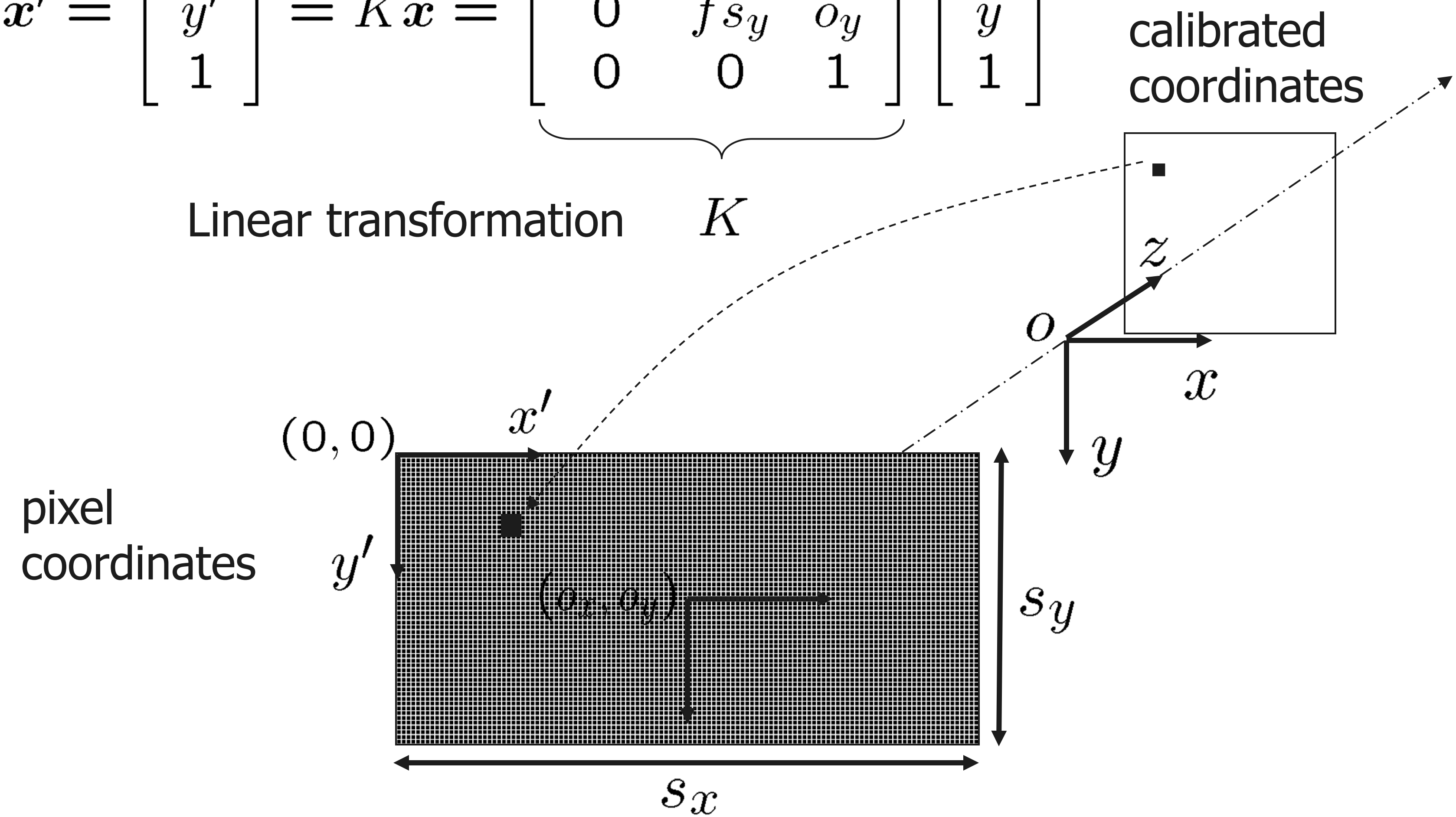
Lecture 5
Uncalibrated Geometry
& Stratification

Uncalibrated Camera

$$\boldsymbol{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = K\boldsymbol{x} = \underbrace{\begin{bmatrix} fs_x & fs_\theta & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix}}_{K} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Linear transformation K

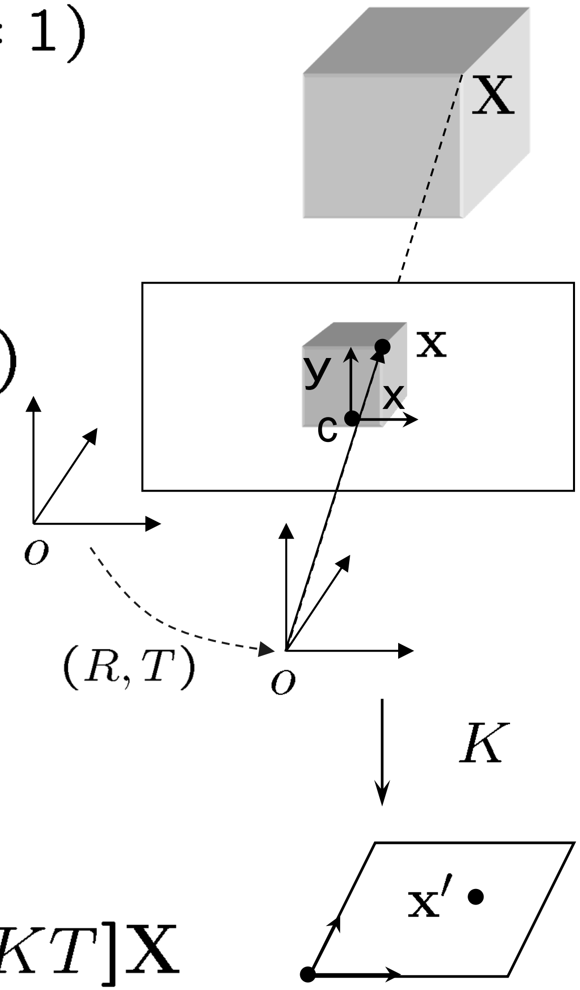


Overview

- Calibration with a rig
- Uncalibrated epipolar geometry
- Ambiguities in image formation
- Stratified reconstruction
- Autocalibration with partial scene knowledge

Uncalibrated Camera

$$\mathbf{X} = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1)$$



Calibrated camera

- Image plane coordinates $\mathbf{x} = [x, y, 1]^T$
- Camera extrinsic parameters $g = (R, T)$
- Perspective projection $\lambda \mathbf{x} = [R, T]\mathbf{X}$

Uncalibrated camera

- Pixel coordinates $\mathbf{x}' = K\mathbf{x}$
- Projection matrix $\lambda \mathbf{x}' = \Pi \mathbf{X} = [KR, KT]\mathbf{X}$

