, `https://en.wikipedia.org/wiki/Lipschitz_continuity#/media/File:Lipschitz_Visualisierung.gif` and `https://ximera.osu.edu/mooculus/calculus1/asymptotesAsLimits/digInVerticalAsymptotes`

4.2.1 Nonlinear Systems

There are three different types of nonlinear systems:

1. General Nonlinear System: $\dot{x} = F(x, u)$
2. Control-Affine Nonlinear System: $\dot{x} = f(x) + g(x)u$
3. Autonomous Nonlinear System: $\dot{x} = f(x)$

In general, most nonlinear systems will be control-affine, however in few cases where they are not the system can often be approximated to that form. Say you have an autonomous nonlinear system $\dot{x} = u^2$. We can define 2 states $z = \begin{bmatrix} x \\ u \end{bmatrix}$. Taking the derivative we get:

$$\dot{z} = \begin{bmatrix} u^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \tag{4.3}$$

which is now control-affine with respect to v.

4.2.2 Equilibrium Points

We define an equilibrium point to be a point (x, u) satisfying

$$\dot{x}|_{(x_{eq}, u_{eq})} = F(x_{eq}, u_{eq}) = 0 \tag{4.4}$$

In less mathematical terms, an equilibrium point is a point at which a system is stable enough to converge. In an autonomous linear system, the equilibrium points can be the following:

- The origin.
- Lines or hyperplanes corresponding to marginally stable modes (such as the null space).

4.2.2.1 Examples

Here are three systems that we will use to demonstrate the connection between stability and equilibrium points.

$$\begin{aligned} \Sigma_1 &: \dot{r} = -r(r^2 - 1), \quad \dot{\theta} = 1 \\ \Sigma_2 &: \dot{r} = r(r^2 - 1), \quad \dot{\theta} = 1 \\ \Sigma_3 &: \dot{r} = r(r^2 - 1)^2, \quad \dot{\theta} = 1 \end{aligned} \tag{4.5}$$

For each of the systems, we want to know if the equilibrium point $r = 0$ is stable. In system 1, plugging in a slightly positive number results in a positive number, immediately demonstrating the system is unstable. Next, in system 2, plugging in a slightly positive number results in a negative number while a slightly negative number results in a positive output. Thus, the system is stable. Finally, plugging a slightly positive number into system 3 results in a very small positive output. Therefore the system will not rapidly expand like system 1 does but also is not stable; thus the system is semi-stable Careful readers might note these systems are actually undefined at $\theta = 0$ since arctan at $x = 0, y = 0$ is undefined. However, by defining $\dot{x} = f_1(x, y), \dot{y} = f_2(x, y) \implies x = 0, y = 0$. Thus, it is okay since this situation is essentially the same as when $r = 0$.

4.3 Stability

We now launch fully into our discussion of Lyapunov Stability. We will cover three types of stability and then end with a discussion of Lyapunov's direct and indirect methods.

4.3.1 Stability in the Sense of Lyapunov

We say x_e is **stable in the sense of Lyapunov** if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that:

$$||x(t, x_0) - x_e|| < \epsilon \tag{4.6}$$

whenever $||x_0 - x_e|| < \delta$. Said another way, if you start in circle δ in Fig. 3, you will never leave circle ε. Here (and in the rest of these notes), we assume the origin is the equilibrium point. Another important fact to note is that SISL is not global condition. As a result, δ and ε should always be minimized to provide the strongest possible bounds. For example, if δ_1 and ε_1 are smaller than δ_2 and ε_2, then if x_e is SISL with respect to δ_1 and ε_1, then it must also be SISL with respect to δ_2 and ε_2 by inspection.

