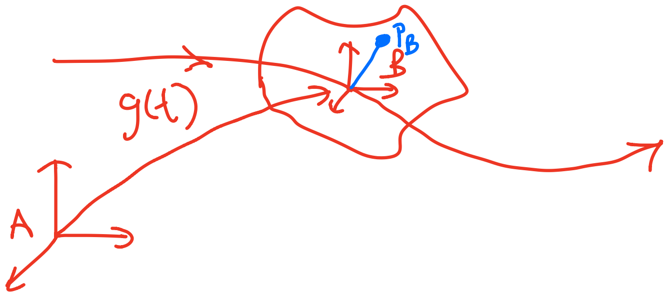
106B Discussion 2: Review of velocities and Jacobians

$$g(t) \in SE(3)$$

$$\begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$



$$g(t) = g_{AB}(t)$$

$$p_A = g_{AB}(t)\, p_B$$

$$\dot{p}_A = \dot{g}_{AB}(t)\, \underbrace{p_B}_{g_{AB}^{-1}(t)\, p_A}$$

$$\dot{p}_A = \begin{bmatrix} \quad & \quad \end{bmatrix} p_A$$

$$\dot{p}_A = \underbrace{\dot{g}_{AB}(t)\, g_{AB}^{-1}(t)}_{\hat{V}^s_{AB}}\, p_A$$

$$\hat{V}^s_{AB} = \begin{bmatrix} \hat{\omega}^s_{AB} & v^s_{AB} \\ 0 & 0 \end{bmatrix} \in se(3) \longrightarrow V^s_{AB} = \begin{bmatrix} v^s_{AB} \\ \omega^s_{AB} \end{bmatrix} \in \mathbb{R}^6$$

<u>spatial rigid body velocity of $g_{AB}(t)$</u>

$$\hat{V}^b_{AB} = g_{AB}^{-1}\, \dot{g}_{AB} \in se(3)$$

$$\begin{bmatrix} \hat{\omega}^b_{AB} & v^b_{AB} \\ 0 & 0 \end{bmatrix}^{\vee} \longleftrightarrow V^b_{AB} = \begin{bmatrix} v^b_{AB} \\ \omega^b_{AB} \end{bmatrix}^{\wedge}$$

"body frame" velocity of $g_{AB}(t)$

Adjoint transformation on $se(3)$

twist $\hat{\xi}_B \in se(3)$ written in frame B.

we can re-write it in frame A by

$$\underbrace{\hat{\xi}_A p_A}_{} = g_{AB} \underbrace{\hat{\xi}_B g_{AB}^{-1} p_A}_{p_B}$$

$$\hat{\xi} \longmapsto g \hat{\xi} g^{-1}$$

$$se(3) \to se(3)$$

"Adjoint transformation"

$$\hat{\xi}_A = g \hat{\xi}_B g^{-1} \leftarrow$$

$$\xi_A = Ad_g \xi_B \leftarrow$$

$$SE(3) = \left\{ g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \right\}$$

$$se(3) = \left\{ \xi = V = \underbrace{\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}}_{} \right\}$$

$$\updownarrow$$

$$\mathbb{R}^6 : \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$Ad_g \in \mathbb{R}^{6\times 6} : \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}, \text{ where } g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Problem 1 :

given: constant twist $\hat{\xi} \in se(3) = (v, \omega)$

$$\boxed{g(t) = e^{\hat{\xi} t} g_0} \leftarrow \frac{d}{dt} = \hat{\xi} e^{\hat{\xi} t} g_0$$

2) $\hat{V}^s_{ab} = \dot{g} g^{-1} = \left(\hat{\xi} \cancel{e^{\hat{\xi} t}} \cancel{g_0}\right)\left(\cancel{g_0^{-1}} \cancel{e^{-\hat{\xi} t}}\right)$

$$= \hat{\xi}$$

3) $\hat{V}^b_{ab} = g^{-1} \dot{g} = \left(g_0^{-1} e^{-\hat{\xi} t}\right)\left(\hat{\xi} e^{\hat{\xi} t} g_0\right)$