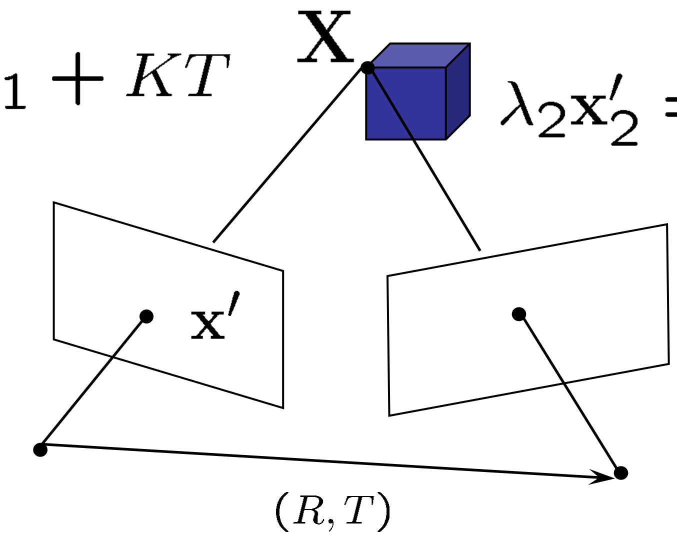
Taxonomy on Uncalibrated Reconstruction

$$\lambda \mathbf{x}' = [KR, KT]\mathbf{X}$$

- K is known, back to calibrated case $\quad \mathbf{x} = K^{-1}\mathbf{x}'$
- K is unknown
  - Calibration with complete scene knowledge (a rig) – estimate K
  - Uncalibrated reconstruction despite the lack of knowledge of K
  - Autocalibration (recover K from uncalibrated images)
- Use partial knowledge K
  - Parallel lines, vanishing points, planar motion, constant intrinsic
- Ambiguities, stratification (multiple views)

Uncalibrated Epipolar Geometry

$$\lambda_2 K \mathbf{x}_2 = K R \lambda_1 \mathbf{x}_1 + K T$$



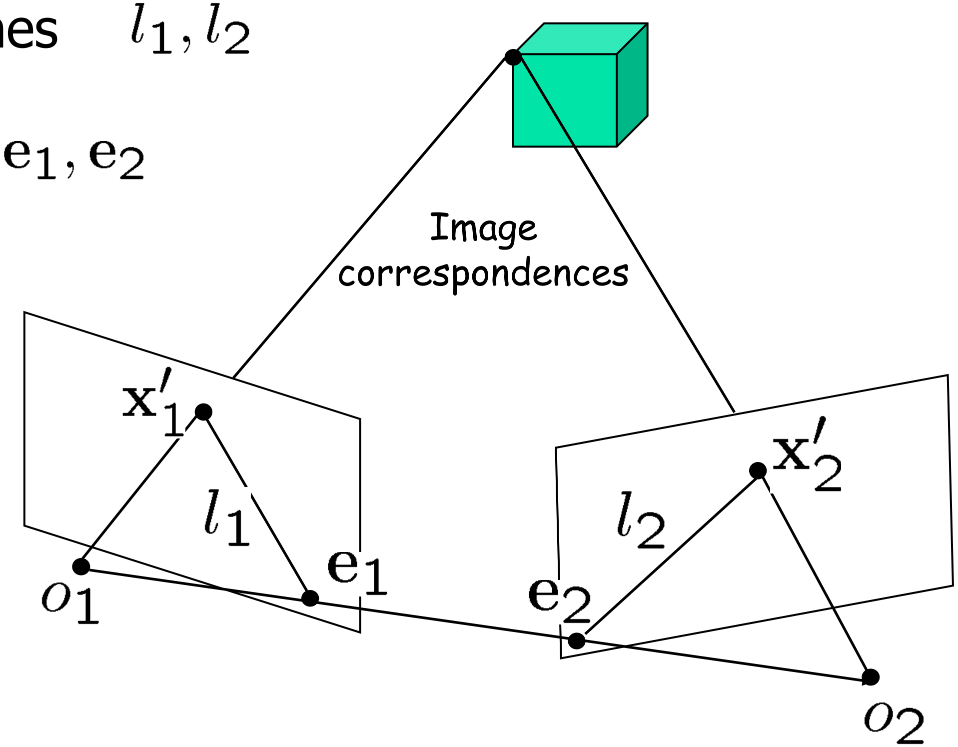
$$\lambda_2 \mathbf{x}'_2 = K R K^{-1} \lambda_1 \mathbf{x}'_1 + T'$$

- Epipolar constraint $\quad {\mathbf{x}'}^T_2 \underbrace{K^{-T} \hat{T} R K^{-1}} {\mathbf{x}'}_1 = 0$
- Fundamental matrix $\quad F = K^{-T} \hat{T} R K^{-1}$
- Equivalent forms of $\quad F = K^{-T} \hat{T} R K^{-1} = \hat{T}' K R K^{-1}$

Properties of the Fundamental Matrix

$$\mathbf{x}'^T_2 F \mathbf{x}'_1 = 0$$

- Epipolar lines l_1, l_2
- Epipoles $\mathbf{e}_1, \mathbf{e}_2$



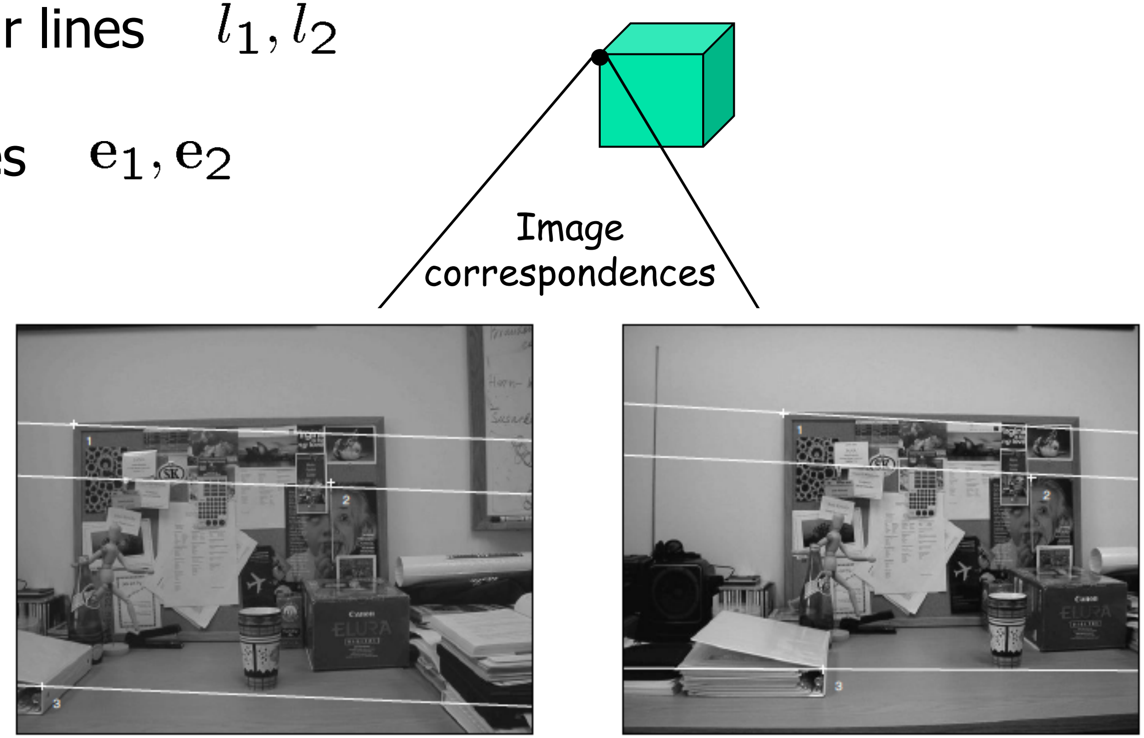
$$l_1 \sim F^T \mathbf{x}'_2 \qquad l_i^T \mathbf{x}'_i = 0 \qquad l_2 \sim F \mathbf{x}'_1$$

