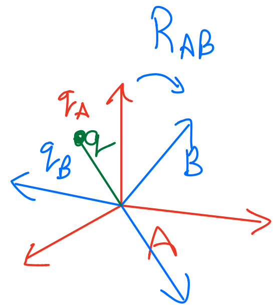
Discussion 2: Exponential Coordinates

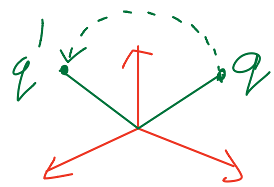
Given a rotation matrix R, there are two ways of interpreting it:



1. As a transformation between two reference frames A, B.

$$q_A = R_{AB}\, q_B$$

→ reference frame changes, point remains static.



2. As an action on 3D space.

$$q : \longmapsto q'$$

$$q' = Rq$$

→ reference frame static, point moves.

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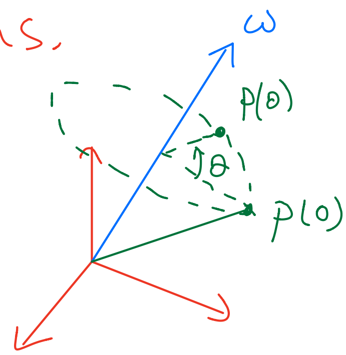
2) $R(\omega, \theta)$: a rotation about an axis ω ($\|\omega\|=1$) by θ radians.

$$R(\omega, \theta) \stackrel{?}{=}$$

$$R_x(\theta) = \underbrace{R(\vec{x}, \theta)}$$



$$p(\theta) = \underbrace{R(\omega, \theta)\, p(0)}$$

Matrix Exponential:

Scalar exp: $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

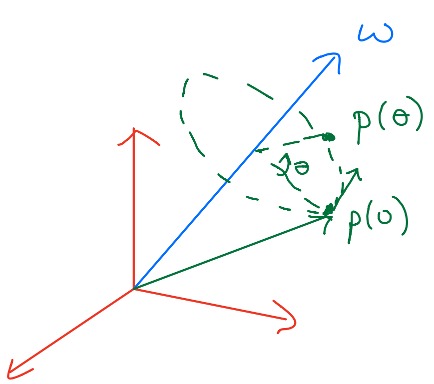
Matrix exp: $f: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ $\quad f(A) = e^A = I + A + \frac{A^2}{2!} + \cdots$

Scalar ODE: $\dot{x} = ax, \quad x(0) = x_0 \longrightarrow x(t) = e^{at} x_0$

Vector ODE: $\boxed{\dot{x} = Ax}, \quad x(0) = x_0 \longrightarrow x(t) = e^{At} x_0$

$\mathbb{R}^n$ $\quad \mathbb{R}^{n\times n}\ \mathbb{R}^n$

$\|\omega\| = 1$



p rotates about ω at a rate of $\dot{\theta} = 1$ rad per second

$\omega = (\omega_1\ \omega_2\ \omega_3)$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} \dot{p} &= \omega \times p \\ \dot{p} &= \underbrace{\hat{\omega}}\, p \end{aligned}\right\}$$

for any v: $\hat{\omega} v = \omega \times v$

$$p(t) = e^{\hat{\omega} t} p(0)$$

$$p(\theta) = e^{\hat{\omega}\theta} p(0) \qquad p(\theta) = R(\omega, \theta) p(0)$$

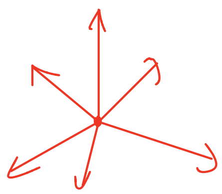
$$\underbrace{R(\omega,\theta) = e^{\hat{\omega}\theta}} = \underbrace{I + \hat{\omega}\sin(\theta) + \hat{\omega}^2(1-\cos(\theta))}_{\text{Rodriguez' Formula}}$$

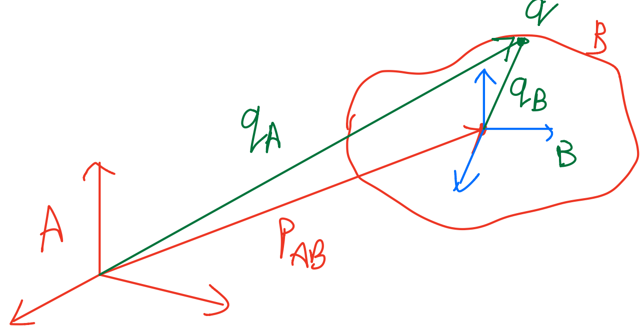
$$R_x(\theta) = e^{\hat{x}\theta}$$
$$R_y(\theta) = e^{\hat{y}\theta}$$
$$R_z(\theta) = e^{\hat{z}\theta}$$

Euler's theorem: For any rotation matrix R, there exists a unit axis w, and angle theta, such that R = R(w, theta)

When we find (ω, θ) such that $R = R(\omega,\theta) = e^{\hat{\omega}\theta}$ we call (ω, θ) the exponential coordinates of R.

