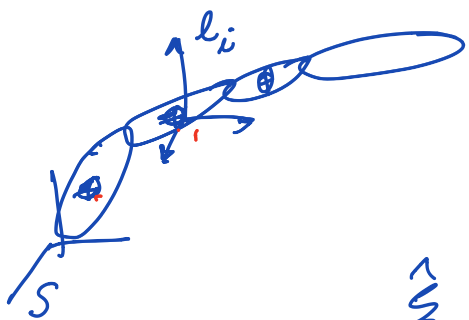
# LECTURE NOTES    OCT 29, 2020

## REVIEW OF DYNAMICS




$l_i$ coordinate frame at center of mass of $i$th link.

$$g_{sl_i}(\theta) = e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_i \theta_i} g_{sl_i}(0)$$

$$V^b_{sl_i} = J^b_{sl_i}(\theta)\,\dot{\theta}$$

$$= \begin{bmatrix} \xi_1^\dagger & \xi_2^\dagger & \cdots & \xi_i^\dagger & 0 & \cdots & 0 \end{bmatrix} \dot{\theta}$$

$$\xi_j^\dagger = Ad^{-1}_{\left(e^{\hat{\xi}_j \theta_j} \cdots e^{\hat{\xi}_i \theta_i} g_{sl_i}(0)\right)} \xi_j$$

Commutes with $\xi_j$

$$= Ad^{-1}_{\left(e^{\hat{\xi}_{j+1} \theta_{j+1}} \cdots e^{\hat{\xi}_i \theta_i} g_{sl_i}(0)\right)} \xi_j$$

K.E. $$T_i(\theta, \dot{\theta}) = \frac{1}{2} \left(V^b_{sl_i}\right)^T \begin{bmatrix} m_i I_3 & 0 \\ 0 & \mathcal{I}_i \end{bmatrix} V^b_{sl_i}$$

$$= \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 \cdots \dot{\theta}_n \end{bmatrix} \begin{bmatrix} \xi_1^{\dagger T} \\ \vdots \\ \xi_i^{\dagger T} \end{bmatrix} \begin{bmatrix} \mathcal{M}_i \end{bmatrix} \begin{bmatrix} \xi_1^\dagger \cdots \xi_i^\dagger\, 0 \cdots 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \end{bmatrix}$$

6x6

$$\Rightarrow \frac{1}{2}\dot{\theta}^T \boxed{J_i^T(\theta)\,\mathcal{M}_i\, J_i(\theta)}\,\dot{\theta}$$ SYMMETRIC

$$T(\theta,\dot{\theta}) = \sum_{i=1}^{n} T_i(\theta,\dot{\theta}_i)$$

$$= \frac{1}{2}\dot{\theta}^T M(\theta)\,\dot{\theta}$$

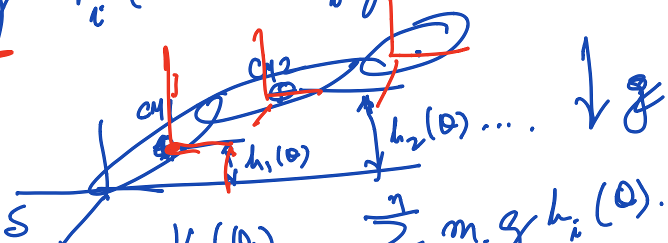
$$M(\theta) = \sum_{i=1}^{n} J_i^T(\theta)\,\mathcal{M}_i\, J_i(\theta)$$

$$M^T(\theta) = M(\theta)$$

$M(\theta)$ Manipulator Inertia matrix $\in \mathbb{R}^{n\times n}$ $\rightarrow$ Symmetric

$$m_{ij}(\theta) = m_{ji}(\theta)$$

PE. gravity $V_i(\theta) = m_i g h_i(\theta)$




$$V(\theta) = \sum_{i=1}^{n} m_i g h_i(\theta).$$

$h_i(\theta)$ z coordinate of $\mathcal{L}_i$ $\quad e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_i]\theta_i} p_i$ loc. of $CM_i$

$p_i$ coordinate of $CM_i$ in S frame.

$$L(\theta,\dot{\theta}) = \sum_{i=1}^{n} T_i(\theta,\dot{\theta}) - V_i(\theta)$$