$$F \mathbf{e}_1 = 0 \qquad l_i^T \mathbf{e}_i = 0 \qquad \mathbf{e}_2^T F = 0$$

Properties of the Fundamental Matrix

$$\mathbf{x'}_2^T F \mathbf{x'}_1 = 0$$

- Epipolar lines l_1, l_2
- Epipoles $\mathbf{e}_1, \mathbf{e}_2$



$l_1 \sim F^T \mathbf{x}'_2$	$l_i^T \mathbf{x}'_i = 0$	$l_2 \sim F \mathbf{x}'_1$
$F\mathbf{e}_1 = 0$	$l_i^T \mathbf{e}_i = 0$	$\mathbf{e}_2^T F = 0$

Properties of the Fundamental Matrix

A nonzero matrix $F \in \mathbb{R}^{3\times 3}$ is a fundamental matrix if F has a singular value decomposition (SVD) $F = U\Sigma V^T$ with

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, 0\}$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}_+$.

There is little structure in the matrix F except that

$$\det(F) = 0$$

Estimating Fundamental Matrix

- Find such F that the epipolar error is minimized

$$min_F \sum_{j=1}^{n} \mathbf{x}_2'^{jT} F \mathbf{x}_1'^{j} \longleftarrow \text{Pixel coordinates}$$

- Fundamental matrix can be estimated up to scale

- Denote $\mathbf{a} = \mathbf{x}_1' \otimes \mathbf{x}_2'$

$$\mathbf{a} = [x_1x_2, x_1y_2, x_1z_2, y_1x_2, y_1y_2, y_1z_2, z_1x_2, z_1y_2, z_1z_2]^T$$

$$F^s = [f_1, f_4, f_7, f_2, f_5, f_8, f_3, f_6, f_9]^T$$

- Rewrite

$$\mathbf{a}^T F^s = 0$$

- Collect constraints from all points

$$\chi F^s = 0$$

$$min_F \sum_{j=1}^{n} \mathbf{x}_2^{jT} F \mathbf{x}_1^{j} \Longrightarrow min_{F^s} \|\chi F^s\|^2$$

Two view linear algorithm – 8-point algorithm

- Solve the LLSE problem:

$$min_F \sum_{j=1}^{n} \mathbf{x}_2^{\prime jT} F \mathbf{x}_1^{\prime j} \Rightarrow \chi F^s = 0$$

- Solution eigenvector associated with smallest eigenvalue of $\chi^T \chi$

- Compute SVD of F recovered from data

$$F = U \Sigma V^T \quad \Sigma = diag(\sigma_1, \sigma_2, \sigma_3)$$

- Project onto the essential manifold:

$$\Sigma' = diag(\sigma_1, \sigma_2, 0) \quad F = U \Sigma' V^T$$

- F cannot be unambiguously decomposed into pose and calibration

$$F = K^{-T} \hat{T} R K^{-1}$$

Calibrated vs. Uncalibrated Space

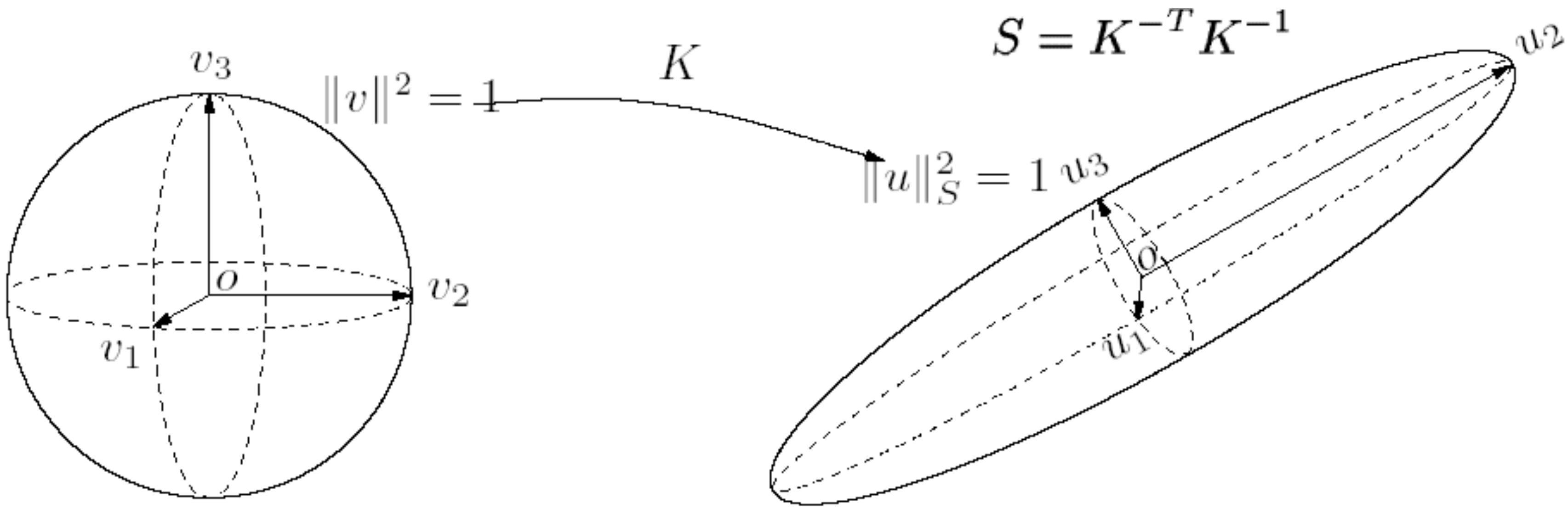
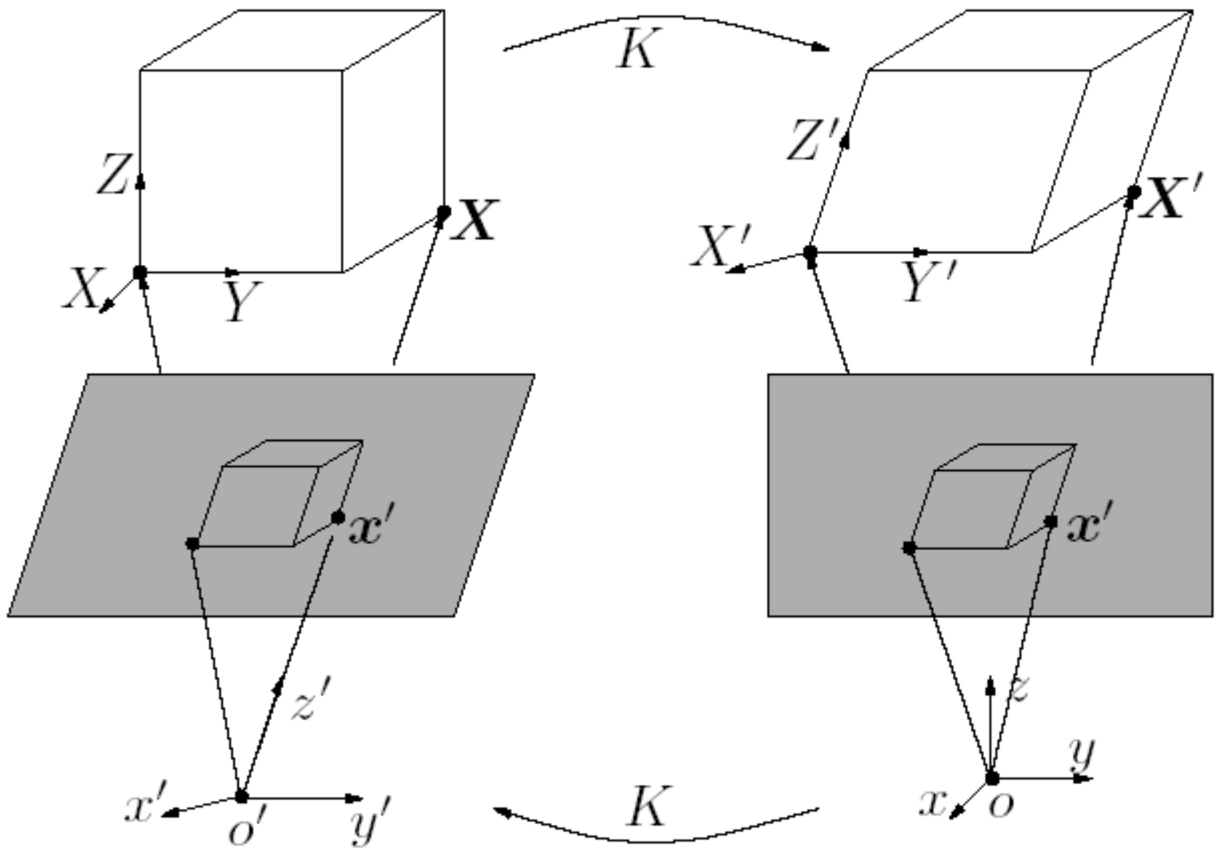


