EECS C106B / 206B Robotic Manipulation and Interaction **Spring 2020**

Lecture 13: Grasping 2: Contact Forces

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13.1 Week Preview

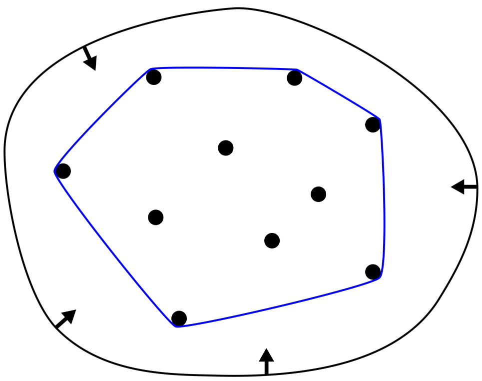
- Next week: Soft robotics, Manipulating Objects
- Next Thursday, guest lecture on soft robotics by Professor from Stanford

13.2 Additional Resources

- A Mathematical Introduction to Robotic Manipulation Chapter 5
- David Montana: The Kinematics of Contact and Grasp

13.3 Convex Hull and Force-Closure

- A convex hull of a set is a the smallest convex set that encloses all the points
- Rubber Band Analogy: If we stretch a rubber band over the points, it will ”shrink” to become the convex hull



- Geometrically: Any point on the boundary, connected to another point on boundary line does not cross boundary. In other words, any non-negative linear combination is in the set.
- Algebraically:
$$S = \{a_1, \cdots, a_n\}, a_i \in \mathbb{R}^m,\ i = 1, \cdots, m$$
$$\text{Convex Hull of } S = \left\{ \sum_{i=1}^{n} w_i a_i \middle| \sum_{i=1}^{n} w_i = 1, w_i \geq 0, \forall i \right\}$$
- **Definition 5.2. Force-closure grasp** A grasp is a force-closure grasp if given any external wrench $F_e \in \mathbb{R}^p$ applied to the object, there exist contact forces $f_c \in FC$ such that $Gf_c = -F_e$.
- **Proposition 5.3. Convexity conditions for force-closure grasps** Consider a fixed contact grasp which contains only frictionless point contacts. Let $G \in R^{p \times m}$ be the associated grasp matrix and let $\{G_i\}$ denote the columns of G. The following statements are equivalent:
  - The grasp is force-closure.
  - The columns of G positively span $\mathbb{R}^p$
  - The convex hull of $\{G_i\}$ contains a neighborhood of the origin.
  - There does not exist a vector $v \in \mathbb{R}^p, v \neq 0$, such that for $i = 1, \cdots, m$, $v \cdots G_i \geq 0$.

13.4 Static Grasping

- To have static grasping such that things don't slip or fall, we need to ensure a static grasp and have balanced forces.
- Consider a rigid body in $\mathbb{R}^3$ constrained by friction-less point contacts so that there is only normal force
- r_i denotes vector from reference frame origin to contact point i
- $\hat{n}_i$ denotes vector normal to body at contact point i toward interior of body
- F_i describes contact force at contact i: $F_i = \hat{n}_i \times i$

$$\begin{bmatrix} n_i & \cdots & n_n \\ r_1 \times n_1 & \cdots & r_n \times n_n \end{bmatrix}_{6\times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{\text{position of contacts}} = \begin{bmatrix} -f_e \\ -m_e \end{bmatrix} \begin{matrix} \text{(external translation force)} \\ \text{(external torsional force)} \end{matrix}$$

- Forces should be balanced so that object does not move. This allows it to then be picked up and manipulated.

