EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2020**

Lecture 7: Differential Geometry and Nonholonomic Control

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7.1 Unicycle Model

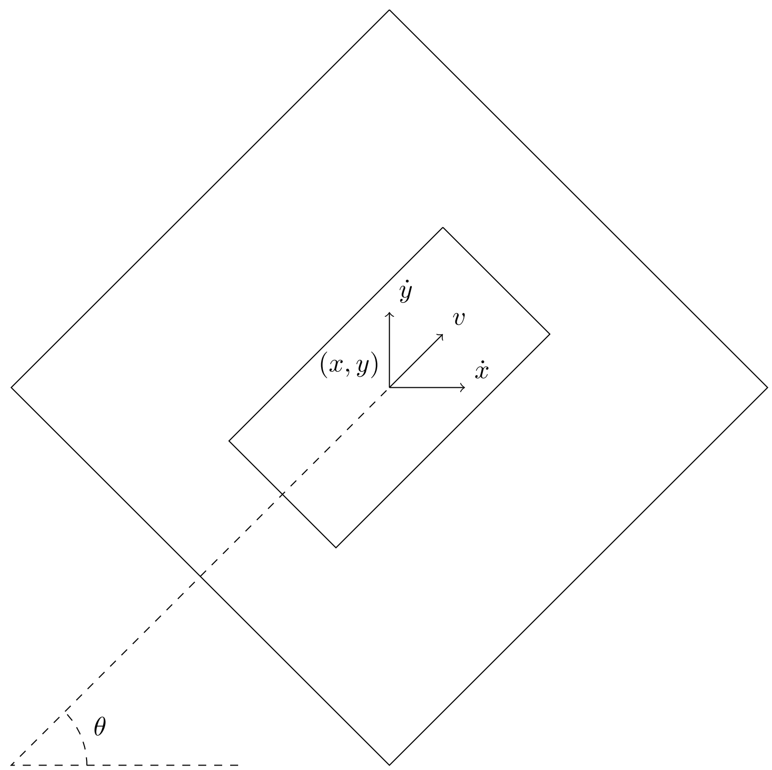


Figure 1: Unicycle Robot Model

The unicycle model consists of a single wheel with linear motion v permitted only along the axis of the wheel. Rotation is allowed about the center of the wheel. If we constrain the wheel to roll without slipping, then:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

This can be rewritten as:

$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0$$

Equivalently:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \in \mathcal{N}(\begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix})$$

If we have control inputs u_1, u_2 then we can write the system dynamics as a linear combination of the control inputs:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

If $|\dot{\theta}|$ is small, then this can be approximated to:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 \\ \theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

However, we have fewer controls than there are degrees of freedom.

7.2 Jumping Model

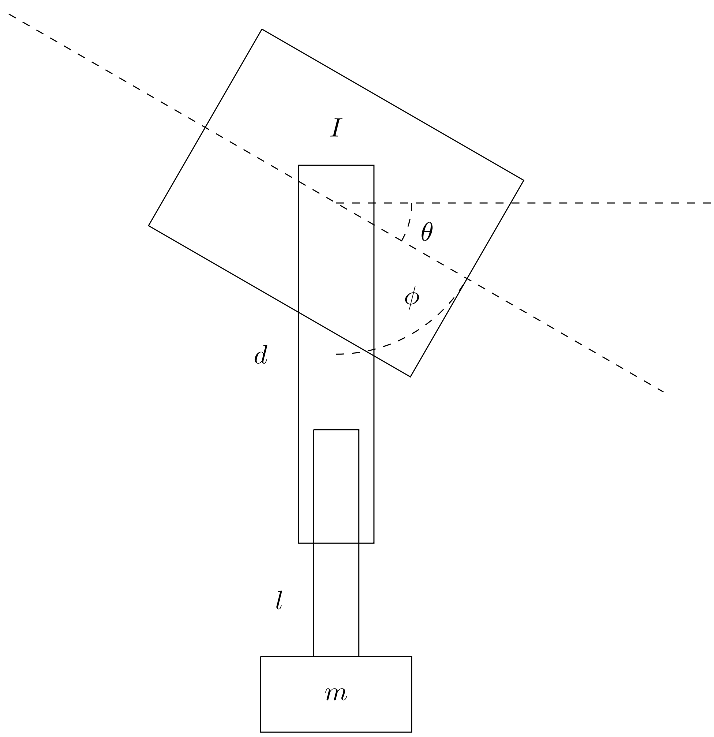


Figure 2: Jumping Robot Model

The jumping model has a constraint derived from the conservation of angular momentum:

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\phi}) = 0 \rightarrow \begin{bmatrix} I + m(l+d)^2 & m(l+d)^2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{l} \end{bmatrix} = 0$$

We can influence the system dynamics with control inputs u_1, u_2:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{l} \end{bmatrix} = \begin{bmatrix} \frac{-m(l+d)^2}{I+m(l+d)^2} \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

7.3 Falling Cat Model

If you hold a cat upside with no angular momentum and release it, it will land right-side up (assuming it has enough time before it hits the ground). It achieves this by rotating its tail in the air to reorient its body. This is an example of the conservation of angular momentum constraint. Here is a video of the experiment (no cats were harmed).

