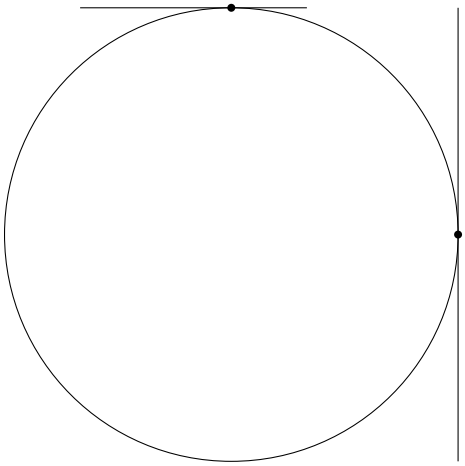
EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2020**

Lecture 1: (Jacobians, Velocities, Dynamics)

Scribes: Tomas Santos, Varun Bhatia

1.1 Lecture Goals

Apply geometric formalism to rigid body transformations. Recall that previously we learned that rotations are lie groups, and so(n) is its lie algebra, which is the tangent space at the identity element. One can find any element in the Lie Group by taking the matrix exponential of an element in the Lie Algebra.



- The vertical tangent line on the right side represents the Lie Algebra for so(2)
- The circle represents the Lie Group of SO(2)
- Need to rotate the top tangent line along Lie Algebra

$$R_{AB} => q_B \to q_A$$

$$\dot{R_{AB}} => q_B \to v_A$$

$$R_{AB}^{-1}\dot{R_{AB}} => q_B \to v_b \, and \, q_A \to v_A$$

$$R_{BA}\dot{R_{AB}} = \dot{R_{BB}} = w_{AB}^{\wedge b}$$

$$\dot{R_{AB}}R_{BA} = \dot{R_{AA}}$$

1.2 Rigid Body Transformation

For all rigid body transformations, the lengths is preserved and the orientation is preserved between all of the points of the rigid body. The following formula shows the affine transformation consisting of a rotation followed by a translation.

$$q_a = R_{ab}q_b + P_{ab}$$

To neatly package the affine transformation we use homogeneous coordinates. These are purely a construct that allow for the math to become a bit more compact and more suitable for linear algebra. Notice the 1 and the 0's in the bottom row of the matrix below. A 1 implies a point and a 0 implies a vector. These numbers are again constructs, but happen to help with regulating our equations. Two points subtracted from each other is equal to a vector. This makes sense because $1 - 1 = 0$ Vectors can be added and subtracted which coincides with adding and subtracting 0's resulting in more 0's. We can also and and subtract vectors from points, which remain points, but we cannot add points because that is both undefined and it would result in a 2 in the bottom row which is not allowed.

1. Used to convert affine transformation into a linear transformation

$$q_A = R_{AB}q_B + p_{AB}$$

$$\begin{bmatrix} q_A \\ 1 \end{bmatrix} = \begin{bmatrix} R_{AB} & p_{AB} \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} q_B \\ 1 \end{bmatrix}$$

2. For a point include a 1 at the bottom:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3. For a vector include a 0 at the bottom:

$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

1.3 Rigid Body Transformations as a Group

Similarly to how rotations are a Lie Group, the Special Euclidean Group $SE(n)$ represents the group of rigid body transformations. It is formally described by the following equation:

$$SE(n) := \mathbb{R}^n \times SO(n)$$

The group is defined by closure, associativity, and the presence of an identity element and inverse elements for all elements of the group. In a similar manner to how the $so(3)$ is the Lie Algebra of the $SO(3)$, $se(3)$ is the Lie Algebra of the $SE(3)$ group. The representation of $se(3)$ is ξ which is defined by:

$$\xi = \begin{bmatrix} v \\ w \end{bmatrix}$$

- The Lie Algebra to SE(n) is se(n)

$$\xi^\wedge = \begin{bmatrix} w^\wedge & v \\ 0 & 0 \end{bmatrix}, w^\wedge \in SO(n), v \in R^n$$

- The nxn matrix can be defined by n(n-1)/2 + n variables

$$\xi = \begin{bmatrix} v \\ w \end{bmatrix} \rightarrow Twist$$

$$g^{\cdot} = \xi^\wedge g \; and \; solve \; the \; ODE \; to \; get \; g' = e^{\xi^\wedge T} \; and \; if \; ||w|| = 1, g' = e^{\xi^\wedge \theta}$$

1.4 Velocities of a Rigid Body

Recall that a twist is a mathematical structure that represents the generator of any rigid body motion. It's essentially the linear and angular motion that when integrated produces the composite rigid body motion. Additionally, any rigid body motion can be represented by a screw motion about some axis.

$$\hat{\xi}^s_{ab} = \hat{V}^s_{ab} = \dot{g_{ab}} g^{-1}_{ab}$$

This is the spatial velocity formula, and the only difference for the body velocity is which side the inverse rigid body motion is placed.

$$\hat{\xi}^b_{ab} = \hat{V}^b_{ab} = g^{-1}_{ab} \dot{g_{ab}}$$

1.5 Forward Kinematics

Forward Kinematics equations can be derived from velocities by moving each joint one at a time, using the differential equation:

$$\dot{g_{st}} = \hat{\xi} g_{st}$$

$$g_{st}(T) = e^{\hat{\xi} T} g_{st}(0)$$

Recall:

