Linear Theory of Elasticity as basis for Soft Robotics

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Slides prepared form various sources

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Linear Elasticity, definition

- **Linear elasticity** is the mathematical study of how solid objects deform and become internally stressed due to prescribed loading conditions. **Linear elasticity** models materials as continua. **Linear elasticity** is a simplification of the more general nonlinear **theory** of **elasticity** and is a branch of continuum mechanics. This is in contrast to solid mechanics.
- One new concept: Constitutive equation is a relation between two physical entities such as; stress or forces to strain or deformation.

Strain displacement equation and Newton equation of motion Constitutive equation

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right]$$

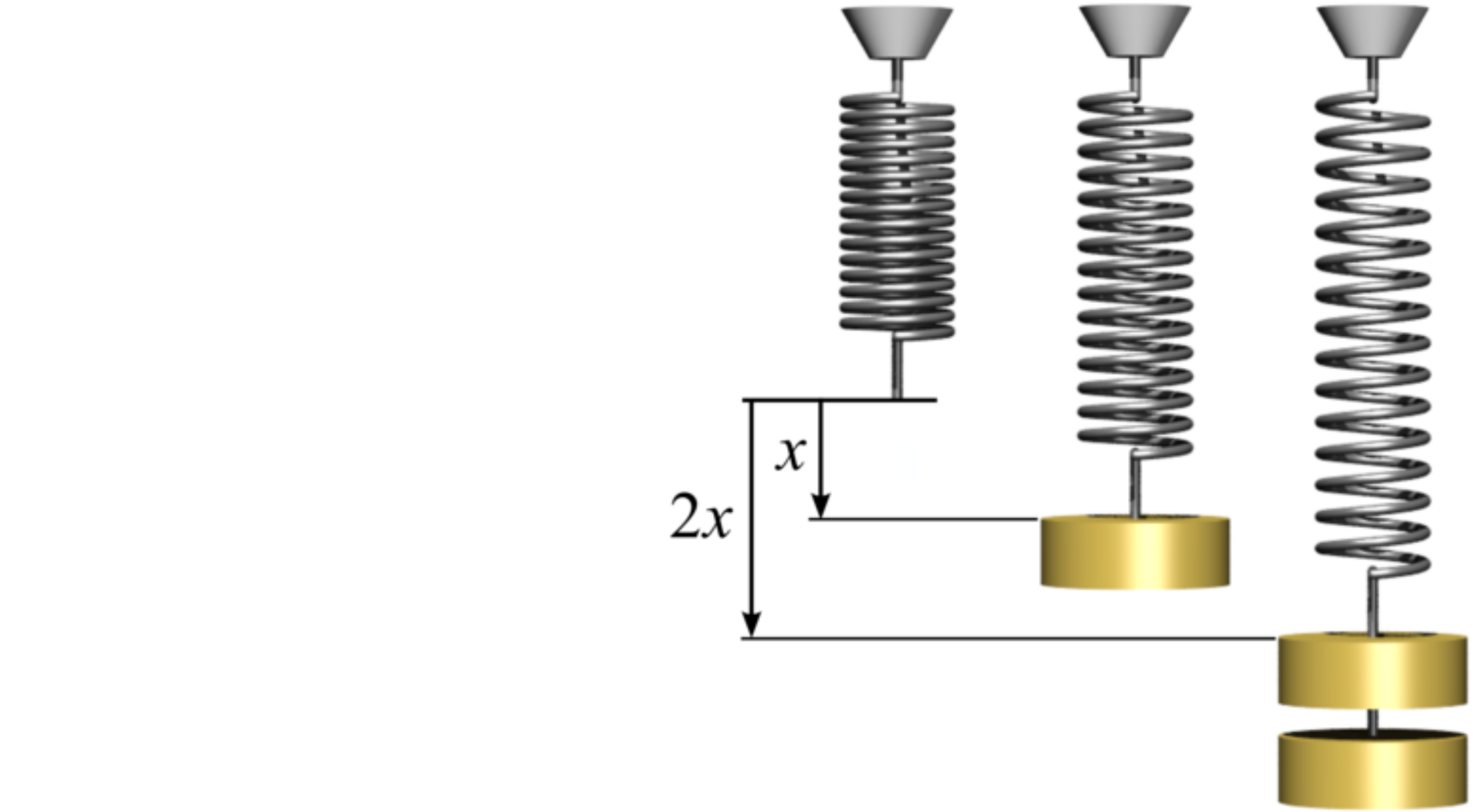
$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{F} = \rho\ddot{\mathbf{u}}$$

