Dynamical Systems

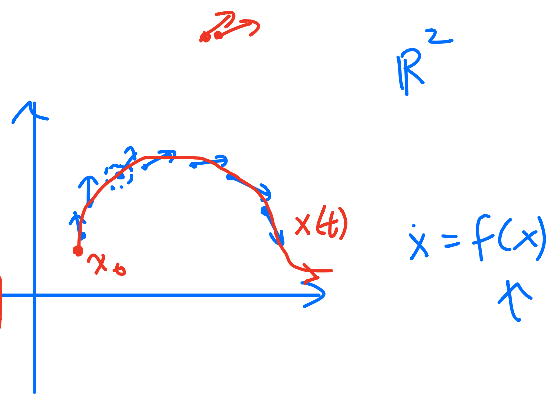
$x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ "nice"

$$\boxed{\dot{x} = f(x)}$$ "Autonomous system"

$\dot{x} = f(x, t)$ "time varying"

$\dot{x} = f(x, u)$ "controlled system"

$\dot{x} = f(x, u, t)$

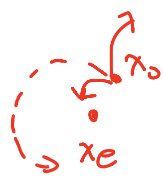


"Lipschitz continuous"

$\exists K \quad \|f(x) - f(y)\| \leq K\|x - y\|$

Equilibrium points $\quad \dot{x} = f(x)$

$x_e \in \mathbb{R}^n$: $\quad f(x_e) = 0 \qquad \dot{x} = 0$

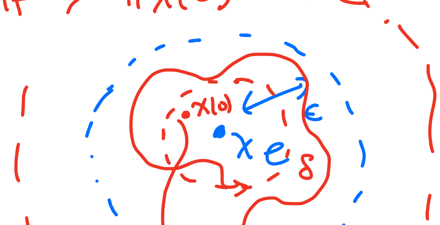


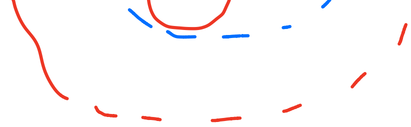
STABILITY OF EQUILIBRIA :

① Stability in the sense of Lyapunov "SISL"

$\forall \epsilon > 0, \quad \exists \delta > 0, \quad$ s.t. $\quad \|x(t) - x_e\| < \epsilon$

if $\|x(0) - x_e\| < \delta$





② Asymptotically stable : (AS)

→ SISL

→ $x(t) \to x_e$ as $t \to \infty$ if $x(0) \in D$

where D is a neighbourhood of x_e

"Globally AS" if $D = \mathbb{R}^n$

③ Exponentially stable :

$$\exists M, \alpha \quad : \quad \|x(t) - x_e\| \leq M e^{-\alpha(t-t_0)} \|x_0 - x_e\|$$

Ⓐ Lyapunov's Indirect Method $\quad \dot{x} = f(x)$

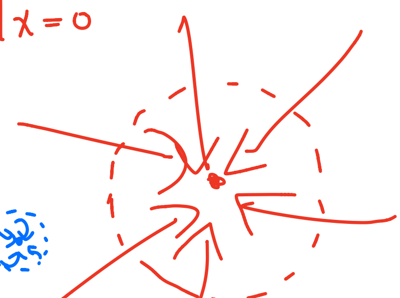
Assume $x_e = 0$

→ Judge stability of 0 by linearizing f near 0.

$$f(x) \approx \overset{0}{\cancel{f(0)}} + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + \cdots$$

close to the origin, $f(x) \approx Ax \quad A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$

so our system $\dot{x} = f(x)$ behaves like the linear system $\dot{x} = Ax$



Conditions: $\{ \lambda_i = \text{eig}(A) \}$

→ If $\text{Re}(\lambda) < 0$ → locally AS

→ If any $\text{Re}(\lambda) > 0$ → unstable

→ else, if we have some evals on $i\omega$ axis ($\text{Re}(\lambda) = 0$) → inconclusive.

Problem 1 - Lyapunov's Indirect Method: Modified Van Der Pol Oscillator

Consider the following model for an oscillator with nonlinear damping.

$$\boxed{\ddot{x} + \mu(1 - x^2)\dot{x} + x = 0} \leftarrow \ddot{x} = -x - \mu(1-x^2)\dot{x} \tag{0.1}$$

where μ is a scalar damping coefficient.

1. By choosing a good set of state variables, write the above model in state space form.
2. Find all equilibria of this system.
3. Linearize the system about the equilibria. Using the indirect method of Lyapunov, comment on the stability of the equilibria for the cases where $\mu > 0$ and $\mu = 0$.

State space form: $\boxed{\dot{x} = f(x)}$

1) $x = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, $x_1 = x$, $x_2 = \dot{x}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - \mu(1 - x_1^2) x_2 \end{bmatrix} \leftarrow f(x)$$

2) Find equilibria

solve $f(x) = 0$ for x

$x_2 = 0$, $x_1 = 0$

the only equilibrium is the origin.

3) Want to approximate $\dot{x} = f(x)$ by $\dot{x} = Ax$

where $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$

$f: \mathbb{R}^n \to \mathbb{R}^n$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \qquad \left(\frac{\partial f}{\partial x}\right)_{ij} = \frac{\partial f_i}{\partial x_j}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 + 2\mu x_1 x_2 & \mu x_1^2 - \mu \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix}$$

$\approx$: $\dot{x} = Ax$

$$\mathrm{char}_A(\lambda) = \det(A - \lambda I)$$

$$= \left|\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda - \mu \end{pmatrix}\right| = \lambda^2 + \mu\lambda + 1$$

roots : $\lambda = \dfrac{-\mu \pm \sqrt{\mu^2 - 4}}{2}$

Q: $\mathrm{Re}(\lambda)$? $\mu > 0$

If $\mu^2 - 4 < 0 \Rightarrow \lambda = \underbrace{\dfrac{-\mu}{2}} \pm (\dots) i$

$\rightarrow \mathrm{Re}(\lambda) < 0$ "AS"

If $\mu^2 - 4 > 0 \Rightarrow \lambda_1 = -\dfrac{\mu}{2} - \dfrac{\sqrt{\mu^2 - 4}}{2} < 0$

$\lambda_2 = -\dfrac{\mu}{2} + \dfrac{\sqrt{\mu^2 - 4}}{2} < 0$

$\rightarrow \mathrm{Re}(\lambda) < 0$ "A" stable

DIRECT METHOD OF LYAPUNOV

$\dot{x} = f(x), \quad x_e = 0$

let D be some nbhd of 0 :

$\exists\, V : D \rightarrow \mathbb{R}$: $\quad \forall x \in D, x \neq 0 : V(x) > 0$

$\dot{V}(0) = 0 \qquad x = 0 : V(0) = 0$

1. $\dot{V} \leq 0 \quad x \in D \Rightarrow$ SISL
2. $\dot{V} < 0 \quad x \in D \ (x \neq 0) \Rightarrow$ AS

(3) $\dot{V} \le -\gamma V \quad x \in D \Rightarrow$ ES

Global? if $D = \mathbb{R}^n$. V is "radially unbounded"

$V \to \infty$ as $\|x\| \to \infty$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x} f(x) \leftarrow \text{projection of } \frac{\partial V}{\partial x} \text{ onto flow } f(x)$$