EECS/BioE C106A/206A Introduction to Robotics

Lost Section 6

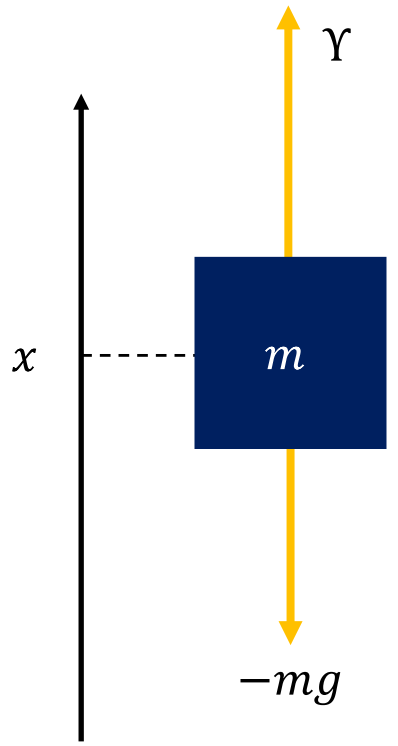
Nov 6 Fri 7 – 9 PM

For any issue in this material, please email to donggun_lee@berkeley.edu

Dynamics

- Lagrange-Euler equation
- Lagrangian (Kinetic energy – Potential energy)

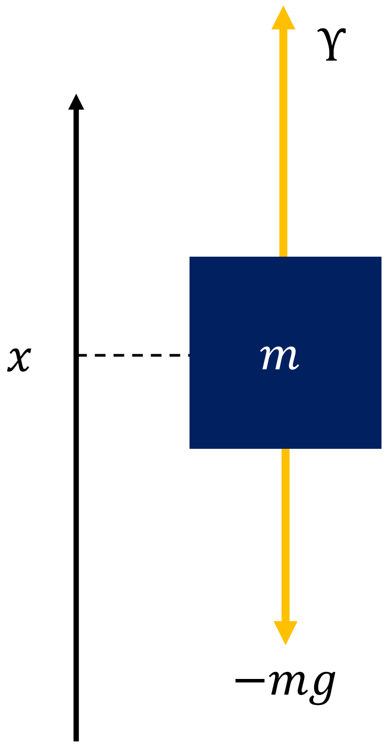
Kinetic energy and Potential energy



Kinetic energy: the energy that a body possesses by virtue of begin in motion
(ex) mechanical kinetic energy, electrical energy, thermal energy

Potential energy: the energy stored by a body by virtue of its position relative to others in force field
(ex) gravitational field, magnetic and electric field, chemical potential energy, elastic potential energy,

Kinetic energy and Potential energy



Kinetic energy

$$T = \frac{1}{2} m\dot{x}^2$$

Potential energy

$$V = mgx$$

Lagrangian

$$L = T - V$$

Euler-Lagrange equation

$$\Upsilon = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}$$
$$= m\ddot{x} + mg$$

If $\Upsilon = 0,$
$$m\ddot{x} = -mg$$

For angle x_i, Υ_{i} is a joint torque acting on the i-th axis or body.

For position x_i, Υ_{i} is a joint force acting on the i-th axis.

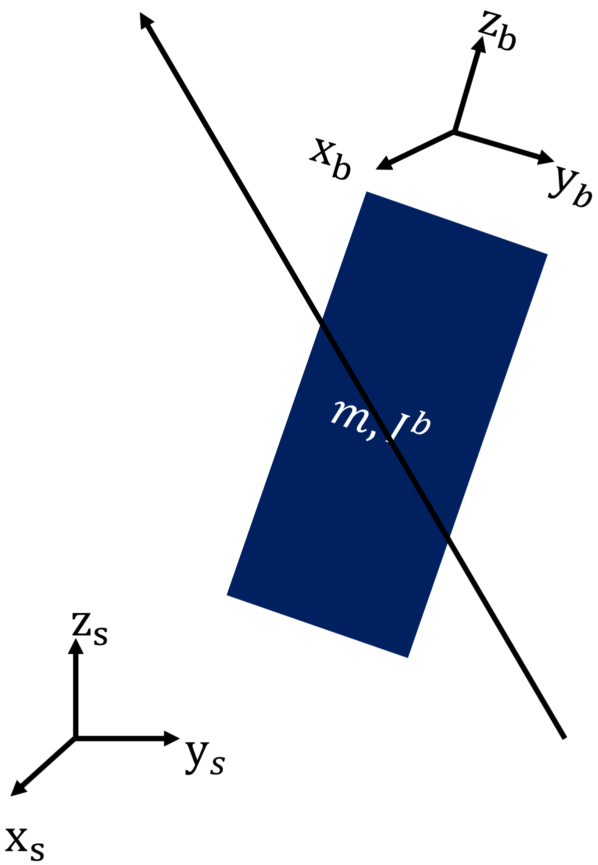
Energy

$$E = T + V$$

$$\dot{E} = m\dot{x}\ddot{x} + mg\dot{x}$$
$$= \dot{x}(m\ddot{x} + mg) = 0$$

: energy conservation

Kinetic energy



$p^s \in \mathbb{R}^3$: the position of the center of the body w.r.t. the spatial frame
$p^b \in \mathbb{R}^3$: the position of the center of the body w.r.t. the body frame

