EECS 106A/206A
Jacobians and Singularities

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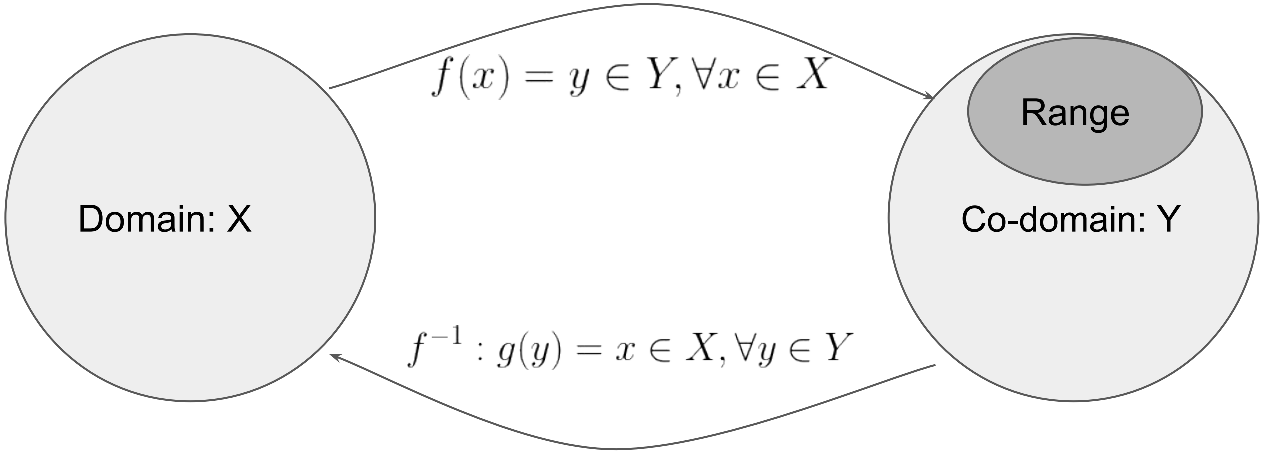
Linear Algebra Review

What is a Function?



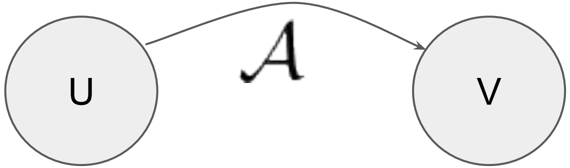
Injective (one-to-one): $f(x_1) = f(x_2) \implies x_1 = x_2, \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

Surjective (onto): $\forall y \in Y, \exists x \in X \ni y = f(x)$ the range is the codomain

Bijective (both): Only bijective functions have inverses

What is a Linear Map (aka Linear Operator)?

A linear operator $\mathcal{A} : U \to V$ is just a linear function*



Range Space: $\mathcal{R}(\mathcal{A}) := \{v | v = \mathcal{A}(u), u \in U\} \subset V$

Null Space: $\mathcal{N}(\mathcal{A}) := \{u | \mathcal{A}(u) = \emptyset v\} \subset U$

The null space and range space are themselves linear operators.

All linear operators can be expressed by matrix multiplication!

$$\mathcal{A} : v = Au$$

We can also examine all matrices as linear operators

*at least for our purposes

What is a matrix

A matrix is a collection of vectors: $A = [v_1, v_2, \cdots v_m], v_i \in \mathbb{R}^n, u \in \mathbb{R}^m$

Thus, left-multiplying a vector by the matrix is just a weighted sum of the columns.

$$Ax = x_1 v_1 + x_2 v_2 \cdots + x_m v_m$$

A set of vectors is *linearly dependent* iff $\exists \{\alpha_1 \cdots \alpha_m\} \ s.t. \sum_{i=1}^{m} \alpha_i v_i = 0$

Since $\mathcal{R}(\mathcal{A})$ and $\mathcal{N}(\mathcal{A})$ are linear operators, they're also matrices

- Rank(A) := dimension (number of columns) of $\mathcal{R}(\mathcal{A})$
- Nullity(A) := dimension of $\mathcal{N}(\mathcal{A})$

Rank Nullity Theorem:

Let $\mathcal{A} : U \to V$

$$\text{Rank}(\mathcal{A}) + \text{Nullity}(\mathcal{A}) = \dim(U)$$

This is one of the most important theorems in linear algebra!

