

## Discussion #4

Author: Valmik Prabhu

### Problem 1 - Pfaffian Constraints

When we study nonholonomic systems, we are concerned with velocity constraints, or constraints on the derivative of our system state. You can represent a velocity constraint as the level set of any function incorporating state and velocity  $h(q, \dot{q}) = 0$ , but we are specifically interested in the subset of velocity constraints that can be represented as

$$A(q)\dot{q} = 0$$

$A$  is a  $k \times n$  matrix, where  $n$  is the number of states, and  $k$  is the number of constraints. This kind of constraint is called a Pfaffian constraint.

MLS uses two notations to describe Pfaffian constraints. It describes a set of Pfaffian constraints as  $A(q)\dot{q} = 0$ , and a singular constraint as  $w(q)\dot{q} = 0$ . In this case,

$$A = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{bmatrix}$$

1. Let's reconsider the Raibert hopper discussed in lecture

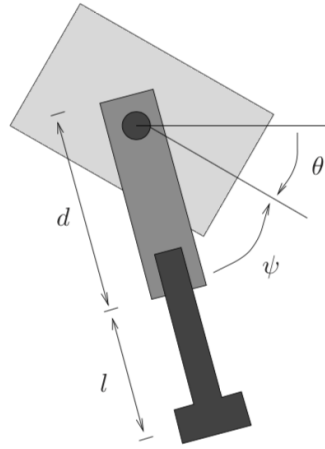


Figure 1: A simplified model of a Raibert Hopper

This system has three state variables  $q = (\phi, l, \theta)$ , and two inputs  $\dot{\phi}$  and  $\dot{l}$ . The constraint on this system is conservation of angular momentum, the initial value of which we assume is zero.

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\phi}) = 0$$

- (a) Express the conservation of angular momentum constraint as a Pfaffian constraint  $A(q)\dot{q} = 0$ .
- (b) Setting our system inputs as  $u_1 = \dot{\phi}$  and  $u_2 = \dot{l}$ , express the dynamics of the system in the form

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

Where  $g_1(q)$  and  $g_2(q)$  form a basis for the null space of  $A(q)$

2. Now let's examine the a Dubins car, or unicycle model robot.

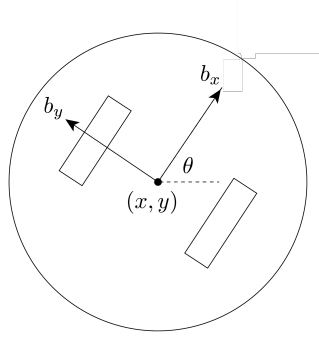


Figure 2: A unicycle model robot

This system has three state variables  $q = (x, y, \theta)$ , and two control inputs  $v = \dot{x}_b$  and  $\omega = \dot{\theta}$ . The constraint on this system is that sideways motion  $\dot{y}_b$  is zero.

- Express the sideways motion constraint as a Pfaffian constraint in terms of the state variables  $q$ :  $A(q)\dot{q} = 0$ .
- Setting our system inputs as  $u_1 = v$  and  $u_2 = \omega$ , express the dynamics of the system in the form

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

Where  $g_1(q)$  and  $g_2(q)$  form a basis for the null space of  $A(q)$

## Problem 2 - Lie Brackets

The *Lie Bracket*, or commutator, describes the degree to which two elements *commute* under some operation. For vector fields, the Lie Bracket is defined as

$$[f(q), g(q)] = \frac{\partial g(q)}{\partial x} f(q) - \frac{\partial f(q)}{\partial x} g(q)$$

The Lie Bracket has the following properties:

- Anti-symmetry (skew-symmetry):  $[X, Y] = -[Y, X]$
- The Jacobi Identity:  $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ .

- Express the Lie Bracket using Lie derivatives (this is a bit of an abuse of notation, but it's fine for our purposes).
- Prove that the Lie Bracket is anti-symmetric.
- (At home) Prove that the Lie Bracket satisfies the Jacobi Identity.
- Lets define  $q = (x, y, z)$

$$g_1(q) = \begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix}, \quad g_2(q) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- What is  $[g_1, g_2]$ ? Is it linearly independent from  $g_1, g_2$ ?
- What is  $[g_1, [g_1, g_2]]$ ? Is it linearly dependent from  $g_1, g_2, [g_1, g_2]$ ?

