EECS/BioE C106A/206A Introduction to Robotics

Lost Section 1

Sep 18 Fri 7 – 9 PM

For any issue in this material, please email to donggun_lee@berkeley.edu

Contents

- Theory
  - Rigid body transformation
  - Twist, exponential coordinates
  - Screw

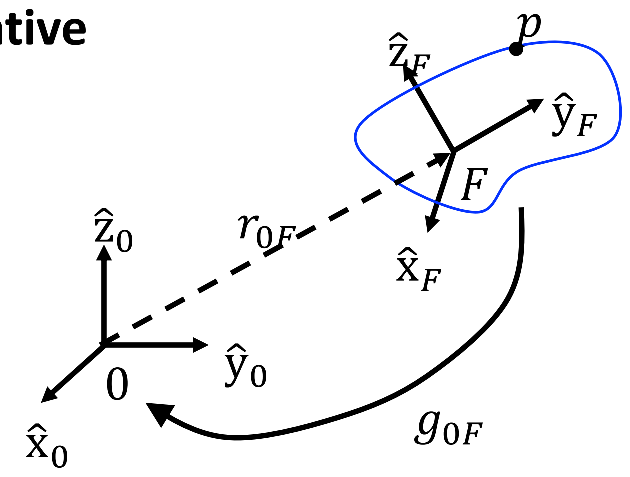
- Example
  - The satellites system (Homework1 Problem 4)

One message in this section:

Check a frame in which a coordinate is defined.

Rigid body transformation (relative+absolute)

relative



$$g_{0F} = \begin{bmatrix} R_{0F} = e^{\hat{\omega}\theta} & r_{0F} \\ 0 & 1 \end{bmatrix}$$

ω: the coordinate for the rotation axis of frame 0

r_{0F}: the coordinate for the translation of frame 0

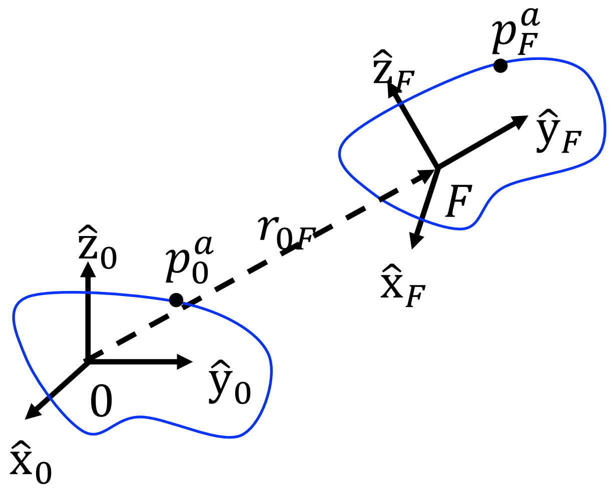
p: a point

p_F: the coordinate for p of frame F

p_0: the coordinate for p of frame 0

$$\begin{bmatrix} p_0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{0F} & r_{0F} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_F \\ 1 \end{bmatrix}$$

absolute



ω: frame 0

r_{0F}: frame 0

p_0^a: the coordinate of frame 0, for the fixed point on frame 0

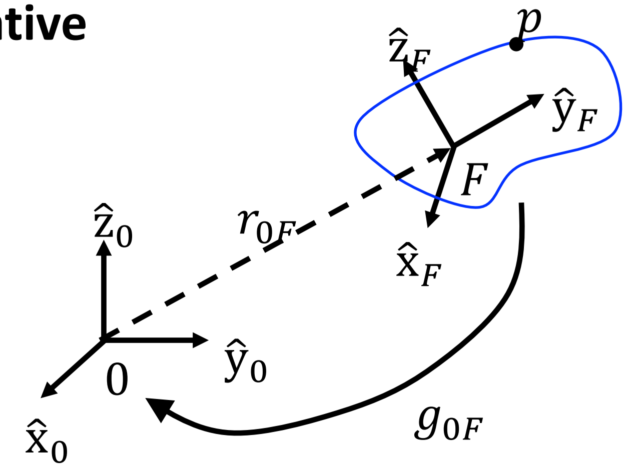
p_F: the coordinate of frame 0, for the fixed point on frame F

$$p_F^a = R_{0F}\, p_0^a + r_{0F}$$

$$\begin{bmatrix} p_F^a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{0F} & r_{0F} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_0^a \\ 1 \end{bmatrix} = g_{0F} \begin{bmatrix} p_0^a \\ 1 \end{bmatrix}$$

