# EECS/BioE C106A/206A
# Introduction to Robotics

# Lost Section 5

Oct 30 Fri 7 – 9 PM



For any issue in this material, please email to donggun_lee@berkeley.edu

# Velocities and Jacobian, and Singularities

- The spatial velocity and Jacobian: $V^s = J^s\dot{\theta}$
- The body velocity and Jacobian: $V^b = J^b\dot{\theta}$
- Singularities

$$\xi = (v, \omega)^T \in \mathbb{R}^6$$

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

# Spatial Jacobian: $V^s = J^s \dot{\theta}$

$$\hat{V}^s = \dot{g} g^{-1} = \sum_{i=1}^{n} \left( \frac{\partial g}{\partial \theta_i} \dot{\theta}_i \right) g^{-1} = \begin{bmatrix} \frac{\partial g}{\partial \theta_1} g^{-1} & \frac{\partial g}{\partial \theta_2} g^{-1} & \cdots & \frac{\partial g}{\partial \theta_n} g^{-1} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

Then,

$$V^s = \underbrace{\begin{bmatrix} \left( \frac{\partial g}{\partial \theta_1} g^{-1} \right)^\vee & \left( \frac{\partial g}{\partial \theta_2} g^{-1} \right)^\vee & \cdots & \left( \frac{\partial g}{\partial \theta_n} g^{-1} \right)^\vee \end{bmatrix}}_{J^s = [\xi_1' \quad \xi_2' \quad \cdots \quad \xi_n']} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

$$J^s = \begin{bmatrix} \boxed{\begin{matrix} -\omega_1' \times q_1' \\ \omega_1' \end{matrix}} & \boxed{\begin{matrix} v_2' \\ 0 \end{matrix}} & \cdots & \begin{matrix} -\omega_n' \times q_n' \\ \omega_n' \end{matrix} \end{bmatrix}$$

revolute joint prismatic joint

$\xi = (v, \omega)^T \in \mathbb{R}^6$

$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$

# Body Jacobian: $V^b = J^b \dot{\theta}$

$$\hat{V}^b = g^{-1}\dot{g} = \sum_{i=1}^{n} g^{-1}\left(\frac{\partial g}{\partial \theta_i}\dot{\theta}_i\right) = \begin{bmatrix} g^{-1}\frac{\partial g}{\partial \theta_1} & g^{-1}\frac{\partial g}{\partial \theta_2} & \cdots & g^{-1}\frac{\partial g}{\partial \theta_n} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

Then,

$$V^b = \underbrace{\begin{bmatrix} \left(g^{-1}\frac{\partial g}{\partial \theta_1}\right)^\vee & \left(g^{-1}\frac{\partial g}{\partial \theta_2}\right)^\vee & \cdots & \left(g^{-1}\frac{\partial g}{\partial \theta_n}\right)^\vee \end{bmatrix}}_{J^b = \begin{bmatrix} \xi_1^\dagger & \xi_2^\dagger & \cdots & \xi_n^\dagger \end{bmatrix}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

$$J^b = \begin{bmatrix} -\omega_1^\dagger \times q_1^\dagger & v_2^\dagger & \cdots & -\omega_n^\dagger \times q_n^\dagger \\ \omega_1^\dagger & 0 & & \omega_n^\dagger \end{bmatrix}$$

revolute joint prismatic joint

Another way

$$J^b(\theta) = \mathrm{Ad}_{g^{-1}(\theta)} J^s(\theta)$$

Proof:

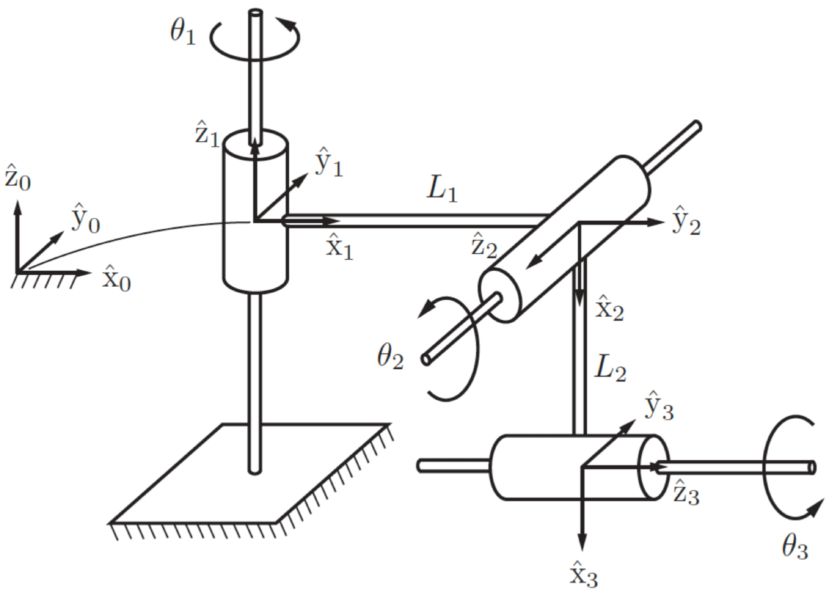
$$\hat{V}^b = \left(J^b\dot{\theta}\right)^\wedge = \sum_i \hat{\xi}_i^\dagger \dot{\theta}_i$$