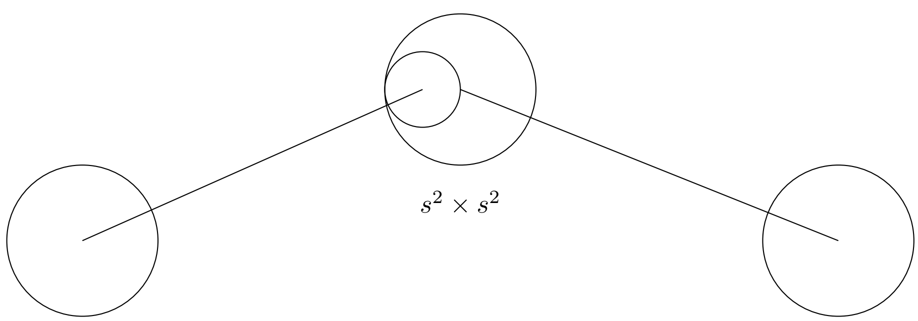


Figure 3: Falling Cat Model

7.4 Holonomic Constraints

Definition 7.4.1. A **constraint** of a mechanical system restricts the motion of the system by limiting the set of paths that the system can follow. A constraint is said to be **holonomic** if it restricts the motion of the system to a smooth hypersurface in the unconstrained configuration space Q.

Holonomic constraints can be represented locally as algebraic constraints in the configuration space:

$$h_i(q) = 0, i = 1, ..., k. \text{ where } h : \mathcal{R}^3 \rightarrow \mathcal{R}^1, p \in \mathcal{R}^n$$

Each h_i is a mapping from Q to R which restricts the motion of the system. We assume that the constraints are linearly independent and hence the matrix

$$\frac{\partial h}{\partial q} = \begin{bmatrix} \frac{\partial h_1}{\partial q_1} & \cdots & \frac{\partial h_1}{\partial q_3} \\ & \ddots & \\ \frac{\partial h_k}{\partial q_k} & \cdots & \frac{\partial h_k}{\partial q_k} \end{bmatrix}$$

is full rank (MLS 6.1).

The smooth hypersurface that the motion of a system is restricted to is called a ***manifold*** $\in \mathcal{R}^{n-k}$. We further define Pfaffian constraints: $J(\theta, x)\dot{\theta} = G^T(\theta, x)\dot{x}$, where $q = (\theta, x) \in \mathcal{R}^n$. We write this in the form:

$$\omega_i(q)\dot{q} = 0, i = 1, ..., k$$

where each ω_i describes one constraint in the direction in which $\dot{q}$ is permitted to take values. We say that a set of k Pfaffian constraints is integrable if there exist functions $h_i : \mathcal{R}^n \to \mathcal{R}, i = 1, ..., k$ such that $h_i(q(t)) = 0 \Longleftrightarrow \omega_i(q)\dot{q} = 0, i = 1, ..., k$.

Therefore, a set of Pfaffian constraints is **integrable** if it is equivalent to a set of holonomic constraints. In addition, a set of Pfaffian constraints is said to be **nonholonomic** if it is not equivalent to a set of holonomic constraints (MLS 7.1).

It is not easy to check whether a constraint is holonomic or nonholonomic. Consider a single velocity constraint:

$$\omega(q)\dot{q} = 0 \Longleftrightarrow h(q) = 0.$$

Then we differentiate $h(q)$ w.r.t time. If the Pfaffian constraint is holonomic, then:

$$\textstyle\sum_{j=1}^n \omega_j(q)\dot{q}_j = 0 \implies \sum_{j=1}^n \frac{\partial h}{\partial q_j}\dot{q}_j = 0$$

$$\tfrac{\partial}{\partial q} h(q) = 0$$

$$\begin{bmatrix} \frac{\partial h}{\partial q_1} & \frac{\partial h}{\partial q_2} \cdots \frac{\partial h}{\partial q_n} \end{bmatrix} \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \\ \cdots \\ \dot{q_n} \end{bmatrix} = 0$$

$$\tfrac{\partial \omega_i}{\partial q_j} = \tfrac{\partial \omega_j}{\partial q_i}$$

$$\tfrac{\partial^2 h}{\partial q_j \partial q_i} = \tfrac{\partial^2 h}{\partial q_i \partial q_j}$$

