

Steering Nonholonomic Systems

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From Murray, Li, Sastry, Chapter 8
Sastry, Chapter 13, Nonlinear Systems

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The Nonholonomic Integrator

Optimal Control

Chained Form Systems

Steering Using Sinusoids

Conversion to Chained Form

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Nonholonomic Integrator

Consider the so-called nonholonomic integrator:

$$\begin{aligned}\dot{q}_1 &= u_1 \\ \dot{q}_2 &= u_2 \\ \dot{q}_3 &= q_1 u_2 - q_2 u_1\end{aligned}$$

This system has

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ -q_2 \end{bmatrix} \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ q_1 \end{bmatrix} \quad [G_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

This system is written by dropping the dt as

$$dq_3 = q_1 dq_2 - q_2 dq_1$$

Optimal Control

The optimal input minimizing the cost function

$$\int_0^1 \|u(t)\|^2 dt$$

from an initial $q(0)$ to a final $q(1)$ was shown by Brockett to be sinusoidal. The frequency λ of the optimal input is striking when $q_1(0) = q_1(1)$ and $q_2(0) = q_2(1)$ to be $2n\pi$ with $n = 0, \pm 1, \pm 2, \dots$. The generalization of this system to $m > 2$ inputs is stated as a control system on $q \in \mathbb{R}^m \times Y \in so(m)$ as

$$\begin{aligned}\dot{q} &= u \\ \dot{Y} &= qu^T - uq^T\end{aligned}$$

If $q(0) = q(1)$ and $Y(1) \in so(m)$ is given then it can be shown that the optimal input is multiples of 2π , that is

$$\begin{array}{ll} 2\pi, 4\pi, \dots, \frac{m}{2} & m \text{ even} \\ 2\pi, 4\pi, \dots, \frac{m-1}{2} & m \text{ odd} \end{array}$$

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Definition of a 1-chain form

The preceding discussion motivates some generalizations. First the non-holonomic integrator extended to dimension n

$$\begin{aligned}\dot{q}_1 &= u_1 \\ \dot{q}_2 &= u_2 \\ \dot{q}_3 &= q_2 u_1 \\ \dot{q}_4 &= q_3 u_1 \\ &\vdots \\ \dot{q}_n &= q_{n-1} u_1\end{aligned}$$

These are called **chained form** or **Goursat normal form systems**.

Controllability of a One Chain system

$$[g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad [g_1, [\dots, [g_1, \text{ k times, } g_2] \dots]] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (-1)^k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

By way of notation we define

$$ad_{g_1} g_2 = [g_1, g_2] \quad ad_{g_1}^{k+1} = [g_1, ad_{g_1}^k g_2]$$

Thus

$$\text{Span} = \{g_1, g_2, ad_{g_1}^k g_2, k = 1, \dots, n-2\} = \mathfrak{R}^n$$

Steering Chained Form Systems

Algorithm 3. Steering chained form systems

1. Steer q_1 and q_2 to their desired values.
2. For each q_{k+2} , $k \geq 1$, steer q_k to its final value using $u_1 = a \sin 2\pi t$, $u_2 = b \cos 2\pi kt$, where a and b satisfy

$$q_{k+2}(1) - q_{k+2}(0) = \left(\frac{a}{4\pi}\right)^k \frac{b}{k!}.$$

We use sinusoidal signals $u_1(t) = \sin 2\pi t$ and $u_2 = \cos 2\pi kt$ to steer q_k at $t = 1$ without changing the preceding $q_i, i < k$ for 1 second.