Rigid body transformation (relative+absolute)

relative



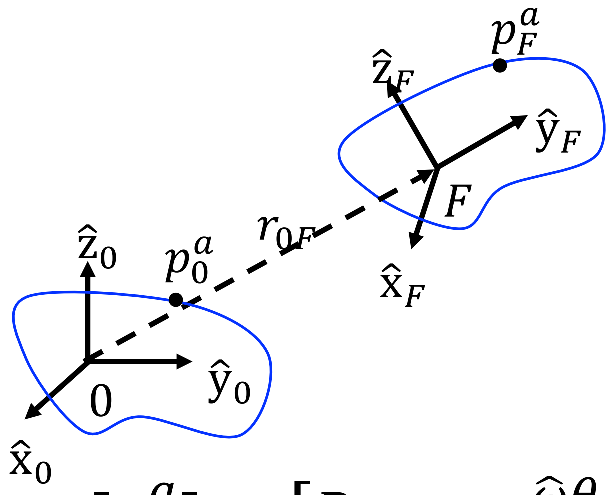
$$\begin{bmatrix} p_0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{0F} = e^{\widehat{\omega}\theta} & r_{0F} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_F \\ 1 \end{bmatrix}$$

ω: the coordinate for the rotation axis of frame 0
r_{0F}: the coordinate for the translation of frame 0

p: a point
p_F: the coordinate for p of frame F
p_0: the coordinate for p of frame 0

There is a single p, but multiple coordinates p_F, p_0.
$(R, \omega, r_{0F}, p_F, p_0)$ are defined in the relative frame.

absolute



$$\begin{bmatrix} p_F^a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{0F} = e^{\widehat{\omega}\theta} & r_{0F} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_0^a \\ 1 \end{bmatrix}$$

ω: frame 0
r_{0F}: frame 0

p_0^a: the coordinate of frame 0, for the fixed point on frame 0
p_F^a: the coordinate of frame 0, for the fixed point on frame F

There are multiple p_0^a, p_F^a.
$(R, \omega, r_{0F}, p_0^a, p_F^a)$ are defined in the frame 0.

Exponential Coordinates, Twist + Screw (absolute)

$(v, \widehat{\omega}) \in se(3)$: twist

$\xi := (v, \omega) \in \mathbb{R}^6$: the twist coordinates of $\hat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}$

(ξ, θ): exponential coordinate

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \hat{\xi}\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p(t) \\ 1 \end{bmatrix}$$

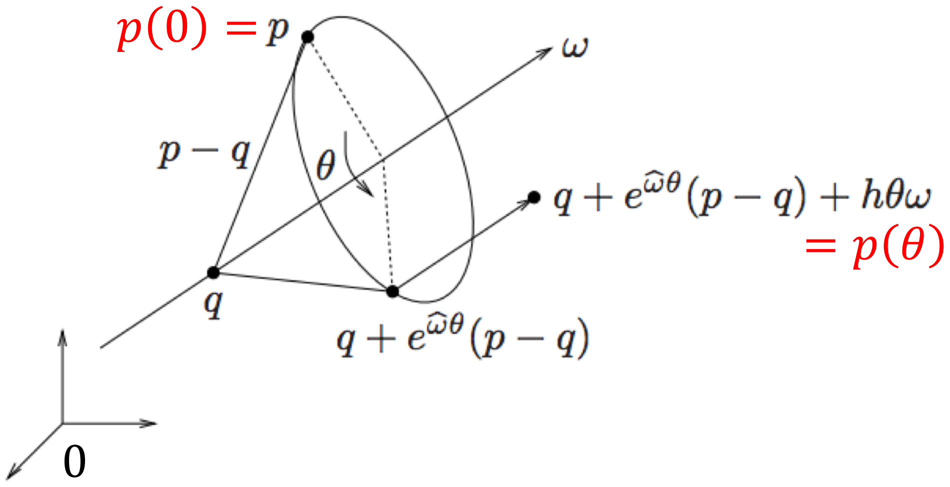
$p(t)$: frame 0
ω: the coordinate for the rotation axis of frame 0

Then, $$\begin{bmatrix} p(\theta) \\ 1 \end{bmatrix} = e^{\hat{\xi}\theta}\begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

Here, we only have a single frame: frame 0.

$$g = e^{\hat{\xi}\theta} = \begin{cases} \begin{bmatrix} e^{\widehat{\omega}\theta} & v\theta \\ 0 & 1 \end{bmatrix} & \omega = 0 \\ \begin{bmatrix} e^{\widehat{\omega}\theta} & (I - e^{\widehat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} & \omega \neq 0, \| \omega \| = 1 \end{cases}$$