If it is possible to find a set of functions $h_i(q)$ for $i = 1, ..., p$, where $p < k$, the motion of the system is restricted to level surfaces of h, namely to sets of the form:

$$\{q : h_1(q) = c_1, ..., h_q(q) = c_p\}.$$

- If $p = k$, then the constraints are **holonomic**
- If $p < k$, then the constraints are **partially holonomic**, meaning the constraints are not completely equivalent to a set of holonomic constraints but the reachable points of the system are still restricted
- If $p > k$, then the constraints are completely **nonholonomic**, meaning the constraints do not restrict the reachable configuration

Therefore, choosing a basis for the null space of the constraints, denoted by $g_j(q) \in \mathcal{R}^n$, $i = 1, ..., n-k$, we can get the structure of nonholonomic systems satisfying linear velocity constraints of the form:

$$\omega_i(q)g_j(q) = 0 \quad i = 1, ..., k.$$

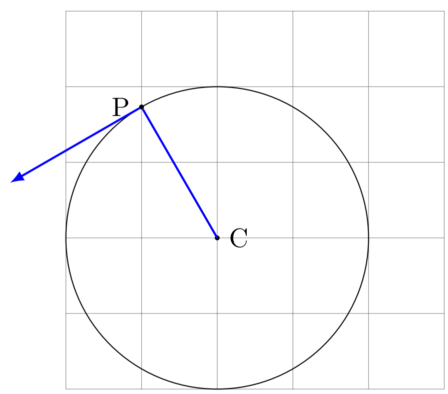
The valid trajectories of the system can be written as the solutions of the control system:

$$\dot{q} = g_1(q)u_1 + ... + g_{n-k}(q)u_{n-k}$$

with controls u_i to be chosen freely. Notice that this is an underactuated system since the dimension is $n-k$, which is less than n.

7.4.1 Example

Given $q_1^2 + q_2^2 = c$, $P = (q_1, q_2)$, derive the holonomic constraint $h(q)$.



$$2q_1\dot{q}_1 + 2q_2\dot{q}_2 = 0$$

$$\begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = 0$$

$$\frac{\partial \omega_1}{\partial q_2} = \frac{\partial \omega_2}{\partial q_1}$$

$$\frac{\partial h}{\partial q_1} = q_1$$

$$\frac{\partial h}{\partial q_2} = q_2$$

$$h(q) = \frac{q_1^2}{2} + q_1 q_2$$

Note: If a constraint is < 0 or > 0, then it is inactive, meaning there is no constraint at all. There is a large room for $\dot{q}_1$ and $\dot{q}_2$.

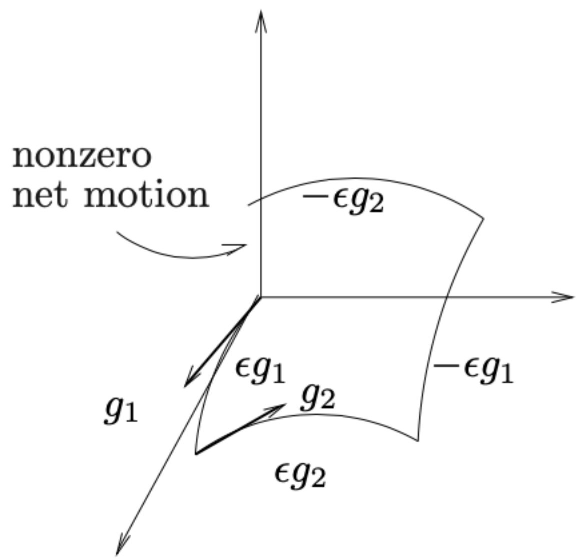
7.5 Lie Bracket

Definition 7.5.1. A **vector field** is a smooth map which assigns each point $q \in \mathcal{R}^n$ a tangent vector $f(q) \in T_q\mathcal{R}^n$ where $T_q\mathcal{R}^n$ referring to the tangent space to $\mathcal{R}^n$ at a point $q \in \mathcal{R}^n$.

Given two vector fields g_1 and g_2, we use the following input sequence:

$$u_1 = +1,\ u_2 = 0\ for\ \epsilon\ seconds$$
$$u_1 = 0,\ u_2 = +1\ for\ \epsilon\ seconds$$
$$u_1 = -1,\ u_2 = 0\ for\ \epsilon\ seconds$$
$$u_1 = 0,\ u_2 = -1\ for\ \epsilon\ seconds$$

We got the motion in the direction of the Lie bracket $[g_1, g_2]$ as illustrated in the following graph:



Notice that we do not finish where we started!