Steering Calculation

$$q_1 = \frac{u}{2\pi} (1 - \cos 2\pi t),$$

$$q_2 = \frac{b}{2\pi k} \sin 2\pi k t$$

$$\begin{aligned} q_3 &= \int \frac{ab}{2\pi k} \sin 2\pi k t \sin 2\pi t \, dt \\ &= \frac{1}{2} \frac{ab}{2\pi k} \left(\frac{\sin 2\pi(k-1)t}{2\pi(k-1)} - \frac{\sin 2\pi(k+1)t}{2\pi(k+1)} \right) \end{aligned}$$

$$\begin{aligned} q_4 &= \frac{1}{2} \frac{a^2 b}{2\pi k \cdot 2\pi(k-1)} \int \sin 2\pi(k-1)t \cdot \sin 2\pi t \, dt + \dots \\ &= \frac{1}{2^2} \frac{a^2 b}{2\pi k \cdot 2\pi(k-1) \cdot 2\pi(k-2)} \sin 2\pi(k-2)t + \dots \end{aligned}$$

\vdots

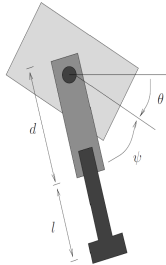
$$\begin{aligned} q_{k+2} &= \int \frac{1}{2^{k-1}} \frac{a^k b}{2\pi k \cdot 2\pi(k-1) \cdots 2\pi} \sin^2 2\pi t \, dt + \dots \\ &= \frac{1}{2^{k-1}} \frac{a^k b}{(2\pi)^k k!} \frac{t}{2} + \dots \end{aligned}$$

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With $q = (\psi, l, \theta)^T$, the control system of the hopper is

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(l+d)^2}{l+m(l+d)^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

If you define $\alpha = \theta + \frac{nm d^2}{l+m d^2} \psi$, we get for suitably defined f

$$\dot{\alpha} := f(l)u_1$$

Fourier Technique

Now choose

$$u_1 = a_1 \sin(2\pi t)$$

$$u_2 = a_2 \cos(2\pi t)$$

By Fourier series after integrating for l we get

$$f(l) = f\left(\frac{a_2}{2\pi} \sin 2\pi t\right) = \beta_1 \sin 2\pi t + \beta_2 \sin 4\pi t + \dots$$

Using this in the equation for $\dot{\alpha}$ gives

$$\alpha(1) - \alpha(0) = \frac{1}{2} a_1 \beta_1$$

After 1 second, ψ, l are back to their initial values but α has changed by $\frac{1}{2} a_1 \beta_1 (\alpha_2)$! By suitably choosing a_1, a_2 we can make this be π radians (a flip).!

Steering the kinematic car

The control systems for the car is

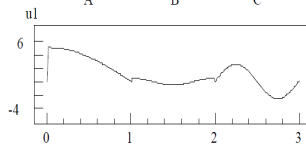
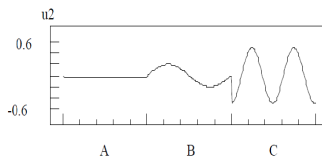
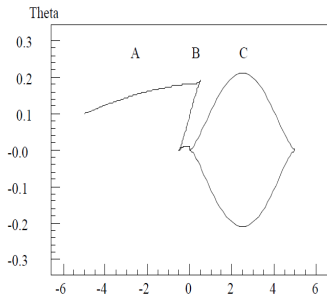
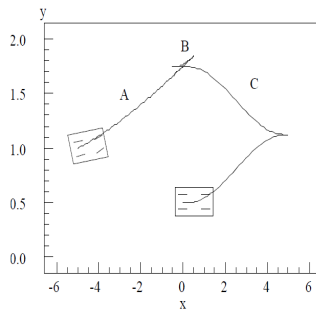
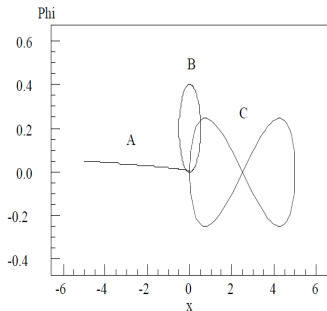
$$\begin{aligned}\dot{x} &= \cos \theta u_1 \\ \dot{y} &= \sin \theta u_1 \\ \dot{\theta} &= \frac{1}{l} \tan \phi u_1 \\ \dot{\phi} &= u_2\end{aligned}$$