13.5 Grasping with Friction

planar_friction.png

- Planar Friction Example
- Forces must be kept in balance
- Interior angle of cone is 2α, where $\mu = \tan(\alpha)$
- $e_1, e_2, \in \mathbb{R}^2$ are vectors aligned with edges of the left friction cone.
- $e_3, e_4, \in \mathbb{R}^2$ are vectors aligned with edges of the right friction cone.

$$e_1 = \begin{bmatrix} 1 \\ \mu \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ -\mu \end{bmatrix}, e_3 = \begin{bmatrix} -1 \\ \mu \end{bmatrix}, e_4 = \begin{bmatrix} -1 \\ \mu \end{bmatrix}$$

$$f_a = e_1 x_1 + e_2 x_2, \; f_b = e_3 x_3 + e_4 x_4$$

- Force Closure Condition: For arbitrary external force $f_e \in \mathbb{R}^2$ and external moment/torsion $m_e \in \mathbb{R}$ there must exist contact forces f_a, f_b, such that :

$$f_a + f_b = -f_e, \; m_a + m_b = -m_e$$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ \mu & -\mu & -\mu & \mu \\ -\mu r & \mu r & -\mu r & \mu r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -f_e \\ -m_e \end{bmatrix}$$

13.6 Kinematics of Contact, Rolling Contact Kinematics

- Concerned with Geometry and Velocity: no Dynamics
- Differential Geometry & Surface Models

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  - Given an object in $\mathbb{R}^3$, the surface can be described using a *local coordinate chart* $c : U \subset \mathbb{R}^2 \to \mathbb{R}^3$
  - The map $c(u, v)$ takes point $(u, v) \in \mathbb{R}^3$ to a point $x \in \mathbb{R}^3$ on the surface of the object in coordinates of O.
  - Like on a sphere you can do 2D localization even though position is in 3D using the tangent plane.
  - We want to find a tangent direction to point of contact to find the directions of friction.
  - **Definition:** A surface S is *regular* if for each point $p \in S$ there exists a neighborhood $V \subset \mathbb{R}^3$, an open set $U \subset \mathbb{R}^2$, and a map $c : U \to V \cap S$ such that:
    1. c is differentiable.
    2. c is a homeomorphism from U to $V \cap S$. That is, c is continuous, bijective (one-to-one and onto), and has a continuous inverse.
    3. For every $\alpha = (u, v) \in U$, the map $\frac{\partial c}{\partial \alpha}(\alpha) : \mathbb{R}^2 \to \mathbb{R}^3$ is injective (one-to-one).
  - At any point on the object, we can define a tangent plane which consists of vectors tangent to the surface of the object at that point. This is spanned by:

$$c_u := \frac{\partial c(u, v)}{\partial u}, c_v := \frac{\partial c(u, v)}{\partial v}$$

  - The coordinate chart is an orthogonal coordinate chart if c_u and c_v are orthogonal.
  - **Theorem 5.9**. Locally, there exists an orthogonal chart for all regular surfaces.

– First Fundamental Form: describes how the inner product of two tangent vectors is related to the natural inner product

$$I_p = \begin{bmatrix} c_u^\intercal c_u & c_u^\intercal c_v \\ c_v^\intercal c_u & c_v^\intercal c_v \end{bmatrix} == \begin{bmatrix} ||c_u||^2 & 0 \\ 0 & ||c_v||^2 \end{bmatrix} \text{ if orthogonal parameterization}$$

– Metric Tensor:

$$I_p = M_p \cdot M_p$$

$$M_p = I_p^{\frac{1}{2}} = \begin{bmatrix} ||c_u|| & 0 \\ 0 & ||c_v|| \end{bmatrix} \text{if orthogonal parameterization}$$

– Gauss Map: gives the unit normal at each point on the surface S. For smooth, orientable surfaces, the Gauss map is a well defined, differentiable mapping. Describes the rate of change of the normal vector projected onto the tangent plane.

$$N : S \to \S^2 : N(u,v) = \frac{c_u \times c_v}{||c_u \times c_v||} := n$$

– Second Fundamental Form: a measure of the curvature of a surface.

