

Structure of Nonholonomic Systems

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Outline

Nonholonomic Systems

Integrability, Holonomy and Non holonomy
Equivalent Control Systems

Examples

Hopper
Space Robot
Rolling Penny: Unicycle
Front Wheel Drive Car
Car with N trailers
Firetruck

Controllability

The Lie Bracket, Frobenius Theorem
Chow's Theorem

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Pfaffian Constraints

Some systems are characterized by having constraints on their velocities. Thus, for example if the state $q \in \mathbb{R}^n$ then a set of constraints of the form

$$\omega_i(q)\dot{q} = 0 \quad i = 1, \dots, k$$

with $\omega_i^T(q) \in \mathbb{R}^n$ is referred to as a system of Pfaffian constraints. We will assume that the rows $\omega_i(q)$ are linearly independent at q so that the k constraints are independent.

- The first question that we ask is whether the constraints may be converted into constraints of the form

$$h_i(q) = 0 \quad i = 1, \dots, k$$

which would say that the state space q is constrained to lie in a manifold of dimension $n - k$.

- We may be encouraged by the fact that $h(q) = 0 \Leftrightarrow dh(q)\dot{q} = 0$ for a single constraint.
- The answer to this question is neither easy nor obvious!

Single Pfaffian Constraints

A single constraint

$$\omega(q)\dot{q} = \sum_{i=1}^n \omega_i(q)\dot{q}_i = 0$$

is said to be **integrable** if there exists a function $h : \mathfrak{R}^n \rightarrow \mathfrak{R}$ such that

$$\omega(q)\dot{q} = 0 \Leftrightarrow h(q) = 0$$

That is

$$\sum_{j=1}^n \omega_j(q)\dot{q}_j \Rightarrow \sum_{j=1}^n \frac{\partial h}{\partial q_j} \dot{q}_j = 0$$

This implies that there exists some function $\alpha(q)$ called the integrating factor such that

$$\alpha(q)\omega_j(q) = \frac{\partial h}{\partial q_j}(q), \quad j = 1, \dots, n$$

Integrability

From the equality of the mixed partials of h , that is

$$\frac{\partial^2 h}{\partial q_i \partial q_j} = \frac{\partial^2 h}{\partial q_j \partial q_i}$$

it follows that

$$\frac{\partial(\alpha \omega_j)}{\partial q_i} = \frac{\partial(\alpha \omega_i)}{\partial q_j} \quad i, j = 1, \dots, n$$

The problem about this condition is that it relies on finding the integrating factor $\alpha(q)$. This becomes even harder when there are k constraints because you need to not only check the integrability of each constraint but also that of linear combinations of the constraints

$$\sum_{i=1}^k \alpha_i(q) \omega_i(q) \dot{q}$$

Holonomic, Nonholonomic

The set of Pfaffian constraints $\omega_i(q), i = 1, \dots, k$ is said to be **holonomic** if there exists functions $h_i(q), i = 1, \dots, k$ such that

$$\omega_i(q)\dot{q} = 0 \Leftrightarrow h_i(q) = c_i, \quad i = 1, \dots, k$$

That is the number of constraints on q are precisely k and thus q lies on a manifold of dimension $(n - k)$. On the other hand if there are only $p < k$ functions such that

$$\omega_i(q)\dot{q} = 0 \Leftrightarrow h_i(q) = c_i, \quad i = 1, \dots, p.$$

the Pfaffian system is said to be **nonholonomic**. If $p = 0$ the Pfaffian system is said to be **completely nonholonomic**. For nonholonomic systems there are fewer than k constraints on the state space q . For completely nonholonomic systems there are *NO* constraints on q . If $0 < p < k$ the constraints are called **partially nonholonomic**.

Equivalent Control Systems

If Pfaffian constraints give you the directions that the body coordinates q cannot move, how about the directions that they can move? To this end, we construct the right null space of the constraints, denoted $g_j(q), j = 1, \dots, n - k =: m$. That is

$$\omega_i(q)g_j(q) = 0 \quad \begin{array}{l} i = 1, \dots, k \\ j = 1, \dots, n - k \end{array}$$

Then the allowable trajectories satisfying the Pfaffian constraints are the trajectories of the control system

$$\dot{q} = g_1(q)u_1 + \dots + g_m(q)u_m$$

for suitably chosen inputs $u_1(\cdot), \dots, u_m(\cdot), i = 1, \dots, m$. This is a *drift free* control system.