LAGRANGIAN FOLLOWS FROM KINEMATICS

$$= \frac{1}{2}\underbrace{\dot{\theta}^T M(\theta)\,\dot{\theta}}_{1 \times n,\ n \times n,\ n\times 1} - V(\theta)$$

EULER LAGRANGE

$$L(\theta,\dot{\theta}) = \frac{1}{2}\sum_{j,k=1}^{n} M_{jk}(\theta)\,\dot{\theta}_j\,\dot{\theta}_k - V(\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_i}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot\theta_i}\, \frac{1}{2}\sum_{j,k=1}^{n} M_{jk}(\theta)\,\dot\theta_j \dot\theta_k\right)$$

$$= \frac{d}{dt}\frac{1}{2}\left(\sum_{k=1}^{n} M_{ik}(\theta)\,\dot\theta_k + \sum_{j=1}^{n} M_{ji}(\theta)\,\dot\theta_j\right)$$

$$= \frac{d}{dt}\frac{1}{2}\left(\sum_{j=1}^{n} M_{ij}(\theta)\,\dot\theta_j + \sum_{j=1}^{n} M_{ij}(\theta)\,\dot\theta_j\right)$$

Because $M_{ij}(\theta) = M_{ji}(\theta)$

$$= \frac{d}{dt}\left(\sum_{j=1}^{n} M_{ij}\,\dot\theta_j\right)$$

$$= \sum_{j=1}^{n} M_{ij}(\theta)\,\ddot\theta_j + \sum_{j=1}^{n} \dot M_{ij}(\theta)\,\dot\theta_j$$

$$\frac{\partial L}{\partial \theta_i} = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial M_{jk}}{\partial \theta_i}\,\dot\theta_k \dot\theta_j - \frac{\partial V}{\partial \theta_i}$$

Thus Euler Lagrange equations read

$$\sum_{j=1}^{n} M_{ij}(\theta)\,\ddot\theta_j + \sum_{j=1}^{n} \dot M_{ij}(\theta)\,\dot\theta_j - \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial M_{jk}}{\partial \theta_i}\,\dot\theta_k\dot\theta_j + \frac{\partial V}{\partial \theta_i}$$

$$\sum_{j=1}^{n} M_{ij}(\theta)\,\ddot\theta_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial \theta_k}\,\dot\theta_k\dot\theta_j - \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial M_{jk}}{\partial \theta_i}\,\dot\theta_k\dot\theta_j + \frac{\partial V}{\partial \theta_i} = \tau_i$$

EXT. FORCES

$$\sum_{j=1}^{n} M_{ij}(\theta)\,\ddot\theta_j + \boxed{\sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma_{ijk}\,\dot\theta_j\dot\theta_k} + \frac{\partial V}{\partial \theta_i} = \tau_i$$

CORIOLIS

$$\sum_{j,k=1}^{n} \Gamma_{ijk}\dot\theta_j\dot\theta_k = \sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial\theta_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial\theta_i}\right)\dot\theta_j\dot\theta_k$$

$$= \sum_{j,k=1}^{n}\left(\frac{1}{2}\frac{\partial M_{ij}}{\partial\theta_k} + \frac{1}{2}\frac{\partial M_{ik}}{\partial\theta_j} - \frac{1}{2}\frac{\partial M_{jk}}{\partial\theta_i}\right)\dot\theta_j\dot\theta_k$$

$\Gamma_{ijk}$

**Define** $$C_{ij}(\theta,\dot\theta) \doteq \sum_{k=1}^{n}\Gamma_{ijk}\dot\theta_k$$

$$= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial\theta_k} + \frac{\partial M_{ik}}{\partial\theta_j} - \frac{\partial M_{jk}}{\partial\theta_i}\right)\dot\theta_k$$

Then

$$\sum_{j=1}^{n} M_{ij}(\theta)\ddot\theta_j + \sum_{j=1}^{n} C_{ij}(\theta,\dot\theta)\dot\theta_j + \frac{\partial V}{\partial\theta_i} \doteq \tau_i$$

