EECS 106B/206B
Robotic Manipulation and Interaction

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Stress and Strain

Stress:

- Force per unit area

$$\sigma = \frac{dF}{dA} \approx \frac{F}{A}$$

Strain:

- Change in length per unit length

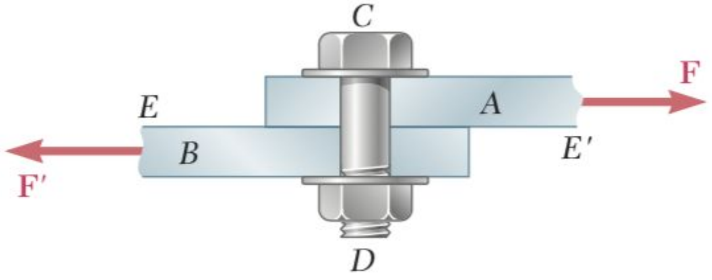
$$\epsilon = \frac{d\delta}{dx} \approx \frac{\delta}{x}$$

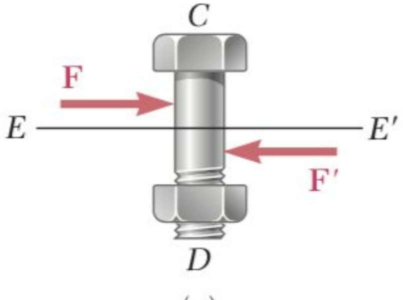
Normal Stress vs Shear Stress

Examine a plane passing through your object:

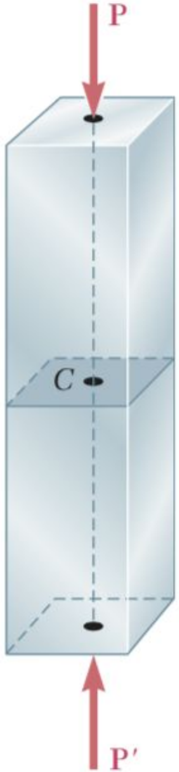
- Normal stress is stress perpendicular to the plane
- Shear stress is stress in the plane

$$\text{Normal: } \sigma \qquad \text{Shear: } \tau$$





(a)



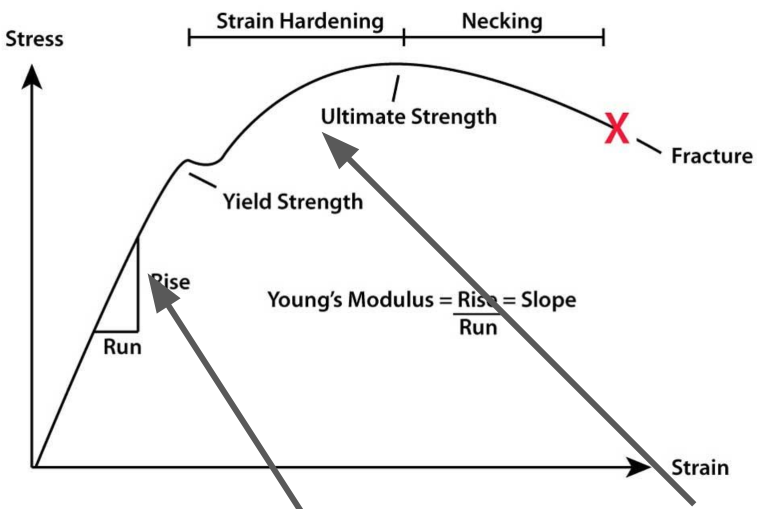
Normal Strain vs Shear Strain

Examine a plane passing through your object

- Normal strains are perpendicular to that plane
- Shear strains are in the plane

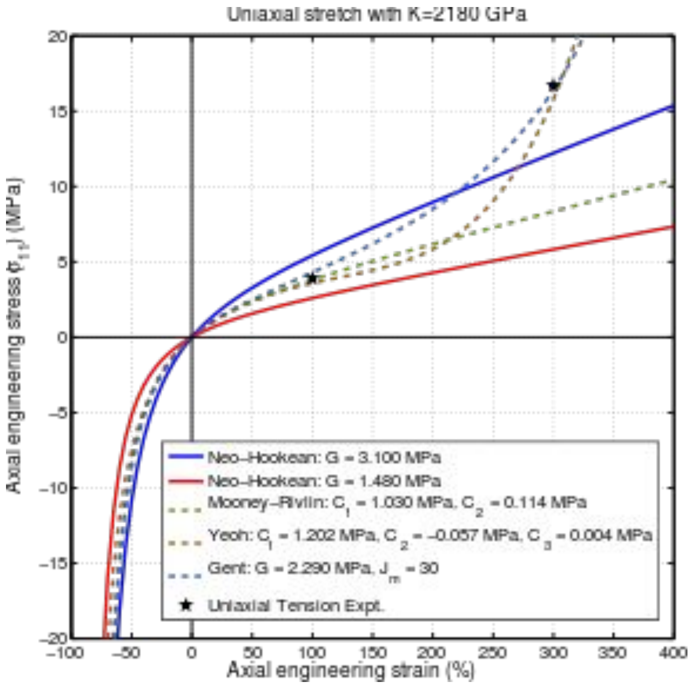
$$\text{Normal: } \epsilon \quad \text{Shear: } \gamma$$