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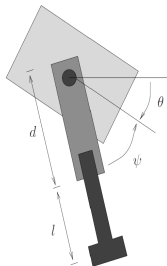
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Example 1: Raibert's hopper



The one legged hopper was originally designed by Marc Raibert to mimic a kangaroo. It has a prismatic joint in the leg and a revolute joint at the hip. The hopping is emulated by the prismatic joint and the swinging of the leg by the hip joint. The hopper has a stance phase on the ground and a flying phase in the air.

Angular Momentum Constraints to Control

When it is in the air angular momentum is conserved. I is the moment of inertia of the body, the leg mass m is concentrated at the foot. The formula for the angular momentum set to zero is

$$I\dot{\theta} + m(l+d)^2(\dot{\theta} + \dot{\psi}) = (I + m(l+d)^2)\dot{\theta} + m(l+d)^2\dot{\psi} = 0$$

If $q = (\psi, l, \theta)^T$ then an equivalent control system for describing it is found by finding a basis for the null space of

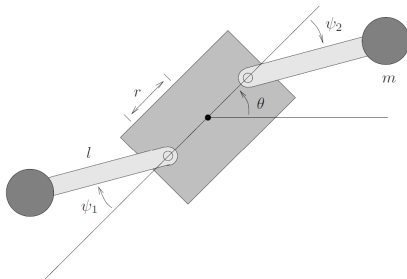
$$\omega_1(q) = \begin{bmatrix} m(l+d)^2 & 0 & I + m(l+d)^2 \end{bmatrix}$$

An especially convenient one is

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(l+d)^2}{I+m(l+d)^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

Planar Space Robot

This is a simplified model of a robot in space with two arms connected to the body through revolute joints.



The mass and moment of inertia of the central body are M, I and the mass of each arm is m concentrated at the ends of the arms of length l .

Angular Momentum Constraints to Control

In MLS, page 335 there is a detailed derivation of the Lagrangian equations for the Space Robot. The Lagrangian does not depend on the body angle θ . Hence (this is the statement of angular momentum conservation)

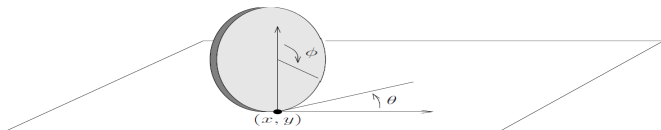
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} = 0 = a_{13}(\psi) \dot{\psi}_1 + a_{23}(\psi) \dot{\psi}_2 + a_{33}(\psi) \dot{\theta}$$

Setting $q = (\psi_1, \psi_2, \theta)^T$ we get the equivalent control system

$$\dot{q} = \begin{bmatrix} 1 \\ 0 \\ -\frac{a_{13}}{a_{33}} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ -\frac{a_{23}}{a_{33}} \end{bmatrix} u_2$$

Rolling without slipping

A second source of nonholonomy is from constraints that arise from discs, wheels which roll without slipping. Consider a penny rolling on a surface:



Here x, y are the location of the contact point on the plane θ is the angle that the disk makes with the horizontal, ϕ is the angle made by a fixed line on the disk relative to the vertical axis. ρ is the radius of the disk.