If there is friction $\beta\dot\theta_i$ include it with $\tau_i$

$$\begin{bmatrix} M_{11}(\theta) & \cdots & M_{1n}(\theta) \\ M_{21}(\theta) & \cdots & M_{2n}(\theta) \\ M_{n1}(\theta) & \cdots & M_{nn}(\theta)\end{bmatrix}\begin{bmatrix}\ddot\theta_1\\ \vdots \\ \ddot\theta_n\end{bmatrix} + \begin{bmatrix} C(\theta,\dot\theta)\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \vdots \\ \dot\theta_n\end{bmatrix} + \begin{bmatrix}\frac{\partial V}{\partial\theta_1}\\ \vdots \\ \frac{\partial V}{\partial\theta_n}\end{bmatrix} = \begin{bmatrix}\tau_1\\ \vdots \\ \tau_n\end{bmatrix} - \begin{bmatrix}\beta_1\dot\theta_1\\ \vdots \\ \beta_n\dot\theta_n\end{bmatrix}$$

$$M(\theta)\ddot\theta + C(\theta,\dot\theta)\dot\theta + \frac{\partial V}{\partial\theta} + \begin{bmatrix}\beta_1 & & 0\\ & \ddots & \\ 0 & & \beta_n\end{bmatrix}\dot\theta = \tau$$

There are many choices of $C(\theta,\dot\theta)$ to make

$C(\theta,\dot\theta)\dot\theta$ = Coriolis Forces

However choosing $c_{ij}(\theta,\dot\theta) := \sum_{k=1}^{n} \Gamma_{ijk}\,\dot\theta_k$ ✓

gives many desirable properties for control

PROPOSITION

$\frac{d}{dt}M(\theta) - 2C(\theta,\dot\theta)$

→ [ $\dot M - 2C$ is a skew symmetric matrix ]

Proof

$$(\dot M - 2C)_{ij} = \dot m_{ij}(\theta) - 2c_{ij}(\theta,\dot\theta)$$

$$= \sum_{k=1}^{n}\left(\cancel{\frac{\partial m_{ij}}{\partial\theta_k}\dot\theta_k - \frac{\partial m_{ij}}{\partial\theta_k}\dot\theta_k} - \frac{\partial m_{ik}}{\partial\theta_j}\dot\theta_k + \frac{\partial m_{kj}}{\partial\theta_i}\dot\theta_k\right)$$

$$= \sum_{k=1}^{n}\left(-\frac{\partial m_{ik}}{\partial\theta_j} + \frac{\partial m_{kj}}{\partial\theta_i}\right)\dot\theta_k \quad ✓$$

$$(\dot M - 2C)_{ji} = \sum_{k=1}^{n}\left(-\frac{\partial m_{jk}}{\partial\theta_i} + \frac{\partial m_{ki}}{\partial\theta_j}\right)\dot\theta_k$$

But $m_{jk} = m_{kj}$ & $m_{ik} = m_{ki}$

$$= \sum_{k=1}^{n}\left(-\frac{\partial m_{kj}}{\partial\theta_i} + \frac{\partial m_{ik}}{\partial\theta_j}\right)\dot\theta_k$$

Hence $(\dot M - 2C)_{ij} = -(\dot M - 2C)_{ji}$

PASSIVE

FRICTION

SUMMARY

$n$ 2nd order

$$\boxed{M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + N(\theta,\dot{\theta}) = \tau}$$

$$N_i(\theta,\dot{\theta}) = \frac{\partial V}{\partial \theta_i} + \beta_i \dot{\theta}_i + \text{STICTION } f \ldots$$

VISCOUS FRICTION

non conservative forces

SEC 5.1

CONTROL [without doing EECS 128!]

Path Planner says given $\theta_d(t) : t \in [0,T]$, find $\tau(t)$ (INPUTS) to get the robot to follow $\theta_d(t)$ ($\theta$ desired $(t)$)

Try $\tau_d(t) = M(\theta_d)\ddot{\theta}_d + C(\theta_d,\dot{\theta}_d)\dot{\theta}_d + N(\theta_d,\dot{\theta}_d)(t)$

Then we get

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + N(\theta,\dot{\theta}) = M(\theta_d)\ddot{\theta}_d + C(\theta_d,\dot{\theta}_d)\dot{\theta}_d + N(\theta_d,\dot{\theta}_d)$$

$\theta - \theta_d = e$

$\dot{\theta} - \dot{\theta}_d = \dot{e}$

PROBLEM: ① HARD TO ANALYZE

② $\theta(0) \neq \theta_d(0)$ $\quad \dot{\theta}(0) \neq \dot{\theta}_d(0)$

TRY $\tau = M(\theta)\ddot{\theta}_d + C(\theta,\dot{\theta})\dot{\theta} + N(\theta,\dot{\theta})$