$$= g^{-1}_{(t)} \hat{\xi} g(t)$$

$$= g_0^{-1} \cancel{e^{-\hat{\xi} t}} \cancel{e^{\hat{\xi} t}} \hat{\xi} g_0$$

$$\hat{V}^b_{ab} = g_0^{-1} \hat{\xi} g_0$$

$$g(t) = e^{\hat{\xi} t} g_0$$

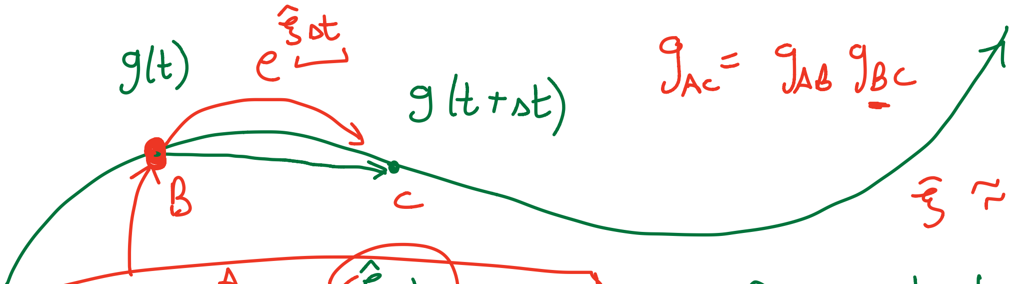
given: $g \in SE(3)$

↳ $\log(g) = \hat{\xi} \in se(3)$

$$e^{\hat{\xi}} = g$$

$g(t) = e^{\hat{\xi} t} I$ ← uniform motion according to velocity ξ

after 1 second: $g(1) = e^{\hat{\xi}} = \underline{g}$



$SE(3)$

$g_{AC} = g_{AB}\, g_{BC}$

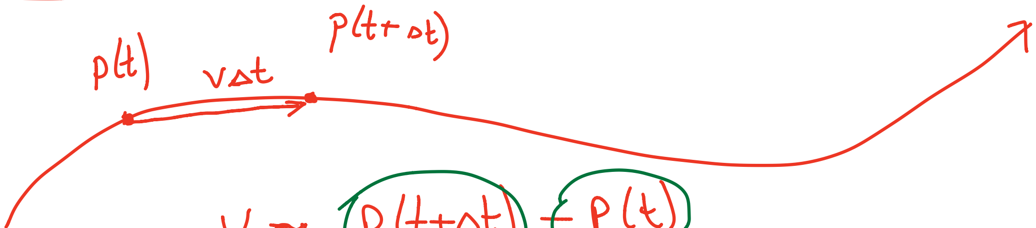
$\hat{\xi} \approx \frac{1}{\Delta t} \log\left(g^{-1}(t)\, g(t+\Delta t)\right)$

$\boxed{g_{AB}(t+\Delta t) = e^{\hat{\xi}\Delta t} g_{AB}(t)} \Rightarrow \hat{\xi} \approx \frac{1}{\Delta t} \log\left(g(t+\Delta t)\, g(t)^{-1}\right)$

$\lim_{\Delta t \to 0} \quad \hat{V}^s = \dot{g} g^{-1}$

$\boxed{g_{AB'}(t+\Delta t) = g_{AB}(t)\, e^{\hat{\xi}\Delta t}}$ $B \to B'$

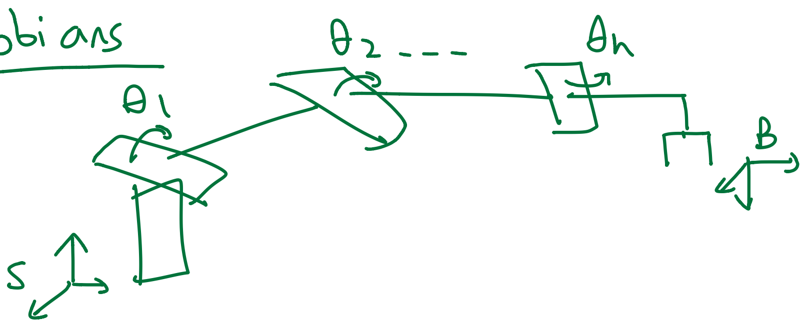
$\mathbb{R}^3$



$v \approx \frac{p(t+\Delta t) - p(t)}{\Delta t}$

$\lim_{\Delta t \to 0} : \quad \dot{p}(t)$

Jacobians



Forward "position" kinematics* : $(\theta_1 \dots \theta_n) \to g_{sB}(\theta)$

$= e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{sB}(0)$

Forward "velocity" kinematics: $(\dot{\theta}_1, \dot{\theta}_2, \cdots, \dot{\theta}_n) \to \hat{V}_{SB}$?

Jacobian $J^S(\theta) \in \mathbb{R}^{6\times n}$

$$V^S_{SB} = J^S(\theta)\,\dot{\theta} \qquad \text{(spatial jacobian)}$$

$$J^b: \qquad V^b_{SB} = J^b\,\dot{\theta}$$

$$J^S = Ad_{g_{SB}}\, J^b$$

Inverse "velocity" kinematics:

desired velocity V^S_{SB}. what joint velocities do I need?

Solve for $\dot{\theta}$: $V^S_{SB} = J^S\dot{\theta}$

$$\Rightarrow \quad \dot{\theta} = \underbrace{J^{S+}}\, V^S_{SB}$$

$$A \quad [\qquad\qquad]^{6\times n}$$

$$Ax = b$$

x_0 : $Ax_0 = b$

$x_s \in \mathcal{N}(A)$: $x_0 + x_s$ is also a sol.

$$A(x_0 + x_s) = Ax_0 + Ax_s$$

$$b + 0 = b$$