Rolling Constraint to Control

If the disk rolls without slipping we have with $q = (x, y, \theta, \phi)^T \in \mathbb{R}^4$

$$\begin{aligned}\dot{x} - \rho \cos \theta \dot{\phi} &= 0 \\ \dot{y} - \rho \sin \theta \dot{\phi} &= 0\end{aligned}$$

This may be written as

$$\begin{bmatrix} 1 & 0 & 0 & -\rho \cos \theta \\ 0 & 1 & 0 & -\rho \sin \theta \end{bmatrix} \dot{q} = 0$$

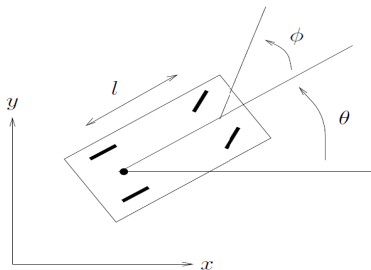
Thus, there are 2 Pfaffian constraints on \mathbb{R}^4 . A convenient choice of control system, with $\dot{\theta} = u_1$ and $\dot{\phi} = u_2$ is

$$\dot{q} = \begin{bmatrix} \rho \cos \theta \\ \rho \sin \theta \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2$$

This is a two input control system.

Front Wheel Drive Car

Here is a picture of a front wheel drive car. The steering angle is ϕ , the angle of the car body is θ and the position of the midpoint of the rear axle is x, y .



This is sometimes referred to as the kinematic model of a car. It is used frequently in the analysis of self-driving cars and their motion plans.

Rolling without Slipping to Car Model

The rolling without slipping constraints for the front wheels and back wheels are a statement that the velocity perpendicular to the direction that the velocity of the wheels perpendicular to the direction they are pointing is 0:

$$\begin{aligned}\sin(\theta + \phi)\dot{x} - \cos(\theta + \phi)\dot{y} - l \cos \phi \dot{\theta} &= 0 \\ \sin \theta \dot{x} - \cos \theta \dot{y} &= 0\end{aligned}$$

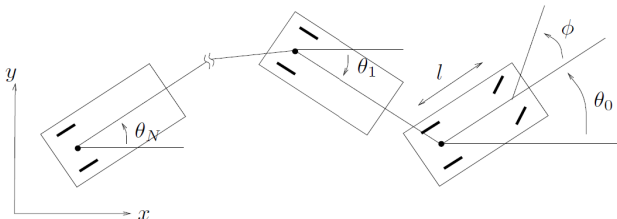
Using the steering velocity as $u_2 = \dot{\phi}$ and $q = (x, y, \theta, \phi)^T \in \mathbb{R}^4$ gives the control system

$$\dot{q} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ \frac{1}{l} \tan \phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

u_1 has the interpretation of the driving input and u_2 as the steering input.

Car with N Trailers

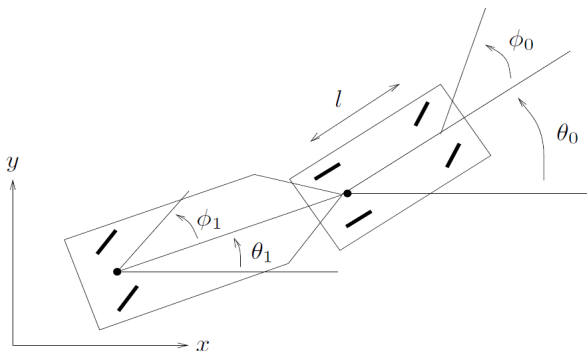
The figure shows a car with N trailers attached. The hitch of each trailer is attached to the center of the rear axle of the previous trailer. The wheels of the individual trailers are aligned with the body of the trailer.



Satisfy yourself that $q = (x, y, \phi, \theta_0, \dots, \theta_N)^T \in \mathbb{R}^{N+4}$. There are $N+2$ sets of wheels which roll without slipping to give $N+2$ Pfaffian constraints. (see Exercise 6 in Chapter 7 of MLS).

A Firetruck

The figure shows a kinematic model of a fire truck. You may have noticed that there is a driver in the front and one more at the back of the ladder. It is not unlike a car with one trailer, except that the rear axle is also steerable.



How many Pfaffian constraints are there? What is the dimension of q . (See Exercise 7 in Chapter 7 of MLS).

