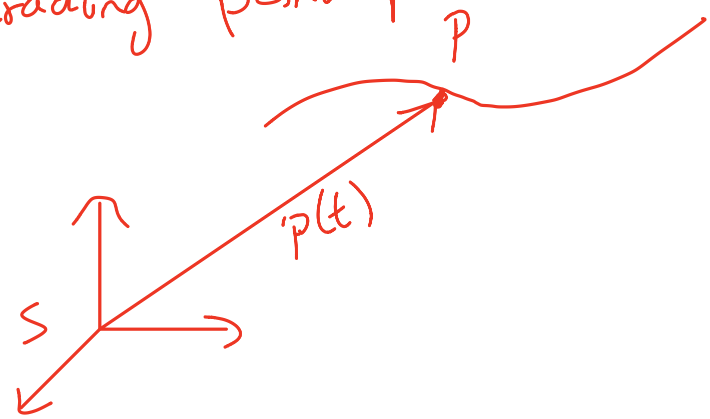
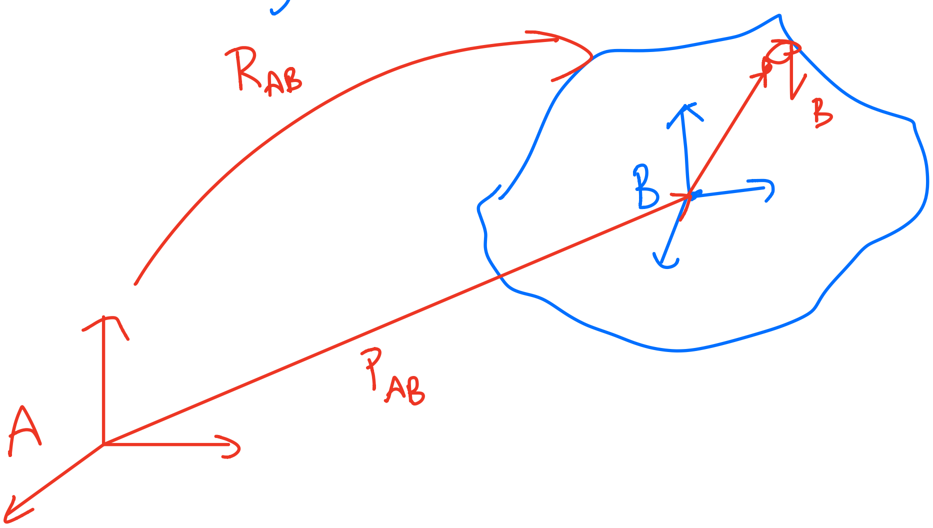
Discussion 1: Review of SO(3) and SE(3) Kinematics

Pre-106A physics

→ tracking point p:



→ tracking rigid bodies.



$$q_A = R_{AB} \underbrace{q_B} + p_{AB}$$

$$\underbrace{\begin{bmatrix} q_A \\ 1 \end{bmatrix}}_{4\times 1} = \underbrace{\begin{bmatrix} R_{AB} & p_{AB} \\ 0 & 1 \end{bmatrix}}_{\substack{4\times 4 \\ g_{AB}}} \underbrace{\begin{bmatrix} q_B \\ 1 \end{bmatrix}}_{4\times 1}$$

Homogeneous coordinates

a point $p \in \mathbb{R}^3 \to$ homogeneous coords as $\begin{bmatrix} p \\ 1 \end{bmatrix}$

a vector $v \in \mathbb{R}^3 \to$ homogeneous coords as $\begin{bmatrix} v \\ 0 \end{bmatrix}$

Rotation matrix R:

(1) As a transformation b/w two ref frames

$R = R_{AB}$: $P_A = R_{AB} P_B$

(2) As an action on $\mathbb{R}^3$.

$p \xrightarrow{R} p'$

new location

$p' = Rp$

Exponential Coordinates

Matrix Exponential:

Scalar exp: $e^x : \mathbb{R} \to \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \leftarrow$$

Matrix exponential: $e^A : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$

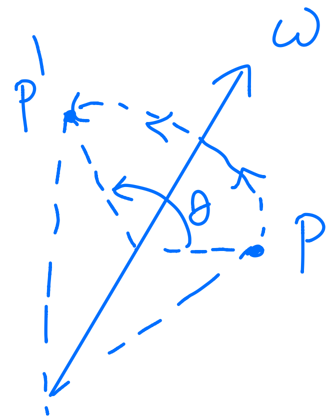
$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \leftarrow$$

$R(\omega, \theta)$ = rotates a point about the unit axis ω by θ radians

$\nearrow$ $\uparrow$

$\mathbb{R}^3$ $\mathbb{R}$

$\|\omega\| = 1$



$$R(\omega, \theta) = e^{\hat{\omega}\theta} = \underbrace{I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)}_{\text{Rodriguez formula } (\|\omega\|=1)}$$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \text{ where } \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\hat{\omega} v = \omega \times v$$

Euler's Theorem: Given any rotation matrix R, $\exists\ \omega \in \mathbb{R}^3$, with $\|\omega\| = 1$ and $\theta \in \mathbb{R}$ s.t. $R = R(\omega, \theta)$

$$= e^{\hat{\omega}\theta}$$

We call (ω, θ) the "exponential coordinates" of R.

