EE106A Discussion 4: Inverse Kinematics

1   Inverse kinematics

In forward kinematics, we found the expression for $g_{st}(\theta)$ as a function of θ. Now, in inverse kinematics, we are given a desired configuration of the tool frame g_d, and we wish to find the θ for which

$$e^{\widehat{\xi_1}\theta_1}...e^{\widehat{\xi_n}\theta_n}g_{st}(0) = g_{st}(\theta) = g_d \tag{1}$$

2   Padan-Kahan subproblems

To solve the inverse kinematics problem, one technique is to distill it into the following three simpler subproblems for which we know the solutions.

2.1   Subproblem 1: Rotation about a single axis

Let ξ be a zero-pitch twist along ω with unit magnitude, and p, $q \in \mathbb{R}^3$ be two points. Find θ such that

$$e^{\widehat{\xi}\theta}p = q \tag{2}$$

- *Define $u = (p - r)$ and $v = (q - r)$ where r is a point on the axis. Find an expression that relates u and v.*
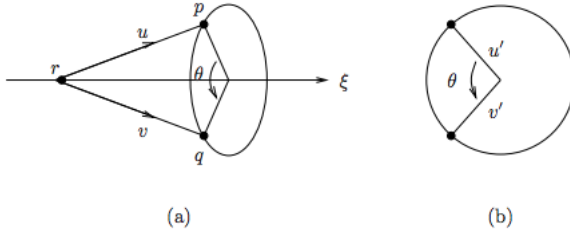

(a)   (b)

Figure 1: Subproblem 1: a) Rotate p about the axis of ξ until it is coincident with q. b) Projection of u and v onto the plane perpendicular to the twist axis.

- *Find expressions for u' and v', the projected u and v on the plane perpendicular to the rotation axis.*

- *Write the necessary conditions for there to be a solution.*

- *Find the solution for θ given that it exists.*

2.2 Subproblem 2: Rotation about two subsequent axes

Let ξ_1 and ξ_2 be two zero-pitch, unit magnitude twists with intersecting axes, and $p, q \in \mathbb{R}^3$ be two points. Find θ_1 and θ_2 such that

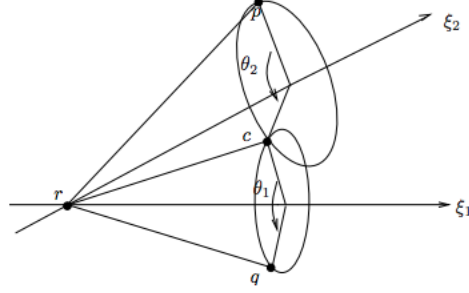$$e^{\widehat{\xi}_1 \theta_1} e^{\widehat{\xi}_2 \theta_2} p = q \qquad (3)$$

Figure 2: Subproblem 2: Rotate p around the axis of ξ_2, then around the axis of ξ_1 such that the final location is coincident with q.

- *Geometrically, when does there exist zero, one, or multiple solutions to this subproblem?*

Let r be the intersection of the two axes, and c be the intermediate point at which p is rotated about ω_2 by θ_2. Define vectors $u = (p - r)$, $v = (q - r)$, and $z = (c - r)$.

- *Write the expression for z in terms of a transformations applied to u and v.*

Similarly to Subproblem 1, it is true that

$$\omega_2^T u = \omega_2^T z \qquad (4)$$
$$\omega_1^T v = \omega_1^T z \qquad (5)$$
$$||u|| = ||z|| = ||v|| \qquad (6)$$

We can express z as a linear combination of the linearly independent vectors ω_1, ω_2, and $\omega_1 \times \omega_2$:

$$z = \alpha \omega_1 + \beta \omega_2 + \gamma (\omega_1 \times \omega_2) \qquad (7)$$

The solutions to these coefficients α, β, and γ are found by using the expressions in Eqs. 4 to 6 (see textbook for full details). There are either zero, one, or two real solutions to these coefficients. If a solution exists, we have z, and hence c.

What's left is to solve

$$e^{\widehat{\xi}_2 \theta_2} p = c \qquad (8)$$

and

$$e^{-\widehat{\xi}_1 \theta_1} q = c \qquad (9)$$

which requires us to solve Subproblem 1 twice.