$$II_p = \begin{bmatrix} c_u^\intercal n_u & c_u^\intercal n_v \\ c_v^\intercal n_u & c_v^\intercal n_v \end{bmatrix}, \text{ where } n_u := \frac{\partial n}{\partial u},\ n_v := \frac{\partial n}{\partial v}$$

– Curvature Tensor:

$$K_p = M_p^{-1\intercal} II_p M_p^{-1} = \begin{bmatrix} \frac{c_u^\intercal n_u}{||c_u||^2} & \frac{c_u^\intercal n_v}{||c_u||||c_v||} \\ \frac{c_v^\intercal n_u}{||c_v||||c_u||} & \frac{c_v^\intercal n_v}{||c_v||^2} \end{bmatrix}$$

– Normalized Gauss Frame:

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} \frac{c_u}{||c_u||} & \frac{c_v}{||c_v||} & n \end{bmatrix},\ K_p = \begin{bmatrix} x^\intercal \\ y^\intercal \end{bmatrix} \begin{bmatrix} \frac{n_u}{||c_u||} & \frac{n_v}{||c_v||} \end{bmatrix}$$

– Torsion Form:

$$T_p = y^\intercal \begin{bmatrix} \frac{x_u}{||c_u||} & \frac{x_v}{||c_v||} \end{bmatrix} = \begin{bmatrix} \frac{c_v^\intercal c_{uu}}{||c_u||^2||c_v||} & \frac{c_v^\intercal c_{uv}}{||c_u||||c_v||^2} \end{bmatrix},\ x_u = \frac{\partial x}{\partial u}, x_v = \frac{\partial x}{\partial v}, c_{uu} = \frac{\partial^2 c}{\partial u^2}, c_{uv} = \frac{\partial^2 c}{\partial u \partial v}$$

– Geometric Parameters of a Surface: (M_p, K_p, T_p)

• Sphere Example:

5-20.png

$$c(u,v) = \begin{bmatrix} \rho\cos(u)\cos(v) \\ \rho\cos(u)\sin(v) \\ \rho\sin(u) \end{bmatrix}$$

$$U = \left\{(u,v)\middle| -\frac{\pi}{2} < u < \frac{\pi}{2},\ -\pi < v < \pi\right\}$$

$$c_u = \frac{\partial c(u,v)}{\partial u} = \begin{bmatrix} -\rho\sin(u)\cos(v) \\ -\rho\sin(u)\sin(v) \\ \rho\cos(v) \end{bmatrix},\ c_v = \frac{\partial c(u,v)}{\partial v} = \begin{bmatrix} -\rho\cos(u)\sin(v) \\ \rho\cos(u)\cos(v) \\ 0 \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix},\ M = \begin{bmatrix} \rho & 0 \\ 0 & \rho\cos(u) \end{bmatrix},\ T = \begin{bmatrix} 0 & -\frac{1}{\rho}\tan(u) \end{bmatrix}$$

Lemma: Induced velocity of the contact frame The body velcoity of the contact frame C relative to the reference frame O of the object is given by $V_{oc}^b = (v_{oc}, \omega_{oc})$ where

$$v_{oc} = \begin{bmatrix} M\dot{\alpha} \\ 0 \end{bmatrix}$$

$$\hat{\omega}_{oc} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -TM\dot{\alpha} & KM\dot{\alpha} \\ TM\dot{\alpha} & 0 & \\ -(KM\dot{\alpha})^{\mathsf{T}} & & 0 \end{bmatrix}$$

and M, K, and T are the geometric parameters of the surface relative to the coordinate chart (c, U).