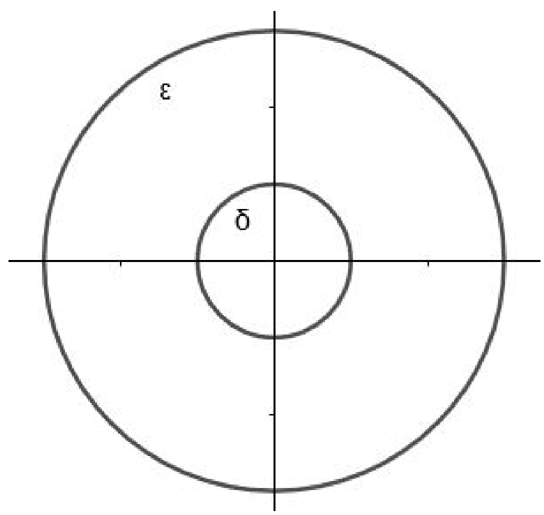

Fig. 3

4.3.2 Asymptotic Stability

x_e is **asymptotically stable** if:

- It is stable in the sense of Lyapunov.
- There exists some $\nu > 0$ such that when $||x_0 - x_e|| < \nu$, $\lim_{t\to\infty} ||x(t, x_0) - x_e|| = 0$.

Additionally, if $\nu = \infty$, then the system is globally asymptotically stable.

4.3.3 Exponential Stability

Finally, the state $x_e \equiv 0$ is called **exponentially stable** with **rate of convergence** α if $x_e \equiv 0$ is stable and $\exists M, \alpha > 0$ such that $||x(t)|| \leq Me^{-\alpha(t-t_0)} \cdot |x_0|$.

4.4 Lyapunov's Direct Method

Having defined various types of stability, we now turn to Lyapunov's direct method. We define an energy function V (a Lyapunov function) and consider its derivative. We also define a set D around the origin (which we assume is an equilibrium point). Then, the following holds true:

$$\begin{aligned}
&V : D \to \mathbb{R} \\
&V > 0, \forall x \in D, x \neq 0 \\
&\dot{V} \leq 0, \forall x \in D \implies SISL \\
&\dot{V} < 0, \forall x \in D, x \neq 0 \implies AS \\
&\dot{V} \leq -\gamma V, \forall x \in D, x \neq 0 \implies ES
\end{aligned} \tag{4.7}$$

4.4.1 Level Sets and Convergence

We also define a level set of a function $V(x)$ to be $\{x|V(x) = r\}$. We can demonstrate the importance of the level set in the following example and Fig. 4. Let D be a ball as shown below and α, β be two different level sets. We can see α is not fully contained inside D. As a result, even though most of α is within D, we cannot provide any stability guarantees for this level set. Conversely, because β is completely within D, we can guarantee stability. In this case β is the largest level set that fits completely within D, we call this the **Region of Convergence**.

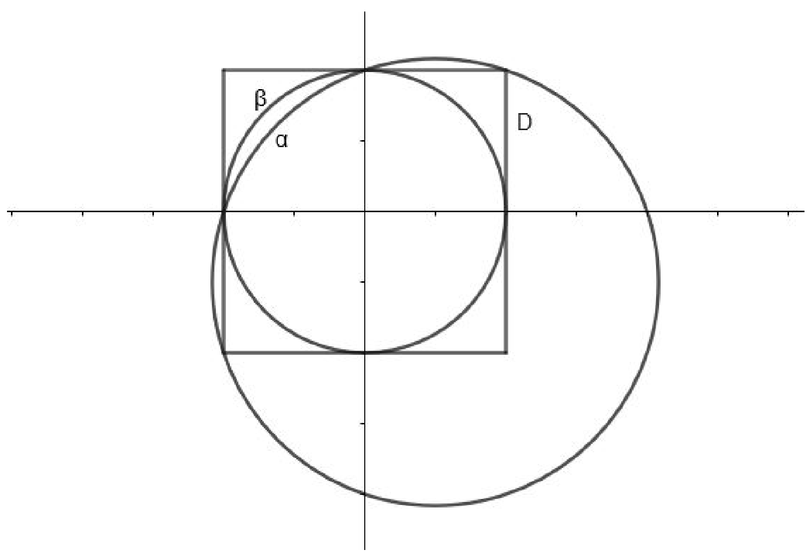

Fig. 4

4.4.2 Proving Global Stability

To prove gloabl stability, two additional conditions must hold:

1. The set D must be the entire state space of the system.
2. V must be *radially unbounded*, which means that $||x|| \to \infty \implies V(x) \to \infty$.