Figure 6.1. Effect of the matrix K as a map $K : v \mapsto u = Kv$, where points on the sphere $\|v\|^2 = 1$ is mapped to points on an ellipsoid $\|u\|_S^2 = 1$ (a "unit sphere" under the metric S). Principal axes of the ellipsoid are exactly the eigenvalues of S.

Calibrated vs. Uncalibrated Space



Distances and angles are modified by S

Motion in the distorted space

$$\mathbf{X}(t) = R(t)\mathbf{X}(t_0) + T(t)$$

$$K\mathbf{X}(t) = KR(t)\mathbf{X}(t_0) + KT(t)$$

Calibrated space

Uncalibrated space

$$\mathbf{X}(t) = R(t)\mathbf{X}(t_0) + T(t)$$

$$\mathbf{X}'(t) = KR(t)K^{-1}\mathbf{X}'(t_0) + KT(t)$$

- Uncalibrated coordinates are related by

$$G' = \left\{ g' = \begin{bmatrix} KRK^{-1} & T' \\ 0 & 1 \end{bmatrix} \mid T' \in \Re^3, R \in SO(3) \right\}$$

- Conjugate of the Euclidean group

What Does F Tell Us?

- F can be inferred from point matches (eight-point algorithm)
- Cannot extract motion, structure and calibration from one fundamental matrix (two views)
- F allows reconstruction up to a projective transformation (as we will see soon)
- F encodes all the geometric information among two views when no additional information is available

Decomposing the Fundamental Matrix

$$F = K^{-T}\widehat{T}RK^{-1} = \widehat{T}'KRK^{-1}$$

- Decomposition of the fundamental matrix into a skew symmetric matrix and a nonsingular matrix

$$F \mapsto \Pi = [R', T'] \quad \Rightarrow \quad F = \widehat{T'}R'.$$

- Decomposition of F is not unique

$$\mathbf{x}_2'^{}\widehat{T}'(T'v^T + KRK^{-1})\mathbf{x}_1' = 0 \quad T' = KT$$

- Unknown parameters - ambiguity

$$v = [v_1, v_2, v_3]^T \in \Re^3, \quad v_4 \in \Re$$

- Corresponding projection matrix

$$\Pi = [KRK^{-1} + T'v^T, v_4T']$$

Projective Reconstruction

- From points, extract F, followed by computation of projection matrices Π_{ip} and structure $\mathbf{X}_p$
- Canonical decomposition

$$F \quad \mapsto \quad \Pi_{1p} = [I, \; 0], \; \Pi_{2p} = [(\widehat{T'})^T F, \; T']$$

- Given projection matrices – recover structure $\mathbf{X}_p$

$$\begin{aligned} \lambda_1 \mathbf{x}_1' &= \Pi_{1p} \mathbf{X}_p &= [I, 0] \mathbf{X}_p, \\ \lambda_2 \mathbf{x}_2' &= \Pi_{2p} \mathbf{X}_p &= [(\widehat{T'})^T F, T'] \mathbf{X}_p. \end{aligned}$$

- Projective ambiguity – non-singular 4x4 matrix H_p

$$\lambda_i \mathbf{x}_i' = \boxed{\Pi_{ip} H^{-1} H \mathbf{X}_p}$$

$$\lambda_i \mathbf{x}_i' = \tilde{\Pi}_{1p} \tilde{\mathbf{X}}_p$$

Both Π_{ip} and $\tilde{\Pi}_{ip}$ are consistent with the epipolar geometry – give the same fundamental matrix

Projective Reconstruction

- Given projection matrices recover projective structure

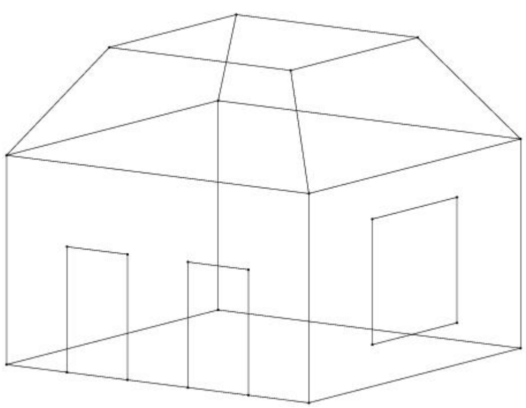
$$(x_1\pi_1^{3T})\mathbf{X}_p = \pi_1^{1T}\mathbf{X}_p, \qquad (y_1\pi_1^{3T})\mathbf{X}_p = \pi_1^{2T}\mathbf{X}_p,$$
$$(x_2\pi_2^{3T})\mathbf{X}_p = \pi_2^{1T}\mathbf{X}_p, \qquad (y_2\pi_2^{3T})\mathbf{X}_p = \pi_2^{2T}\mathbf{X}_p,$$