$$p^s = Rp^b$$

$\dot{p}^s \in \mathbb{R}^3$: the velocity of the center of the body w.r.t. the spatial frame
$\dot{p}^b \in \mathbb{R}^3$: the velocity of the center of the body w.r.t. the body frame

$$\dot{p}^s = R\dot{p}^b$$

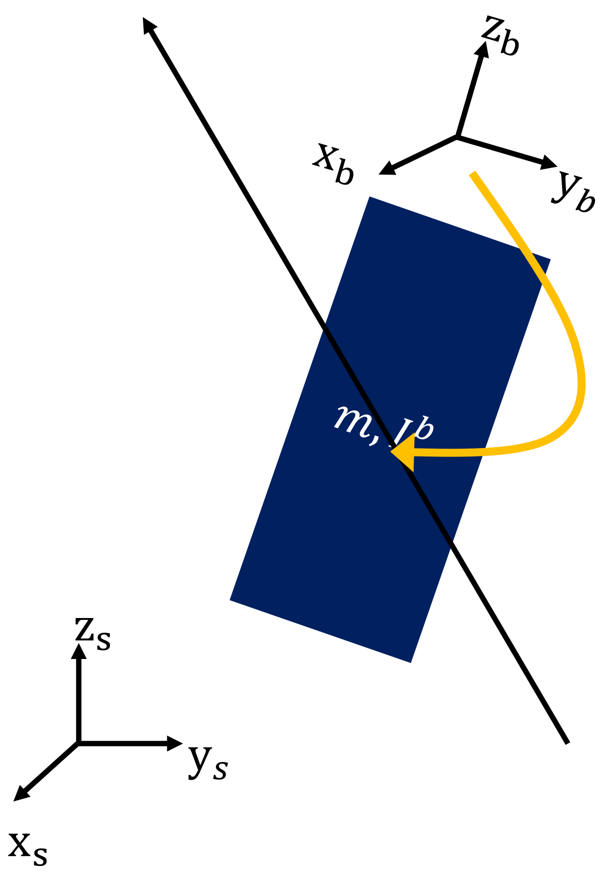
$\overline{\omega}^s \in \mathbb{R}^3$: the angular velocity of the center of the body w.r.t. the spatial frame
$\overline{\omega}^b \in \mathbb{R}^3$: the angular velocity of the center of the body w.r.t. the body frame

$$\overline{\omega}^s = R\overline{\omega}^b$$

$I^s \in \mathbb{R}^{3\times3}$: the inertia matrix w.r.t. the spatial frame
$I^b \in \mathbb{R}^{3\times3}$: the inertia matrix w.r.t. the body frame

$$I^s = RI^bR^{\mathrm{T}}$$

Kinetic energy



$$T = \frac{1}{2} m \|\dot{p}^s\|^2 + \frac{1}{2} \overline{\omega}^{s\mathrm{T}} I^s \overline{\omega}^s \quad \text{: method 1}$$

$$= \frac{1}{2} m \left\|\dot{p}^b\right\|^2 + \frac{1}{2} \overline{\omega}^{b\mathrm{T}} I^b \overline{\omega}^b$$

$$= \frac{1}{2} V^{b\mathrm{T}} \mathcal{M} V^b, \text{where } \mathcal{M} := \begin{bmatrix} mI & 0 \\ 0 & I^b \end{bmatrix} \quad \text{: method 2}$$