Exponential Coordinates, Twist + **Screw (absolute)**



Case 1. $\omega \neq 0$

$$\begin{bmatrix} p(\theta) \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\widehat{\omega}\theta} & (I - e^{\widehat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

Case 2. $\omega = 0$: pure translation $h = \infty$

$$\begin{bmatrix} p(\theta) \\ 1 \end{bmatrix} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$, where v is a velocity vector and a unit vector.

p: the coordinate of the point of frame 0
ω: the coordinate for the rotation axis of frame 0
q: the coordinate for the center of the rotation of frame 0

h: pitch
θ: rotation angle

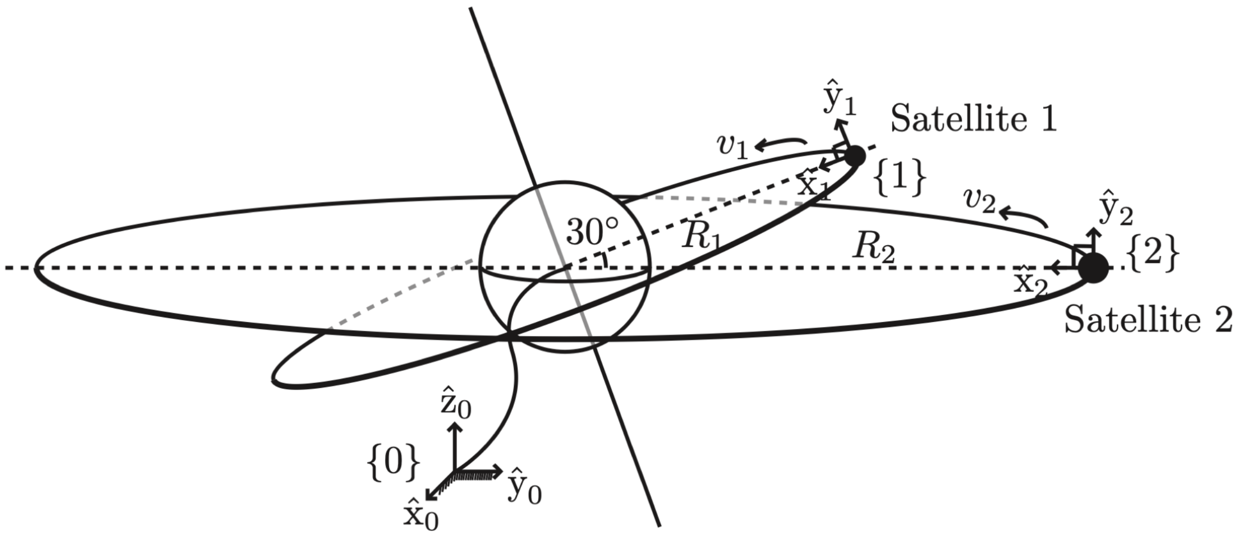
Here, we only have a single frame: frame 0.

