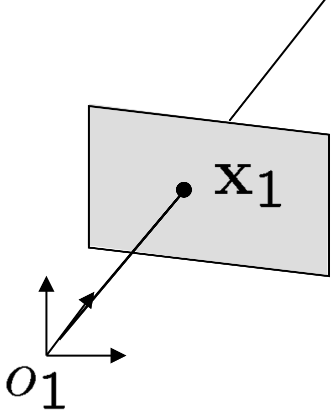
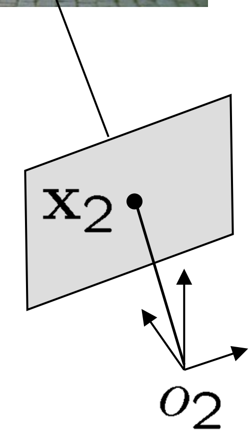
Lecture 4

Two-View Geometry

General Formulation



Given two views of the scene recover the unknown camera displacement and 3D scene structure



Pinhole Camera Model

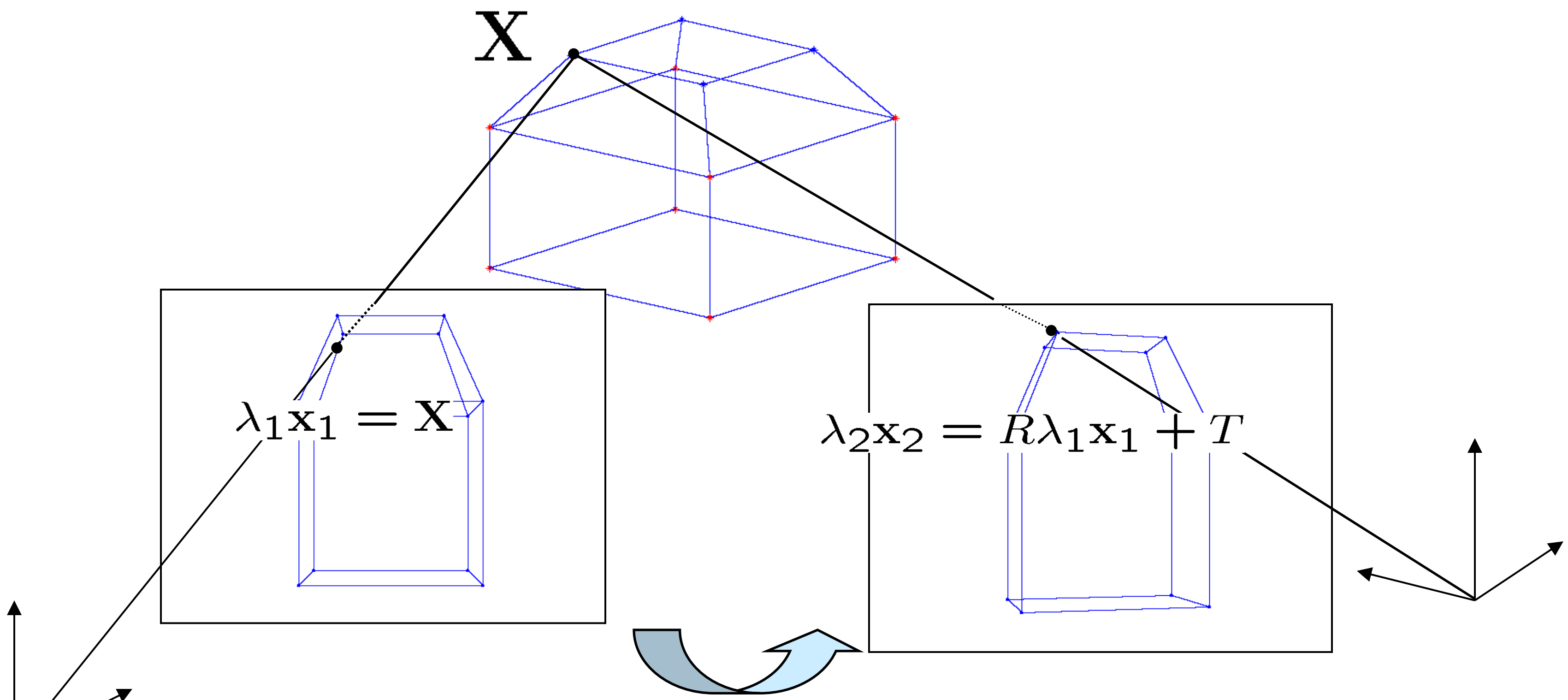
- 3D points $\mathbf{X} = [X, Y, Z, W]^T \in \Re^4, \quad (W = 1)$
- Image points $\mathbf{x} = [x, y, z]^T \in \Re^3, \quad (z = 1)$
- Perspective Projection $\lambda \mathbf{x} = \mathbf{X}$

