EECS 106B/206B

Robotic Manipulation and Interaction

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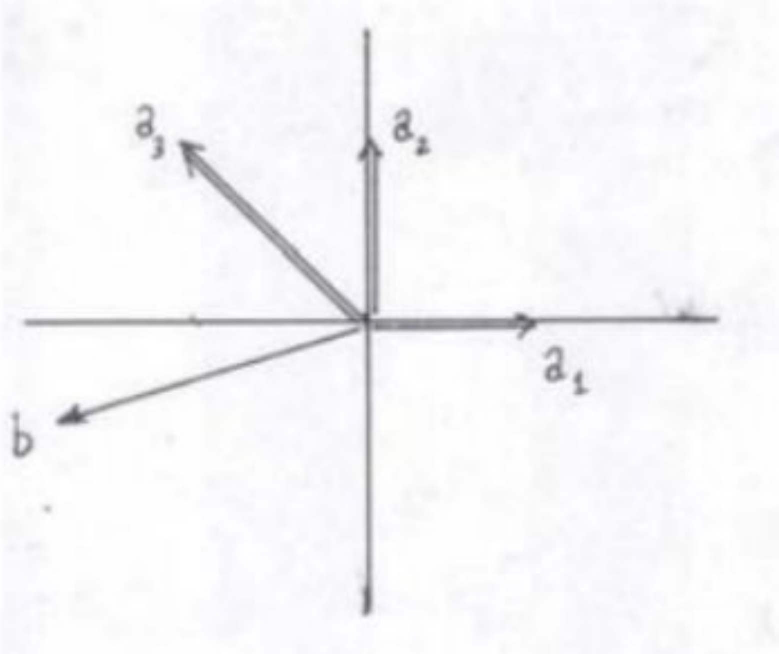
CONVEX HULL

Planar Example



Not in force closure

This can formulated as:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Which is rewritten as

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = b$$

Solving for x, we see that at least one weight x_i must be negative.

Convex Hull

Determining force closure reduces to Ax = b s.t. x ≥ 0

We will describe convex hull as

$$S = \{a_1, \ldots, a_n\}, \quad a_i \in \mathbb{R}^m, \ i = 1, \ldots, n,$$

$$\text{Convex hull of } S \ = \left\{ \sum_{i=1}^{n} w_i a_i \ \middle| \ \sum_{i=1}^{n} w_i = 1, \ w_i \geq 0 \text{ for all } i \right\}$$

Convex Hull

The requirement that the nonnegative linear combinations of the columns of A span R^m can now be equivalently stated as follows: there must exist some m-dimensional open ball in R^m, centered at the origin, that lies in the interior of the convex hull of the columns of A.

Figure 3.6: The convex hull of various sets of points in $\mathbb{R}^2$.

Convex Hull

Consider a rigid body in R^3 constrained by n frictionless point contacts.
r_i denotes the vector from the reference frame origin to contact point i
$\hat{n}_i$ denotes a vector normal to the body at contact point i directed toward the interior of the body
F_i describes the contact force at contact i; $F_i = \hat{n}_i$ x i

$$\begin{bmatrix} n_1 & \cdots & n_n \\ r_1 \times n_1 & \cdots & r_n \times n_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -f_e \\ -m_e \end{bmatrix}$$

Convex Hull

Grasp is in force closure if there exits non-negative x in R^n for arbitrary a and b satisfying Ax=b

Geometrically, the grasp is force closure if and only if there exists a six-dimensional open ball centered at the origin that lies in the interior of the convex hull formed by the columns of A.

$$\begin{bmatrix} n_1 & \cdots & n_n \\ r_1 \times n_1 & \cdots & r_n \times n_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -f_e \\ -m_e \end{bmatrix}$$

Grasps with Friction

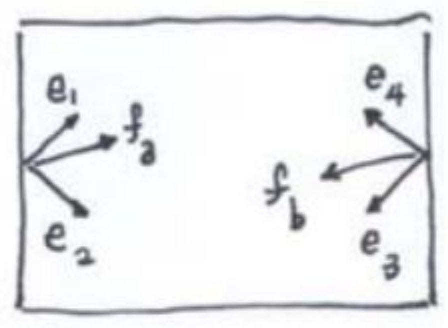
Planar Example

Figure 3.8: (a) A planar rigid body constrained by two point contacts with friction. (b) Force decomposition diagram.

