Discussion 9: Lagrangian Dynamics

The Lagrangian formulation is an alternative to using Newton's laws to find the equations of motion of a system. It is often easier to write down the Lagrangian of a system than it is to isolate all the forces in the system and write down the F = ma constraints for all of them.

To find the equations of motion using Lagrangian Dynamics:

1. Pick a set of coordinates q that together parameterize the full state of the system. These are the *generalized coordinates.*
2. Write down T = the total kinetic energy of the system in terms of q. ← also $\dot{q}$
3. Write down V = the total potential energy of the system in terms of q. ←
4. Define tha Lagrangian $\boxed{L = T - V}$ $L(q, \dot{q})$
5. Then, the system must satisfy

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \Upsilon} \leftarrow \begin{matrix}\Upsilon_1 \\ \Upsilon_2 \\ \vdots \\ \Upsilon_n\end{matrix}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \Upsilon_i}$$

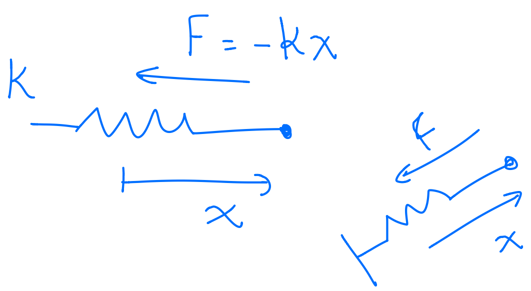
Where the RHS is a vector of *generalized forces* which captures the action of external and non-conservative forces acting in the direction of each generalized coordinate.

Common Sources of Potential Energy

- Gravity: the center of mass of each rigid body or point mass in the system will contribute a term to the potential energy.

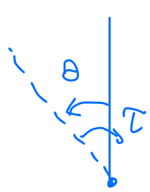
$$V = mgh$$

- Linear springs with spring constant k will contribute a potential energy proportional to the square of the displacement from the spring's neutral position.



$$V = \frac{1}{2} k x^2$$

- Torsional springs are also often seen in robotics: these are springs that produce a *torque* proportional to an angular deviation from a neutral position about some pivot. The



Spring constant

$$\tau = -k\theta$$

$$V = \frac{1}{2} k \theta^2$$

Common Sources of Kinetic Energy

- Linear kinetic energy: a mass m moving at speed v has a translational kinetic energy associated with it.

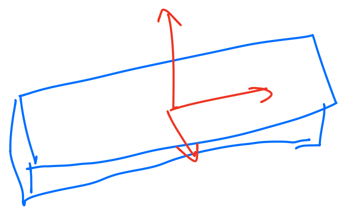
$$T = \frac{1}{2} m v^2$$

- Rotational kinetic energy: a rigid body with moment of inertia matrix I (with respect to the body frame at the center of mass) rotating with angular body velocity w has a rotational kinetic energy associated with it.

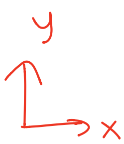
$$\boxed{T = \frac{1}{2} \underline{\omega}^T I_c \underline{\omega}}$$

$$\hat{\omega} = R^T \dot{R}$$

$$I_c \in \mathbb{R}^{3\times 3} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ \vdots & I_{yy} & I_{yz} \\ \cdot & \cdot & I_{zz} \end{bmatrix}$$

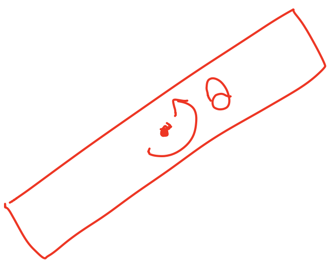


$$= \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$



for 2D constrained rotations

$$T = \frac{1}{2} \underline{I} \dot{\theta}^2$$



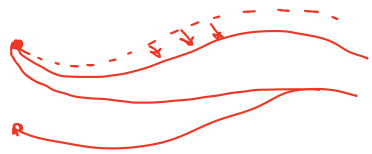
General form of the dynamics

$$T = \frac{1}{2}\dot{q}^T M(q)\,\dot{q}$$

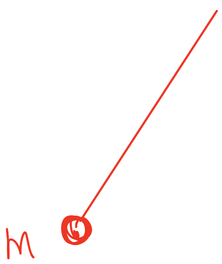
The dynamics of any system can be written in the form