$$\lambda = Z \quad x = \frac{X}{Z} \quad y = \frac{Y}{Z}$$

- Rigid Body Motion $\Pi = [R, T] \in \Re^{3\times 4}$
- Rigid Body Motion + Projective projection

$$\lambda \mathbf{x} = \Pi \mathbf{X} = [R, T]\mathbf{X}$$

Rigid Body Motion – Two views



$$\boxed{\lambda_2 \mathbf{x}_2 = R\lambda_1 \mathbf{x}_1 + T}$$

3D Structure and Motion Recovery

Euclidean transformation

$$\underbrace{\lambda_2}_{}\underbrace{\mathbf{x}_2}_{} = \underbrace{R\lambda_1}_{}\underbrace{\mathbf{x}_1}_{} + \underbrace{T}_{}$$

measurements unknowns

$$\sum_{j=1}^{n} \|\mathbf{x}_1^j - \pi(R_1, T_1, \mathbf{X})\|^2 + \|\mathbf{x}_2^j - \pi(R_2, T_2, \mathbf{X})\|^2$$

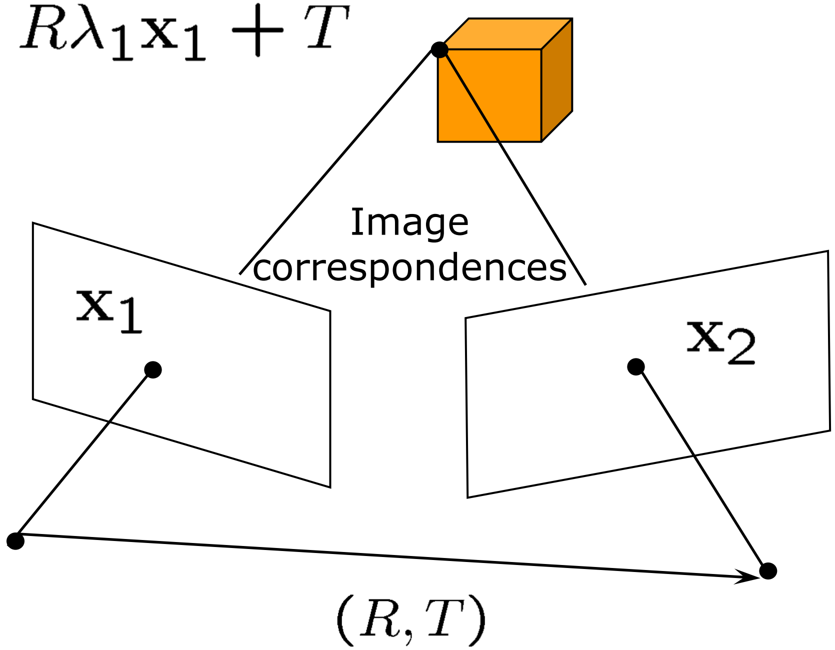
Find such Rotation and Translation and Depth that the reprojection error is minimized