Planar Example

Consider a rigid body in R^3 constrained by 2 point contacts with friction

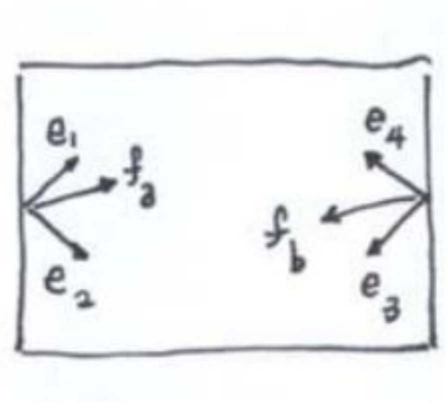
- Interior angle of friction cone: 2α (where $\mu = \tan(\alpha)$)
- e_1, e_2 in R^2 are vectors whose directions are aligned along the two edges of the left friction cone
- e_3, e_4 are the same along the right friction cone



$$e_1 = \begin{bmatrix} 1 \\ \mu \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 \\ -\mu \end{bmatrix}, \quad e_3 = \begin{bmatrix} -1 \\ -\mu \end{bmatrix}, \quad e_4 = \begin{bmatrix} -1 \\ \mu \end{bmatrix}$$

Planar Example

Assuming the contact forces at the two contacts (denoted f_a and f_b, respectively) both lie inside their respective friction cones, f_a and f_b can then be written

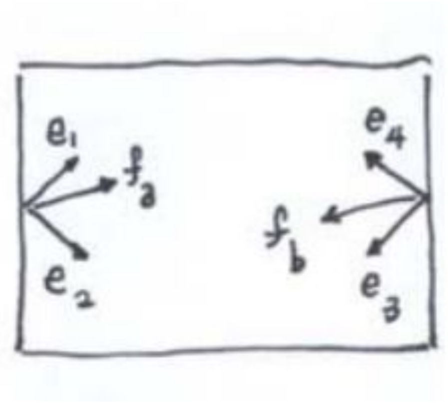


$$
\begin{aligned}
f_a &= e_1 x_1 + e_2 x_2 \\
f_b &= e_3 x_3 + e_4 x_4
\end{aligned}
$$

Planar Example

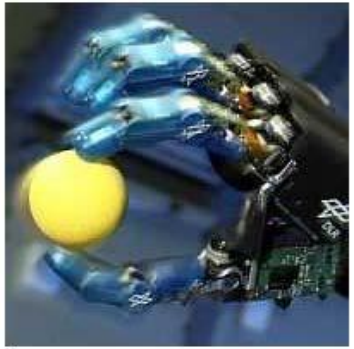
Force Closure condition is as follows:
For any arbitrary external force $f_e \in R^2$ and external moment $m_e \in R$, there must exist contact forces f_a and f_b such that

$$\begin{aligned} f_a + f_b &= -f_e \\ m_a + m_b &= -m_e. \end{aligned}$$

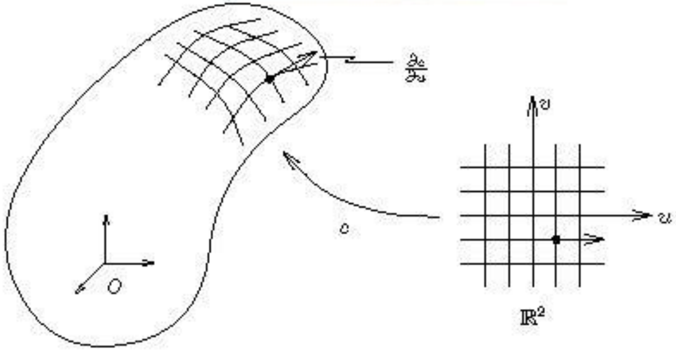


$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ \mu & -\mu & -\mu & \mu \\ -\mu r & \mu r & -\mu r & \mu r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -f_e \\ -m_e \end{bmatrix}$$