- Computing Matrix Exponential
  1. $$e^A = I + A + A^2/2 + A^3/6 + ... + A^N/N!$$
  2. Rodrigues' Formula (Rotations):

$$e^{w^\wedge \theta} = I + w^\wedge sin\theta + w^{\wedge 2}(1 - cos\theta), ||w|| = 1$$

- The Inverse Exponential: Matrix Log
  1. Inverse of matrix exponential is matrix log.

$$A = e^{logA}$$

- A Velocity Perspective for a 2-Link Arm

$$\dot{g} = \xi^\wedge g$$

$$\dot{g_{ST}} = \xi_2^\wedge g_{ST}$$

$$g_{ST}(T) = e^{\xi_2^\wedge T_2} g_{ST}(0) = g'_{ST}$$

$$\dot{g'_{ST}} = \xi_1^\wedge g'_{ST} \; so \; g'_{ST}(T_1) = e^{\xi_1^\wedge} g'_{ST}(0)$$

$$g_{ST}(T_1) = e^{\xi_1^\wedge T_1} g'_{ST}(0) = e^{\xi_1^\wedge T_1} g'_{ST}(0) e^{\xi_2^\wedge T_2} g_{ST}(0)$$

  1. if the order of the joints were reversed in the multiplication of exponentials, you'd get the same answer.

1.6 Inverse Kinematics

For inverse kinematics, the Kaden-Pahen solutions that we learned in 106b are purely for pedagogical purposes. In reality, inverse kinematics is an open field of study. Two major questions are 1) how to find a solution at all, and 2) which solution do you pick, especially considering there are infinite solutions for a 7-DOF manipulator.

1.7 The Matrix Adjoint

The adjoint is essentially a change of basis for the nx1 representation of a skew-symmetic matrix. In matrix form we would use this equation:

$$\hat{V}^s_{ab} = g_{ab}\hat{V}^b_{ab}g^{-1}_{ab}$$

The equivalent form of that expression for vectors is:

$$V^s_{ab} = Ad_{g_{ab}}V^b_{ab}$$

The adjoint has the form:

$$\begin{bmatrix} R_{ab} & \hat{p}_{ab}R_{ab} \\ 0 & R_{ab} \end{bmatrix}$$

Recall that the inverse adjoint of g is equal to the adjoint of inverse g.

1.8 The Jacobian

The jacobian, which conceptually can be thought of as a function that maps Joints to the end effector, is composed of the twists of each joint. The jacobian is derived from the equation:

$$\hat{V}^b_{ab} = \dot{g_{ab}}g^{-1}_{ab}$$

The process is to find the derivative of $\dot{g}_{st}$ with respect to each of the θ's. This results in an equation that can be massaged into an adjoint representation and finally the columns of the Jacobian come from the ξ of each joint that are transformed by the adjoint of all of the joints before it. This is conceptually moving each joint by the amount of all of the previous joints' combined rigid body motion.

1.9 Deriving the Jacobian

$$\frac{\partial g_{ST}}{\partial \theta_i} * g^{-1}_{ST} = (\frac{\partial g_{ST}}{\partial \theta_i} * g^{-1}_{ST})^V \rightarrow \textit{the partial derivative represents the product of exponentials}$$

$$\rightarrow \frac{\partial}{\partial \theta_2}(e^{\hat{\xi_1}\theta_1}e^{\hat{\xi_2}\theta_2}e^{\hat{\xi_3}\theta_3}g_{ST}(0)) * e^{-\hat{\xi_3}\theta_3}e^{-\hat{\xi_2}\theta_2}e^{-\hat{\xi_1}\theta_1}$$

$$\rightarrow e^{\hat{\xi_1}\theta_1}\frac{\partial}{\partial \theta_2}(e^{\hat{\xi_2}\theta_2})e^{\hat{\xi_3}\theta_3}g_{ST}(0)g^{-1}_{ST}(0) * e^{-\hat{\xi_3}\theta_3}e^{-\hat{\xi_2}\theta_2}e^{-\hat{\xi_1}\theta_1}$$

$$\rightarrow g_1\frac{\partial}{\partial \theta_2}(e^{\hat{\xi_2}\theta_2}) * g^{-1}_1 = g_1\hat{\xi_2}g^{-1}_1$$

$$\rightarrow \xi_i' = Ad_{g_1 \dots g_{i-1}} \xi_i'^{\wedge}$$

1.10 The Pseudoinverse Jacobian

The pseudoinverse Jacobian is used to convert from joint velocities to spatial or body end effector velocities.

$$V = J(\theta)\dot{\theta} \implies \dot{\theta} = J^{-1}(\theta)V$$

We cannot simply use the Jacobian inverse because often times there is no inverse. Instead we (naively) use the Moore-Penrose pseudoinverse instead. I say naively because an active area of research is finding alternatives to the pseudoinverse that are less susceptible to singularities and are more accurate in general. The Moore-Penrose pseudoinverse is essentially a least squares problem and is defined as:

$$J^{\dagger} = J^T(JJ^T)^{-1}$$

Thus,

$$\dot{\theta} = J^{\dagger}(\theta)V$$