Spatial and body velocities

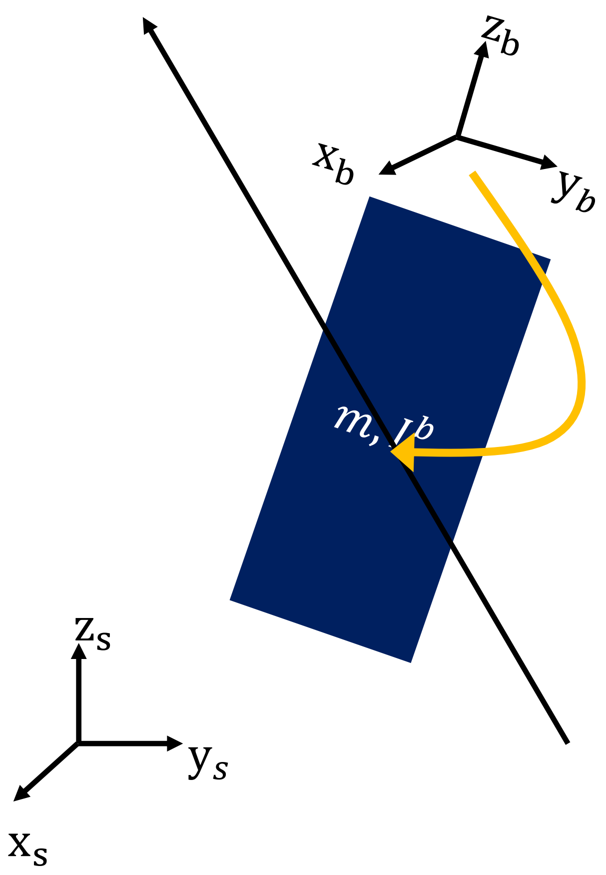
$$\dot{\bar{p}}^s = \hat{V}^s \bar{p}^s \text{ and } \dot{\bar{p}}^b = \hat{V}^b \bar{p}^b, \text{where } \bar{p}^s = [p^s, 1]^\mathrm{T} \text{ and } \bar{p}^b = [p^b, 1]^T$$

We know that $V^s \in \mathbb{R}^6$ is the twist times $\dot{\theta}$ w.r.t. the spatial frame,
$V^b \in \mathbb{R}^6$ is the twist times $\dot{\theta}$ w.r.t. the body frame.

$$V^s = \xi^s \dot{\theta} = [v^s, \omega^s]^\mathrm{T} \dot{\theta}$$
$$V^b = \xi^b \dot{\theta} = [v^b, \omega^b]^\mathrm{T} \dot{\theta}, \text{ where } \dot{\theta} \in \mathbb{R}$$

Note that $\overline{\omega}^s = \omega^s \dot{\theta}$ and $\overline{\omega}^b = \omega^b \dot{\theta}$.

Kinetic energy



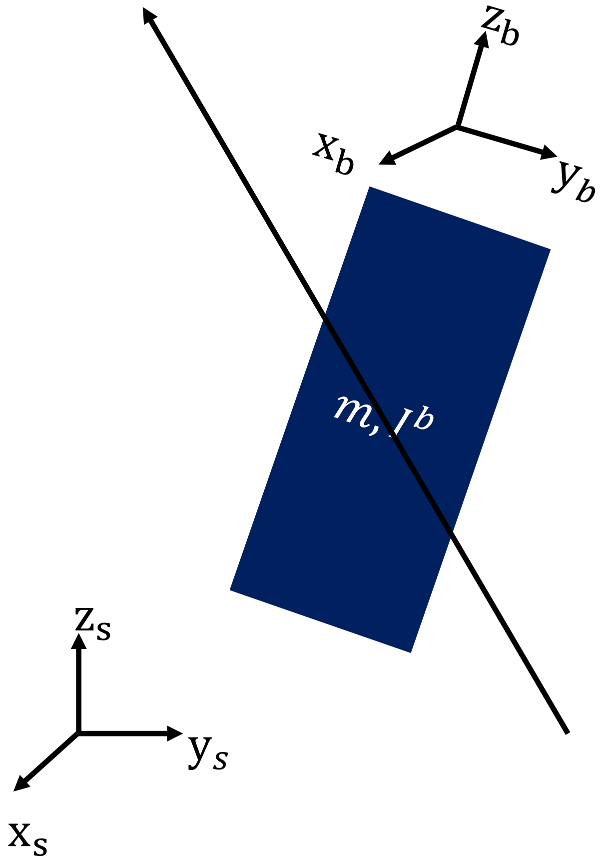
$$
\begin{aligned}
T &= \frac{1}{2} m \|\dot{p}^s\|^2 + \frac{1}{2} \bar{\omega}^{s\,\mathrm{T}} I^s \bar{\omega}^s \\
&= \frac{1}{2} m \left\|\dot{p}^b\right\|^2 + \frac{1}{2} \bar{\omega}^{b\,\mathrm{T}} I^b \bar{\omega}^b \\
&= \frac{1}{2} m \left\|v^b\right\|^2 \dot{\theta}^2 + \frac{1}{2} \omega^{b\,\mathrm{T}} I^b \omega^b \dot{\theta}^2 \\
&= \frac{1}{2} \xi^{b\,\mathrm{T}} \begin{bmatrix} mI & 0 \\ 0 & I^b \end{bmatrix} \xi^b \dot{\theta}^2 = \frac{1}{2} V^{b\,\mathrm{T}} \mathcal{M} V^b
\end{aligned}
$$