□ *Surface Model:*



$$c : U \subset \mathbb{R}^2 \to \mathbb{R}^3, c(U) \subset S$$



$$c_u = \frac{\partial c}{\partial u} \in \mathbb{R}^3$$
$$c_v = \frac{\partial c}{\partial v} \in \mathbb{R}^3$$

First Fundamental form: $I_p = \begin{bmatrix} c_u^T c_u & c_u^T c_v \\ c_v^T c_u & c_v^T c_v \end{bmatrix}$

5.3 Kinematics of Contact

Orthogonal Coordinates Chart: $c_u^T c_v = 0$ (assumption)

$$I_p = \begin{bmatrix} \|c_u\|^2 & 0 \\ 0 & \|c_v\|^2 \end{bmatrix} = M_p \cdot M_p$$

Metric tensor:

$$M_p = \begin{bmatrix} \|c_u\| & 0 \\ 0 & \|c_v\| \end{bmatrix}$$

Gauss map:

$$N : S \to s^2 : N(u, v) = \frac{c_u \times c_v}{\|c_u \times c_v\|} := n$$

2nd Fundamental form:

$$II_p = \begin{bmatrix} c_u^T n_u & c_u^T n_v \\ c_v^T n_u & c_v^T n_v \end{bmatrix}, n_u = \frac{\partial n}{\partial u}, n_v = \frac{\partial n}{\partial v}$$

5.3 Kinematics of Contact

Curvature tensor:

$$K_p = M_p^{-T} II_p M_p^{-1} = \begin{bmatrix} \frac{c_u^T n_u}{\|c_u\|^2} & \frac{c_u^T n_v}{\|c_u\|\|c_v\|} \\ \frac{c_v^T n_u}{\|c_u\|\|c_v\|} & \frac{c_v^T n_v}{\|c_v\|^2} \end{bmatrix}$$

Gauss frame:

$$[x, y, z] = \begin{bmatrix} \frac{c_u}{\|c_u\|} & \frac{c_v}{\|c_v\|} & n \end{bmatrix}, K_p = \begin{bmatrix} x^T \\ y^T \end{bmatrix} \begin{bmatrix} \frac{n_u}{\|c_u\|} & \frac{n_v}{\|c_v\|} \end{bmatrix}$$

Torsion form:

$$T_p = y^T \begin{bmatrix} \frac{x_u}{\|c_u\|} & \frac{x_v}{\|c_v\|} \end{bmatrix}$$

(M_p, K_p, T_p):Geometric parameter of the surface.

5.3 Kinematics of Contact

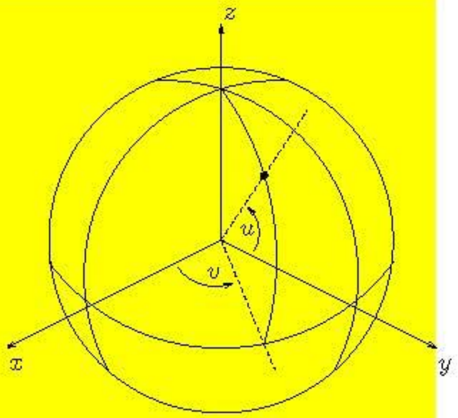
◇ *Example: Geometric parameters of a sphere in $\mathbb{R}^3$*

$$c(u,v) = \begin{bmatrix} \rho\cos u\cos v \\ \rho\cos u\sin v \\ \rho\sin u \end{bmatrix}$$

$$U = \{(u,v) \mid -\tfrac{\pi}{2} < u < \tfrac{\pi}{2}, -\pi < v < \pi\}$$

$$c_u = \begin{bmatrix} -\rho\sin u\cos v \\ -\rho\sin u\sin v \\ \rho\cos v \end{bmatrix}$$

$$c_v = \begin{bmatrix} -\rho\cos u\sin v \\ \rho\cos u\cos v \\ 0 \end{bmatrix}$$