$$\boldsymbol{\sigma} = \mathsf{C} : \boldsymbol{\varepsilon},$$

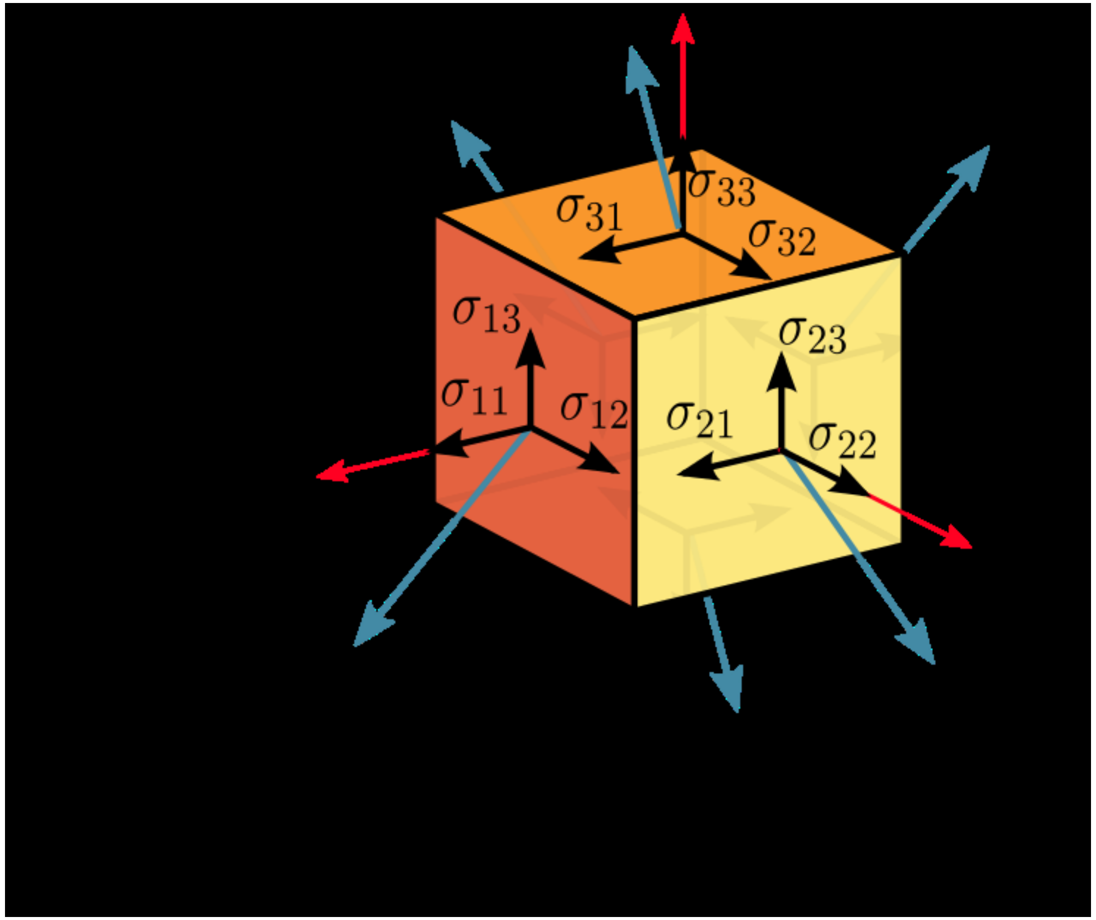
Notation

- Sigma :Cauchy Stress tensor
- Epsilon: Infinitesimal strain tensor
- U: Displacement vector
- C: Fourth order stiffness tensor
- F: body Force/unit volume
- RO: Mass desnity
- A:B is inner product of second order terms
- CONSTITUTIVE Equation in physics is a relation between two physical entities, such as stress and strain

HOOK's Law is first order linear approx. between force and displacement x



Cauchy Stress tensor sigma



Cauchy Tensor

- The tensor consists of nine components
- σ [i, j]
- that completely define the state of stress at a point inside a material in the deformed state, placement, or configuration. The tensor relates a unit-length direction vector n to the stress vector T(n) across an imaginary surface perpendicular to n

Explicit representation of the components of stress tensor

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} \quad \text{or} \quad T_j^{(n)} = \sigma_{ij} n_i.$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

Equation of Motion in Cartesian coordinate system, where the forces are proportional to second derivative of displacement, Cauchy Momentum equation for stress

$$\sigma_{ji,j} + F_i = \rho \partial_{tt} u_i$$

$$\frac{D\mathbf{u}}{Dt} = \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}$$

Introduction to Strain

- A strain is in general a tensor quantity. Physical insight into strains can be gained by observing that a given strain can be decomposed into normal and shear components. The amount of stretch or compression along material line elements or fibers is the *normal strain*, and the amount of distortion associated with the sliding of plane layers over each other is the *shear strain*, within a deforming body.[4] This could be applied by elongation, shortening, or volume changes, or angular distortion

The state of strain related to geometry /deformation

- The state of strain at a material point of a continuum body is defined as the totality of all the changes in length of material lines or fibers, the *normal strain*, which pass through that point and also the totality of all the changes in the angle between pairs of lines initially perpendicular to each other, the *shear strain*, radiating from this point. However, it is sufficient to know the normal and shear components of strain on a set of three mutually perpendicular directions.