Invertible Matrix Theorem

The for an nxn matrix A, the following statements are equivalent:

- A is invertible
- The columns of A are linearly independent
- The reduced row echelon form of A is $I_{n \times n}$
- The rank of A is n
- The range of the rows is $\mathbb{R}^n$
- The range of the columns is $\mathbb{R}^n$
- A doesn't have a null space (only the zero vector)
- The nullity of A is zero
- The determinant of A is nonzero

JACOBIANS

What is a Jacobian?

A jacobian is the gradient of a vector function

$$\mathbf{J} = \left[\frac{\partial \mathbf{f}}{\partial x_1} \cdots \frac{\partial \mathbf{f}}{\partial x_n}\right] = \begin{bmatrix} \frac{\partial \mathbf{f_1}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f_1}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{f_m}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f_m}}{\partial x_n} \end{bmatrix}$$

How do we take the jacobian of a rigid body transform?

g isn't a vector, it's a 4x4 matrix; the jacobian formula doesn't work

$$g = \begin{bmatrix} R & p \\ \mathbf{0} & 1 \end{bmatrix}$$

We know that the derivative of g is a twist, so $\frac{\partial g}{\partial \theta}$ should also be a twist! Thus

$$J = \left[\frac{\partial g}{\partial \theta_1} \cdots \frac{\partial g}{\partial \theta_n} \right] = [\xi_1' \cdots \xi_n']$$

This works because $se(3) \ni \xi$ is the tangent space to $SE(3)$

The rank of the Jacobian

The rank of the Jacobian is the number of *degrees of freedom* of your robot

If R(J) = 6, then you can freely set the end effector’s linear and angular velocity

If R(J) < 6, you can only control some subset of these DOFs

A *singularity* occurs where the rank of the Jacobian drops below its maximum

Some common singularities

The Inverse Jacobian and under/overactuation

If there are six angles and the jacobian is rank 6, the inverse jacobian exists.

$$V = J(\theta)\dot{\theta} \implies \dot{\theta} = J^{-1}(\theta)V$$

Given some desired end effector velocity, we can pick joint velocities.

Underactuated manipulator: rank(J) always < 6 (usually fewer than 6 angles)

Overactuated manipulator: more than 6 angles

Moore-Penrose Pseudoinverse: $J^{\dagger} = J^{T}(JJ^{T})^{-1}$

Least-squares solution to choosing joint velocities: $\dot{\theta} = J^{\dagger}(\theta)V$

Wrenches: The duals of twists

A wrench represents an instantaneous force

$$F = \begin{bmatrix} f \\ \tau \end{bmatrix} \begin{matrix} \text{linear component } f \in \mathbb{R}^3 \\ \text{angular component } \tau \in \mathbb{R}^3 \end{matrix}$$

The work caused by a wrench and a velocity is $W = \int V^T F dt$

Two wrenches are equivalent if they generate the same work for all possible V

To change coordinate frames, we use the transpose adjoint: $F_b = Ad_{g_{ab}}^T F_a$

To get joint torques from a wrench, we use J^T: $\tau = J^T(\theta)F$

APPENDIX

The dual of a linear operator:

The dual of a linear operator is defined $\mathcal{A}^*(\text{or } \mathcal{A}^T) : V^* \to U^*$

It maps the transpose of the codomain to the transpose of the domain.

When you turn this into a matrix, you have: $\mathcal{A}^* : A^* v = u, v^* A = u^*$

One important fact to note is that: $rank(A^*) = rank(A)$

* Here, the star operator refers to the Hermitian, or complex conjugate. This is the generalization of the transpose for complex numbers