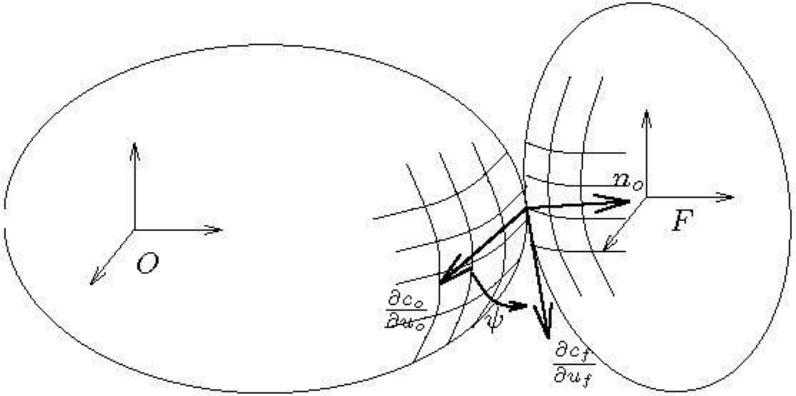


$$c_u^T c_v = 0$$

$$K = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix}, M = \begin{bmatrix} \rho & 0 \\ 0 & \rho\cos u \end{bmatrix}, T = \begin{bmatrix} 0 & \frac{\tan v}{\rho} \end{bmatrix}$$

5.3 Kinematics of Contact

□ *Gauss Frame:*



$$g_{oc}(t) \in SE(3)$$
$$V^b_{oc} = ?$$

$$p_{oc}(t) = p(t) = c(\alpha(t)), R_{oc}(t) = [x(t), y(t), z(t)] = \begin{bmatrix} \frac{c_u}{\|c_u\|} & \frac{c_v}{\|c_v\|} & \frac{c_u \times c_u}{\|c_u \times c_v\|} \end{bmatrix}$$

$$v_{oc} = R_{oc}^T \dot{p}_{oc}(t) = \begin{bmatrix} x^T \\ y^T \\ z^T \end{bmatrix} \frac{\partial c}{\partial \alpha} \dot{\alpha} = \begin{bmatrix} x^T \\ y^T \\ z^T \end{bmatrix} \begin{bmatrix} c_u & c_v \end{bmatrix} \dot{\alpha} = \begin{bmatrix} \|c_u\| & 0 \\ 0 & \|c_v\| \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} M\dot{\alpha} \\ 0 \end{bmatrix}$$

$$\hat{\omega}_{oc} = R_{pc}^T \dot{R}_{oc} = \begin{bmatrix} x^T \\ y^T \\ z^T \end{bmatrix} \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & x^T\dot{y} & x^T\dot{z} \\ y^T\dot{x} & 0 & y^T\dot{z} \\ z^T\dot{x} & z^T\dot{y} & 0 \end{bmatrix}$$

5.3 Kinematics of Contact

$$y^T \dot{x} = y^T [x_u \; x_v] \dot{\alpha} = TM\dot{\alpha}$$

$$\begin{bmatrix} x^T \dot{z} \\ y^T \dot{z} \end{bmatrix} = \begin{bmatrix} x^T \\ y^T \end{bmatrix} \dot{z} = \begin{bmatrix} x^T \\ y^T \end{bmatrix} \begin{bmatrix} n_u & n_v \end{bmatrix} \dot{\alpha} = KM\dot{\alpha}$$

$$\hat{\omega}_{oc} = \left[\begin{array}{cc|c} 0 & -TM\dot{\alpha} & \\ TM\dot{\alpha} & 0 & KM\dot{\alpha} \\ \hline \multicolumn{2}{c|}{-(KM\dot{\alpha})^T} & 0 \end{array}\right]$$

,

$$v_{oc} = \begin{bmatrix} M\dot{\alpha} \\ 0 \end{bmatrix}$$

5.3 Kinematics of Contact

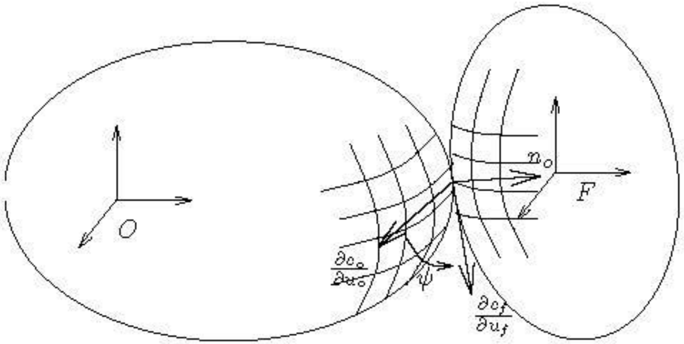
□ *Contact Kinematics:*

$$p_t \in S_0 \mapsto p_f(t) \in S_f$$

Local coordinate:

$$c_0 : U_0 \subset \mathbb{R}^2 \to S_0$$
$$c_f : U_f \subset \mathbb{R}^2 \to S_f$$

$$\alpha_0(t) = c_0^{-1}(p_0(t))$$
$$\alpha_f(t) = c_f^{-1}(p_f(t))$$

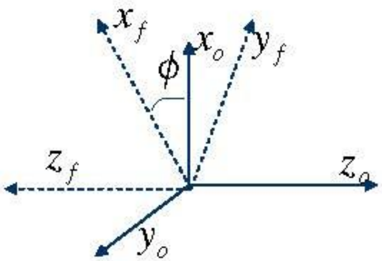


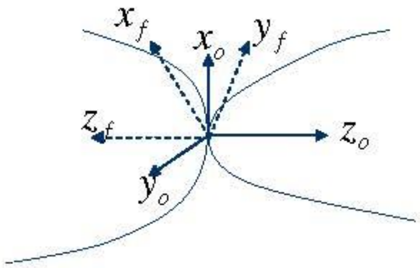
Angle of contact: ϕ
Contact coordinates: $\eta = (\alpha_f, \alpha_0, \phi)$

5.3 Kinematics of Contact

Rotation about the z-axis of C_o by $-\phi$ aligns the x axis of C_f with that of C_o

$$\Rightarrow R_{c_o c_f} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ -\sin\phi & -\cos\phi & 0 \\ 0 & 0 & -1 \end{bmatrix}, p_{c_o c_f} = 0 \in \mathbb{R}^3$$



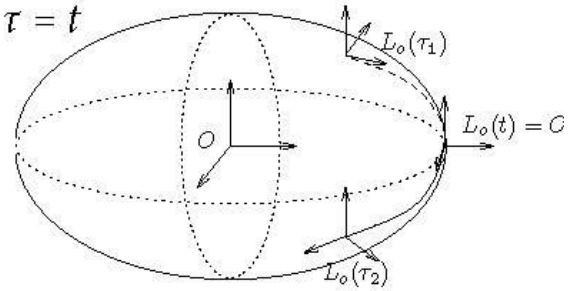


5.3 Kinematics of Contact

Define $L_0(\tau)$:

At $\tau = t$, $L_0(\tau)$ coincide with the Gauss frame at $p_0(t)$.

$L_f(\tau)$: coincide with $C_f(t)$ at $\tau = t$



$$v_{l_0 l_f} = (v_x, v_y, v_z),$$

$$\omega_{l_0 l_f} = (\omega_x, \omega_y, \omega_z),$$

$\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix}$: Rolling velocities

$\begin{bmatrix} v_x \\ v_y \end{bmatrix}$: Sliding velocities

v_z : Linear velocity in the normal direction

$V_{l_0 l_f} = Ad_{g_{f_{l_f}}} V_{of}$: Velocity of the finger relative to the object

5.3 Kinematics of Contact

Define:$\tilde{K}_0 = R_\phi K_0 R_\phi$:Curvature of O relative to C_f

$K_f + \tilde{K}_0$: Relative Curvature.

Theorem 3: Montana Equations of contact

$$\begin{cases} \dot{\alpha}_f = M_f^{-1}(K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} - \tilde{K}_o \begin{bmatrix} v_x \\ v_y \end{bmatrix}\right) \\ \dot{\alpha}_o = M_o^{-1} R (K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} + \tilde{K}_f \begin{bmatrix} v_x \\ v_y \end{bmatrix}\right)_\psi \\ \dot{\psi} = \omega_z + T_f M_f \dot{\alpha}_f + T_o M_o \dot{\alpha}_o \\ v_z = 0 \end{cases}$$