You can measure joint angles $\theta$ (encoders) & their velocities $\dot{\theta}$. Then

$$M(\theta)\ddot{\theta} + \cancel{C(\theta,\dot{\theta})\dot{\theta}} + \cancel{N(\theta,\dot{\theta})} = M(\theta)\ddot{\theta}_d + \cancel{C(\theta,\dot{\theta})\dot{\theta}}$$

$+ N(\theta, \dot\theta)$

$$\Rightarrow \quad M(\theta)\left[\ddot\theta - \ddot\theta_d\right] = 0$$

$M^{-1}(\theta)$ exists $\Rightarrow \ddot\theta = \ddot\theta_d$

$\ddot e = 0$

But again if

$\theta(0) \neq \theta_d(0)$

$\dot\theta(0) \neq \dot\theta_d(0)$

What ?

TRY

$$\tau = M(\theta)\left(\ddot\theta_d + \begin{bmatrix} k_{v_1} & & 0 \\ & \ddots & \\ 0 & & k_{v_n} \end{bmatrix}(\dot\theta_d - \dot\theta) + \begin{bmatrix} k_{p_1} & & 0 \\ & \ddots & \\ 0 & & k_{p_n} \end{bmatrix}(\theta_d - \theta)\right) + C(\theta,\dot\theta) + N(\theta,\dot\theta).$$

Then we get

$$M(\theta)\ddot\theta = M(\theta)\left(\ddot\theta_d - K_v\dot e - K_p e\right)$$

where $e = \theta - \theta_d$ joint angle error

$\dot e = \dot\theta - \dot\theta_d$ joint velocity error

$$\Rightarrow \quad M(\theta)\left(\ddot\theta - \ddot\theta_d + K_v\dot e + K_p e\right) = 0$$

$$\ddot e + K_v \dot e + K_p e = 0$$

$$\Rightarrow \begin{bmatrix} \ddot e_1 \\ \vdots \\ \ddot e_n \end{bmatrix} + \begin{bmatrix} k_{v_1} & & 0 \\ & \ddots & \\ 0 & & k_{v_n} \end{bmatrix}\begin{bmatrix} \dot e_1 \\ \vdots \\ \dot e_n \end{bmatrix} + \begin{bmatrix} k_{p_1} & & 0 \\ & \ddots & \\ 0 & & k_{p_n} \end{bmatrix}\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = 0$$

decoupled linear

ith equation

$$\ddot e_i + K_{v_i}\dot e_i + K_{p_i} e_i = 0 \qquad i = 1, \dots, n$$

$K_{p_i}$ smaller

linear system $s^2 + K_{v_i} s + K_{p_i}$ ✓

Choose $K_{v_i}, K_{p_i}$ so that roots of $s^2 + K_{v_i} s + K_{p_i}$ are in $\mathbb{C}^-$

then $e_i \to 0$ as $t \to \infty$!

The NL control law

$$\tau = M(\theta)\left(\ddot{\theta}_d + K_v(\dot{\theta}_d - \dot{\theta}) + K_p(\theta_d - \theta)\right) + C(\theta,\dot{\theta})\dot{\theta} + N(\theta,\dot{\theta})$$

linearizes and decouples the robot dynamics! AMAZING!

(COMPUTED TORQUE CONTROL LAW)

Even if $C(\theta,\dot{\theta})\dot{\theta}$ & $N(\theta,\dot{\theta})$ are not known exactly the CT control law is really robust!

P.D.

Prop DERIVATIVE

$$\tau = M(\theta)\ddot{\theta}_d + C(\theta,\dot{\theta})\dot{\theta}_d + N(\theta,\dot{\theta}) - K_v\dot{e} - K_p e.$$

P.I.D.

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + N(\theta,\dot{\theta})$$

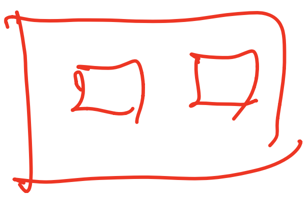
$$\tau = M(\theta)\ddot{\theta}_d + C(\theta, \dot{\theta})\dot{\theta}_d \ldots - K_v \dot{e} - K_p e - K_I \int_0^t e(\tau)\,d\tau$$

NYU Courant Inst. → HKUST → Googol.

FANUC

$10^{100}$ = Googol

Motion Control Boards



C.T.

P.D.

P.I.D.

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