Two views ~ 200 points
6 unknowns – Motion 3 Rotation, 3 Translation
- Structure 200x3 coordinates
- (-) universal scale

Difficult optimization problem

Epipolar Geometry

$$\lambda_2 \mathbf{x}_2 = R\lambda_1 \mathbf{x}_1 + T$$



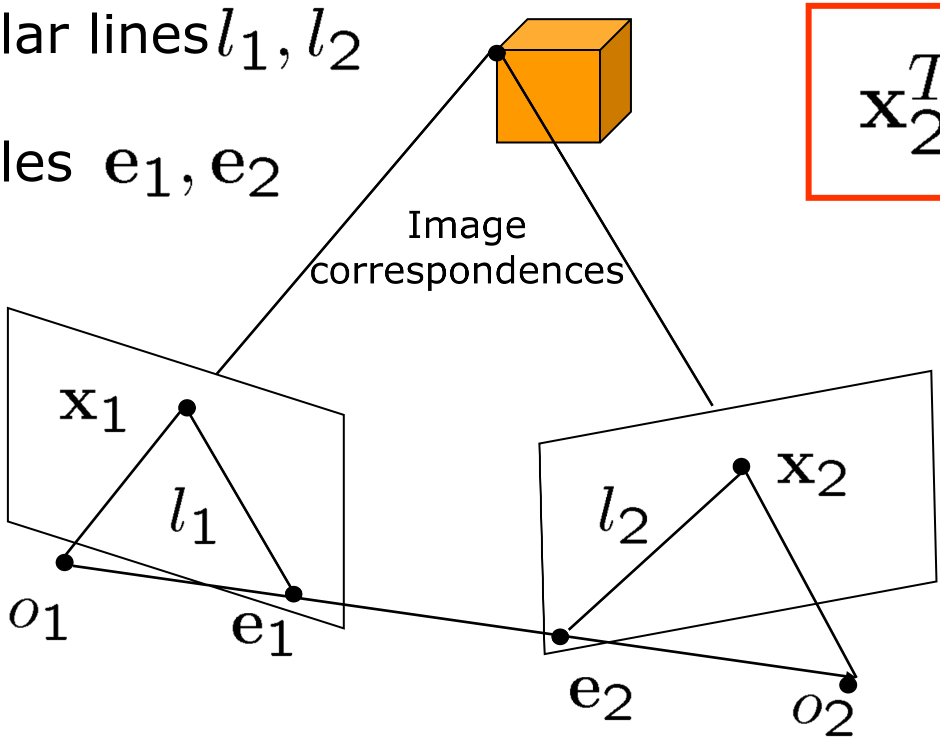
- Algebraic Elimination of Depth [Longuet-Higgins '81]:

$$\mathbf{x}_2^T \underbrace{\widehat{T}R}_{E} \mathbf{x}_1 = 0$$

- Essential matrix $E = \widehat{T}R$

Epipolar Geometry

- Epipolar lines l_1, l_2
- Epipoles $\mathbf{e}_1, \mathbf{e}_2$



$$\boxed{\mathbf{x}_2^T E \mathbf{x}_1 = 0}$$

$$E = \hat{T} R$$

$$l_1 \sim E^T \mathbf{x}_2 \qquad l_i^T \mathbf{x}_i = 0 \qquad l_2 \sim E \mathbf{x}_1$$

$$E \mathbf{e}_1 = 0 \qquad l_i^T \mathbf{e}_i = 0 \qquad \mathbf{e}_2 E^T = 0$$

Characterization of the Essential Matrix

$$\mathbf{x}_2^T \widehat{T} R \mathbf{x}_1 = 0$$

- Essential matrix $E = \widehat{T} R$ Special 3x3 matrix

$$\mathbf{x}_2^T \begin{bmatrix} e_1 & e_2 & e_2 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix} \mathbf{x}_1 = 0$$