Stress-Strain Curve



Plastic Region

Elastic Region



Hyper-elastic Materials

Homogeneous and Isotropic Materials

- Homogeneous materials:
    - Does not depend on position within the material
- Isotropic materials:
    - Properties do not depend on direction within the material

Hooke's Law

In the elastic region of the material, we get spring-like behavior

$$\sigma = E\epsilon$$

E is the Young's Modulus, or stiffness, of the material

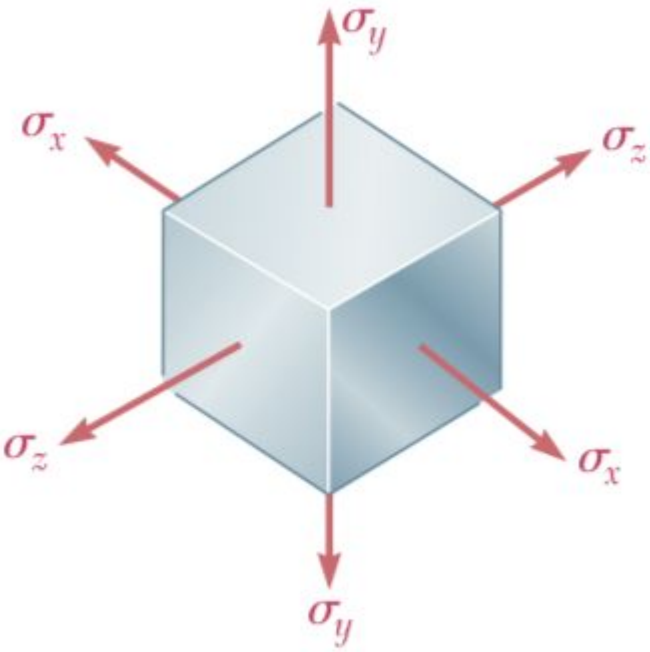
Poisson's Ratio

Pulling along one axis of an object will cause it to extend in that direction, but will cause it to contract in the other directions.

$$\nu = -\frac{\epsilon_{\cdot y}}{\epsilon_x} = -\frac{\epsilon_{\cdot z}}{\epsilon_x}$$