- This is a linear problem and can be solve using linear least-squares

$$M\mathbf{X}_p = 0$$

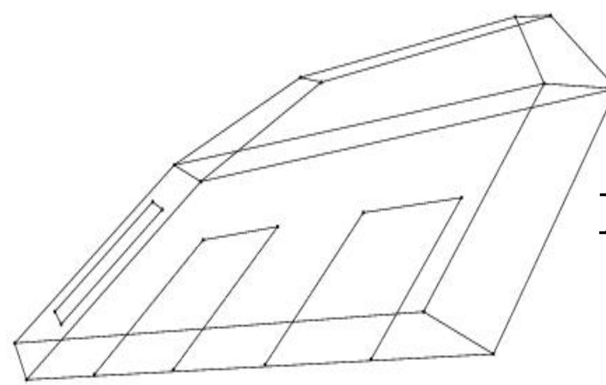
- Projective reconstruction – projective camera matrices and projective structure

$\mathbf{X}_e$



Euclidean Structure

$$\mathbf{X}_e = H\mathbf{X}_p$$



$\mathbf{X}_p$

Projective Structure

Euclidean vs Projective reconstruction

- Euclidean reconstruction – true metric properties of objects
  lenghts (distances), angles, parallelism are preserved
- Unchanged under rigid body transformations
- => Euclidean Geometry – properties of rigid bodies under
  rigid body transformations, similarity transformation

- Projective reconstruction – lengths, angles, parallelism are NOT
  preserved – we get distorted images of objects – their distorted 3D
  counterparts --> 3D projective reconstruction
- => Projective Geometry

Homogeneous Coordinates (RBM)

3-D coordinates are related by: $\boldsymbol{X}_c = R\boldsymbol{X}_w + T,$

Homogeneous coordinates:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \rightarrow \quad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \in \mathbb{R}^4,$$

Homogeneous coordinates are related by:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Homogenous and Projective Coordinates

- Homogenous coordinates in 3D before – attach 1 as the last coordinate – render the transformation as linear transformation
- Projective coordinates – all points are equivalent up to a scale

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \approx \boldsymbol{X} = \begin{bmatrix} WX \\ WY \\ W \end{bmatrix} \in \mathbb{R}^3 \quad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \approx \boldsymbol{X} = \begin{bmatrix} WX \\ WY \\ WZ \\ W \end{bmatrix} \in \mathbb{R}^4$$

2D- projective plane

3D- projective space

- Each point on the plane is represented by a ray in projective space

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} \qquad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} \in \mathbb{R}^4$$

- Ideal points – last coordinate is 0 – ray parallel to the image plane
- points at infinity – never intersects the image plane

Vanishing points – points at infinity

Representation of a 3-D line – in homogeneous coordinates

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{bmatrix}, \quad \mu \in \mathbb{R}$$

When λ -> 1 - vanishing points – last coordinate -> 0

$$\boldsymbol{X} = \begin{bmatrix} X_o + \lambda V_1 \\ Y_o + \lambda V_2 \\ Z_o + \lambda V_3 \\ 1 \end{bmatrix} \qquad \boldsymbol{X} = \begin{bmatrix} X_o/\lambda + V_1 \\ Y_o/\lambda + V_2 \\ Z_o/\lambda + V_3 \\ 1/\lambda \end{bmatrix} \qquad \boldsymbol{X} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{bmatrix}$$

Similarly in the image plane

Ambiguities in the image formation

$$\lambda \mathbf{x}' = K \Pi_0 g \mathbf{X}$$

$$K = \begin{bmatrix} f s_x & f x_y & o_x \\ 0 & f s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

- Potential Ambiguities

$$\lambda \mathbf{x}' = \Pi \mathbf{X} = K \Pi_0 g \mathbf{X} = \underbrace{K R_0^{-1} R_0 \Pi_0 H^{-1}}_{\tilde{\Pi}} \underbrace{H g g_w^{-1} g_w \mathbf{X}}_{\tilde{\mathbf{X}}}$$

- Ambiguity in K (K can be recovered uniquely – Cholesky or QR)

$$\lambda \mathbf{x}' = K \Pi_0 g \mathbf{X} = K R_0 R_0^{-1} [R,\ T] \mathbf{X} \doteq= \tilde{K} \Pi_0 \tilde{g} \mathbf{X}$$

- Structure of the motion parameters

$$g \mathbf{X} = g g_w^{-1} g_w \mathbf{X}$$

- Just an arbitrary choice of reference frame

Ambiguities in Image Formation

Structure of the (uncalibrated) projection matrix $\Pi = [KR, KT]$

$$\lambda \mathbf{x}' = \Pi \mathbf{X} = (\Pi H^{-1})(H\mathbf{X}) = \tilde{\Pi}\tilde{\mathbf{X}}$$

- For any invertible 4 x 4 matrix H
- In the uncalibrated case we cannot distinguish between camera Π imaging word $\mathbf{X}$ from camera $\tilde{\Pi}$ imaging distorted world $\tilde{\mathbf{X}}$
- In general, H is of the following form

$$H^{-1} = \begin{bmatrix} G & b \\ v^T & v_4 \end{bmatrix}$$

- In order to preserve the choice of the first reference frame we can restrict some DOF of H

Structure of the Projective Ambiguity

- 1st frame as reference $\lambda_1 \mathbf{x}'_1 = K_1 \Pi_0 \mathbf{X}_e$

$$\lambda_1 \mathbf{x}'_1 = K_1 \Pi_0 H^{-1} H \mathbf{X}_e = \Pi_{1p} \mathbf{X}_p$$

- Choose the projective reference frame

$\Pi_{1p} = [I_{3\times 3}, 0]$ then ambiguity is

$$H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix}$$

- H^{-1} can be further decomposed as

$$H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \doteq H_a^{-1} H_p^{-1}$$

$$\mathbf{X}_p = H_p \overbrace{H_a \underbrace{g_e \mathbf{X}}_{\mathbf{X}_e}}^{\mathbf{X}_a}$$

Stratified (Euclidean) Reconstruction

- General ambiguity – while preserving choice of first reference frame

$$H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix}$$

- Decomposing the ambiguity into affine and projective one

$$H^{-1} = H_a^{-1} H_p^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix}$$

- Note the different effect of the 4-th homogeneous coordinate

Affine upgrade

- Upgrade projective structure to an affine structure

$$H_p^{-1} = \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \qquad \mathbf{X}_a = H_p^{-1}\mathbf{X}_p$$

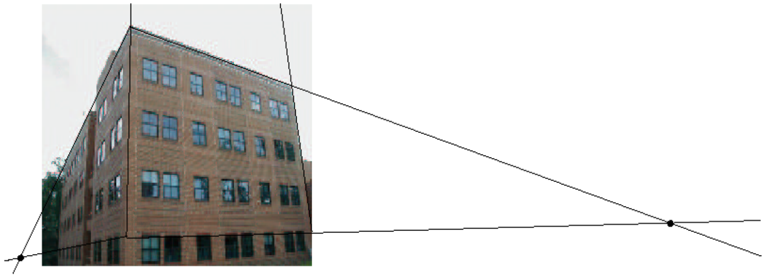
- Exploit partial scene knowledge
  - Vanishing points, no skew, known principal point
- Special motions
  - Pure rotation, pure translation, planar motion, rectilinear motion
- Constant camera parameters (multi-view)

Affine upgrade using vanishing points

How to compute $H_p^{-1} = \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix}$ $\quad \mathbf{X}_a = H_p^{-1}\mathbf{X}_p$

