EECS/ME/BioE 106B Homework 5: Grasping

Spring 2023

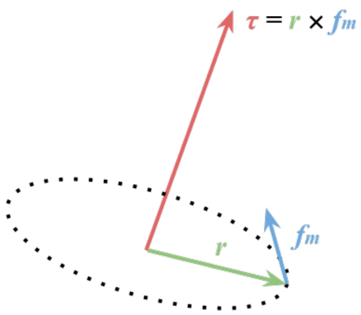
Problem 1: Adjoints and Wrench Transformations

How can we use robots to effectively grasp objects of arbitrary shapes and sizes? To answer this question, we will develop physical models for the interaction between robotic systems and objects. In this question, we'll begin constructing these models by discussing wrenches - a generalization of forces - and their associated transformations.

When we grasp an object, our fingers exert a *force* and a *moment* (*torque*) on the object. Whereas force represents a simple pushing force, moment represents a twisting force on an object. Recall that formally, a moment is defined to be the cross product:

$$\tau = r \times f_m \tag{1}$$

Where f_m is a force vector producing the twisting force and r is a vector from the point of rotation to the point where the force is applied.



By applying a force at a certain distance away from a point of rotation, we get a twisting motion that characterizes the moment. Force and moment may both be represented by three dimensional vectors:

$$f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}, \ \tau = \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} \tag{2}$$

With respect to some coordinate frame. Recall that the magnitude of the force vector indicates the strength of the force, while the direction indicates the direction along which the force is applied. Similarly, the magnitude of the moment vector indicates the strength of the twisting force and the direction of the moment vector represents the axis about which the twisting force is applied.

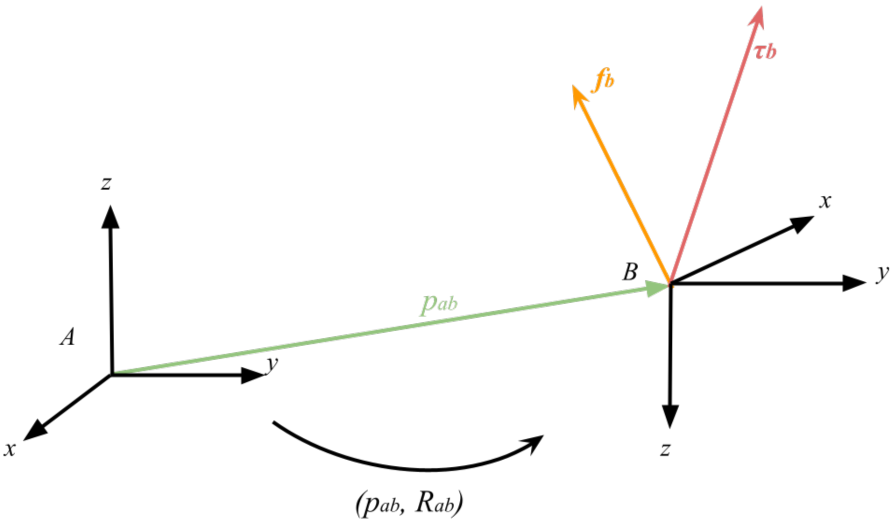
When considering the motion of rigid bodies, we'll combine force and moment into a single six dimensional vector called a *wrench*. Typically, wrenches are denoted by a capital letter F.

$$F = \begin{bmatrix} f \\ \tau \end{bmatrix} \tag{3}$$

Where f and τ are expressed with respect to the same coordinate frame. When examining the effects of these wrenches in different coordinate frames, we must *transform* the wrenches between different frames. Let's derive the form of these wrench transformations.

Questions

1. Consider the following scenario, where we have a force $f_b \in \mathbb{R}^3$ and moment $\tau_b \in \mathbb{R}^3$ are specified with respect to frame B.