: method 2

where $\mathcal{M} := \begin{bmatrix} mI & 0 \\ 0 & I^b \end{bmatrix}$

Here $p^b = 0 \in \mathbb{R}^3$, thus $\dot{\bar{p}}^b = \hat{V}^b \bar{p}^b = \begin{bmatrix} \hat{\omega}^b & v^b \\ 0 & 0 \end{bmatrix} \dot{\theta} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v^b \dot{\theta} \\ 0 \end{bmatrix}$.

Potential energy



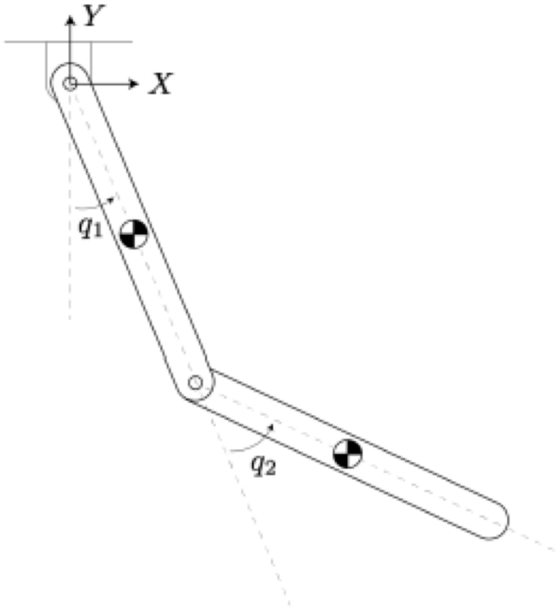
$$V = \int_{(0,0,0)}^{p^s} -F \cdot dx = mg{p^s}^{\mathrm{T}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now, we have the Lagrangian L and ready to find the dynamics via the Euler-Lagrange equation.

Example: double pendulum, Problem 3, HW 9

Assumption: the two links are empty but has a concentrated mass at the centers.

$$I_1^s = I_1^b = I_2^s = I_2^b = 0$$



method 1: $T = \frac{1}{2} m_1 \|\dot{p_1}^s\|^2 + \frac{1}{2} m_2 \|\dot{p_2}^s\|^2$

$$p_1^s = \frac{L}{2}\begin{bmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{bmatrix}, p_2^s = L\begin{bmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{bmatrix} + \frac{L}{2}\begin{bmatrix} \sin(q_1+q_2) \\ -\cos(q_1+q_2) \\ 0 \end{bmatrix}$$

$$\dot{p}_1^s = \frac{L}{2}\dot{q}_1\begin{bmatrix} \cos q_1 \\ \sin q_1 \\ 0 \end{bmatrix}, p_2^s = L\dot{q}_1\begin{bmatrix} \cos q_1 \\ \sin q_1 \\ 0 \end{bmatrix} + \frac{L}{2}(\dot{q}_1+\dot{q}_2)\begin{bmatrix} \cos(q_1+q_2) \\ \sin(q_1+q_2) \\ 0 \end{bmatrix}$$

Then, $\|\dot{p_1}^s\|^2 = \frac{L^2}{4}\dot{q}_1^2, \|\dot{p_2}^s\|^2 = L^2\dot{q}_1^2 + \frac{L^2}{4}(\dot{q}_1+\dot{q}_2)^2 + L^2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos q_2$

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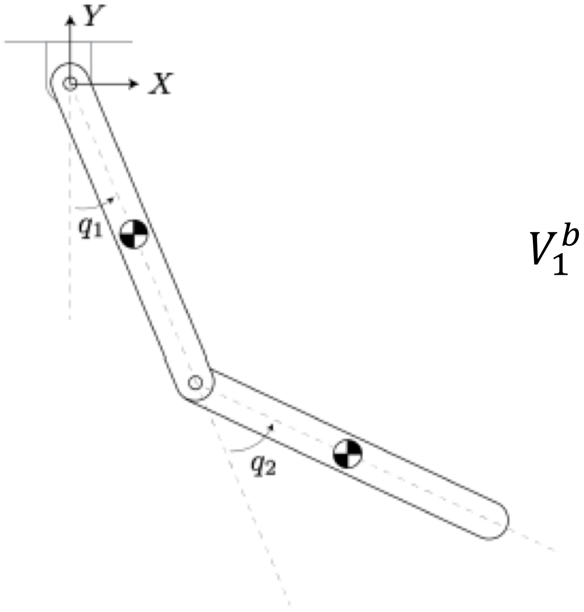