Rigid body transformations:

$$g = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$$

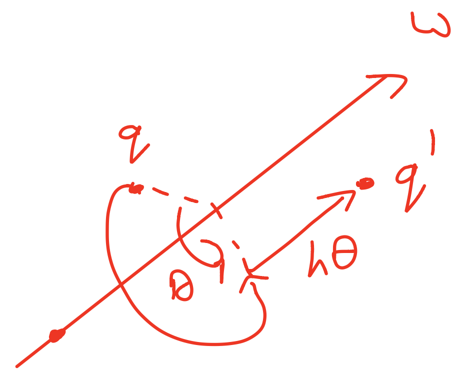
1. As a transformation b/w reference frames

$$g = g_{AB}: \qquad P_A = g_{AB} P_B$$

2. As an action on $\mathbb{R}^3$

$$q' = Rq + P \quad \cong \quad q' = gq$$

Screw Motions



Twists:

$$\hat{\xi} = \underbrace{\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}}_{4\times4} \qquad \xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6$$

$$g = e^{\hat{\xi}\theta}$$

Thm: for any RBT g: $\exists\ \underbrace{\xi}_{\text{"unit" twist.}} \in \mathbb{R}^6 = \begin{bmatrix} v \\ \omega \end{bmatrix}$, and a $\theta \in \mathbb{R}$

s.t. $g = e^{\hat{\xi}\theta}$

$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$ is "unit" twist if:

a. $\omega = 0$ and $\|v\| = 1$

b. $\|\omega\| = 1$

$SO(3)$ = "Special orthogonal group" (set of all rotation matrices)

$$= \{ R \in \mathbb{R}^{3\times3} : R^T R = I, \ \det(R) = 1 \}$$

$$so(3) = \{ \hat{\omega} : \omega \in \mathbb{R}^3 \}$$

$$= \{ A \in \mathbb{R}^{3\times3} : A^T = -A \} \leftarrow \text{skew symmetric matrices.}$$

$SE(3)$ = "special Euclidean group"

$$= \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}, \ R \in SO(3), \ p \in \mathbb{R}^3 \right\}$$

$se(3)$ = "set of all twists"

$$= \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}, \ \omega \in \mathbb{R}^3, v \in \mathbb{R}^3 \right\}$$

Exponential map

$$\exp : so(3) \longrightarrow SO(3)$$
$$\hat{\omega}\theta \qquad e^{\hat{\omega}\theta} = R$$

$$\exp : se(3) \longrightarrow SE(3)$$
$$\hat{\xi}\theta \qquad e^{\hat{\xi}\theta} = g$$

$$\log : SO(3) \to so(3)$$

$$R \qquad \log(R) = \hat{\omega}$$

$$\text{s.t. } e^{\hat{\omega}} = R$$

is this well defined?

$$\log : SE(3) \to se(3)$$

$$g \qquad \log(g) = \hat{\xi}$$

$$\text{s.t. } e^{\hat{\xi}} = g$$

The log can be made well-defined by restricting its output:

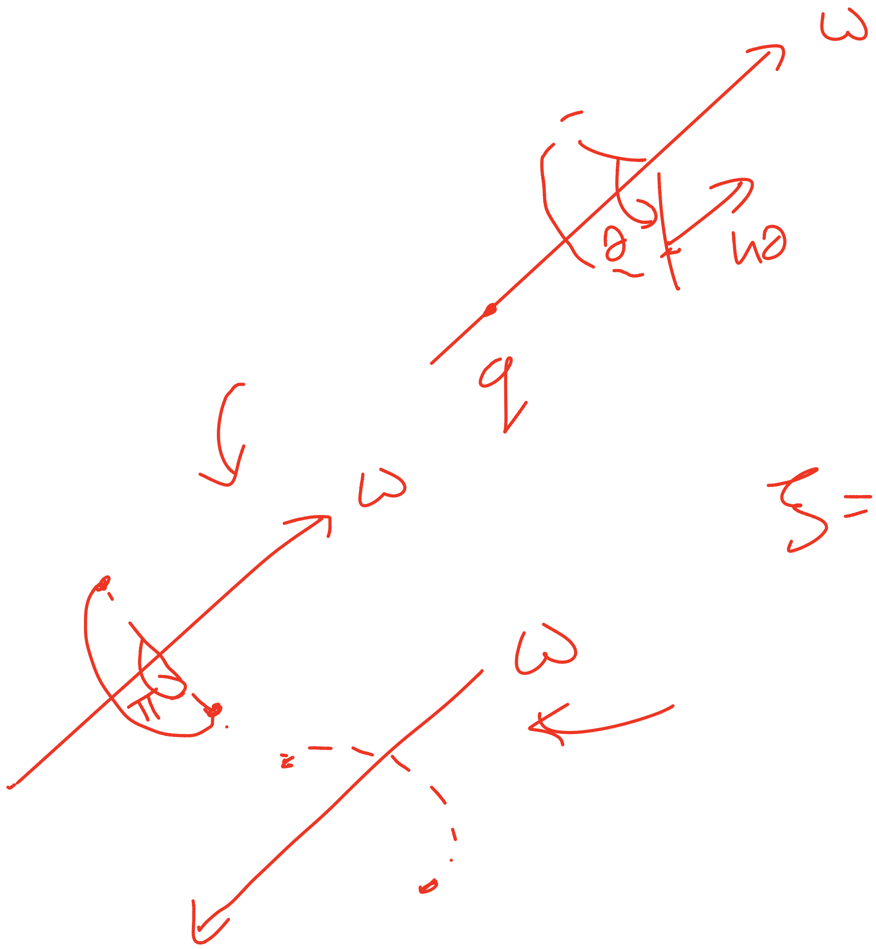
$$\mathrm{Log}(R) = \hat{\omega}, \quad \|\omega\| < \pi$$

$$\mathrm{Log}(g) = \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} : \|\omega\| < \pi$$

$$\log(R) = \hat{\omega}\theta$$

$$\log(g) = \hat{\xi}\theta$$

(ω, θ) are exponential coordinates for R



$$\xi = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix} = V$$