Maps the points $[v, v_4]^T \mathbf{X}_p = 0$

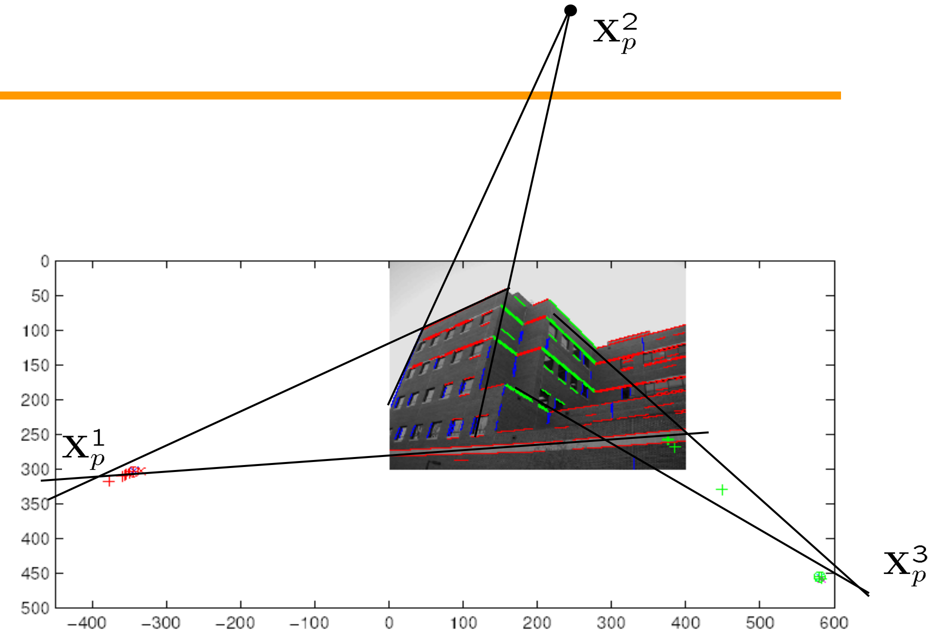
To points with affine coordinates $\mathbf{X}_a = [X, Y, Z, 0]^T$



$\mathbf{X}_a = [X, Y, Z, 0]^T$

Vanishing points – last homogeneous affine coordinate is 0

Affine Upgrade



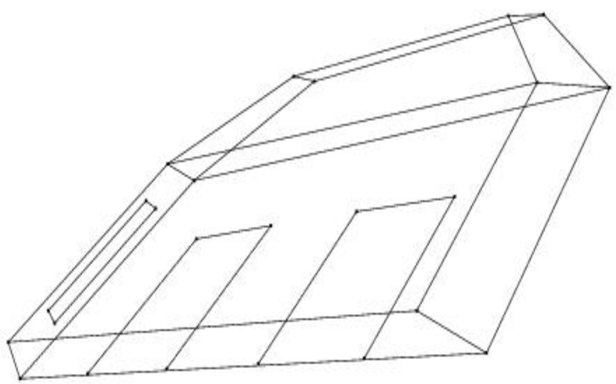
Need at least three vanishing points

$$[v, v_4]^T \mathbf{X}_p^i = 0, i = 1, 2, 3$$

3 equations, 4 unknowns (-1 scale)
Solve for

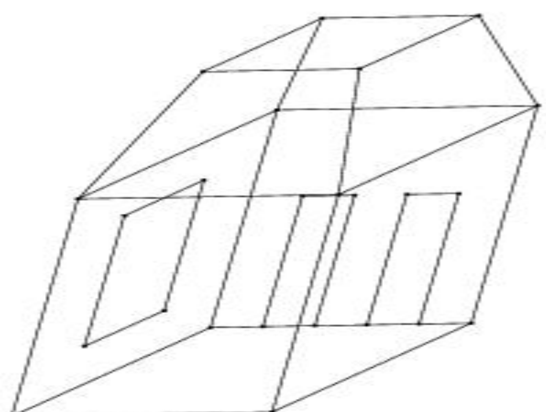
$$[v, v_4] = [v_1, v_2, v_3, v_4]$$

Set up H_p^{-1} and update the projective structure



$\mathbf{X}_p$

$$\mathbf{X}_a = H_p^{-1} \mathbf{X}_p$$



$\mathbf{X}_a$

Euclidean upgrade

- We need to estimate remaining affine ambiguity

$$H_a^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Alternatives:</u>

- In the case of special motions (e.g. pure rotation) – no projective ambiguity – cannot do projective reconstruction

$$\lambda_2 \mathbf{x}_2' = R_a \lambda_1 \mathbf{x}_1'$$

$$R_a = KRK^{-1} \Rightarrow R_a(KK^T)R_a^T = (KK^T).$$

- Estimate KK^T directly (special case of rotating camera – follows)
- Multi-view case – estimate projective and affine ambiguity together
- Use additional constraints of the scene structure (next)
- Autocalibration (Kruppa equations)

Direct Stratification from Multiple Views

From the recovered projective projection matrix

$$\Pi_{ip} = \Pi_{ie} H^{-1} = [B_i, b_i], \qquad B_i \in \mathbb{R}^{3\times 3}, b_i \in \mathbb{R}^3$$

we obtain the absolute quadric contraints

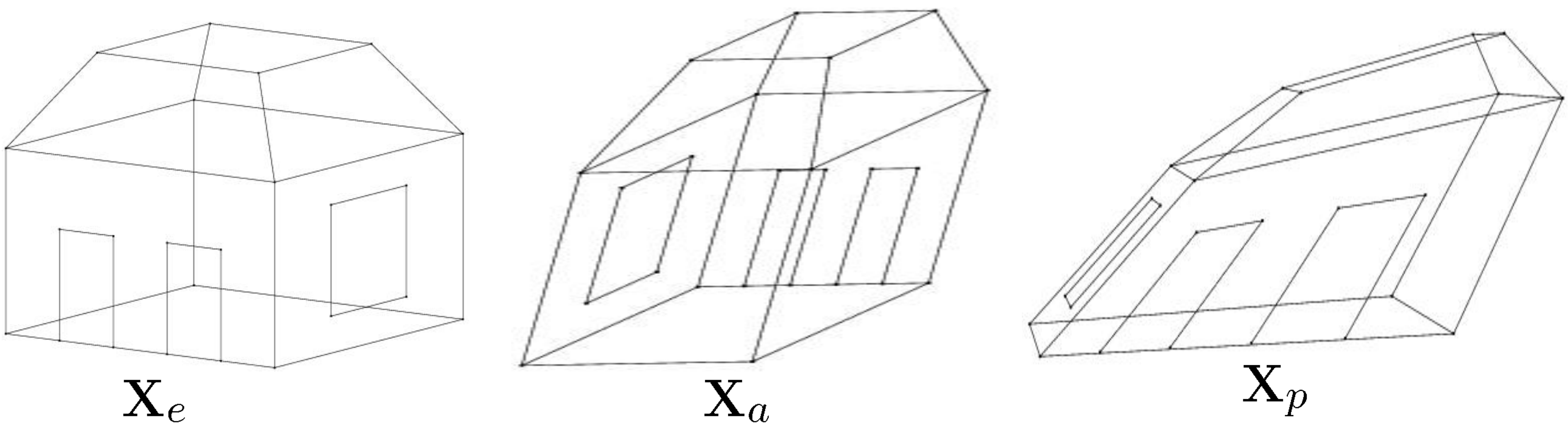
$$(B_i - b_i v^T) K K^T (B_i - b_i v^T)^T = \lambda K K^T$$

Partial knowledge in K (e.g. zero skew, square pixel) renders the above constraints linear and easier to solve.