Rigid Body Transformations





$$(R_{AB}, P_{AB}) = g_{AB}$$

$$\underline{q_A} = \underbrace{R_{AB}}\, \underline{q_B} + \underbrace{P_{AB}}$$

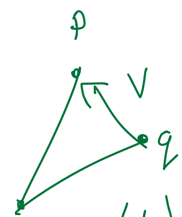
$$\begin{pmatrix} q_A \\ 1 \end{pmatrix} = \underbrace{\left[\begin{array}{c|c} R_{AB} & P_{AB} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]}_{4\times 4} \begin{pmatrix} q_B \\ 1 \end{pmatrix}$$

$$g_{AB} \in SE(3) = \left\{ g := \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},\ \begin{array}{l} R \in SO(3) \\ p \in \mathbb{R}^3 \end{array} \right\}$$

<u>Homogeneous Coordinates</u>

$$p \in \mathbb{R}^3 \longrightarrow \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \in \mathbb{R}^4 \quad \text{(point)}$$

$$v \in \mathbb{R}^3 \longrightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} \in \mathbb{R}^4 \quad \text{(vector)}$$



$$p(t) = \begin{bmatrix} p(t) \\ 1 \end{bmatrix}$$

$$\dot{p} = \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix}$$

Rigid body transformations can be seen as:

1. As transformations between reference frames A, B.

$$g_{AB}: \quad q_A = g_{AB}\, q_B$$

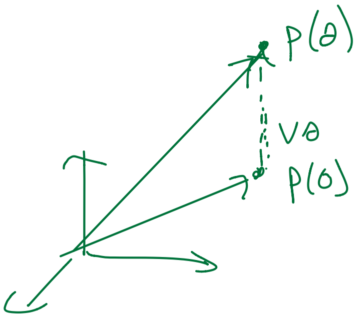
2. As an action on 3D space

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}: \quad \underbrace{q} \longmapsto gq = \underbrace{Rq + p}$$

1) Pure translation

2) pure rotation

1) pure translation: g implements a translation along a unit vector v by θ units.



$$\underline{p(\theta) = p(0) + v\theta}$$

translation according to $\dot{\theta} = 1$ unit per second.

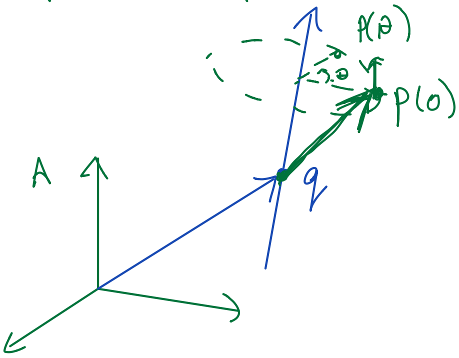
$$\dot{p} = v \quad : \quad \begin{pmatrix} \dot{p} \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0_{3\times 3} & v_{3\times 1} \\ 000 & 0 \end{pmatrix}_{4\times 4}}_{\hat{\xi}} \begin{pmatrix} p \\ 1 \end{pmatrix}$$

$$\underbrace{\dot{p} = \hat{\xi} p}$$

$$p(t) = e^{\hat{\xi} t} p(0)$$

$$\underline{p}(\theta) = \underbrace{e^{\hat{\xi}\theta}}_{g} \underline{p}(0)$$

$$p(\theta) = g\, p(0)$$

2) Pure rotation : 

rotation happens at rate $\dot{\theta} = 1$ rad per second.

$$\dot{p} = \omega \times (p - q)$$

$$\dot{p} = \underbrace{\overbrace{\hat{\omega} p}^{} - \overbrace{\omega \times q}^{}}_{}$$

$$\begin{pmatrix} \dot{p} \\ 0 \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} \hat{\omega} & -\omega \times q \\ \hline 0\,0\,0 & 0 \end{array} \right)}_{\hat{\xi}} \begin{pmatrix} p \\ 1 \end{pmatrix}$$

$$\dot{p} = \hat{\xi} p$$

$$p(t) = e^{\hat{\xi} t} p(0)$$

$$p(\theta) = \underbrace{e^{\hat{\xi}\theta}}_{} p(0)$$

$$g = e^{\hat{\xi}\theta}$$ ← rotation about ω passing thoug q.

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1) pure trans: $\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \rightarrow g = e^{\hat{\xi}\theta}$

2) pure rot : $\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \rightarrow g = e^{\hat{\xi}\theta}$

$so(3)$ $\qquad$ $\mathbb{R}^3$

$\hat{\xi}$ always looks like $\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$

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chasle's theorem:

for any RBT $g \in SE(3)$ exists scalar θ and $\hat{\xi} = \underbrace{\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}}$ for some (ω, v) such that $g = e^{\hat{\xi}\theta}$

so, if we find such a $\xi = (\underline{\omega}, \underline{v})$ and a $\underline{\theta}$: $g = e^{\hat{\xi}\theta}$ then (ξ, θ) are called the exponential coordinates of g

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if I write $g = e^{\hat{\xi}\theta}$ for (ω, v)

$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$: ω : $R = e^{\hat{\omega}\theta}$

$\xi = (v, \omega) \in \mathbb{R}^6$ Twist

$\downarrow$

$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$ Twist hat

$\downarrow$

$e^{\hat{\xi}\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$ rigid body transformation.