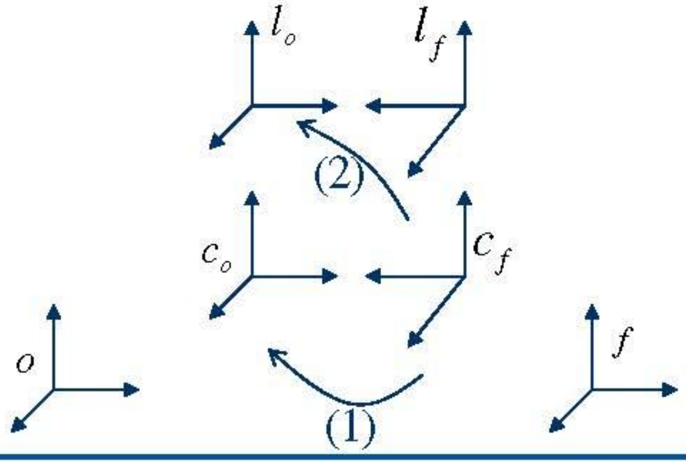
5.3 Kinematics of Contact

Proof of Theorem 3:

$$V_{fl_f} = 0$$

$$V_{fc_f} = Ad_{g^{-1}_{l_f c_f}} V_{fl_f} + V_{l_f c_f} = V_{l_f c_f}$$

$$V_{oc_o} = Ad_{g^{-1}_{l_o c_o}} V_{ol_o} + V_{l_o c_o} = V_{l_o c_o}$$



5.3 Kinematics of Contact

(1). At time t, $P_{l_f c_f} = 0$, $R_{l_f c_f} = I \Rightarrow V_{l_o c_f} = V_{l_o l_f} + V_{l_f c_f}$

$$(2)\ p_{c_o c_f} = 0 \Rightarrow V_{l_o c_f} = \begin{bmatrix} R^T_{c_o c_f} & 0 \\ 0 & R^T_{c_o c_f} \end{bmatrix} V_{l_o c_o} + V_{c_o c_f}$$

$$\Rightarrow V_{l_o l_f} + V_{f c_f} = \begin{bmatrix} R^T_{c_o c_f} & 0 \\ 0 & R^T_{c_o c_f} \end{bmatrix} V_{o c_o} + V_{c_o c_f}$$

$$p_{c_o c_f} = 0 \Rightarrow V_{c_o c_f} = 0 \qquad R_{c_o c_f} = \begin{bmatrix} R_\psi & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \omega_{c_o c_f} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$V_{l_o l_f} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, V_{f c_f} = \begin{bmatrix} M_f \dot{\alpha}_f \\ 0 \end{bmatrix}$$

$$\hat{\omega}_{f c_f} = \left[\begin{array}{cc|c} 0 & -T_f M_f \dot{\alpha}_f & K_f M_f \dot{\alpha}_f \\ T_f M_f \dot{\alpha}_f & 0 & \\ \hline \multicolumn{2}{c|}{-(K_f M_f \dot{\alpha}_f)^T} & 0 \end{array}\right]$$

5.3 Kinematics of Contact

$$v_{oc_o} = \begin{bmatrix} M_o \dot{\alpha}_o \\ 0 \end{bmatrix}, \quad \hat{\omega}_{oc_o} = \begin{bmatrix} 0 & -T_o M_o \dot{\alpha}_o & K_o M_o \dot{\alpha}_o \\ T_o M_o \dot{\alpha}_o & 0 & \\ -(K_o M_o \dot{\alpha}_o)^T & & 0 \end{bmatrix}$$

Linear component: $\begin{bmatrix} M_f \dot{\alpha}_f \\ 0 \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} R_\psi M_o \dot{\alpha}_o \\ 0 \end{bmatrix}$

$$\begin{bmatrix} K_f M_f \dot{\alpha}_f \\ T_f M_f \dot{\alpha}_f \end{bmatrix} + \begin{bmatrix} \omega_y \\ -\omega_x \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} - \begin{bmatrix} R_\psi K_o M_o \dot{\alpha}_o \\ T_o M_o \dot{\alpha}_o \end{bmatrix}$$

$\Rightarrow$ Theorem result $\square$

Corollary: Rolling contact motion.