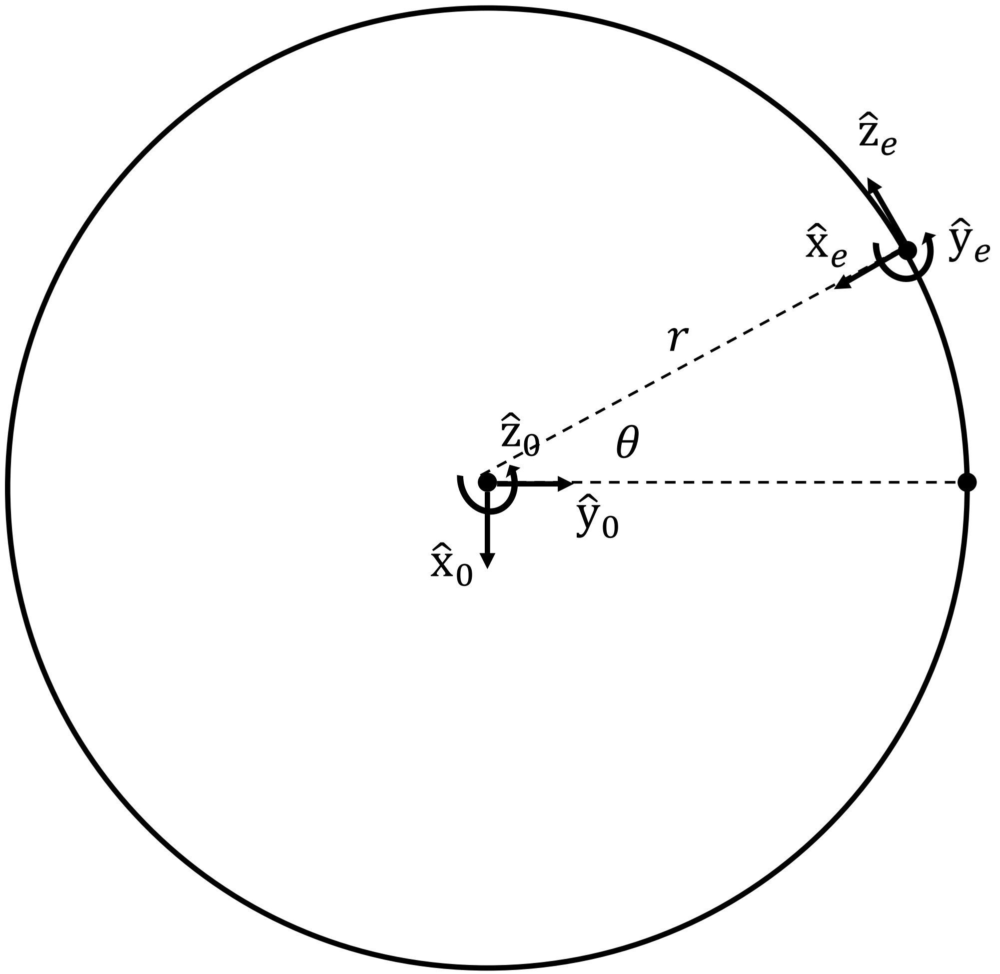
Exponential Coordinates, Twist + Screw (absolute)

	Exponential Coordinates, Twist	Screw
All v, ω, q are defined in the single frame: frame 0.	(v, ω, θ): exponential coordinate	ω, q, h, θ
$\omega \neq 0$	$\begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)q + h\theta\omega \\ 0 & 1 \end{bmatrix}$
$\omega = 0$	$\begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$

If $v = -\omega \times q + h\omega$,
the two transformations are the same.

HW1 Problem 4. Satellite System





Find g_{0e}?

Relative transformation: method 1

rotation θ and translation R (g_{01}) -> coordinate change (g_{1e})

$$g_{0e} = g_{01} g_{1e}$$

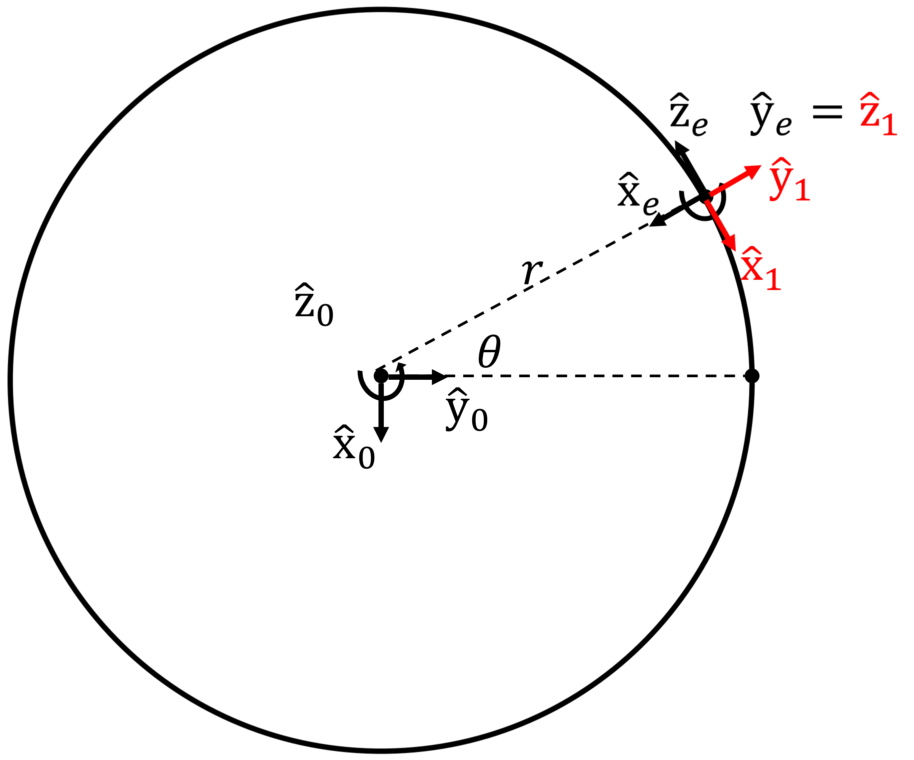
$$g_{1e} = \begin{bmatrix} R_{1e} & p_{1e} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If you want to find ω_{1e} and θ_{1e} use Rodrigues' formula.