Together, $F_b = [f_b, \tau_b] \in \mathbb{R}^6$ represents a wrench specified with respect to frame B. If the vector from the origin of frame A to the origin of frame B is p_{ab}, and the rotation matrix between frames A and B is R_{ab}, prove that we may represent F_b in frame A as:

$$F_a = \begin{bmatrix} R_{ab} & 0 \\ \hat{p}_{ab} R_{ab} & R_{ab} \end{bmatrix} F_b \tag{4}$$

Where $\wedge : \mathbb{R}^3 \to so(3)$ is the skew symmetric hat operator. *Hint: Does f_b apply a moment about the origin of frame A?*

2. Starting from the formula for the adjoint of a transformation $g = (p, R) \in SE(3)$:

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{5}$$

Prove that the transformation matrix defined above between F_b and F_a is the transpose of the adjoint of g_{ab}^{-1}:

$$Ad^T_{g_{ab}^{-1}} = \begin{bmatrix} R_{ab} & 0 \\ \hat{p}_{ab} R_{ab} & R_{ab} \end{bmatrix} \tag{6}$$

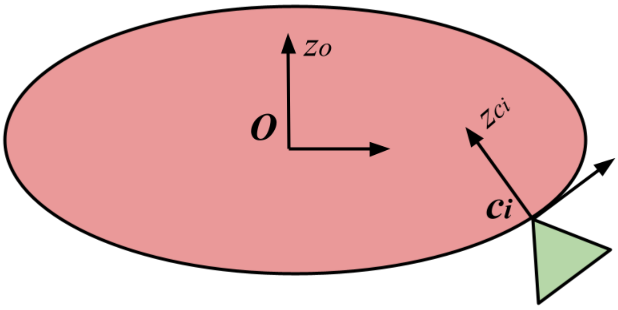
Where $g_{ab} = (p_{ab}, R_{ab}) \in SE(3)$ is the rigid body transformation:

$$g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \tag{7}$$

This proves that the transpose of an adjoint transforms wrenches between coordinate frames! *Hint: For a rotation matrix $R \in SO(3)$ and $p \in \mathbb{R}^3$, recall that $(Rp)^\wedge = R\hat{p}R^T$.*

Problem 2: Finding a Simple Grasp Map

Let's apply the theory of wrenches to study the interactions between robotic hands and rigid bodies. Suppose we have a robotic hand with n fingers, each of which contact a rigid body at a different point. We'll define a few coordinate frames associated with these contacts to develop a systematic method of treating the wrenches the fingers apply to the rigid body.



At the center of mass of the body, we'll define the body frame, which we'll conventionally denote by the letter O. At each of the n contact points between the robotic hand fingers and the rigid body, we'll define a contact frame, c_i, where by convention the z vector of the contact frame points inwards along the surface normal of the body.

How can we model the interactions between the fingers and the body? There are several types of *contact models* we may use to think about how the fingers apply forces to the body. The first, and simplest of these is the frictionless point contact. A frictionless point contact will simply apply a force to the body inwards along the surface normal at the point of contact. If contact i is a frictionless point contact, we can represent the contact wrench in frame i as:

$$F_{c_i} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} f_{c_i} \tag{8}$$

Where $f_{c_i} \geq 0$ is the magnitude of the contact force. Notice that $f_{ci} \geq 0$ since frictionless point contacts can only *push* on a rigid body and not pull.

A second, and more realistic type of contact is a *point contact with friction*. With respect to frame i, the wrench associated with this type of contact is given by:

$$F_{c_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} f_{c_i} \tag{9}$$

Where now, $f_{ci} \in \mathbb{R}^3$ is a 3D vector representing the contact force of the finger on the body. Since there is friction, forces may also be applied in the x and y direction in the contact frame in addition to the z direction.

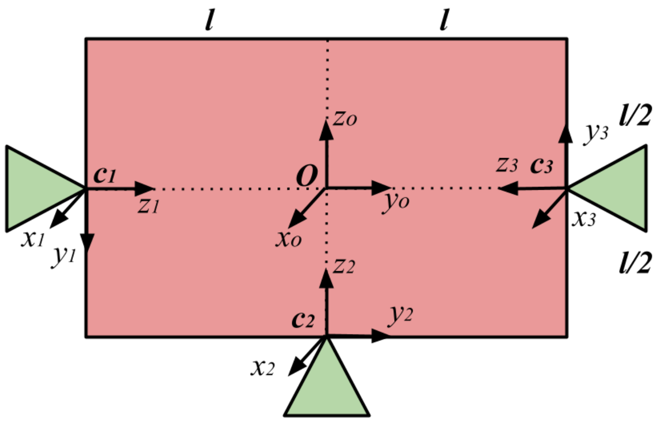
When working with arbitrary types of grasps, we refer to the matrices or vectors multiplied by f_{ci} as *wrench bases*, denoted by B_{c_i}. Each B_{c_i} provides with a mapping from f_{c_i}, the contact force from contact i, to F_{c_i}, the wrench associated with contact i in frame c_i. Thus, we have the general relation:

$$F_{c_i} = B_{c_i} f_{c_i} \tag{10}$$

Questions

Note: You may use a symbolic calculator such as MATLAB symbolic or SymPy if you wish to perform the computations in this section. If you choose to do this, please note where you used the calculator and attach a screenshot of your code.

1. Suppose we have the following points of contact between a rectangular rigid body of side lengths l and $2l$ and a three-fingered hand:



   Each finger contacts the rigid body at the center of one of its sides. If $g_{oc_i} \in SE(3)$ is the rigid body transformation matrix between the contact frame c_i and the body frame O, compute the following adjoint transformations:

$$Ad^T_{g^{-1}_{oc_1}},\ Ad^T_{g^{-1}_{oc_2}},\ Ad^T_{g^{-1}_{oc_3}} \tag{11}$$

   You may leave your answer in terms of the length l, and may assume that each contact frame is rotated by a multiple of 90 degrees with respect to the body frame.

2. With respect to the contact frame c_i, the wrench applied by contact c_i to the body is computed by taking the product $F_{c_i} = B_{c_i} f_{c_i}$. Show that the total wrench applied to the body in the body frame is computed by taking the product:

$$F_o = \begin{bmatrix} Ad^T_{g^{-1}_{oc_1}} B_{c_1} & Ad^T_{g^{-1}_{oc_2}} B_{c_1} & Ad^T_{g^{-1}_{oc_3}} B_{c_3} \end{bmatrix} \begin{bmatrix} f_{c_1} \\ f_{c_2} \\ f_{c_3} \end{bmatrix} \tag{12}$$

   The matrix containing the products of adjoints and wrench bases is known as the *grasp map*, G. This matrix maps from each finger contact force to the net body wrench.

3. If each finger is a point contact with friction, calculate the grasp map matrix G for the system in part 1. If there were a fourth point contact with friction applied to the rigid body, what would the shape of the grasp map matrix be?

Problem 3: Finding an N Finger Grasp Map

Let's generalize our results about grasping and contact modelling. If we have a rigid body and a robot hand with n fingers, we define the grasp map to be the matrix:

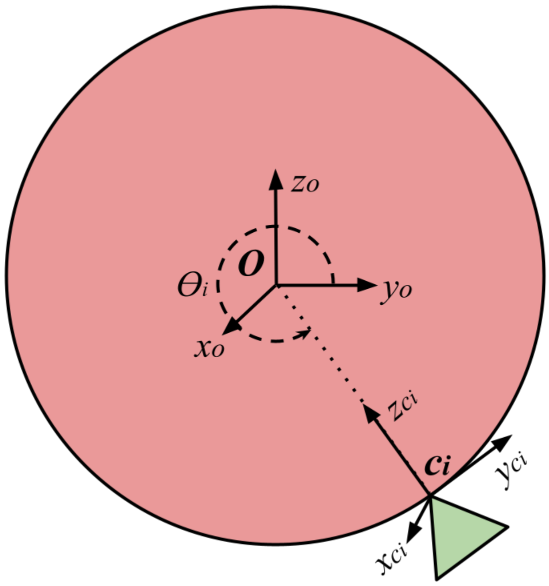
$$G = \begin{bmatrix} Ad^T_{g^{-1}_{oc_1}} B_{c_1} & Ad^T_{g^{-1}_{oc_2}} B_{c_1} & \dots & Ad^T_{g^{-1}_{oc_n}} B_{c_n} \end{bmatrix} \tag{13}$$

Where each c_i represents one of the finger contacts and each B_{c_i} is the wrench basis of the i^{th} contact. Let's derive the grasp map for a circular object in contact with n robotic fingers.