$$\dot{\alpha}_f = M_f^{-1}(K_f + \tilde{K}_o)^{-1} \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} \quad \dot{\alpha}_o = M_o^{-1} R_\psi$$

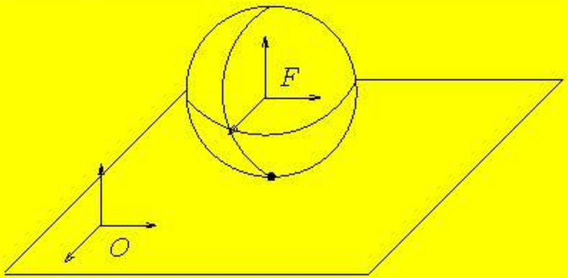
$$(K_f + \tilde{K}_o)^{-1} \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix} \dot{\psi} = T_f M_f \dot{\alpha}_f + T_o M_o \dot{\alpha}_o$$

5.3 Kinematics of Contact

◇ *Example: A sphere rolling on a plane*

$$c_f(u,v) = \begin{bmatrix} \rho \cos u_f \cos v_f \\ \rho \cos u_f \sin v_f \\ \rho \sin u_f \end{bmatrix}$$

$$c_o(u,v) = \begin{bmatrix} u_o \\ v_o \\ 0 \end{bmatrix}$$



$$K_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, K_f = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix}$$

$$M_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_f = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix},$$

$$T_o = \begin{bmatrix} 0 & 0 \end{bmatrix}, T_f = \begin{bmatrix} 0 & -\frac{1}{\rho}\tan u_f \end{bmatrix}$$

5.3 Kinematics of Contact

$$\begin{bmatrix} \dot{u}_f \\ \dot{v}_f \\ \dot{u}_o \\ \dot{v}_o \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ \sec u_f \\ -\rho \sin \psi \\ -\rho \cos \psi \\ -\tan u_f \end{bmatrix} \omega_x + \begin{bmatrix} -1 \\ 0 \\ -\rho \cos \psi \\ \rho \sin \psi \\ 0 \end{bmatrix} \omega_y$$

$$\dot{\eta} = g_1(\eta) \underbrace{\omega_x}_{u_1(t)} + g_2(\eta) \underbrace{\omega_y}_{u_2(t)} \quad (*)$$

Q:Given η_0, η_f, how to find a path u:o$[0, T] \rightarrow \mathbb{R}^2$ so that solution of (*) links η_0 to η_f?

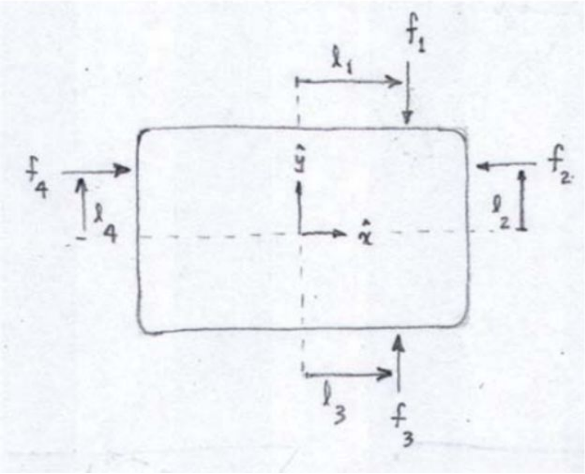
A question of nonholonomic motion planning!

Sources

MLS Ch 3

Static Equilibrium and Force Closure

Consider a rigid object in contact with a set of n stationary and frictionless point contacts. If an arbitrary external force is applied to the object, each of the point contacts may exert some reaction force in the direction of the object surface normal.



Static Equilibrium and Force Closure

A rigid object is considered in force closure if there exists a set of normal contact forces that satisfy static equilibrium (object can resist any external force and moment)

Formally, there must exist n non-negative scalars (x_1, … , x_n) such that

$$f_{\text{ext}} + \sum_{i=1}^{n} x_i \hat{n}_i = 0$$

$$m_{\text{ext}} + \sum_{i=1}^{n} x_i (r_i \times \hat{n}_i) = 0$$