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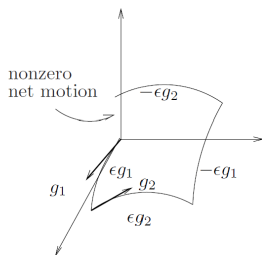
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The Lie Bracket Motion

Consider the control system with $q \in \mathbb{R}^n$

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

You see immediately that you can move in the directions g_1, g_2 at a point q . But there may be more directions. Consider what is called the Lie bracket motion: follow g_1 for ε seconds, followed by g_2 for ε seconds, then $-g_1$ for ε seconds and $-g_2$ for ε seconds as seen in the figure



The Lie Bracket

You might think that after 4ε seconds, you are back to where you started from, but it is amazing that a careful Taylor series expansion will give you

$$q(4\varepsilon) = q(0) + \varepsilon^2 \left(\frac{\partial g_2}{\partial q} g_1(q_0) - \frac{\partial g_1}{\partial q} g_2(q_0) \right) + O(\varepsilon^3)$$

The leading term is a term of $O(\varepsilon^2)$, however and its coefficient measures the extent to which g_1, g_2 do not commute! This is the **Lie Bracket**

$$[f, g](q) = \frac{\partial g}{\partial q} f(q) - \frac{\partial f}{\partial q} g(q)$$

A **Lie product** is a nested set of Lie brackets, for example

$$[[f, g], [f, [f, g]]]$$

Properties of Lie Brackets

Given vector fields f, g, h on \mathfrak{R}^n and smooth functions $\alpha, \beta : \mathfrak{R}^n \rightarrow \mathfrak{R}$, we have

- **Skew Symmetry**

$$[f, g] = -[g, f]$$

- **Jacobi Identity**

$$[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0$$

- **Chain Rule**

$$[\alpha f, \beta g] = \alpha \beta [f, g] + \alpha (L_f \beta) g - \beta (L_g \alpha) f$$

Distributions, Involutivity

A *distribution* $\Delta \subset \mathfrak{X}^n$ assigns a subspace of vector fields at each $q \in \mathfrak{X}^n$. Thus

$$\Delta(q) = \text{span } (g_1(q), \dots, g_m(q))$$

The distribution is said to be **regular** if the dimension of the subspace $\Delta(q)$ does not vary with q . Δ is said to be **involutive** if it is closed under the Lie Bracket that is

$$\Delta \text{ involutive} \quad \Leftrightarrow \quad \forall f, g \in \Delta, \quad [f, g] \in \Delta$$

Frobenius Theorem

A regular distribution Δ of dimension p is said to be **integrable** if there exist functions $h_1, \dots, h_{n-p} : \mathfrak{R}^n \rightarrow \mathfrak{R}$ such that

$$\forall f \in \Delta \quad L_f h_i = \frac{\partial h_i}{\partial q} f(q) = L_f h_i(q) \equiv 0 \quad i = 1, \dots, (n-p)$$

Then, the manifolds M_c parameterized by $c \in \mathfrak{R}^{n-k}$

$$M_c = \{h_1(q) = c_1, \dots, h_{n-k}(q) = c_{n-k}\}$$

are called the integral manifolds of Δ of dimension $n - k$.

Frobenius Theorem

A regular distribution Δ is integrable if and only if Δ is involutive.

Frobenius Theorem and Integrability of Pfaffians

Given a set of Pfaffian constraints $\omega_i(q)\dot{q} = 0, i = 1, \dots, k$, convert it into the equivalent control system

$$\dot{q} = g_1(q)u_1 + \dots g_{n-k}(q)u_{n-k}$$

Then the distribution $\Delta := \text{span}\{g_1, \dots, g_{n-k}\}$ may not be involutive. Take all possible Lie brackets and Lie products of the vector fields in Δ to get its *involutive closure*. This is denoted $\bar{\Delta}$. By definition it is involutive. Let it be regular and have dimension be $p \geq (n - k)$.

By Frobenius' Theorem $\bar{\Delta}$ is integrable. Let the functions h_1, \dots, h_{n-p} define the integral manifolds of $\bar{\Delta}$. Note that $n - p \leq k$. If $p = n$ there are **no integral manifolds**, that is the Pfaffian system is **completely nonholonomic**. If $p < n$ the original Pfaffian system is **partially non-holonomic**. If $p = k$, the system is **holonomic**.