Questions

Note: You may use a symbolic calculator such as MATLAB symbolic or SymPy if you wish to perform the computations in this section. If you choose to do this, please note where you used the calculator and attach a screenshot of your code.

1. Consider the following system. Suppose we have a circular object of radius r that is being held by a robotic hand with n fingers.



Each contact c_i is at an angle θ_i, traced counter clockwise from the y_0 axis of the body frame, O. Find an expression for the adjoint transformation:

$$Ad^T_{g^{-1}_{oc_i}} \in \mathbb{R}^{6\times 6} \tag{14}$$

In terms of θ_i and r. You may assume that the y and z axes of all of the frames are within the same plane, and that the x axes point out of the page. Further, assume that all grasps make contact with the circle at $x = 0$.

2. We know that the grasp map for a system of n grasps may be computed using:

$$G = \begin{bmatrix} Ad^T_{g^{-1}_{oc_1}} B_{c_1} & Ad^T_{g^{-1}_{oc_2}} B_{c_1} & \dots & Ad^T_{g^{-1}_{oc_n}} B_{c_n} \end{bmatrix} \tag{15}$$

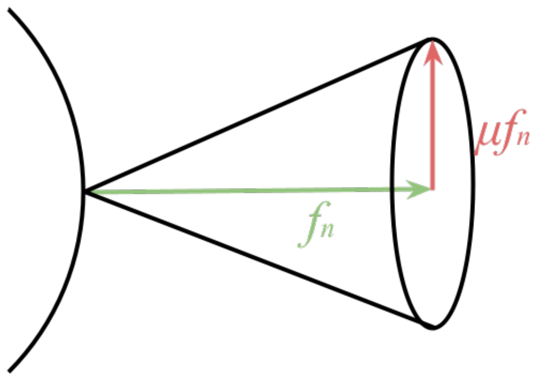
If each contact for the system discussed above is a frictionless point contact, find an expression for:

$$Ad^T_{g^{-1}_{oc_i}} B_{c_i} \tag{16}$$

This gives an expression for each column of the grasp map for our circular object!

Problem 4: Force Closure Grasps

Let's think a little bit more deeply about the contacts between robotic fingers and rigid bodies. When a finger comes into contact with a body, the body exerts a normal force onto the finger along the surface normal of the body at the point of contact.



If the contact is a simple point contact with friction, a static frictional force proportional in magnitude to the normal force will be generated by the contact. This frictional force will be *tangent* to the surface at the point of contact. All possible frictional forces must lie within a set called the *friction cone*, depicted above, where the largest frictional force without the contact slipping is given by:

$$f_f = \mu f_n \tag{17}$$

Where f_n is the magnitude of the normal force, f_f is the magnitude of the frictional force, and μ is a coefficient known as the coefficient of static friction.
For this case, we may mathematically define the friction cone to be the set:

$$FC = \{f \in \mathbb{R}^3 : \sqrt{f_1^2 + f_2^2} \leq \mu f_3, f_3 \geq 0\} \tag{18}$$

Where f_1, f_2 are frictional forces and f_3 is the force normal to the surface in the contact frame. For a robotic hand with n points of contact, we refer to the friction cone associated with each point of contact as FC_{c_i}. Thus, to consider the set of *all* possible combinations of contact-related forces, we define the total friction cone, FC, as:

$$FC = FC_{c_1} \times FC_{c_2} \times ... \times FC_{c_n} \tag{19}$$

Where $\times$ represents the Cartesian product, an operation that returns a set containing all combinations of the set elements. The elements of the total friction cone FC are vectors containing each contact force f_{c_i}.

$$FC = [f_{c1}^T, f_{c_2}^T, ..., f_{c_n}^T]^T \tag{20}$$

Using the friction cone, we may determine if our grasp on the rigid body is secure enough to resist *any* external wrench applied to the rigid body. If G is the grasp map of the robotic hand on the rigid body, we say that the grasp will be able to resist *any* external wrench $F_e \in \mathbb{R}^6$ if:

$$G(FC) = \mathbb{R}^6 \tag{21}$$

Since we must be able to oppose *any* external wrench in $\mathbb{R}^6$ using the contact forces of the fingers. A grasp that has the property $G(FC) = \mathbb{R}^6$ is said to be a *force closure* grasp. Let's develop some conditions to determine if a grasp is force closure!