Here, $\omega_{1e} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\theta_{1e} = \frac{2}{3}\pi$, $p_{1e} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are defined in frame 1.

$$g_{01} = \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & -r\sin\theta \\ \sin\theta & \cos\theta & 0 & r\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, $\omega_{01} = \hat{z}_0$, $\theta_{01} = \theta$, $p_{01} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix}$ are defined in frame 0.



Relative transformation: method 2

rotation θ (g_{01}) -> coordinate change and translation R (g_{1e})

$$g_{0e} = g_{01} g_{1e}$$

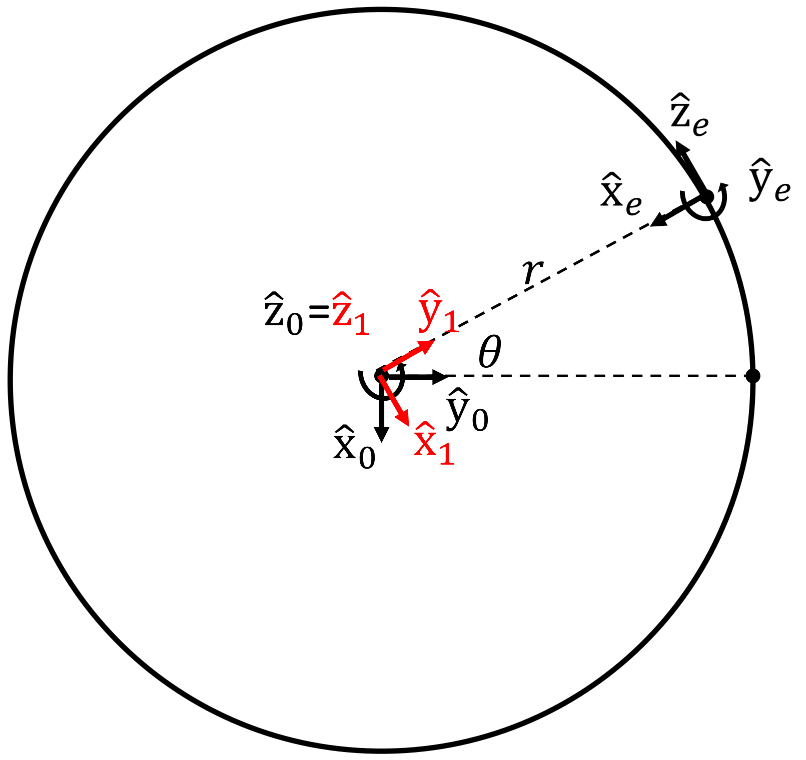
$$g_{1e} = \begin{bmatrix} R_{1e} & p_{1e} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If you want to find ω_{1e} and θ_{1e} use Rodrigues' formula.

Here, $\omega_{1e} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\theta_{1e} = \frac{2}{3}\pi$, $p_{1e} = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$ are defined in frame 1.

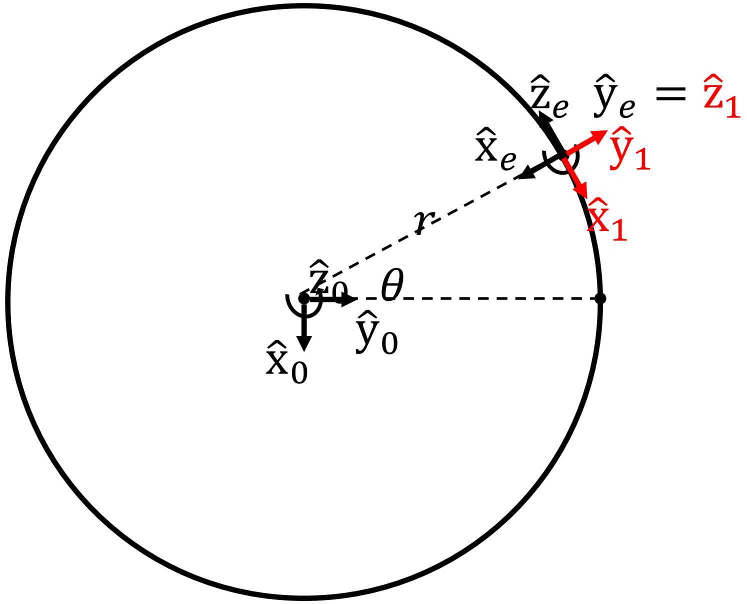
$$g_{01} = \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, $\omega_{01} = \hat{z}_0$, $\theta_{01} = \theta$, $p_{01} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are defined in frame 0.