The strain is an entity without dimension,(I is the identity matrix.
In this equation a general deformation of a body is expressed as x=F(X) where X is the reference point of material points in the body F’ is the first derivative.

$$\boldsymbol{\varepsilon} \doteq \frac{\partial}{\partial \mathbf{X}} (\mathbf{x} - \mathbf{X}) = \boldsymbol{F}' - \boldsymbol{I},$$

Small e is the Engineering normal strain

- The **Cauchy strain** or **engineering strain** is expressed as the ratio of total deformation to the initial dimension of the material body in which the forces are being applied. The *engineering normal strain* or *engineering extensional strain* or *nominal strain* e of a material line element or fiber axially loaded is expressed as the change in length ΔL per unit of the original length L of the line element or fibers. The normal strain is positive if the material fibers are stretched and negative if they are compressed. Thus, we have

$$e = \frac{\Delta L}{L} = \frac{l - L}{L}$$

Strain, cont.

- *L* is the original length of the fiber and *l* is the final length of the fiber. Measures of strain are often expressed in parts per million or micro-strains.
- The *true shear strain* is defined as the change in the angle (in radians) between two material line elements initially perpendicular to each other in the un-deformed or initial configuration. The *engineering shear strain* is defined as the tangent of that angle, and is equal to the length of deformation at its maximum divided by the perpendicular length in the plane of force application which sometimes makes it easier to calculate.

Constitutive equation for Hook’s Law

c[ijkl] is the stiffness tensor

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

Relationship between Stress and Strain

- These are 6 independent equations relating stresses and strains. The requirement of the symmetry of the stress and strain tensors lead to equality of many of the elastic constants, reducing the number of different elements to 21
- An elastostatic boundary value problem for an isotropic-homogeneous media is a system of 15 independent equations and equal number of unknowns (3 equilibrium equations, 6 strain-displacement equations, and 6 constitutive equations). Specifying the boundary conditions, the boundary value problem is completely defined. To solve the system two approaches can be taken according to boundary conditions of the boundary value problem: a **displacement formulation**, and a **stress formulation**.

Equation of motion in spherical coordinates

The strain tensor in spherical coordinates is

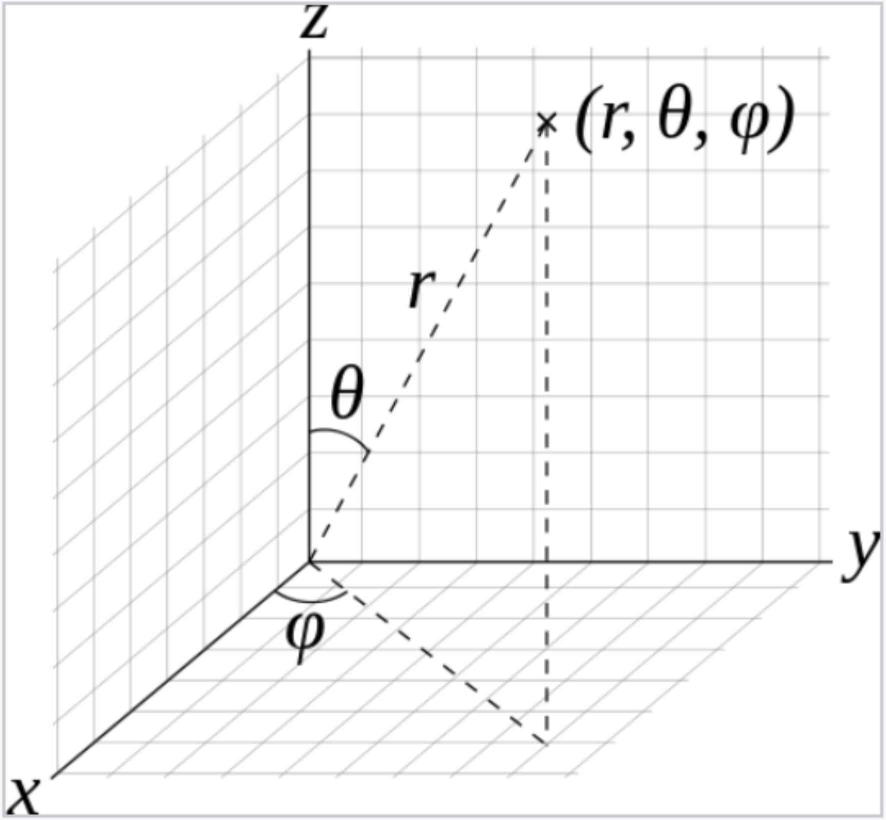
$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)$$