2

2.3 Subproblem 3: Rotation to a given distance

Let ξ be a zero-pitch, unit magnitude twist, p, $q \in \mathbb{R}^3$ be two points, and $\delta > 0$. Find θ such that:

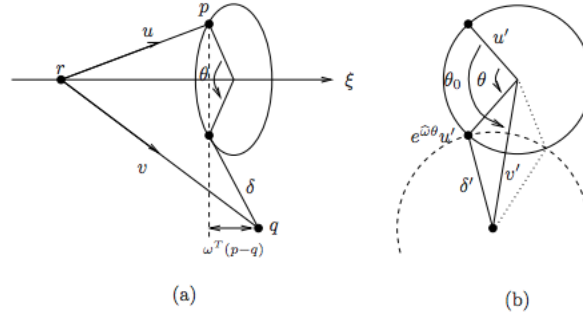$$||q - e^{\widehat{\xi}\theta}p|| = \delta \qquad (10)$$

(a)   (b)

Figure 3: Subproblem 3: a) Rotate p about the axis of ξ until it is a distance δ from point q. b) Projection onto plane perpendicular to axis.

- *Geometrically, when does there exist zero, one, or multiple solutions to this subproblem?*

- *Write the expressions of the projected u and v onto the plane perpendicular to ω, which we call u' and v'.*

- *Write an expression for the distance δ' (the projected δ onto the plane perpendicular to ω) as a function of u' and v'.*

- *Find the solution for θ. Hint: use result derived in Subproblem 1 to find θ_0.*

3 Using PK subproblems to solve inverse kinematics

We want to simplify complete inverse kinematics problems into the three subproblems we know how to solve. The full equation becomes more simplified when we apply the kinematics equations to *special points.*

3.1 Trick 1: Apply equations to a point on the axes

If we have a revolute twist ξ and we have a point p on the twist axis, applying the transformation on that point does nothing to it, ie:

$$e^{\widehat{\xi}\theta}p = p \tag{11}$$

For example, if our IK problem is

$$e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}e^{\widehat{\xi_3}\theta_3} = g \tag{12}$$

then choosing a point p on the axis of ξ_3 yields

$$e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}p = gp \tag{13}$$

and this is simply Subproblem 2.

3.2 Trick 2: Subtract a point from both sides and take the norm

Remember that rigid motions preserve norm. For example, say we wish to solve the same IK problem as in Eq. 12. If the axes of ξ_1 and ξ_2 intersect at a point q, we can select a point p that is not on the axis of ξ_3 and simplify to the following:

$$
\begin{aligned}
\delta := ||gp - q|| &= ||e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}e^{\widehat{\xi_3}\theta_3}p - q|| \\
&= ||e^{\widehat{\xi_1}\theta_1}e^{\widehat{\xi_2}\theta_2}(e^{\widehat{\xi_3}\theta_3}p - q)|| \\
&= ||e^{\widehat{\xi_3}\theta_3}p - q||
\end{aligned}
\tag{14}
$$

which is just Subproblem 3.

4 Elbow manipulator example

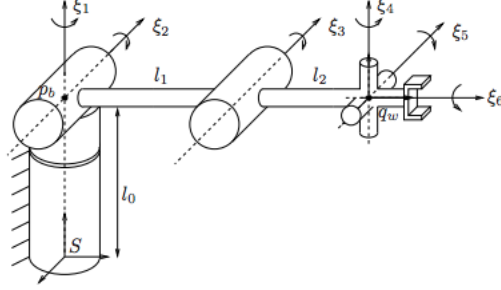Break down the inverse kinematics for the elbow manipulator in Fig. 4 into simpler PK subproblems.

Figure 4: Elbow manipulator.

5   SCARA manipulator example

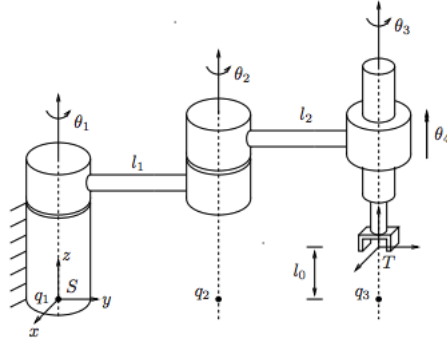Break down the the inverse kinematics for the SCARA manipulator in Fig. 5 into simpler PK sub-problems.

Figure 5: SCARA manipulator.