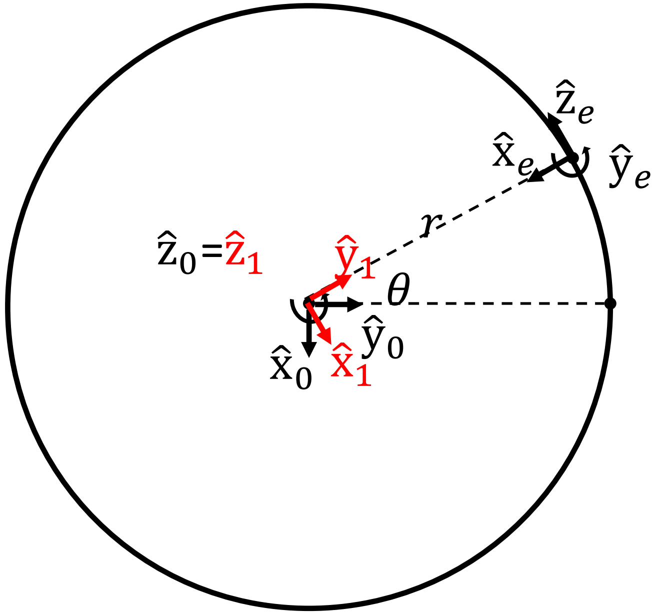


Relative transformation: other methods

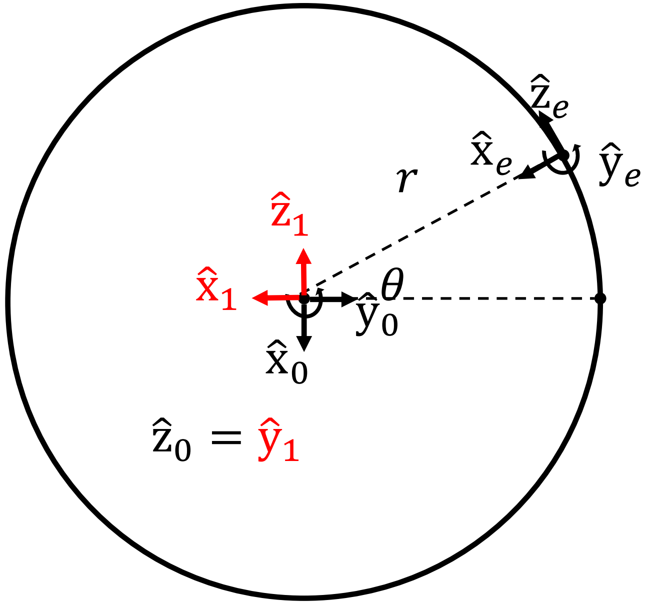
method 1



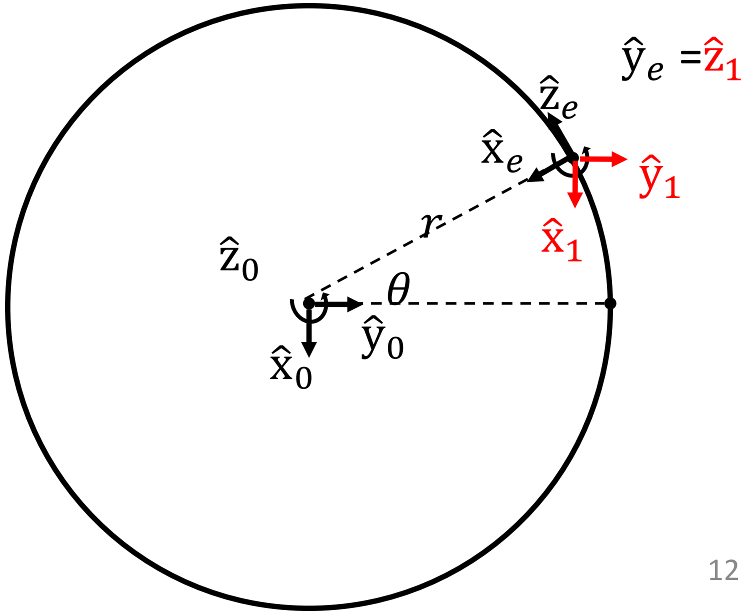
method 2



other method 1



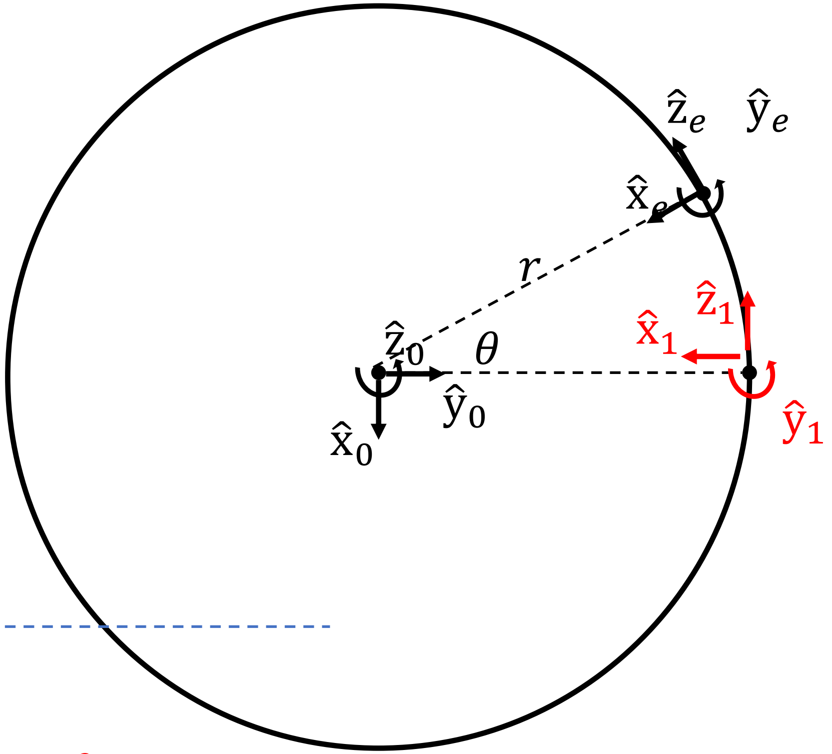
other method 2



Absolute transformation: method 3

translation R and coordinate change (g_1) -> rotation θ (g_2)

$$g_{0e} = g_2 g_1$$ Check the multiplication order.



$$g_1 = \begin{bmatrix} R_1 = e^{\hat{\omega}_1\theta_1} & p_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & r \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using exponential coordinate and screw is algebraically difficult here.

