EE106A Discussion 2: Exponential Coordinates

1 Exponential coordinates for rotation

Definition 1. The matrix exponential of A, e^A, is defined to be

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ... \tag{1}$$

Problem 1. *Find the rotation matrix $R(\omega, \theta)$ for a rotation about some axis ω by amount θ. How is Rodrigues' formula related?*

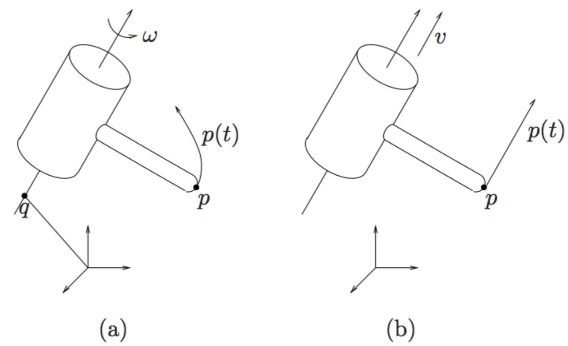

Figure 1: a) A revolute joint and b) a prismatic joint.

2 Exponential coordinates for rigid motion

Robotic rigid body transformations are enabled by *joints*, which connect sets of rigid *links* together. In this class, we focus on two types of joints — *revolute* and *prismatic* joints (see Fig. 1). Revolute joints allow adjacent links to rotate relative to each other about a fixed axis, and prismatic joints allow links to move linearly relative to each other along a fixed axis.

Problem 2. *Write the expressions for the velocity of the point p (ie. $\dot{p}(t)$) when attached to both the revolute and prismatic joints in Fig. 1. Assume that $\omega \in \mathbb{R}^3$, $||\omega|| = 1$, and $q \in \mathbb{R}^3$ is some point along the axis of ω.*

2.1 Twist of revolute joint

Let's transform the above expression for velocity into homogeneous coordinates. Recall that in homogeneous coordinates, append a 0 to vectors and a 1 to points.

Problem 3. *Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a revolute joint.*

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \qquad\qquad \\ \\ \end{bmatrix}}_{=:\,\widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Hint: Recall the skew symmetric matrix $\widehat{w}$ of w:

$$\omega = [\omega_1 \;\; \omega_2 \;\; \omega_3]^T; \quad \widehat{w} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2}$$

2.2 Twist of prismatic joint

Problem 4. *Find $\widehat{\xi}$ to complete the following expression of $\dot{p}(t)$ in homogeneous coordinates for a prismatic joint.*

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \end{bmatrix}}_{=:\, \widehat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

2.3 Vee and wedge operators of a twist

The above quantity we have derived, $\widehat{\xi}$, is called a *twist*. A twist captures the angular and linear velocities of a body. There are two handy operators that we use on twist entities. First, given a twist $\widehat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^4$, the $\vee$ (vee) operator extracts the 6-dimensional vector which parameterizes a twist, where $\xi := (v, \omega)$ are the *twist coordinates* of $\widehat{\xi}$.

$$\begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} v \\ \omega \end{bmatrix} =: \xi \tag{3}$$

The inverse operator, $\wedge$ (wedge), constructs a matrix out of a vector of the twist coordinates:

$$\begin{bmatrix} v \\ \omega \end{bmatrix}^{\wedge} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \tag{4}$$

2.4 Solution to differential equation gives us the exponential map

Problem 5. *Write the general solution to the differential equation $\dot{\bar{p}} = \widehat{\xi}\bar{p}$. Then, make use of the fact that $||\omega|| = 1$ to reparameterize t to be θ. Specifically, find the expression for $p(\theta)$ in terms of $p(0)$.*

This transformation is not the same as the rigid transformations we studied previously in that it is not a mapping from one coordinate frame to another, but rather the mapping of points from their initial coordinates $p(0)$ to their coordinates after a rigid motion parameterized by a joint angle θ is applied. It turns out that the matrix exponential simplifies to:

$$e^{\widehat{\xi}\theta} = \begin{bmatrix} e^{\widehat{\omega}\theta} & (I - e^{\widehat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} \tag{5}$$

3 Screw motion

Chasles' theorem states that any rigid body transformation can be decomposed into an equivalent finite rotation about a fixed axis and a finite translation along that axis. This is what we call a *screw motion* S, which consists of an axis l, a pitch h, and a magnitude M. It is equivalent to a rotation by an amount $\theta = M$ about l followed by a translation by $h\theta = hM$ along l (see Fig. 3). ($h = 0$ corresponds to pure _____________, and $h = \infty$ corresponds to pure _____________).

The transformation g corresponding to S has the following effect on a point p:

$$gp = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega \tag{6}$$

Problem 6. *Convert this transformation to homogeneous coordinates. What do you notice between this expression and the one in Eq. 5?*

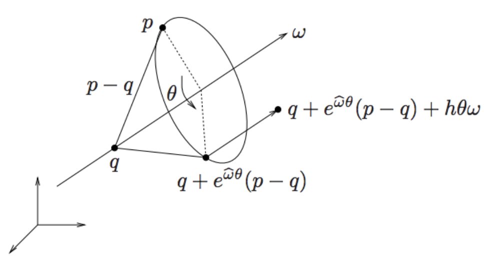

Figure 2: Generalized screw motion.