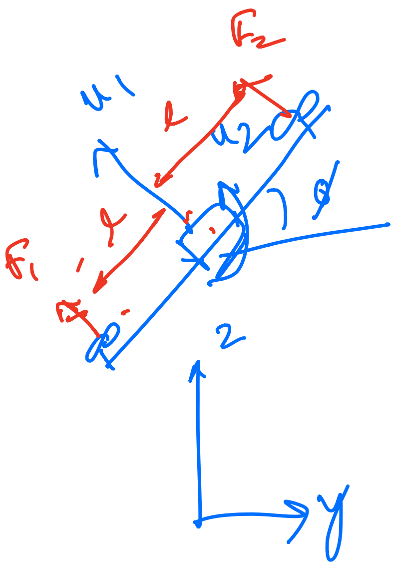
Quadrotor Dynamics

$u_1 = F_1 + F_2$

$u_2 = (F_1 - F_2)\, l$



$$\begin{bmatrix}\ddot{y}\\ \ddot{z}\\ \ddot{\phi}\end{bmatrix} = \begin{bmatrix}0\\ -g\\ 0\end{bmatrix} + \begin{bmatrix}-\frac{1}{m}\sin\phi & 0\\ \frac{1}{m}\cos\phi & 0\\ 0 & \frac{1}{I_{xx}}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{bmatrix} = \begin{bmatrix}y\\ z\\ \phi\\ \dot{y}\\ \dot{z}\\ \dot{\phi}\end{bmatrix}$$

$$\dot{x} = \begin{bmatrix}x_4\\ x_5\\ x_6\\ 0\\ -g\\ 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ -\frac{1}{m}\sin x_3 & 0\\ \frac{1}{m}\cos x_3 & 0\\ 0 & \frac{1}{I_{xx}}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\underset{\mathbb{R}^6}{\dot{x}} = \underset{\mathbb{R}^6}{f(x)} + \underset{6 \times 2}{g(x)}\, u$$

Equilibrium

$$\dot{x} = f(x) + g(x)\,u \qquad g(x) \in \mathbb{R}^{6\times 2}$$

$\bar{x}$ is an equilib if $\exists\ \bar{u}$ such that

$$f(\bar{x}) + g(\bar{x})\,\bar{u} = 0$$

$$\bar{x}_4 = \bar{x}_5 = \bar{x}_6 = 0$$

$$-\frac{1}{m}\sin\bar{x}_3 \cdot \bar{u}_1 = 0 \qquad \frac{1}{m}\cos\bar{x}_3\,\bar{u}_1 = g$$

$$\Rightarrow \bar{x}_3 = 0 \qquad \bar{u}_1 = mg$$

$$0 = \bar{u}_2 / I_{xx} \qquad \bar{u}_2 = 0$$

Linearization

$$x = \bar{x} + \delta x \qquad u = \bar{u} + \delta u$$

$$x(t) = \bar{x} + \delta x(t)$$

$$\cancelto{0}{\dot{\bar{x}}} + \delta\dot{x} = f(\bar{x} + \delta x) + g(\bar{x} + \delta x)(\bar{u} + \delta u)$$

$$= f(\bar{x} + \delta x) + g(\bar{x} + \delta x)\,\bar{u} + g(\bar{x})\,\delta u + \text{h.o.t}$$

$$\delta\dot{x} = \cancelto{0}{f(\bar{x}) + g(\bar{x})\,\bar{u}} + \left[Df(\bar{x}) + D_x\big(g(x)\bar{u}\big)\right]\delta x$$

$$+\, g(\bar{x})\,\delta u$$

$$\delta \dot{x} = A(\bar{u})\,\delta x + B\,\delta u \sim$$

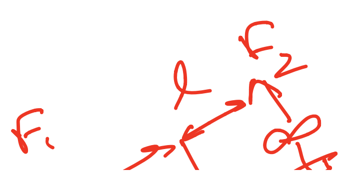
$$A(\bar{u}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{m}\cos\bar{x}_3 u_1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{m}\sin\bar{x}_3 u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin\bar{x}_3 & 0 \\ \frac{1}{m}\cos\bar{x}_3 & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix}$$

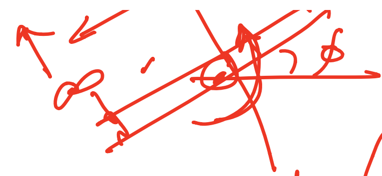
$$A(\bar{u}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix}$$

$$\ddot{y} = -g\phi$$

$$\ddot{z} = \frac{1}{m} u_1$$

$$\ddot{\phi} = \frac{u_2}{I_{xx}}$$





$$u_2 = (F_2 - F_1)\,\ell$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\frac{1}{m} \sin x_3 \, u_1 \\ \dot{x}_5 = g + \frac{1}{m} \cos x_3 \, u_1 \\ \dot{x}_6 = \frac{u_2}{I_{xx}} \end{cases}$$

$\bar{x}$ Equilibrium point

$$\dot{x} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{1}{m}\sin x_3 & 0 \\ \frac{1}{m}\cos x_3 & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\dot{x} = f(x) + g(x)\,u$$

$\bar{x}$ is an equilib point if $\exists\, \bar{u}$

such that $f(\bar{x}) + g(\bar{x})\,\bar{u} = 0$

$\bar{x}_4 = 0$ ✓

$\bar{x}_5 = 0$ ✓

$\bar{x}_6 = 0$

$$\begin{cases} -\frac{1}{m}\sin\bar{x}_3\,\bar{u}_1 = 0 \\ \frac{1}{m}\cos\bar{x}_3\,\bar{u}_1 = g \end{cases}$$

$\bar{x}_3 = 0$

$\bar{u}_1 = mg$

$$\frac{\bar{u}_2}{I_{xx}} = 0$$

$\bar{u}_2 = 0$

$$\boxed{\begin{array}{l} \bar{x}_1,\ \bar{x}_2,\ \text{arbitrary} \\ \bar{x}_3 = 0 = \bar{x}_4 = \bar{x}_5 = \bar{x}_6 \end{array}}$$

$\bar{u}_1 = mg$

$\bar{u}_2 = 0$

$$F_1 = F_2 = \frac{mg}{2}$$