All R_1, ω_1, p_1 are defined in frame 0.

screw

$$g_2 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Warning: $R_2 \neq \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

$$R_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta_2 = \theta\right), q_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Then, } p_2 = (I - R_2)q_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

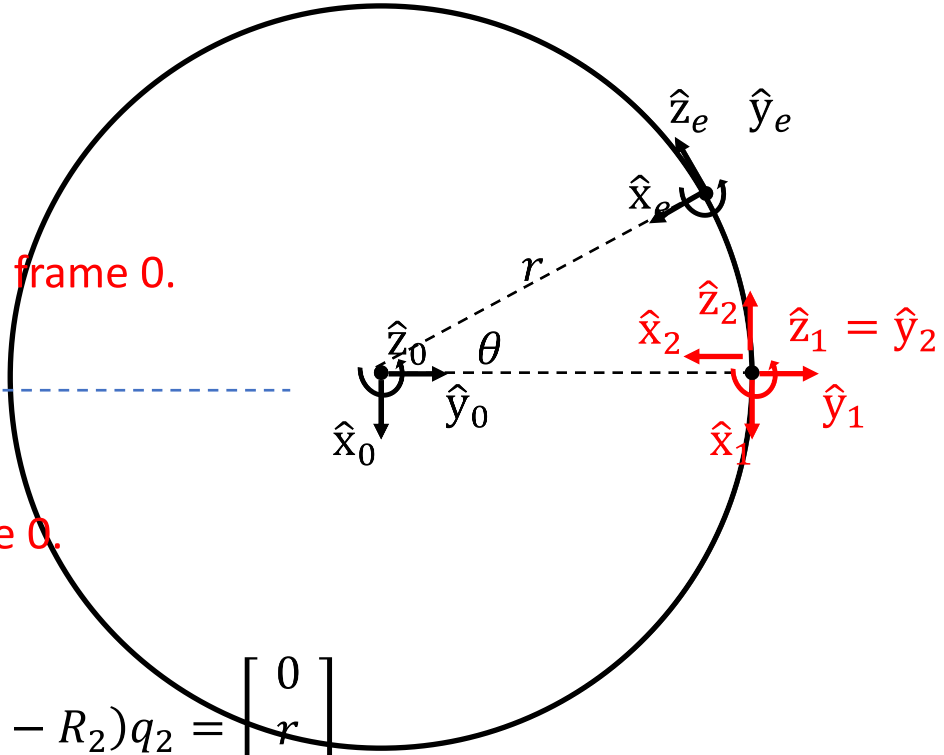
rotation axis　angle　Center of the rotation

All R_2, ω_2, p_2, q_2 are defined in frame 0.