$$\varepsilon_{\phi\phi} = \frac{1}{r\sin\theta}\left(\frac{\partial u_\phi}{\partial \phi} + u_r\sin\theta + u_\theta\cos\theta\right)$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\varepsilon_{\theta\phi} = \frac{1}{2r}\left[\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \left(\frac{\partial u_\phi}{\partial \theta} - u_\phi\cot\theta\right)\right]$$



Spherical coordinates (r, θ, φ) as

Stress equations for Spherical Coordinates

In spherical coordinates (r, θ, ϕ) the equations of motion are[1]

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta) + F_r = \rho\,\frac{\partial^2 u_r}{\partial t^2}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r}[(\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + 3\sigma_{r\theta}] + F_\theta = \rho\,\frac{\partial^2 u_\theta}{\partial t^2}$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}(2\sigma_{\theta\phi}\cot\theta + 3\sigma_{r\phi}) + F_\phi = \rho\,\frac{\partial^2 u_\phi}{\partial t^2}$$

Stiffness tensor for isotropic materials

$$C_{ijkl} = K\,\delta_{ij}\,\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\,\delta_{ij}\,\delta_{kl})$$

Notation

- Delta[i.j] is Kronecker delta
- K bulk modulus or incompressibility
- Mu Shear modulus or rigidity
- If the medium is inhomogeneous, the isotropic model is sensible if either the medium is piecewise-constant or weakly inhomogeneous; in the strongly inhomogeneous smooth model, anisotropy has to be accounted for. If the medium is homogeneous, then the elastic moduli will be independent of the position in the medium

This equation separates stress into two parts: scalar pressure part and shear forces.
The simpler version has Lamme first parameter (Lambda)

$$\sigma_{ij} = K\delta_{ij}\varepsilon_{kk} + 2\mu(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}).$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

Strain as an expression of Stresses, and Young modulus and Poisson ratio

$$\varepsilon_{ij} = \frac{1}{9K}\delta_{ij}\sigma_{kk} + \frac{1}{2\mu}\left(\sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk}\right)$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right]$$

Elastostatics

- Elastostatics is the study of linear elasticity under the conditions of equilibrium, in which all forces on the elastic body sum to zero, and the displacements are not a function of time. The equilibrium equations are then
- e. The equilibrium equations are then
- σ [j i , j] + F [i] = 0.

Stress formulation

- In this case, the surface tractions are prescribed everywhere on the surface boundary. In this approach, the strains and displacements are eliminated leaving the stresses as the unknowns to be solved for in the governing equations. Once the stress field is found, the strains are then found using the constitutive equations.
- There are six independent components of the stress tensor which need to be determined, yet in the displacement formulation, there are only three components of the displacement vector which need to be determined. This means that there are some constraints which must be placed upon the stress tensor, to reduce the number of degrees of freedom to three. Using the constitutive equations, these constraints are derived directly from corresponding constraints which must hold for the strain tensor, which also has six independent components

Engineering equation for strain

Engineering notation [hide]

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} = 2\frac{\partial^2 \epsilon_{zx}}{\partial z \partial x}$$

$$\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z}\right)$$

Stress equation

The strains in this equation are then expressed in terms of the stresses using the constitutive equations, which yields the corresponding constraints on the stress tensor. These constraints on the stress tensor are known as the *Beltrami-Michell* equations of compatibility:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} + F_{i,j} + F_{j,i} + \frac{\nu}{1-\nu}\delta_{i,j}F_{k,k} = 0.$$

In the special situation where the body force is homogeneous, the above equations reduce to

$$(1+\nu)\sigma_{ij,kk} + \sigma_{kk,ij} = 0.$$ [6]

A necessary, but insufficient, condition for compatibility under this situation is $\boldsymbol{\nabla}^4\boldsymbol{\sigma} = \mathbf{0}$ or $\sigma_{ij,kk\ell\ell} = 0$.[1]

These constraints, along with the equilibrium equation (or equation of motion for elastodynamics) allow the calculation of the stress tensor field. Once the stress field has been calculated from these equations, the strains can be obtained from the constitutive equations, and the displacement field from the strain-displacement equations.

An alternative solution technique is to express the stress tensor in terms of stress functions which automatically yield a solution to the equilibrium equation. The stress functions then obey a single differential equation which corresponds to the compatibility equations.