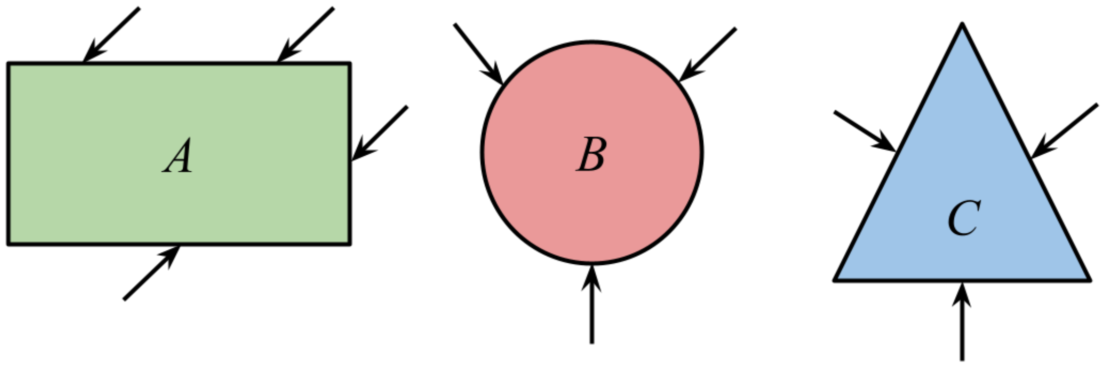
Questions

In this question, we'll consider the case where we have n frictionless point contacts on a rigid body. The friction cone for each frictionless point contact c_i is simply the set of scalar normal forces:

$$FC_{c_i} = \{f \in \mathbb{R} : f \geq 0\} \tag{22}$$

As we now don't have to deal with frictional forces. Note that $f \geq 0$ since a frictionless point contact can only push on a rigid body, not pull.

1. Let's gain an intuitive feel for what a force closure grasp is. Consider the following planar systems, where external forces may only be applied within the plane of the page and external moments may only be applied along the axis coming out of the page.

   If each arrow represents a frictionless point contact, determine if the grasps above are force closure without performing any calculations. If the grasp is not force closure, sketch a force vector or moment vector that the grasp would not be able to resist. You may assume B is a perfect circle. *Hint: Imagine pulling and turning the object in each direction.*

2. Suppose we have a set of n frictionless point contacts on a rigid body. What is the shape of the grasp map G for a system of n frictionless point contacts? *Hint: What is the wrench basis for a frictionless point contact?*

3. The friction cone FC for a set of n frictionless point contacts is the set of vectors:

   $$FC = \{[f_1, ..., f_n]^T : f_i \geq 0\} \subseteq \mathbb{R}^n \tag{23}$$

   Let $G_1, ..., G_n$ be the columns of the grasp map G for this grasp. Prove that this grasp is force closure if and only if for all $v \in \mathbb{R}^6$ there exist positive constants $\alpha_i \geq 0$ such that:

   $$v = \alpha_1 G_1 + ... + \alpha_n G_n. \tag{24}$$

   Hint: A grasp is force closure if and only if $G(FC) = \mathbb{R}^6$. Split up G into its columns $G_1, ..., G_n$.

4. Prove that the minimum number of vectors $v_1, ..., v_k \in \mathbb{R}^n$ such that for all $v \in \mathbb{R}^n$, there exist constants $\alpha_i \geq 0$ where:

   $$v = \alpha_1 v_1 + ... + \alpha_k v_k \tag{25}$$

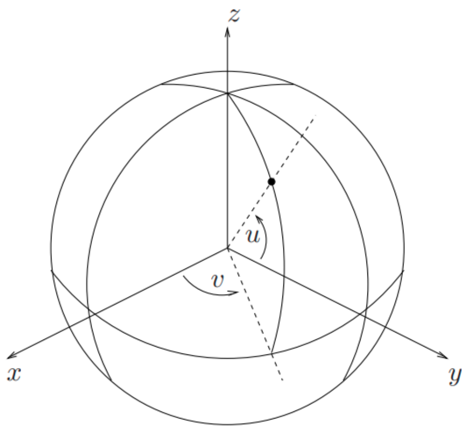
   Is $k = n + 1$. Using this result, what is the minimum number of frictionless point contacts required to have a force closure grasp of a 3D (non-planar) rigid body with applied wrenches $F_o \in \mathbb{R}^6$? *Hints: Start by proving a set of $k \leq n$ vectors cannot span $\mathbb{R}^n$ in this manner. Then, consider a basis $\{v_1, ..., v_n\}$ and $v_{n+1} = -\lambda_1 v_1 - ... - \lambda_n v_n$, $\lambda_i > 0$.*