- Lets define  $q = (x, y, z)$

$$g_1(q) = \begin{bmatrix} xz \\ yz \\ 0 \end{bmatrix}, \quad g_2(q) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- (a) What is  $[g_1, g_2]$ ? Is it linearly independent from  $g_1, g_2$ ?
- (b) What is  $[g_1, [g_1, g_2]]$ ? Is it linearly dependent from  $g_1, g_2, [g_1, g_2]$ ?

### Problem 3 - Constraint Identification

Given a physical system, you should be able to predict whether the constraints are holonomic or nonholonomic. Consider the following systems. What are the constraints? Are they holonomic or not? How many parameters do you need to minimally represent the system?

1.

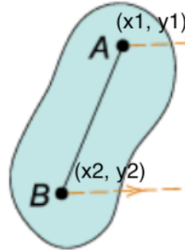


Figure 3: A planar rigid body

2.

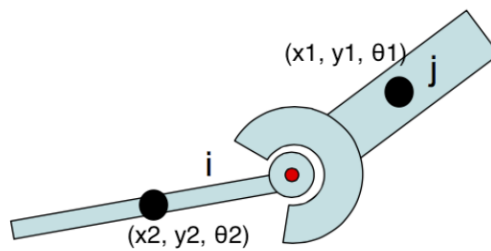


Figure 4: Two planar rigid bodies connected by a revolute joint

3.

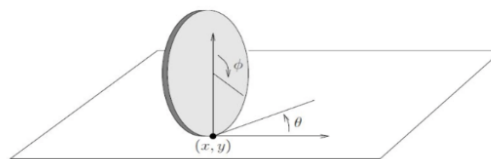


Figure 5: A disk rolling on a plane

4.

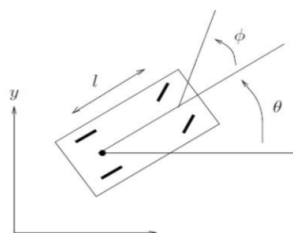


Figure 6: A bicycle model

5.

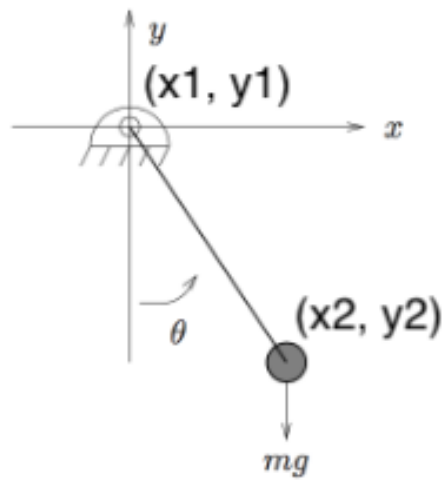


Figure 7: A simple pendulum

6.

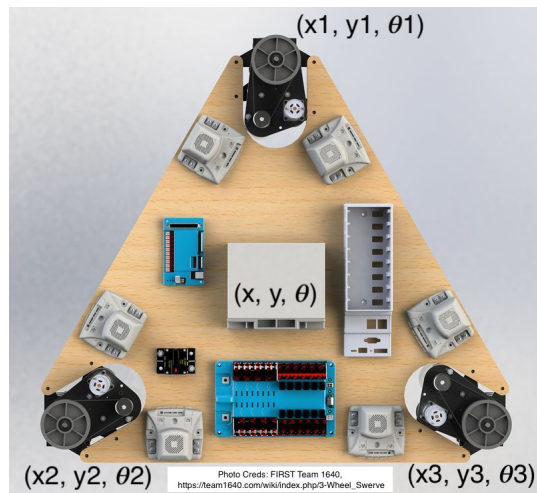


Figure 8: A swerve drive with three wheels

7.

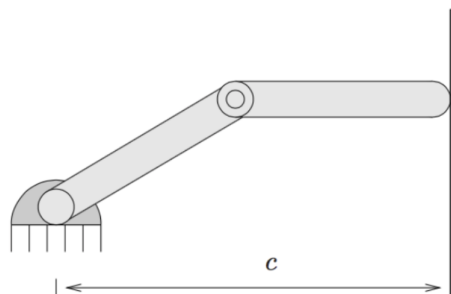


Figure 9: A robot touching a wall