SO(3)

$a \times b \qquad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad a \times b$

$\hat{a}\, b$

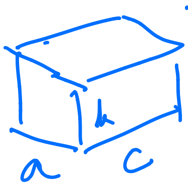
$$\hat{a} \rightsquigarrow \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$

so(3)

Property ① $a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a)$

$\boxed{\text{Check}}$ $\quad a^T (b \times c) = c^T (a \times b)$

$$[a_1 \ a_2 \ a_3] \begin{bmatrix} -b_3 c_2 + b_2 c_3 \\ b_3 c_1 - b_1 c_3 \\ -\underline{b_3} c_1 + b_1 c_2 \end{bmatrix}$$



VOL

→ ② $(a \times b) \times c + (c \times a) \times b + (b \times c) \times a = 0$

JACOBI IDENTITY

$R \in SO(3)$ $\qquad \widehat{Rv} = R\hat{v}R^T$ $\qquad R \in SO(3)$

FACT 1 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v \in \mathbb{R}^3$

Proof:- $R^T = \begin{bmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{bmatrix}$ $\qquad R = \begin{bmatrix} - & r_1^T & - \\ - & r_2^T & - \\ - & r_3^T & - \end{bmatrix}$

$v \in \mathbb{R}^3 \xrightarrow{\wedge} \hat{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$

so (3)

Skew symmetric^T

$= -$(skew symmetric)

$$R\hat{v}R^T$$

$$= R\hat{v}\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$$

$$= R\begin{bmatrix} \hat{v}r_1 & \hat{v}r_2 & \hat{v}r_3 \end{bmatrix}$$

$$= R\begin{bmatrix} | & | & | \\ v\times r_1 & v\times r_2 & v\times r_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}\begin{bmatrix} | & | & | \\ v\times r_1 & v\times r_2 & v\times r_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} r_1^T(v\times r_1) & r_1^T(v\times r_2) & r_1^T(v\times r_3) \\ r_2^T(v\times r_1) & r_2^T(v\times r_2) & r_2^T(v\times r_3) \\ r_3^T(v\times r_1) & r_3^T(v\times r_2) & r_3^T(v\times r_3) \end{bmatrix}$$

0 $\quad -v^T r_3$ $\quad v^T r_2$ $\quad v^T r_3$ $\quad -v^T r_1$ $\quad 0$ $\quad 0$ $\quad -v^T r_2$ $\quad v^T r_1$

$r_2 \cdot (v \times r_1)$

$= r_1 \cdot (r_2 \times v)$

$= v \cdot (r_1 \times r_2)$

$= v \cdot r_3$

$$\text{RHS} = \widehat{Rv}$$

$$= \widehat{\begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \end{bmatrix} v}$$

$$= \begin{bmatrix} r_1^T v \\ r_2^T v \\ r_3^T v \end{bmatrix} = RHS \qquad \square$$

$$\widehat{Rv} = R\hat{v}R^T$$

FACT 2

$$Rv \times Rw = R(v \times w)$$

R preserves angles

Proof:

$$LHS = \widehat{Rv} \cdot Rw$$
$$= R\hat{v}\underbrace{R^T R}_{I} w$$
$$= R\hat{v}w$$
$$= R(v \times w)$$

RODRIGUEZ

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta) \qquad \|\omega\| = 1$$

Proof: $e^{\hat{a}}$

$$a = \omega\theta \qquad \boxed{\|\omega\| = 1}$$
$$\theta = \|a\|$$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2!} + \frac{\hat{\omega}^3\theta^3}{3!} + \frac{\hat{\omega}^4\theta^4}{4!} + \dots$$

$$\hat{\omega}^2 = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Skew Symmetric

$$= \begin{bmatrix} -(\omega_3^2+\omega_2^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_2^2+\omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{bmatrix}$$

Symmetric

$$= \begin{bmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{bmatrix} - \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{=1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\omega}^2 = \omega\omega^T - I$$

$$\hat{\omega}^3 = \hat{\omega}(\omega\omega^T - I) = -\hat{\omega}$$

$$\hat{\omega}\omega\omega^T - \hat{\omega} = -\hat{\omega}$$

$\omega \times \omega$ $= 0$

$$\hat{\omega}^4 = -\hat{\omega}^2$$

$$\hat{\omega}^5 = \hat{\omega}$$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{\hat{\omega}^2\theta^2}{2!} - \frac{\hat{\omega}\theta^3}{3!} - \frac{\hat{\omega}^2\theta^4}{4!} \cdots$$

$$= I + \hat{\omega}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)$$

$$+ \hat{\omega}^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} \cdots\right)$$

$$\boxed{e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)}$$

RODRIGUEZ

$=$