4.4.3 Pendulum Example

We end our discussion of Lyapunov's Direct Method with an example using a pendulum. Consider the pendulum below in Fig. 5.

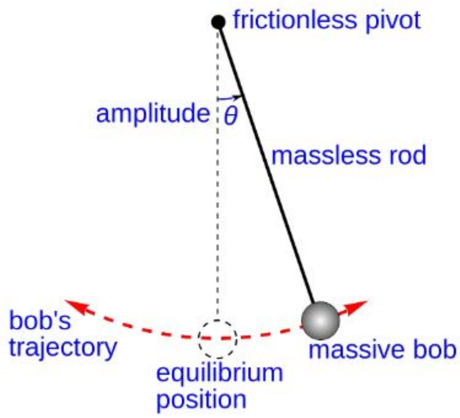

Fig. 5 [2]

We want to determine if the pendulum is stable. First we establish our potential and kinetic energy functions:

$$\ddot{\theta} = -g\sin\theta \tag{4.8}$$
$$PE = 1 - \cos\theta \tag{4.9}$$
$$KE = \dot{\theta}^2 \tag{4.10}$$

For purposes of this example, lets use just the potential energy as our Lyapunov function. Thus:

$$x = -g\sin\theta \tag{4.11}$$
$$V = g(1 - \cos\theta) \tag{4.12}$$
$$D = x \in (-2\pi, 2\pi) \tag{4.13}$$

Finally, we solve for $\dot{V}$ to apply Lyapunov's Direct Method:

$$\dot{V} = \frac{\partial V}{\partial \theta}\dot{\theta} + \frac{\partial V}{\partial \theta}\ddot{\theta} = g\sin(\theta)\dot{\theta} \tag{4.14}$$
$$\dot{V}(\frac{\pi}{4}, 1) = g\frac{\sqrt{2}}{2} \tag{4.15}$$
$$\dot{V} \nleq 0 \in D \implies \text{Failure, must try indirect.} \tag{4.16}$$

Thus we see a problem with Lyapunov's Direct Method. If the chosen V fails any of the conditions, a new one must be selected to determine stability. However, there is no easy way beyond guessing to determine what is a valid V. Thus, we turn to Lyapunov's Indirect Method. In this example, adding the kinetic energy of the system back into V makes the function pass the required conditions.

[2]Figure from:Chetvorno, `https://en.wikipedia.org/wiki/Pendulum#/media/File:Simple_gravity_pendulum.svg`

4.5 Lyapunov's Indirect Method

Lyapunov's Indirect Method offers an alternate way to determine the stability of a system. Say we have a nonlinear system Σ. We can linearlize Σ about the equilibrium point and check the system's eigenvalues:

- If all eigenvalues have negative real parts, Σ is locally exponentially stable.
- If any eigenvalue has a positive real part, Σ is unstable.
- If any eigenvalue has a zero real part, we cannot make any conclusions.

Lyapunov's Indirect Method relies on the Taylor expansion of the function:

$$\dot{x} = f(x_e q, u_e q) + \underbrace{Ax}_{\text{This term dominates}} + \text{...higher order terms} \tag{4.17}$$

Since the Ax term dominates, finding the eigenvalues results in a test for stability. However the weakness of the Indirect Method versus the Direct Method is that the Indirect Method does not provide a radius of convergence, thus there is no way to determine how close the stable region is to the equilibrium point.

4.6 Quadratic Lyapunov Function

We conclude these notes with a brief mention of the Quadratic Lyapunov Function. This is the easiest Lyapunov function to describe:

$$V = x^T P x \tag{4.18}$$

Since the quadratic Lyapunov function works for all linear systems, it will provide an indication of local stability for all nonlinear systems (provided their linearizations are not marginally stable). This should always be the first Lyapunov function tried.