method 2: $T = \frac{1}{2} {V_1^b}^{\mathrm{T}} \mathcal{M}_1 V_1^b + \frac{1}{2} {V_2^b}^{\mathrm{T}} \mathcal{M}_2 V_2^b$

$$V_1^b = J_{s1}^b \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad J_{s1}^b = [\,{}^1\xi_1^\dagger,\ {}^1\xi_2^\dagger] = \begin{bmatrix} {}^1v_1^\dagger & {}^1v_2^\dagger \\ {}^1\omega_1^\dagger & {}^1\omega_2^\dagger \end{bmatrix}$$

$${}^1q_1^\dagger = \begin{bmatrix} 0 \\ \frac{L}{2} \\ 0 \end{bmatrix},\ {}^1\omega_1^\dagger = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ {}^1v_1^\dagger = \begin{bmatrix} \frac{L}{2} \\ 0 \\ 0 \end{bmatrix}$$

$${}^1\xi_2^\dagger = 0 \in \mathbb{R}^6$$

$$V_2^b = J_{s2}^b \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad J_{s2}^b = [\,{}^2\xi_1^\dagger,\ {}^2\xi_2^\dagger] = \begin{bmatrix} {}^2v_1^\dagger & {}^2v_2^\dagger \\ {}^2\omega_1^\dagger & {}^2\omega_2^\dagger \end{bmatrix}$$

$${}^2q_1^\dagger = \begin{bmatrix} L\sin q_2 \\ \frac{L}{2} + L\cos q_2 \\ 0 \end{bmatrix},\ {}^2\omega_1^\dagger = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ {}^2v_1^\dagger = \begin{bmatrix} \frac{L}{2} + L\cos q_2 \\ -L\sin q_2 \\ 0 \end{bmatrix}$$

$${}^2q_2^\dagger = \begin{bmatrix} 0 \\ \frac{L}{2} \\ 0 \end{bmatrix},\ {}^2\omega_2^\dagger = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ {}^2v_2^\dagger = \begin{bmatrix} \frac{L}{2} \\ 0 \\ 0 \end{bmatrix}$$

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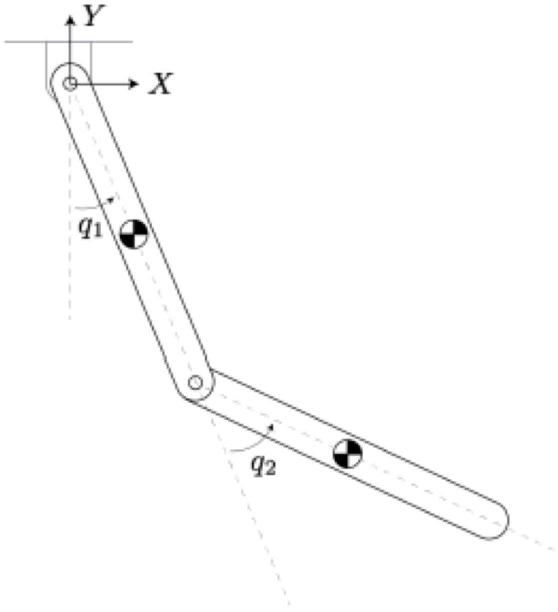

method 2: $T = \frac{1}{2} {V_1^b}^{\mathrm{T}} \mathcal{M}_1 V_1^b + \frac{1}{2} {V_2^b}^{\mathrm{T}} \mathcal{M}_2 V_2^b$

$$= \frac{1}{2} m_1 \left\| \begin{bmatrix} {}^1v_1^\dagger & {}^1v_2^\dagger \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \right\|^2 + \frac{1}{2} m_2 \left\| \begin{bmatrix} {}^2v_1^\dagger & {}^2v_2^\dagger \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \right\|^2$$

$$= \frac{1}{2} m_1 \frac{L^2}{4} \dot{q}_1^2 + \frac{1}{2} m_2 \left\| \begin{bmatrix} \frac{L}{2} + L\cos q_2 & \frac{L}{2} \\ -L\sin q_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \right\|^2$$

$$= \frac{1}{2} m_1 \frac{L^2}{4} \dot{q}_1^2 + \frac{1}{2} m_2 \left(L^2 \dot{q}_1^2 + \frac{L^2}{4} (\dot{q}_1 + \dot{q}_2)^2 + L\dot{q}_1(\dot{q}_1 + \dot{q}_2)\cos q_2 \right)$$

The same as method 1