INVERSE KINEMATICS

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n}\underbrace{g_{ST}(0)}_{\text{KNOWN}} = \underbrace{g_{ST}(\theta)}_{\text{GIVEN}}$$

Solve for $\theta_1, \ldots, \theta_n$

BRUTE FORCE

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} = g_{ST}(\theta)\, g_{ST}^{-1}(0)$$

$$\begin{bmatrix} R(\theta) & p(\theta) \\ 0\,0\,0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 & p_1 \\ 0\,0\,0 & 1 \end{bmatrix}$$

$9 + 3 = 12$

$R_{11}(\theta) = (R_1)_{11}$

$\vdots$

$R_{33}(\theta) = -$

$p_1(\theta) =$

$p_2(\theta) =$

$p_3(\theta) =$

$12 + 6$

x_1 $\quad \cos\theta_1 \quad \sin\theta_1 \quad y_1$

x_2 $\quad \cos\theta_2 \quad \sin\theta_2 \quad y_2$

$\vdots$

$x_n \qquad\qquad y_n$

$\boxed{n = 6}$

Prods of x_i, y_j, $i = 1 \ldots n$, $j = 1, \ldots n$

$x_i^2 + y_i^2 = 1 \quad i = 1, \ldots, 6$

18

$6\, x_i \quad 6\, y_i \qquad 12$ variables $- 18$

12 independent

$f_1(x_1, \ldots x_6, y_1 \ldots y_6) = .$ given

$\vdots$

$f_{12}(x_1 \ldots x_6, y_1, \ldots y_6) = \_$

Solve 12 nonlinear equations for x_i, y_i

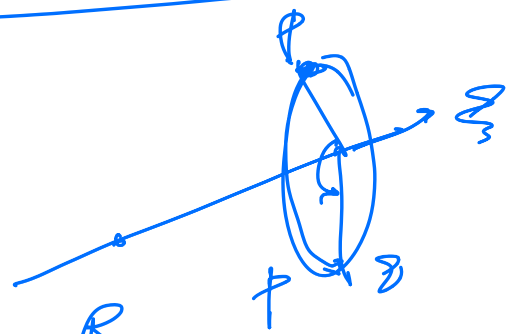
ALGEBRAIC

$x_i , y_i \in \mathbb{C}$

REAL ALGEBRAIC ROOTS

GEOMETRIC

SUB PROBLEMS (1)



R $\quad p$

9 (6) 3

$n > 6$

– Redundant manipulators

∞ many solutions $\quad g \in SE(3)$, 6

$\boxed{n-6} \rightarrow 6$

$n = 7 \quad S \subset \mathbb{R}^7$

1 dim

$\theta_1' = \theta_1^0 \; \ldots \; \theta_2 \ldots \; \theta_7$

$\theta_1 = \theta_1^0 + \delta \quad \theta_2 \ldots \; \theta_7$

$$SE(3) = \begin{bmatrix} R & p \\ 0\,0\,0 & 1 \end{bmatrix} \Big/ \; R \in SO(3) \subset \mathbb{R}^{3\times 3}, \; p \in \mathbb{R}^3$$

$$9 \quad \overset{6}{\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}} \qquad r_i^T r_j = \delta_{ij}$$

$$6$$

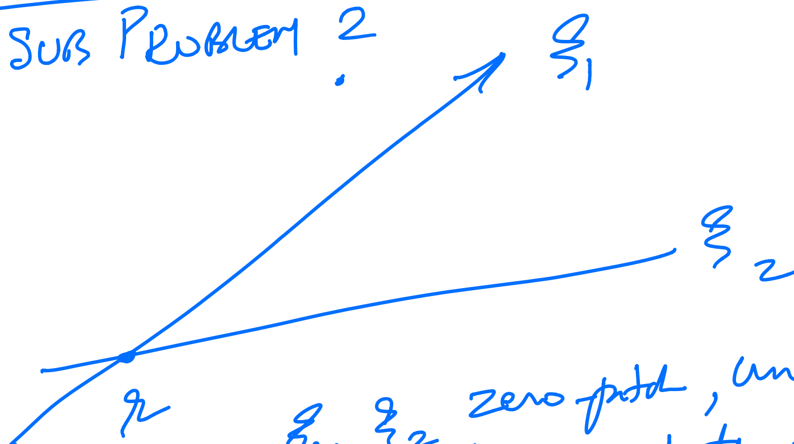
$$9 - 6 = 3 \text{ dim}$$

6 dimensional

$\theta_1 \ldots \theta_n$

$n = 6$ d.o.f.