Change coordinates $z_1 = x, z_2 = \phi, z_3 = \sin \theta, z_4 = y$, inputs $v_1 = \cos \theta u_1, v_2 = u_2$ to get

$$\begin{aligned}\dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= \frac{1}{l} \tan z_2 v_1 \\ \dot{z}_4 &= \frac{z_2}{\sqrt{1-z_3^2}} v_2\end{aligned}$$

Now the linear terms in the last two equations match those of the one chain system and it can be steered using the algorithm with the Fourier series method and double frequency sinusoids

Parking a Car



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Conversion to One Chained Form

Given a control system of the form in \mathbb{R}^n with two inputs

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

If (and only if) the following three distributions are regular (constant dimension) and involutive

$$\Delta_0 = \text{span}\{g_1, g_2, \text{ad}_{g_1}g_2, \dots, \text{ad}_{g_1}^{n-2}g_2\} = \mathbb{R}^n$$

$$\Delta_1 = \text{span}\{g_2, \text{ad}_{g_1}g_2, \dots, \text{ad}_{g_1}^{n-2}g_2\}$$

$$\Delta_2 = \text{span}\{g_2, \text{ad}_{g_1}g_2, \dots, \text{ad}_{g_1}^{n-3}g_2\}$$

then there exists a choice of coordinates $z_1(q), \dots, z_n(q)$ and inputs $v = \alpha(q) + \beta(q)u$ such that

$$\dot{z}_1 = v_1$$

$$\dot{z}_2 = v_2$$

$$\dot{z}_3 = z_2 v_1$$

$$\vdots$$

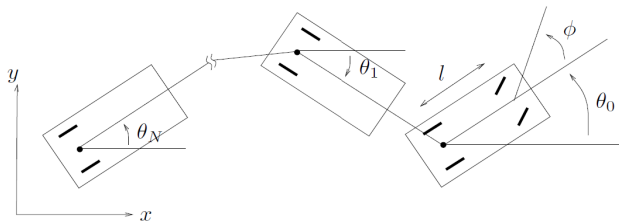
$$\dot{z}_n = z_{n-1} v_1$$

The proof is in Sastry (1999), Chapter 13 and is constructive

Conversion to One Chained form

Amazingly, all of the examples: so far: cars, unicycles, pennies with two inputs can be converted into a single chain form exactly and steered as above !!

Amazingly so is the the **Car with N Trailers**



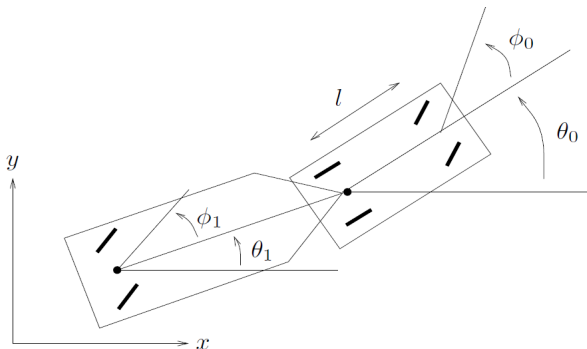
The state space $q = (x, y, \phi, \theta_0, \dots, \theta_N)^T \in \mathbb{R}^{N+4}$. The coordinates z_1, z_2 in the chained form are the x, y coordinates of the last trailer wheel base!!

Multi Chained Forms

When there are $m \geq 3$ inputs for

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 + \cdots + g_m(q)u_m$$

there are analogous necessary and sufficient conditions for the conversion into $m-1$ chains. For example, the **firetruck**. The driver in front has two inputs: drive and steer and the one at the back of the ladder can steer. This can be converted into a two chain system.



Thank you for your attention. Questions?

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