$$p_{oc} = p(t) = c(\alpha(t))$$

$$R_{oc} = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = \begin{bmatrix} \frac{c_u}{||c_u||} & \frac{c_v}{||c_v||} & \frac{c_u \times c_v}{||c_u \times c_v||} \end{bmatrix}$$

$$v_{oc} = R_{oc}^{\mathsf{T}} \dot{p}_{oc} = \begin{bmatrix} x^{\mathsf{T}} \\ y^{\mathsf{T}} \\ z^{\mathsf{T}} \end{bmatrix} \frac{\partial c}{\partial \alpha} \dot{\alpha} = \begin{bmatrix} x^{\mathsf{T}} c_u & x^{\mathsf{T}} c_v \\ y^{\mathsf{T}} c_u & y^{\mathsf{T}} c_v \\ z^{\mathsf{T}} c_u & z^{\mathsf{T}} c_v \end{bmatrix} \dot{\alpha} = \begin{bmatrix} M\dot{\alpha} \\ 0 \end{bmatrix}$$

$$\hat{\omega}_{oc} = R_{oc}^{\mathsf{T}} \dot{R}_{oc} = \begin{bmatrix} x^{\mathsf{T}} \\ y^{\mathsf{T}} \\ z^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & x^{\mathsf{T}}\dot{y} & x^{\mathsf{T}}\dot{z} \\ y^{\mathsf{T}}\dot{x} & 0 & y^{\mathsf{T}}\dot{z} \\ z^{\mathsf{T}}\dot{x} & z^{\mathsf{T}}\dot{y} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -TM\dot{\alpha} & KM\dot{\alpha} \\ TM\dot{\alpha} & 0 & \\ -(KM\dot{\alpha})^{\mathsf{T}} & & 0 \end{bmatrix}$$

- Contact Kinematics

5-21.png

  - Surfaces S_o, S_f, touching at a point. $p_o(t) \in S_o,\ p_f(t) \in S_f$
  - We have two charts: $(c_o, U_o), (c_f, U_f)$ with local coordinates $\alpha_o = c_o^{-1}(p_o) \in U_o, \alpha_f = c_f^{-1}(p_f) \in U_f$
  - ψ is angle of contact, defined as angle between $\frac{\partial c_f}{\partial u_f}$ and $\frac{\partial c_o}{\partial u_o}$.
  - Contact Coordinates: $\eta = (\alpha_f, \alpha_O, \psi)$
  - $g_{of} \in SE(3)$ describe the relative position and orientation of S_f with respect to S_o. Assume that $g_{of} \in W \subset SE(3)$
  - C_o, C_f are coordinate frames that coincide with Gauss frame at $p_o(t), p_f(t)$ for all $t \in I$, where I is interval of interest.
  - $L_o(\tau), L_f(\tau)$ are local frames fixed to O, F respectively and coincide at time $t = \tau$ with $p_o(t), p_f(t)$.

- Let $v_{l_o l_f} = (v_x, v_y, v_z)$ be components of translational velocity of $L_f(t)$ relative to $L_o(t)$ at time t. (v_x, v_y) are linear velocities (sliding velocities) along the tangent plane, v_z is linear velocity in the direction contact normal.
- Let $\omega_{l_o l_f} = (\omega_x, \omega_y, \omega_z)$ be the (body) rotational velocity. (ω_x, ω_y) are rolling velocities along tangent plane at the point of contact, ω_z is rotational velocity about the contact normal.
- While bodies are in contact $v_z = 0$. If pure rolling contact, $v_x = v_y = 0, \omega_z = 0$. If pure sliding contact, $\omega_x = \omega_y = \omega_z = 0$.
- Kinematic equations of contact

$$\begin{aligned}
\dot{\alpha}_f &= M_f^{-1}(K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix}-\omega_y \\ \omega_x\end{bmatrix} - \tilde{K}_o\begin{bmatrix}v_x \\ v_y\end{bmatrix}\right) \\
\dot{\alpha}_o &= M_o^{-1}R_\psi(K_f + \tilde{K}_o)^{-1}\left(\begin{bmatrix}-\omega_y \\ \omega_x\end{bmatrix} - K_f\begin{bmatrix}v_x \\ v_y\end{bmatrix}\right) \\
\dot{\psi} &= \omega_z + T_f M_f \dot{\alpha}_f + T_o M_o \dot{\alpha}_o \\
0 &= v_z
\end{aligned}$$