$$= g^{-1}\dot{g} = g^{-1}\hat{V}^s g = g^{-1}\left[\sum_i \hat{\xi}_i' \dot{\theta}_i\right] g = \sum_i g^{-1}\hat{\xi}_i' g \dot{\theta}_i$$

Since the above equation holds for all $\dot{\theta}$,

$\hat{\xi}_i^\dagger = g^{-1}\hat{\xi}_i' g$ and $\xi_i^\dagger = g^{-1}\xi_i' g$

# Example




**Spatial Jacobian at $\theta = 0$**

$$q_1' = [0,0,0]^T, \omega_1' = [0,0,1]^T$$
$$q_2' = [L_1, 0,0]^T, \omega_2' = [0,-1,0]^T$$
$$q_3' = [0,0,-L_2]^T, \omega_3' = [1,0,0]^T$$

$$J^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_2 \\ 0 & -L_1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{\theta}$$

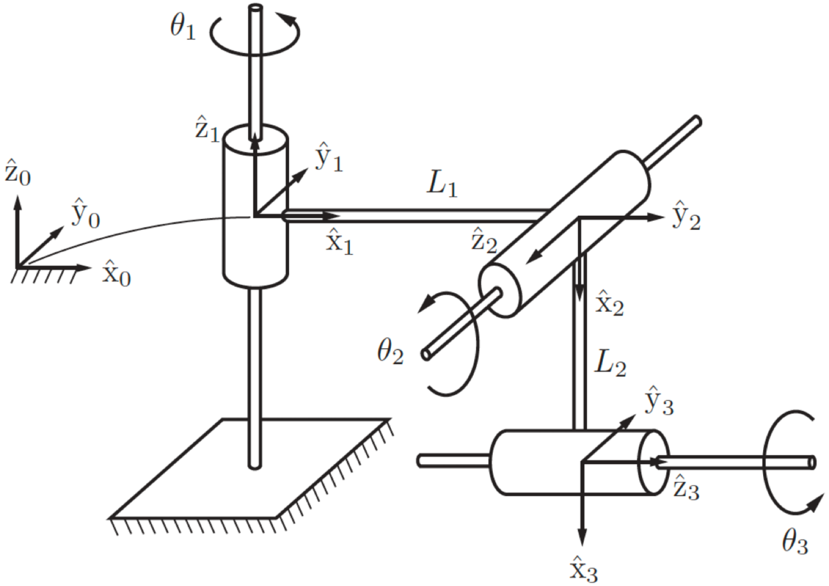
**Body Jacobian at $\theta = 0$**

$$q_1^\dagger = [0,0,-L_1]^T, \omega_1^\dagger = [-1,0,0]^T$$
$$q_2^\dagger = [-L_2, 0,0]^T, \omega_2^\dagger = [0,-1,0]^T$$
$$q_3^\dagger = [0,0,0]^T, \omega_3^\dagger = [0,0,1]^T$$

$$J^b = \begin{bmatrix} 0 & 0 & 0 \\ L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

You can choose other points for $q_1', q_2', q_3', q_1^\dagger, q_2^\dagger, q_3^\dagger$.

# Example




**Spatial Jacobian at $\theta = 0$**

$$J^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_2 \\ 0 & -L_1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \dot{\theta}$$

**Body Jacobian at $\theta = 0$**

$$J^b = \begin{bmatrix} 0 & -L_2 & 0 \\ 0 & 0 & 0 \\ -L_1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

Using the rigid body transformation $g(0) = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

we can check $J^b = \mathrm{Ad}_{g^{-1}} J^s$.