The projection matrices can be recovered from the multiple-view rank method to be introduced later.

Geometric Stratification

	Camera projection	3-D structure
Euclid.	$\Pi_{1e} = [K, 0],\ \Pi_{2e} = [KR,\ KT]$	$\boldsymbol{X}_e = g_e \boldsymbol{X} = \begin{bmatrix} R_e & T_e \\ 0 & 1 \end{bmatrix} \boldsymbol{X}$
Affine	$\Pi_{2a} = [KRK^{-1},\ KT]$	$\boldsymbol{X}_a = H_a \boldsymbol{X}_e = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{X}_e$
Project.	$\Pi_{2p} = [KRK^{-1} + KTv^T, v_4 KT]$	$\boldsymbol{X}_p = H_p \boldsymbol{X}_a = \begin{bmatrix} I & 0 \\ -v^T v_4^{-1} & v_4^{-1} \end{bmatrix} \boldsymbol{X}_a$



Overview of the methods

	Knowledge assumption	Invariants to be utilized
1.	No prior knowledge	Incidence relations
2.	Partial knowledge of the pose	Kruppa's equations
3.	Partial knowledge of the scene	Orthogonality/parallelism
4.	Full knowledge of the scene	Full metric properties
	Characteristics	**Section index**
1.	Stratification: two or multiple views	Section 6.4 or 9.3
2.	Autocalibration under special motions	Section 6.A.2
3.	Autocalibration using vanishing points	Section 6.5
4.	Calibration with known structure	Section 6.6

Summary

	Calibrated case	Uncalibrated case
Image point	$\boldsymbol{x}$	$\boldsymbol{x}' = K\boldsymbol{x}$
Camera (motion)	$g = (R, T)$	$g' = (KRK^{-1}, KT)$
Epipolar constraint	$\boldsymbol{x}_2^T E \boldsymbol{x}_1 = 0$	$(\boldsymbol{x}_2')^T F \boldsymbol{x}_1' = 0$
Fundamental matrix	$E = \widehat{T}R$	$F = \widehat{T'}KRK^{-1}, T' = KT$
Epipoles	$E\boldsymbol{e}_1 = 0,\ \boldsymbol{e}_2^T E = 0$	$F\boldsymbol{e}_1 = 0,\ \boldsymbol{e}_2^T F = 0$
Epipolar lines	$\boldsymbol{\ell}_1 = E^T \boldsymbol{x}_2,\ \boldsymbol{\ell}_2 = E\boldsymbol{x}_1$	$\boldsymbol{\ell}_1 = F^T \boldsymbol{x}_2',\ \boldsymbol{\ell}_2 = F\boldsymbol{x}_1'$
Decomposition	$E \mapsto [R,\ T]$	$F \mapsto [(\widehat{T'})^T F,\ T']$
Reconstruction	Euclidean: $\boldsymbol{X}_e$	Projective: $\boldsymbol{X}_p = H\boldsymbol{X}_e$

Summary of (Auto)calibration Methods

	Euclidean	Affine	Projective
Structure	$\mathbf{X}_e = g_e \mathbf{X}$	$\mathbf{X}_a = H_a \mathbf{X}_e$	$\mathbf{X}_p = H_p \mathbf{X}_a$
Transformation	$g_e = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$	$H_a = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}$	$H_p = \begin{bmatrix} I & 0 \\ -v^T v_4^{-1} & v_4^{-1} \end{bmatrix}$
Projection	$\Pi_e = [KR, \; KT]$	$\Pi_a = \Pi_e H_a^{-1}$	$\Pi_p = \Pi_a H_p^{-1}$
3-step upgrade	$\mathbf{X}_e \leftarrow \mathbf{X}_a$	$\mathbf{X}_a \leftarrow \mathbf{X}_p$	$\mathbf{X}_p \leftarrow \{\mathbf{x}_1', \mathbf{x}_2'\}$
Info. needed	Calibration K	Plane at infinity $\pi_\infty^T \doteq [v^T, v_4]$	Fundamental matrix F
Methods	Lyapunov eqn.	Vanishing points	Canonical decomposition
	Pure rotation	Pure translation	
	Kruppa's eqn.	Modulus constraint	
2-step upgrade	$\mathbf{X}_e \leftarrow \mathbf{X}_p$		$\mathbf{X}_p \leftarrow \{\mathbf{x}_i'\}_{i=1}^m$
Info. needed	Calibration K and $\pi_\infty^T = [v^T, v_4]$		Multiple-view matrix*
Methods	Absolute quadric constraint		Rank conditions*
1-step upgrade	$\{\mathbf{x}_i\}_{i=1}^m \leftarrow \{\mathbf{x}_i'\}_{i=1}^m$		
Info. needed	Calibration K		
Methods	Orthogonality & parallelism, symmetry or calibration rig		

Calibration with a Rig

Use the fact that both 3-D and 2-D coordinates of feature points on a pre-fabricated object (e.g., a cube) are known.

Calibration with a Rig

- Given 3-D coordinates on known object $\mathbf{X}$

$$\lambda \mathbf{x}' = [KR, KT]\mathbf{X} \Rightarrow \lambda \mathbf{x}' = \Pi \mathbf{X}$$

$$\lambda \begin{bmatrix} x^i \\ y^i \\ 1 \end{bmatrix} = \begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix} \begin{bmatrix} X^i \\ Y^i \\ Z^i \\ 1 \end{bmatrix}$$

- Eliminate scale, two linear constraints per point:

$$\begin{aligned} x^i(\pi_3^T \mathbf{X}) &= \pi_1^T \mathbf{X}, \\ y^i(\pi_3^T \mathbf{X}) &= \pi_2^T \mathbf{X} \end{aligned}$$

- Recover projection matrix $\Pi = [KR, KT] = [R', T']$

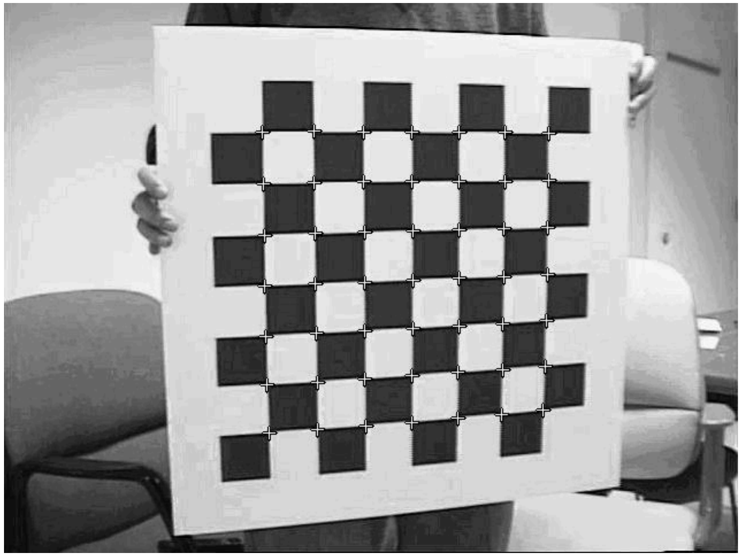
$$\Pi^s = [\pi_{11}, \pi_{21}, \pi_{31}, \pi_{12}, \pi_{22}, \pi_{32}, \pi_{13}, \pi_{23}, \pi_{33}, \pi_{14}, \pi_{24}, \pi_{34}]^T$$