Problem 5: Surface Geometry & Kinematics

In the real world, rigid objects and robotic fingers are most accurately modeled as *surfaces.* Instead of having a simple point contact between an object and a surface that remains stationary in time, a robotic finger might roll across a surface as time passes.
To precisely understand how this rolling motion occurs, we must develop an understanding of the geometry of the surfaces we wish to grasp. In this question, we'll develop some fundamental aspects of surface geometry that may be applied to describe the rolling motion of surfaces.

Questions

Note: You may use a symbolic calculator such as MATLAB symbolic or SymPy if you wish to perform the computations in this section. If you choose to do this, please note where you used the calculator and attach a screenshot of your code.
In this question, we'll consider the geometry of a sphere, pictured below:



As this sphere is a surface embedded in three dimensions, we only need two coordinates, u and v, to describe the location of a point on the surface. Let's explore the consequences of this (u, v) surface parameterization, and see what it tells us about the geometric properties of the shape.

1. First, we'd like to understand how we can move between our two dimension parameterization of the surface $(u, v) \in \mathbb{R}^2$ and the actual three-dimensional coordinates corresponding to (u, v). To accomplish this, we define a special type of function called a *chart*:

$$c : (u, v) \in \mathbb{R}^2 \to \mathbb{R}^3 \tag{26}$$

With respect to the object coordinate frame, defined to be at the center of the sphere in the image above, a chart of the sphere with respect to (u, v) is:

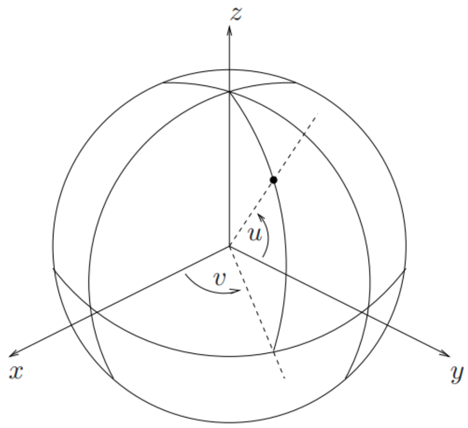
$$\begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} = \begin{bmatrix} \rho \cos u \cos v \\ \rho \cos u \sin v \\ \rho \sin u \end{bmatrix} \tag{27}$$

Where ρ is the radius of the sphere.
It may be shown that the plane tangent to the surface at each (u, v) is spanned by:

$$c_u = \frac{\partial c}{\partial u}, \; c_v = \frac{\partial c}{\partial v} \tag{28}$$

Find the vectors c_u and c_v using the chart provided and prove that they are orthogonal.

2. Using the cross product between c_u, c_v, we can identify a unit normal vector n to the surface. We can use the $\{c_u, c_v, n\}$ orthogonal coordinate frame to define the contact frames of robotic fingers on the surface of the object. Assuming n points outwards on the sphere, sketch the $\{c_u, c_v, n\}$ coordinate frame at the marked point on the sphere:



3. The tangent space to a three-dimensional surface at a point is defined to be the two dimensional plane tangent to the surface at that point. If $x, y \in \mathbb{R}^2$ are two vectors in the tangent space of the sphere at $p = (u, v)$, we define the inner product between these vectors to be:

$$x^T \begin{bmatrix} c_u^T c_u & c_u^T c_v \\ c_v^T c_u & c_v^T c_v \end{bmatrix} y = x^T I_p y \tag{29}$$

The matrix I_p is known as the *first fundamental form* matrix for a surface, and contains $||c_u||^2$ and $||c_v||^2$. Related to I_p, we define the *metric tensor* to be the matrix $M_p \in \mathbb{R}^{2\times 2}$ such that:

$$I_p = M_p M_p \tag{30}$$

Show that M_p for the sphere is computed:

$$M_p = \begin{bmatrix} \rho & 0 \\ 0 & \rho \cos u \end{bmatrix} \tag{31}$$

Note: Metric is a name for a measure of length - this gives the metric tensor its name.