Theorem 1a (Essential Matrix Characterization)
A non-zero matrix E is an essential matrix <u>iff</u> its SVD: $E = U \Sigma V^T$
satisfies: $\Sigma = diag([\sigma_1, \sigma_2, \sigma_3])$ with $\sigma_1 = \sigma_2 \neq 0$ and $\sigma_3 = 0$
and $U, V \in SO(3)$

Estimating the Essential Matrix

- Estimate Essential matrix $E = \hat{T}R$
- Decompose Essential matrix into R, T

$$\mathbf{x}_2^T \hat{T} R \mathbf{x}_1 = 0$$

- Given n pairs of image correspondences:
- Find such Rotation and Translation that the epipolar error is minimized

$$min_E \sum_{j=1}^{n} \mathbf{x}_2^{jT} E \mathbf{x}_1^j$$

- Space of all Essential Matrices is 5 dimensional
- 3 Degrees of Freedom – Rotation
- 2 Degrees of Freedom – Translation (up to scale !)

Pose Recovery from the Essential Matrix

Essential matrix $E = \hat{T}R$

Theorem 1a (Pose Recovery)
There are two relative poses (R, T) with $T \in \mathcal{R}^3$ and $R \in SO(3)$ corresponding to a non-zero matrix essential matrix.

$$E = U\Sigma V^T$$

$$\begin{aligned}(\hat{T}_1, R_1) &= (UR_Z(+\tfrac{\pi}{2})\Sigma U^T, UR_Z^T(+\tfrac{\pi}{2})V^T) \\ (\hat{T}_2, R_2) &= (UR_Z(-\tfrac{\pi}{2})\Sigma U^T, UR_Z^T(-\tfrac{\pi}{2})V^T)\end{aligned}$$

$$\Sigma = diag([1,1,0]) \qquad R_z(+\tfrac{\pi}{2}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Twisted pair ambiguity $(R_2, T_2) = (e^{\hat{u}\pi}R_1, -T_1)$

Estimating Essential Matrix

$$\mathbf{x}_2^T \hat{T} R \mathbf{x}_1 = 0$$

- Denote $\mathbf{a} = \mathbf{x}_1 \otimes \mathbf{x}_2$

$$\mathbf{a} = [x_1x_2, x_1y_2, x_1z_2, y_1x_2, y_1y_2, y_1z_2, z_1x_2, z_1y_2, z_1z_2]^T$$

$$E^s = [e_1, e_4, e_7, e_2, e_5, e_8, e_3, e_6, e_9]^T$$

- Rewrite $\mathbf{a}^T E^s = 0$

- Collect constraints from all points

$$\chi E^s = 0$$

$$min_E \sum_{j=1}^n \mathbf{x}_2^{jT} E \mathbf{x}_1^j \quad \Longrightarrow \quad min_{E^s} \|\chi E^s\|^2$$

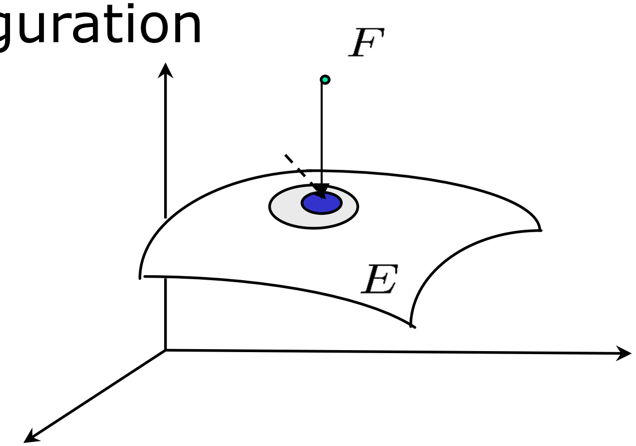
Estimating Essential Matrix

$$min_E \sum_{j=1}^{n} (\mathbf{x}_2^{jT} E \mathbf{x}_1^j)^2 \quad \Longrightarrow \quad min_{E^s} \|\chi E^s\|^2$$