Note that $\mathrm{Ad}_{g(0)} = \begin{bmatrix} 0 & 0 & 1 & 0 & L_2 & 0 \\ 0 & 1 & 0 & L_1 & 0 & -L_2 \\ -1 & 0 & 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$, $\mathrm{Ad}_{g^{-1}(0)} = \begin{bmatrix} 0 & 0 & -1 & 0 & L_1 & 0 \\ 0 & 1 & 0 & L_2 & 0 & L_1 \\ 1 & 0 & 0 & 0 & -L_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

# Singularity

The Jacobian $J^s$ (or $J^b$) is singular if $J^s$ is not full rank.

- check the linear dependency of the columns in $J^s$ with a particular $\theta$
- *find a particular $\theta$ such that the robot cannot achieve a certain direction of translational or angular velocity*
- find a particular θ such that two axes are on the same line

*Why this is true?*

The velocity of the end effector and the corresponding Jacobian ($\bar{J}$): $\begin{bmatrix} v^s(\theta) \\ \omega^s(\theta) \end{bmatrix} = \bar{J}(\theta)\dot{\theta}$

Note that $\bar{J}(\theta) \neq J^s(\theta)$ since $\bar{v}^s(\theta) = \hat{V}^s(\theta)\bar{p}^s(\theta)$. Here $\bar{v}^s = [v^s, 0]^T$ and $\bar{p}^s = [p^s, 1]^T$

$V^s(\theta) = J^s(\theta)\dot{\theta} = [\xi_1' \quad \dots \quad \xi_n']\,\dot{\theta} = \begin{bmatrix} v_1' & \dots & v_n' \\ \omega_1' & \dots & \omega_n' \end{bmatrix}\dot{\theta}$ and $\hat{V}^s(\theta) = \sum_i \hat{\xi}_i'\dot{\theta}_i$

Then, $\bar{J}(\theta)\dot{\theta} = \begin{bmatrix} v^s(\theta) \\ \omega^s(\theta) \end{bmatrix}$, where $\bar{v}^s(\theta) = \hat{V}^s(\theta)\bar{p}^s(\theta) = \sum_i \hat{\xi}_i'\,\bar{p}^s(\theta)\dot{\theta}_i = [\hat{\xi}_1'\bar{p}^s(\theta) \quad \dots \quad \hat{\xi}_n'\bar{p}^s(\theta)]\dot{\theta}$, and $\omega^s(\theta) = [\omega_1' \quad \dots \quad \omega_n']\dot{\theta}$.

Thus,

$$\bar{J}(\theta) = \begin{bmatrix} I_3 & -\hat{p}^s(\theta) \\ 0_3 & I_3 \end{bmatrix} J^s(\theta).$$

Since $\begin{bmatrix} I_3 & -\hat{p}^s(\theta) \\ 0_3 & I_3 \end{bmatrix} \in \mathbb{R}^{6\times 6}$ is always invertible,

$$\mathrm{rank}\big(\bar{J}(\theta)\big) = \mathrm{rank}\big(J^s(\theta)\big).$$

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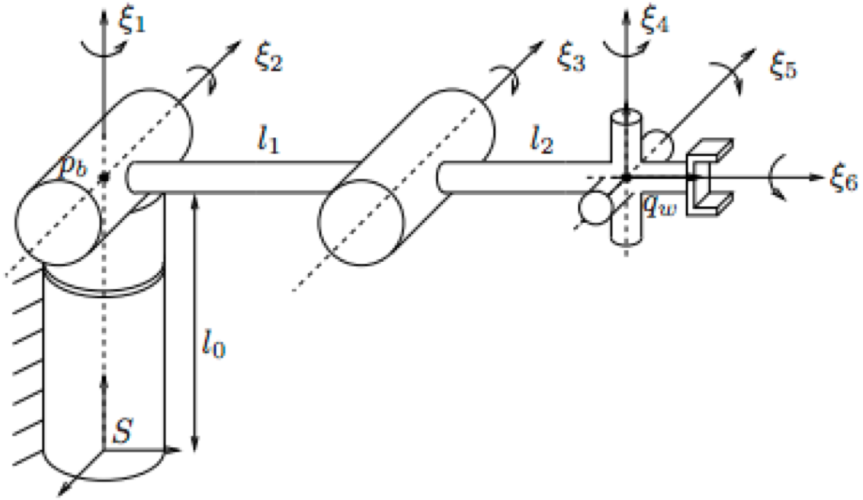
- *find a particular $\theta$ such that the robot cannot achieve a certain direction of translational or angular velocity*
- find a particular θ such that two axes are on the same line




- The *y*-translational velocity cannot be achieved at the zero configuration.
- Six joints must cannot achieve the six dimensional velocities (translation + rotation)
- Thus, we have singularity at the zero configuration.