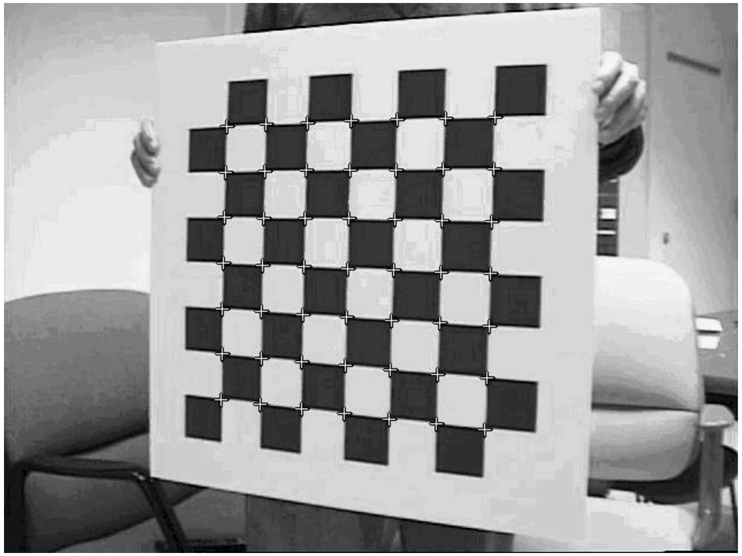
$$\min \|M\Pi^s\|^2 \quad \text{subject to} \quad \|\Pi^s\|^2 = 1$$

- Factor the KR into $R \in SO(3)$ and K using QR decomposition
- Solve for translation $T = K^{-1}T'$

Calibration with a Planar Rig

Use the fact that both 3-D and 2-D coordinates of feature points on a pre-fabricated plane are known.

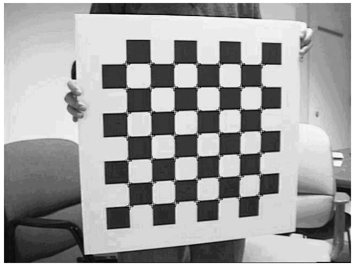




Calibration with a Planar Rig

- Special world frame on the plane

$$\lambda \mathbf{x}' = \Pi \mathbf{X} = [KR, KT]\mathbf{X} \qquad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix}$$



- Homography from the plane to the image

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = K[r_1, r_2, T] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

- Two linear constraints on the calibration $S = K^{-T}K^{-1}$ per image

$$H \doteq K[r_1, r_2, T] \quad \in \mathbb{R}^{3\times 3} \quad K^{-1}[h_1, h_2] \sim [r_1, r_2]$$

$$h_1^T K^{-T} K^{-1} h_2 = 0, \quad h_1^T K^{-T} K^{-1} h_1 = h_2^T K^{-T} K^{-1} h_2.$$

Calibration with Scene Structure: vanishing points

- Vanishing points – intersections of the parallel lines

$$v_i = l_1 \times l_2 = \widehat{l_1} l_2$$

- Vanishing points of three orthogonal directions

$$\mathbf{v}_1 = KRe_1, \quad \mathbf{v}_2 = KRe_2, \quad \mathbf{v}_3 = KRe_1$$

- Orthogonal directions – inner product is zero

$$\mathbf{v}_i^T S \mathbf{v}_j = \mathbf{v}_i^T K^{-T} K^{-1} \mathbf{v}_j = e_i^T R^T R e_j = e_i^T e_j = 0, \quad i \neq j,$$

- Provide directly constraints on matrix $S = K^{-T} K^{-1}$

- S – has 5 degrees of freedom, 3 vanishing points gives three linear constraints (need additional assumption on K)
- Assume zero skew and aspect ratio = 1

Calibration with Motions – Pure Rotation

- Calibrated two views related by rotation only

$$\lambda_2 \mathbf{x}_2 = R \lambda_1 \mathbf{x}_1$$

- Mapping to a reference view – rotation can be estimated

$$\widehat{\mathbf{x}_2} R \mathbf{x}_1 = 0$$

- Mapping to a cylindrical surface

Calibration with Motions: Pure Rotation

- Uncalibrated two views related by a pure rotation:

$$\lambda_2 K \mathbf{x}_2 = \lambda_1 K R K^{-1} K \mathbf{x}_1 \qquad \widehat{\mathbf{x}'_2} K R K^{-1} \mathbf{x}'_1 = 0$$

- Conjugate rotation $C = KRK^{-1}$ can be estimated

- Given C, we have three linear constraints:

$$S^{-1} - C S^{-1} C^T = 0 \text{ where } S^{-1} = KK^T$$

- Given two rotations around linearly independent axes – S, K can be estimated using linear techniques
- Applications – image mosaics

Calibration with Motions: General Motions

The fundamental matrix

$$F = K^{-T}\widehat{T}RK^{-1} = \widehat{T}'KRK^{-1}$$

satisfies the Kruppa's equations

$$FKK^TF^T = \widehat{T'}KK^T\widehat{T'}^T$$

If the fundamental matrix is known up to scale

$$FKK^TF^T = \lambda^2\widehat{T'}KK^T\widehat{T'}^T$$

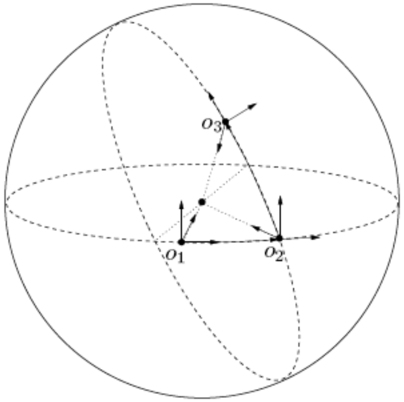
This give two nonlinear constraints on $S^{-1} = KK^T$

Solution to Kruppa's equations can be sensitive to noises.

Calibration with Motions: Special Motions

Under special motions,

1. ω is parallel to T (i.e. the screw motion), and
2. ω is perpendicular to T (e.g., the planar motion).



The scale λ can be determined, hence the Kruppa's equations become linear in $S^{-1} = KK^T$.

$$FKK^TF^T = \lambda^2 \widehat{T'} KK^T \widehat{T'}^T$$

Each Kruppa equation gives two linear constraints on

$$S^{-1} = KK^T$$

Calibration with Motions: Special Motions

Cases	Type of constraints	# of constraints on S^{-1}
$T = 0$	Lyapunov equation (linear)	3
$R \perp T$	Normalized Kruppa (linear)	2
$R \parallel T$	Normalized Kruppa (linear)	2
Others	Unnormalized Kruppa (nonlinear)	2