Solution

- Eigenvector associated with the smallest eigenvalue of $\chi^T \chi$
- if $rank(\chi^T \chi) < 8$ degenerate configuration

Projection on to Essential Space



Theorem 2a (Project to Essential Manifold)

If the SVD of a matrix $F \in \mathcal{R}^{3\times3}$ is given by $F = U diag(\sigma_1, \sigma_2, \sigma_3) V^T$ then the essential matrix E which minimizes the Frobenius distance $\|E - F\|_f^2$ is given by $E = U diag(\sigma, \sigma, 0) V^T$ with $\sigma = \frac{\sigma_1 + \sigma_2}{2}$

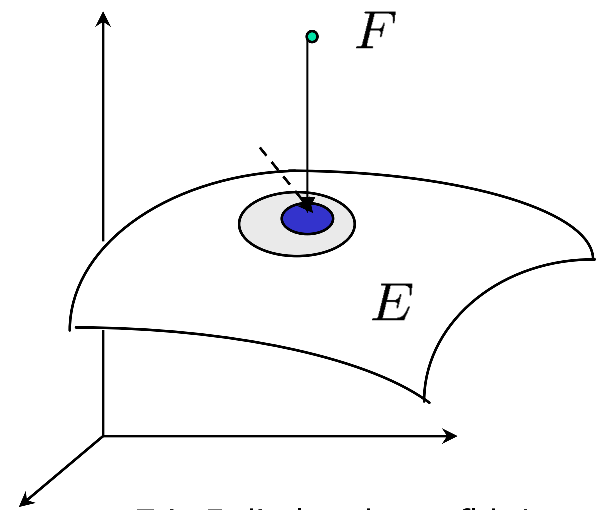
Two view linear algorithm

$$E = \{\hat{T}R | R \in SO(2), T \in S^2\}$$

- Solve the LLSE problem:

$$min_E \sum_{j=1}^{n} \mathbf{x}_2^{jT} E \mathbf{x}_1^j$$

$\chi E^s = 0$ followed by projection



E is 5 diml. sub. mnfld. in

- 8-point linear algorithm

- Project onto the essential manifold:

SVD: $F = U\Sigma V^T$

$$\Sigma' = diag(1, 1, 0)$$

$$E = U\Sigma' V^T$$

- Recover the unknown pose:

$$\boxed{(\hat{T}, R) = (UR_Z(\pm\tfrac{\pi}{2})\Sigma U^T, UR_Z^T(\pm\tfrac{\pi}{2})V^T)}$$

Pose Recovery

- There are exactly two pairs (R,T) corresponding to each essential matrix E .
- There are also two pairs (R,T) corresponding to each essential matrix $-E$.
- Positive depth constraint - used to disambiguate the physically impossible solutions
- Translation has to be non-zero
- Points have to be in general position
  - degenerate configurations – planar points
  - quadratic surface
- Linear 8-point algorithm
- Nonlinear 5-point algorithms yield up to 10 solutions

3D structure recovery

$$\lambda_2 \underline{\mathbf{x}_2} = R \underline{\lambda_1} \mathbf{x}_1 \underline{+} \gamma T$$

- Eliminate one of the scale's

$$\lambda_1^j \widehat{\mathbf{x}_2^j} R \mathbf{x}_1^j + \gamma \widehat{\mathbf{x}_2^j} T = 0, \quad j = 1, 2, \ldots, n$$