4. Let's bring our discussion back to kinematics. Suppose we have a trajectory $p(t) \in \mathbb{R}^3$ that traces out a path on the surface of the sphere, defined:

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \rho\cos(\omega_u t)\cos(\omega_v t) \\ \rho\cos(\omega_u t)\sin(\omega_v t) \\ \rho\sin(\omega_u t) \end{bmatrix} \tag{32}$$

Using the chart c, convert this 3D path into $\alpha(t) = (u(t), v(t)) \in \mathbb{R}^2$ coordinates. Then, calculate $\dot{\alpha}(t)$.

5. Let's define two other objects, the *curvature tensor* K_p and the *torsion form* T_p, which tell us about curvature and its rate of change along the surface. For the sphere, it may be shown that K_p and T_p are computed:

$$K_p = \begin{bmatrix} 1/\rho & 0 \\ 0 & 1/\rho \end{bmatrix}, \; T_p = \begin{bmatrix} 0 & -(1/\rho)\tan u \end{bmatrix} \tag{33}$$

Consider the following theorem concerning the use of M_p, K_p, T_p in kinematics.

Theorem 1 ***Body Velocity for a Path***
The body velocity of the contact frame C (given by the vectors $\{c_u, c_v, n\}$) with respect to the body frame O, is given by $V_{oc}^b = (v_{oc}, \omega_{oc})$, where:

$$v_{oc} = \begin{bmatrix} M_p\dot{\alpha} \\ 0 \end{bmatrix} \tag{34}$$

$$\hat{\omega}_{oc} = \left[\begin{array}{cc|c} 0 & -TM\dot{\alpha} & \\ TM\dot{\alpha} & 0 & \smash{\raisebox{1ex}{$KM\dot{\alpha}$}} \\ \hline \multicolumn{2}{c|}{-(KM\dot{\alpha})^T} & 0 \end{array}\right] \tag{35}$$

Use this theorem to calculate v_{oc} and $\hat{\omega}_{oc}$ for the provided path along the sphere. This gives us an expression for rigid body velocity along the path!

Problem 6: Lagrange Multipliers for Constrained Motion

When grasping objects using robotic systems, it's not only important for us to have an understanding of the kinematics of grasping, but it's also valuable to understand the *dynamics* of grasping. By studying the dynamics associated with grasping, we can derive differential equations of motion that help us understand how contact forces impact the motion of a rigid body. When studying the dynamics of grasping, we must formally account for the presence of constraints - both holonomic and nonholonomic, on the dynamics of our system. In this question, we'll perform an analysis of a simple system under the influence of a constraint.
Recall from our study of kinematic constraints that a set of k constraints may be written in Pfaffian form as:

$$A(q)\dot{q} = 0 \tag{36}$$

Where $q \in \mathbb{R}^n$ is the vector of generalized coordinates used to describe the motion of the system in question and $A(q) \in \mathbb{R}^{k \times n}$. If this constraint is *holonomic*, or integrable, there exists a vector-valued function $h(q) \in \mathbb{R}^k$ such that:

$$A(q)\dot{q} = 0 \leftrightarrow h(q) = 0 \tag{37}$$

A *constraint force*, Γ, is a generalized force $\Gamma \in \mathbb{R}^n$ that ensures the system obeys its kinematic constraints. The constraint force Γ for a Pfaffian constraint $A(q)\dot{q} = 0$ is of the form:

$$\Gamma = A^T(q)\lambda \tag{38}$$

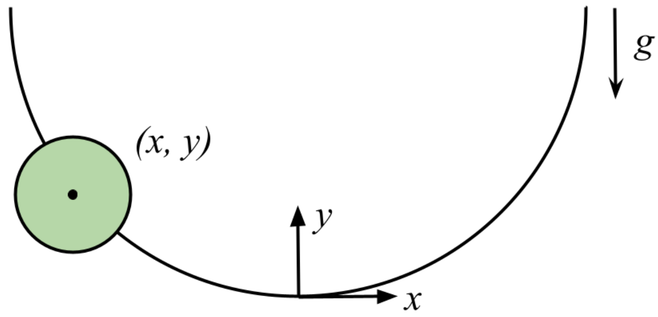
Where $\lambda \in \mathbb{R}^k$ is a vector containing the relative strengths of each constraint force. Each λ_i inside the vector λ is an unknown called a *Lagrange multiplier*. When writing out the constraint force in this form, the value of each Lagrange multiplier is unknown - each must be solved for by looking at the dynamics and constraint equations!
To consider the effect of the constraint forces on the system, we treat the constraint forces as external forces on the system. Assuming the constraint force does no work on the system, we may write the set of Lagrange's equations in the presence of a constraint force as:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) - \frac{\partial L}{\partial q} + A^T(q)\lambda = \Upsilon \tag{39}$$

Where Υ is a vector of other external forces applied to the system and q is the vector of generalized coordinates of the unconstrained system, and L is the unconstrained Lagrangian - the Lagrangian as if no constraints existed on the system.
Once the set of n Lagrange's equations have been computed, the Lagrange multipliers may be solved for by looking at the constraint equation $A(q)\dot{q} = 0$ and the Lagrangian equations of motion. Typically, each Lagrange multiplier will be a function of q, $\dot{q}$, and Υ.
Note that the method of Lagrange multipliers is not the only way of dealing with constrained systems! Rather, it gives us a systematic method of working from the *unconstrained* system towards the constrained system. Other methods relying on directly substituting constraints into the Lagrangian may also be used.
Let's analyze and apply the method of Lagrange multipliers to determine the equations of motion of a simple constrained system!

Questions

Consider the following planar system, in which a particle slides without friction along a parabolic path of equation $y = x^2$:



The center of mass of the particle is described by the coordinates (x, y).

1. Let's begin by thinking about the form of a constraint force for a holonomic constraint. A holonomic constraint $h(q) = 0$ is a vector-valued function composed of k scalar constraint functions, $h_i(q)$, where $q \in \mathbb{R}^n$ is a vector of generalized coordinates.

$$h(q) = \begin{bmatrix} h_i(q) \\ \vdots \\ h_k(q) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{40}$$

   Recall that for a Pfaffian constraint $A(q)\dot{q} = 0$, the associated constraint force $\Gamma \in \mathbb{R}^n$ is of the form $\Gamma = A^T(q)\lambda$, where $\lambda \in \mathbb{R}^k$ is a vector of scalar Lagrange multipliers.
   Each constraint $h_i(q) = 0$ represents a surface in space. Show that for a holonomic constraint $h(q) = 0$, the constraint force Γ is a linear combination of the surface normals of the constraint surfaces $h_i(q) = 0$. *Hint: From calculus, $\partial h_i / \partial q$ is normal to the surface $h_i(q) = 0$.*

2. Let's turn our attention to the particle sliding on the parabola. Show that the Lagrangian of the unconstrained particle system is computed:

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \tag{41}$$

   Where m is the mass of the particle.

3. This system is constrained to travel along the parabolic path $y = x^2$. Show that if $q = [x, y]^T$, this constraint may be represented in Pfaffian form as:

$$A(q)\dot{q} = \begin{bmatrix} -2x & 1 \end{bmatrix} \dot{q} = 0 \tag{42}$$

4. If the constraint force on the system is of the form $\Gamma = A^T(q)\lambda \in \mathbb{R}^3, \lambda \in \mathbb{R}^2$, use the Lagrange multiplier form of Lagrange's equations:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) - \frac{\partial L}{\partial q} + A^T(q)\lambda = \Upsilon \tag{43}$$

   To find the equations of motion of the system in terms of λ. You may assume $\Upsilon = 0$.

5. Use the equations of motion and the constraint equation $A(q)\dot{q} = 0$ to solve for the Lagrange multiplier λ, and $\Gamma = A^T(q)\lambda$ in terms of $q, \dot{q}, m, I$.