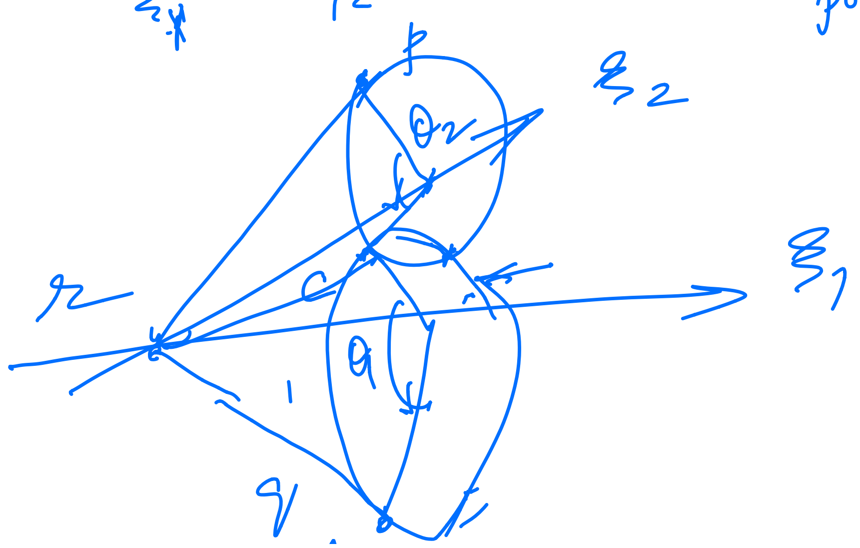
SUB PROBLEM 2 :



ξ_1, ξ_2 zero pitch, unit magnitude twists with intersecting axes

$r \in l_{\xi_1} \cap l_{\xi_2}$

Find (if possible) θ_1, θ_2



$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q$$

$$e^{\hat{\xi}_2\theta_2} p = c = e^{-\hat{\xi}_1\theta_1} q$$

Case 1: If $\omega_1 \parallel \omega_2$ $\quad \xi_1 = \xi_2$

$$e^{\hat{\xi}_1(\theta_1+\theta_2)} p = q$$

SUB PROBLEM 1

Case 2 $\quad \omega_1$ not parallel to ω_2

$\omega_1 \times \omega_2 \neq 0$

$$e^{\hat{\xi}_2\theta_2} p - r = c - r = e^{-\hat{\xi}_1\theta_1} q - r$$

$$e^{\hat{\xi}_2\theta_2} r = r$$

$$e^{\hat{\xi}_2\theta_2}\underbrace{(p - r)}_{u} = \underbrace{c - r}_{z} = e^{-\hat{\xi}_1\theta_1}\underbrace{(q - r)}_{v}$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} - \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} e^{\hat{\omega}_2\theta_2} & X \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v \\ 0 \end{bmatrix}$$

$$\mathbb{R}^3$$

$$\underbrace{e^{\hat{\omega}_2\theta_2} u}_{} = z = \underbrace{e^{-\hat{\omega}_1\theta_1} v}_{}$$

$$\left.\begin{array}{l} \omega_1^T z = \omega_1^T v \\ \|z\|^2 = \|v\|^2 \end{array}\right]$$

$$\left[\begin{array}{l} \omega_2^T u = \omega_2^T z \\ \|u\|^2 = \|z\|^2 \end{array}\right.$$

$\mathbb{R}^3 \ni \underbrace{\omega_1, \omega_2, \omega_1\times\omega_2}$ linearly indep.

$$z = \alpha\omega_1 + \beta\omega_2 + \gamma\,\omega_1\times\omega_2$$

$$\omega_2^T z = \alpha\,\omega_2^T\omega_1 + \beta\,\omega_2^T\omega_2 + \underbrace{\gamma\,\omega_2^T(\omega_1\times\omega_2)}_{0}$$