- Solve LLSE problem

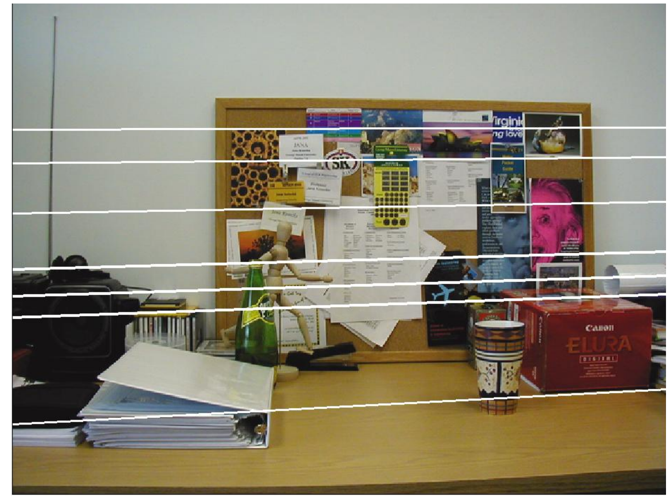
$$M^j \bar{\lambda}^j \doteq \left[\widehat{\mathbf{x}_2^j} R \mathbf{x}_1^j, \ \widehat{\mathbf{x}_2^j} T \right] \begin{bmatrix} \lambda_1^j \\ \gamma \end{bmatrix} = 0$$

If the configuration is non-critical, the Euclidean structure of then points and motion of the camera can be reconstructed up to a universal scale.