We should check all translational and angular velocities.

# Singularity

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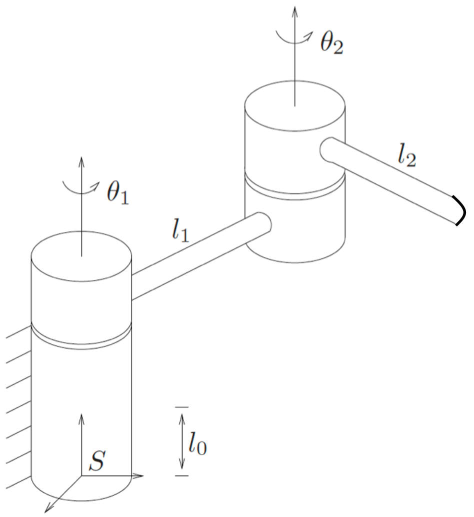
- check the linear dependency of the columns in $J^s$ with a particular $\theta$
- *<u>find a particular $\theta$ such that the robot cannot achieve a certain direction of translational or angular velocity</u>*
- find a particular θ such that two axes are on the same line




$$J^s = \begin{bmatrix} 0 & l_1 \cos\theta_1 \\ 0 & l_1 \sin\theta_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

- In this problem, for any $\theta$, the robot is not singular.

- Note: at $\theta = 0$, $y$-directional velocity cannot be achieved. However, the robot is not singular. We should also consider the angular velocity of the robot, which always span 1-dimensional space.

Warning: We should check all translational and angular velocities.

# Singularity

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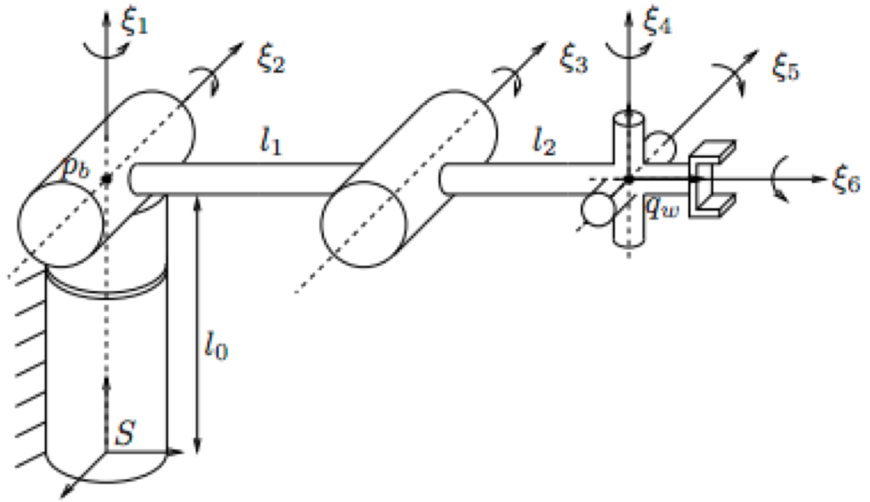


Goal: find $\theta_2$ and $j$ such that $\xi_1 = \pm\xi_j$

Answer: $\theta_2 = \pm\frac{\pi}{2}$ and $j = 6$.

# Singularity

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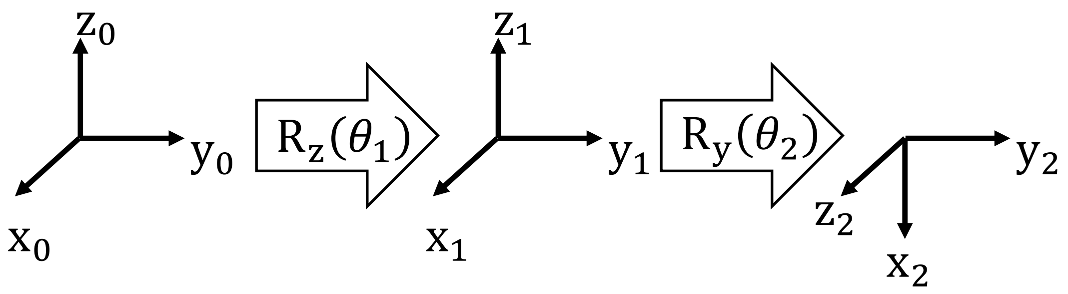
Euler ZYX




Goal: find $\theta_2$ such that $\mathrm{z}_0 = \pm \mathrm{x}_2$.

Answer: $\theta_2 = \pm \frac{\pi}{2}$