Controllability

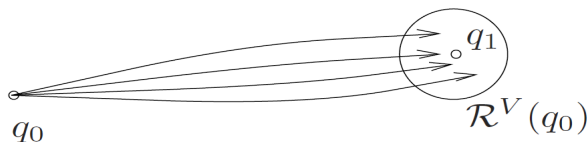
The reachable states of a nonlinear control system

$$\dot{q} = g_1(q)u_1 + \cdots + g_m(q)u_m$$

from an initial state $q_0 \in V$ are defined by first defining $\mathcal{R}^V(q_0, t)$ to be the set of all states that you can steer the system to at t seconds starting from q_0 and staying inside V . Then,

$$\mathcal{R}^V(q_0 \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}^V(q_0, t)$$

is called the **reach set**.



Chow's Theorem

Chow's theorem gives the relationship between the reach set and the involutive closure of $\Delta = \text{span}\{q_1, \dots, g_m\}$, referred to as $\bar{\Delta}$.

Chow's Theorem

If $\bar{\Delta}(q) = \mathbb{R}^n$ for all q in a neighborhood of q_0 , then $\mathcal{R}^V(q_0, \leq T)$ has non empty interior.

This says in an understated way that you can steer to a set which has bulk/interior (that is, is of full dimension) and the condition $\bar{\Delta}(q) = \mathbb{R}^n$ is referred to as the **controllability rank condition**.

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Filtration

Given a distribution $\Delta = \text{span}\{g_1, \dots, g_m\}$ define the sequence of distributions

$$\Delta_1 = \Delta \quad \Delta_i = \Delta_{i-1} + [\Delta_1, \Delta_{i-1}]$$

The chain of distributions is called the filtration and roughly speaking Δ_i has the i -th order Lie brackets. $\Delta_i \subset \Delta_{i+1}$. When the rank Δ_i is the same as the rank of Δ_{i+1} , that first value of i at which this happens is called the **degree of nonholonomy**. The dimensions r_i of the Δ_i are called the **growth vector**.

One legged hopper

Recall that with $q = (\psi, l, \theta)^T$ an equivalent control system for describing the hopper is

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

with

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ -\frac{m(l+d)^2}{l+m(l+d)^2} \end{bmatrix} \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ \frac{2lm(l+d)}{l+m(l+d)^2} \end{bmatrix}$$

Thus $\bar{\Delta}$ is of dimension 3, the hopper system is completely nonholonomic with degree of holonomy 2 and growth vector $(2, 3)$.

Rolling Penny

With $q = (x, y, \theta, \phi)^T \in \mathfrak{R}^4$ and $\dot{\theta} = u_1$ and $\dot{\phi} = u_2$ we have

$$g_1 = \begin{bmatrix} \rho \cos \theta \\ \rho \sin \theta \\ 0 \\ 1 \end{bmatrix} \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The Lie products are

$$g_3 = [g_1, g_2] = \begin{bmatrix} \rho \sin \theta \\ -\rho \cos \theta \\ 0 \\ 0 \end{bmatrix} \quad g_4 = [g_2, g_3] = \begin{bmatrix} \rho \cos \theta \\ \rho \sin \theta \\ 0 \\ 0 \end{bmatrix}$$

Since span of g_1, g_2, g_3, g_4 is \mathfrak{R}^4 the rolling penny system is completely non-holonomic with degree of nonholonomy 3 and growth vector $(2, 3, 4)$.

Front wheel drive car

Recall that with u_1 is the driving input and u_2 the steering input

$$\dot{q} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ \frac{1}{l} \tan \phi \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

$$g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{l \cos^2 \phi} \\ 0 \end{bmatrix} \quad g_4 = [g_1, g_3] = \begin{bmatrix} -\frac{\sin \theta}{l \cos^2 \phi} \\ \frac{\cos \theta}{\cos^2 \phi} \\ 0 \\ 0 \end{bmatrix}$$

Except at $\phi = \pm\pi$ the front wheel drive car is completely non-holonomic with degree of nonholonomy 3 and growth vector (2,3,4). The vector fields are named: g_1 is **drive**, g_2 is **steer**, g_3 is **wiggle**, and g_4 is **slide**.

Thank you for your attention. Questions?

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