Multiaxial Loading

We now examine a small cube in space

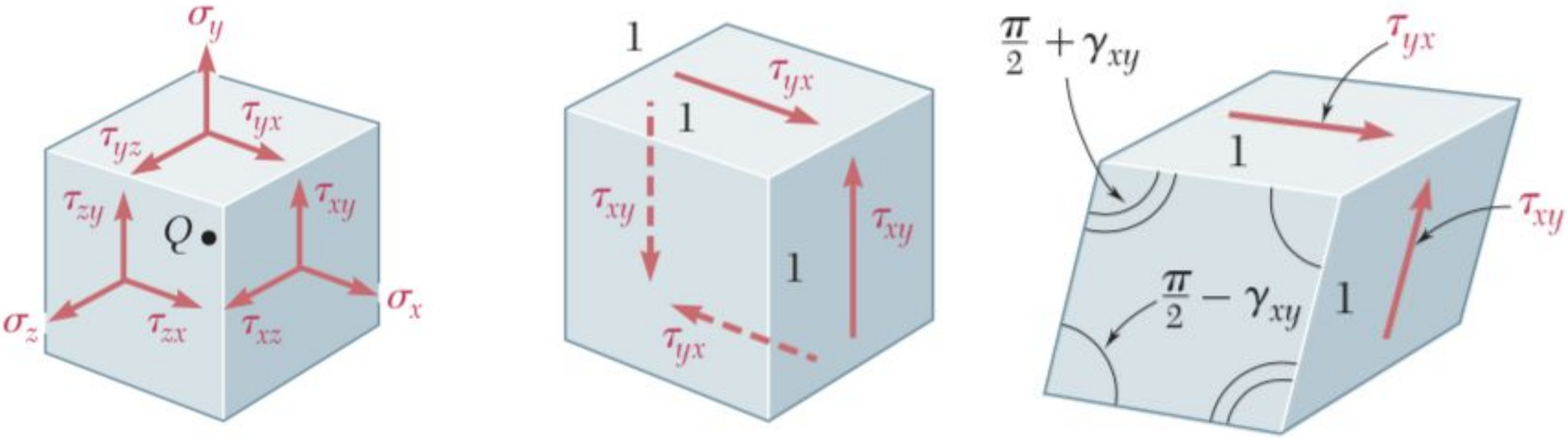


$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

Adding Shear



$$\tau_{xy} = G\gamma_{xy}$$

Sheer stress and shear strain are related through the *shear modulus* G.

Relating modulus and Poisson's ratio

You only need ⅔ of:

- Young's Modulus
- Poisson's Ratio
- Shear Modulus

$$\frac{E}{2G} = 1 + \nu$$

Cauchy Stress Tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

This is structured a lot like an inertia tensor (moment of inertia matrix)

Note that there are only six degrees of freedom (since $\tau_{xz} = -\tau_{zx}$)

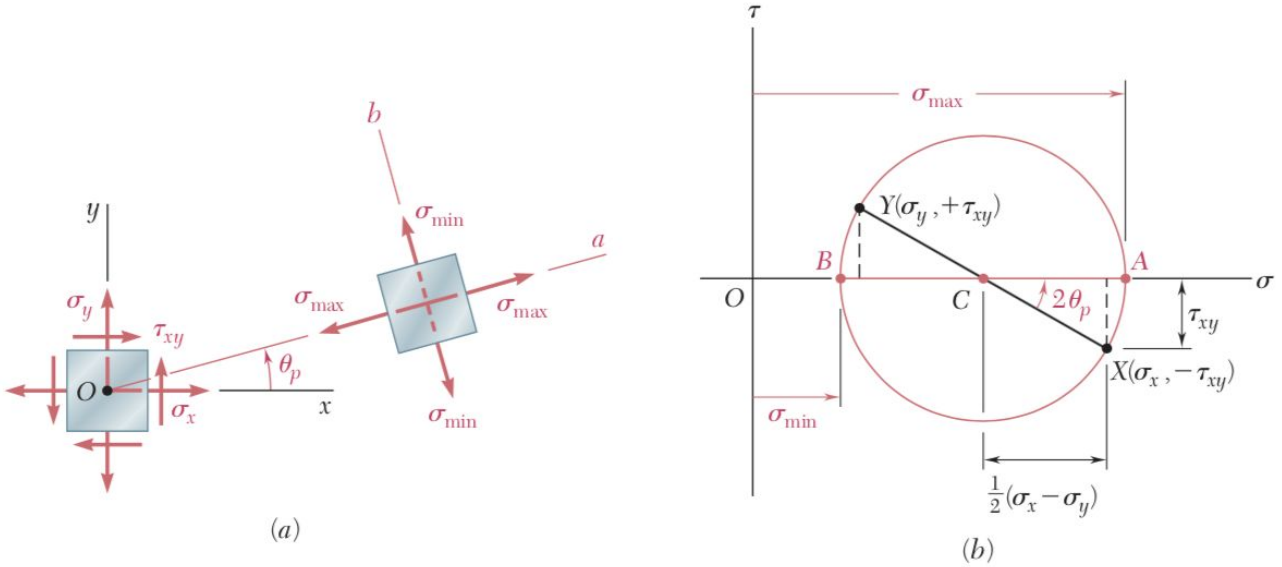
Principal Stresses

There always exists a coordinate frame in which the Cauchy stress tensor looks like

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

This is a matrix diagonalization, so the diagonal entries, or *principal stresses* are the eigenvalues.

Principal Stresses



You can visualize this transformation in a plane with Mohr's Circle

Strain Tensor

You can also express strain as a tensor

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_{zz} \end{bmatrix}$$

Note that here you also have only six degrees of freedom

Generalized Hooke’s Law

You can relate the stress tensor and the strain tensor using the Young’s Modulus, Shear Modulus, and Poisson’s Ratio.

$$\boldsymbol{\sigma} = \boldsymbol{C\epsilon}$$

Since both stress and strain are second order tensors, C must be a fourth order tensor. However, since you can express both sigma and epsilon as vectors of six variables, you can express the transformation between these representations with six equations (or a 6x6 matrix)