$$\omega_2^T u = \omega_2^T z = \alpha(\omega_2^T\omega_1) + \beta$$

$$\quad = \alpha + \beta(\omega_1^T\omega_2)$$

$\checkmark\ w_1^T v = w_1^T z$

Solve for α, β

$$\alpha = \frac{(w_1^T w_2)\, w_2^T u - w_1^T v}{(w_1^T w_2)^2 - 1}$$

$$\beta = \frac{(w_1^T w_2)\, w_1^T v - w_2^T u}{(w_1^T w_2)^2 - 1}$$

$$\|z\|^2 = \|u\|^2 = \|v\|^2$$

$$z = \alpha w_1 + \beta w_2 + \gamma (w_1 \times w_2)$$

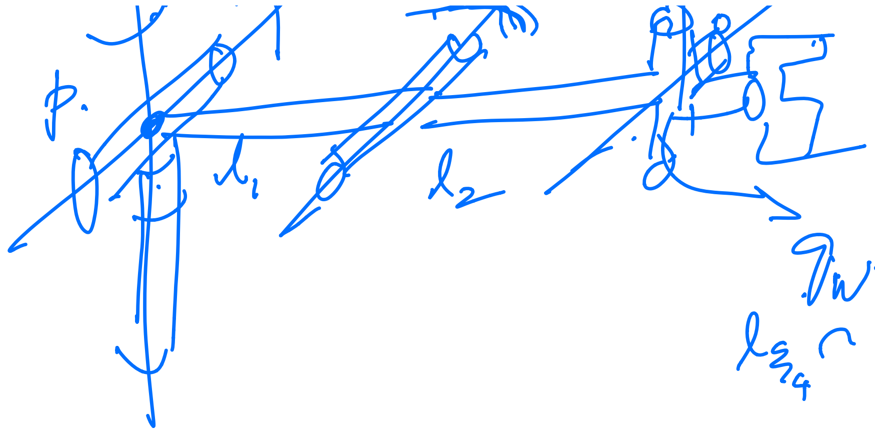
$$\|u\|^2 = z^T z = \alpha^2 w_1^T w_1 + 2\alpha\beta\, w_1^T w_2 + \beta^2 w_2^T w_2 + \gamma^2 \|w_1 \times w_2\|^2 + \underbrace{\cancel{2\alpha\gamma\, w_1^T (w_1 \times w_2)}}_{0} + \underbrace{\cancel{2\beta\gamma\, w_2^T (w_1 \times w_2)}}_{0}$$

$$\gamma^2 = \frac{\overbrace{\|u\|^2 - \alpha^2 - 2\alpha\beta\, w_1^T w_2 - \beta^2}^{\geq 0}}{\|w_1 \times w_2\|^2} \quad (*)$$

NUM > 0	2 SOLNS
$= 0$	1 SOLUTION
< 0	0 SOLUTION

$$z = \alpha w_1 + \beta w_2 + \gamma\, w_1 \times w_2$$

SUB PROBLEM $\quad \underbrace{e^{\hat{\xi}_2 \theta_2} u = z}\ = e^{-\hat{\xi}_1 \theta_1} v \quad$ SUB PROBLEM θ_1



ξ_6

$q_w \in l_{\xi_4} \cap l_{\xi_5} \cap l_{\xi_6}$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_6\theta_6} g_{st}(0) = g_{st}(\theta)$$

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_6\theta_6} = g_{st}(\theta) \cdot g_{st}^{-1}(0) = g$$

$$e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_w = g q_w$$

$$e^{\hat{\xi}_6\theta_6} q_w = q_w$$

$$e^{\hat{\xi}_5\theta_5} q_w = q_w$$

$$e^{\hat{\xi}_4\theta_4} q_w = q_w$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w = g q_w$$

$p \in l_{\xi_1} \cap l_{\xi_2}$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_w - p = g q_w - p$$

$$\boxed{e^{\hat{\xi}_1\theta_1}}\ \boxed{e^{\hat{\xi}_2\theta_2}} \left[e^{\hat{\xi}_3\theta_3} q_w - p \right] = g q_w - p$$

$p \in l_{\xi_1} \cap l_{\xi_2}$

Rotations

WRIST POINT

$$\| e^{\hat{\xi}_3\theta_3} q_w - p \| = \| g q_w - p \|$$

SHOULDER POINT

SUB PROBLEM 3 $\quad \theta_3$ BEND OF ELBOW

2 SOLUTION, 1 SOLUTION, 0

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \underbrace{e^{\hat{\xi}_3\theta_3} q_w}_{} = g q_w.$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p_1 = \underline{g q_w}\ q_1$$

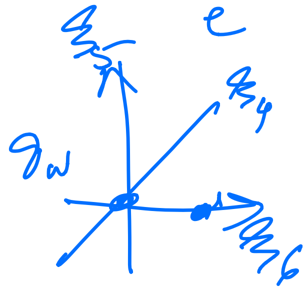
SUB PROBLEM 2. 2 SOLUTION, 1 SOLUTION, 0

θ_1, θ_2

$$\underbrace{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3}} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}$$

$$= g$$

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = \underbrace{e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g}_{\text{KNOWN}}$$



q_2

$q_2 \in l_{\xi_6}$ but $\notin l_{\xi_5} \notin l_{\xi_4}$

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} \underbrace{e^{\hat{\xi}_6\theta_6} q_2} = \boxed{}\, q_2$$

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} q_2 = \underbrace{\boxed{}\, q_2}$$

SUB PROBLEM $\underline{2}$ to SOLVE FOR θ_4, θ_5

2 SOLUTIONS, 1 SOLUTION, 0 SOLUTIONS

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = \boxed{}$$

$q_3 \notin l_{\xi_6}$

$$e^{\hat{\xi}_6\theta_6} q_3 = \underbrace{e^{-\hat{\xi}_5\theta_5} e^{-\hat{\xi}_4\theta_4} \boxed{}\, q_3}$$

SUB PROBLEM 1 to get θ_6.