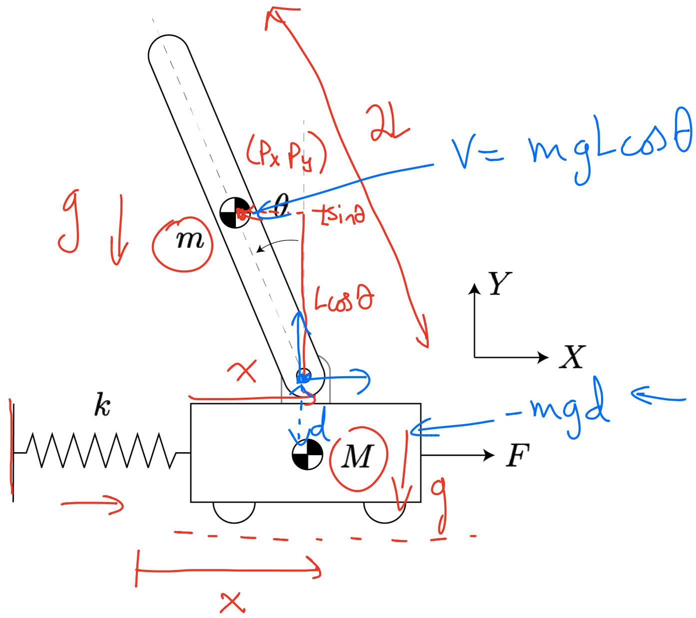
$$M(q)\ddot{q} + \boxed{C(q,\dot{q})\dot{q}} + N(q) = \Upsilon$$

Where M is the *inertia matrix*, C is the *Coriolis matrix,* and N is the *gravity vector.*
These are obtained by simply isolating terms that depend on the second, first, and zeroth derivatives of the state.



Example 3.4: Cartpole with uniform mass rod





① $q = \begin{bmatrix} x \\ \theta \end{bmatrix}$

② T sources: $\frac{1}{2}M\dot{x}^2 + \boxed{\frac{1}{2}I\dot{\theta}^2} + \frac{1}{2}m V_m^2$

$$V_m^2 = {V_m^x}^2 + {V_m^y}^2 = \left(\dot{x}^2 - (2L\cos\theta)\dot{\theta}\dot{x} + L^2\dot{\theta}^2\right)$$

$P_x = x - L\sin\theta \Rightarrow \dot{P}_x = \dot{x} - (L\cos\theta)\dot{\theta}$

$P_y = L\cos\theta \Rightarrow \dot{P}_y = -L(\sin\theta)\dot{\theta}$

$$T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\underbrace{(I + mL^2)}\dot{\theta}^2 - \underbrace{mL(\cos\theta)\dot{\theta}\dot{x}}$$

③ $$V = \frac{1}{2}kx^2 + mgL\cos\theta$$

④ $$L = T - V$$

⑤ $$\frac{\partial L}{\partial q} = \begin{bmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial \theta} \end{bmatrix} = \begin{bmatrix} -kx \\ \underbrace{mL(\cancel{\sin\theta})\dot{\theta}\dot{x}} + mgL\sin\theta \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{x}} \\ \frac{\partial L}{\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} (M+m)\dot{x} - mL(\cos\theta)\dot{\theta} \\ -mL(\cos\theta)\dot{x} + (mL^2 + I)\dot{\theta} \end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} \underbrace{(M+m)}\ddot{x} - \underbrace{mL\cos(\theta)}\ddot{\theta} + \underbrace{mL(\sin\theta)}\dot{\theta}^2 \\ \underbrace{mL(\cancel{\sin\theta})\dot{\theta}\dot{x}} - \underbrace{mL\cos\theta}\,\ddot{x} + \underbrace{(mL^2+I)}\ddot{\theta} \end{bmatrix}$$

⑥ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

$$M(q) = \begin{bmatrix} m+M & -mL\cos\theta \\ -mL\cos\theta & mL^2 + I \end{bmatrix}$$

$$c(q,\dot{q})\dot{q} = \begin{bmatrix} mL\sin\theta\,\dot{\theta}^2 \\ 0 \end{bmatrix}$$

$$N(q) = \begin{bmatrix} kx \\ -mgL\sin\theta \end{bmatrix}$$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = \Upsilon$$

$$\begin{bmatrix} F \\ \tau \end{bmatrix}$$

$$\begin{bmatrix} 0 & mL\sin\theta\,\dot{\theta}^2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} mL\sin\theta\,\dot{\theta}^2 \\ 0 \end{bmatrix}$$