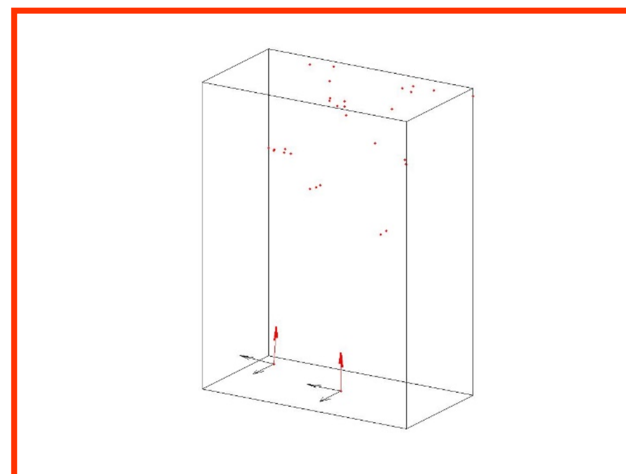
Example- Two views

Point Feature Matching

Example – Epipolar Geometry



Camera Pose
and
Sparse Structure Recovery



Epipolar Geometry – Planar Case

- Plane in first camera coordinate frame

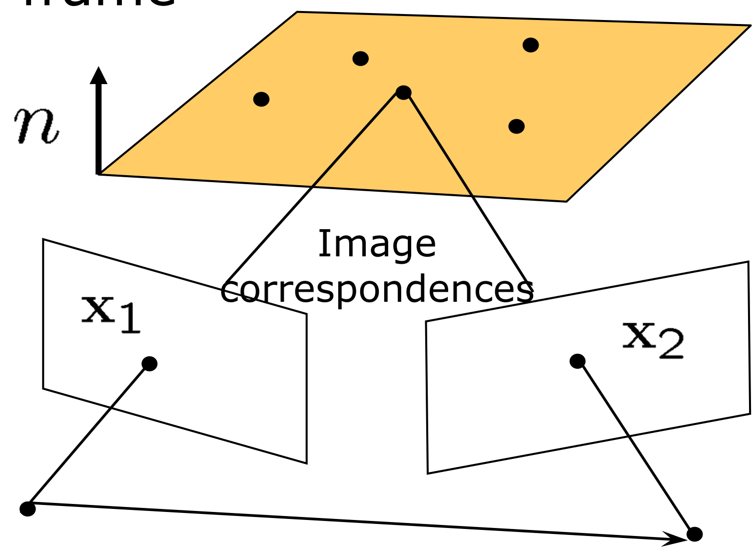
$$aX + bY + cZ + d = 0$$

$$\frac{1}{d} N^T \mathbf{X} = 1$$

$$\lambda_2 \mathbf{x}_2 = R \lambda_1 \mathbf{x}_1 + T$$

$$\lambda_2 \mathbf{x}_2 = (R + \frac{1}{d} T N^T) \lambda_1 \mathbf{x}_1$$

$$\mathbf{x}_2 \sim H \mathbf{x}_1$$



Planar homography

$$\boxed{H = (R + \frac{1}{d} T N^T)}$$

Linear mapping relating two corresponding planar points in two views

Decomposition of H

- Algebraic elimination of depth $\widehat{\mathbf{x}_2} H \mathbf{x}_1 = 0$
- H_L can be estimated linearly $H_L = \lambda H$
- Normalization of $H = H_L / \sigma_3$
- Decomposition of H into 4 solutions $H = (R + \frac{1}{d} T N^T)$

$$
\begin{array}{|c|c|c|c|}
\hline
\begin{array}{c} R_1 = W_1 U_1^T \\ N_1 = \widehat{v_2} u_1 \\ \frac{1}{d} T_1 = (H - R_1) N_1 \end{array} &
\begin{array}{c} R_3 = R_1 \\ N_3 = -N_1 \\ \frac{1}{d} T_3 = -\frac{1}{d} T_1 \end{array} &
\begin{array}{c} R_2 = W_2 U_2^T \\ N_2 = \widehat{v_2} u_2 \\ \frac{1}{d} T_2 = (H - R_2) N_2 \end{array} &
\begin{array}{c} R_4 = R_2 \\ N_4 = -N_2 \\ \frac{1}{d} T_4 = -\frac{1}{d} T_2 \end{array} \\
\hline
\end{array}
$$

- $$H^T H = V \Sigma V^T \quad V = [v_1, v_2, v_3] \quad \Sigma = diag(\sigma_1^2, \sigma_2^2, \sigma_3^2)$$
  $$u_1 \doteq \frac{\sqrt{1-\sigma_3^2}\, v_1 + \sqrt{\sigma_1^2 - 1}\, v_3}{\sqrt{\sigma_1^2 - \sigma_3^2}} \qquad u_2 \doteq \frac{\sqrt{1-\sigma_3^2}\, v_1 - \sqrt{\sigma_1^2 - 1}\, v_3}{\sqrt{\sigma_1^2 - \sigma_3^2}}$$
  $$U_1 = [v_2, u_1, \widehat{v_2} u_1], \quad W_1 = [H v_2, H u_1, \widehat{H v_2} H u_1];$$
  $$U_2 = [v_2, u_2, \widehat{v_2} u_2], \quad W_2 = [H v_2, H u_2, \widehat{H v_2} H u_2].$$

Motion and pose recovery for planar scene

- Given at least 4 point correspondences $\widehat{\mathbf{x}_2^j} H \mathbf{x}_1^j = 0$
- Compute an approximation of the homography matrix H_l^s
- As nullspace of χ

  $\chi H_l^s = 0$ the rows of χ are $\mathbf{a} = \mathbf{x}_1^j \otimes \widehat{\mathbf{x}_2^j}$
- Normalize the homography matrix

  $$H = H_L / \sigma_3$$
- Decompose the homography matrix

  $$H^T H = V \Sigma V^T$$
- Select two physically possible solutions imposing positive depth constraint

Example

Special Rotation Case

- Two view related by rotation only $\lambda_2 \mathbf{x}_2 = R\lambda_1 \mathbf{x}_1$

$$\widehat{\mathbf{x}_2} R \mathbf{x}_1 = 0$$

- Mapping to a reference view

- Mapping to a cylindrical surface

Motion and Structure Recovery – Two Views

- Two views – general motion, general structure

1. Estimate essential matrix
2. Decompose the essential matrix
3. Impose positive depth constraint
4. Recover 3D structure

- Two views – general motion, planar structure

1. Estimate planar homography
2. Normalize and decompose H
3. Recover 3D structure and camera pose