Absolute transformation: method 4

translation R (g_1) -> coordinate change (g_2) -> rotation θ (g_3)

$$g_{0e} = g_3 g_2 g_1$$



$$g_1 = \begin{bmatrix} R_1 = e^{\hat{\omega}_1 \theta_1} & p_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, p_1 = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$$ are defined in frame 0.

screw

$$g_2 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & r \\ 0 & 1 & 0 & -r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All R_2, ω_2, p_2, q_2 are defined in frame 0.

$$R_2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \left(\omega_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \theta_2 = \frac{2}{3}\pi \right), q_2 = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}. \text{ Then, } p_2 = (I - R_2) q_2 = \begin{bmatrix} 0 \\ r \\ -r \end{bmatrix}$$

To find R_2, no need to fine ω_2, θ_2.

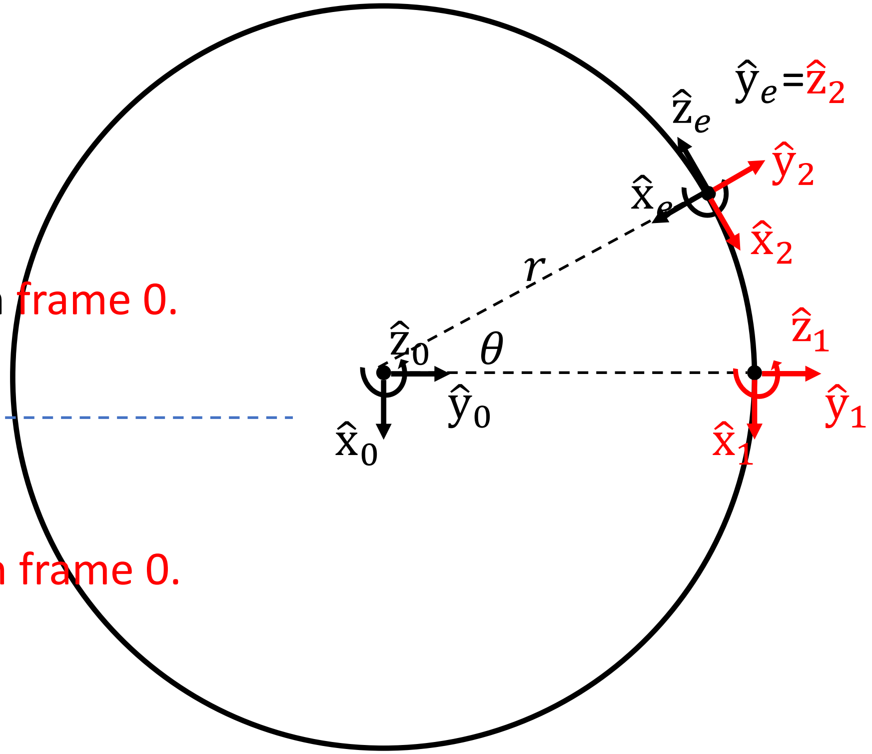
screw

$$g_3 = \begin{bmatrix} R_3 & p_3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Warning: $R_3 \neq \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ All R_3, ω_3, p_3, q_3 are defined in frame 0.

$$R_3 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta_3 = \theta \right), q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Then, } p_3 = (I - R_3) q_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Absolute transformation: method 5. This method requires a complex algebra. Take a look for practice.
translation R (g_1) -> rotation θ (g_2) -> coordinate change (g_3)



$$g_{0e} = g_3 g_2 g_1$$

$$g_1 = \begin{bmatrix} R_1 = e^{\hat{\omega}_1\theta_1} & p_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, p_1 = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix}$$ are defined in frame 0.

screw

$$g_2 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All R_2, ω_2, p_2, q_2 are defined in frame 0.

$$R_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \theta_2 = \theta\right), q_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Then, $p_2 = 0$

screw

$$g_3 = \begin{bmatrix} R_3 & p_3 \\ 0 & 1 \end{bmatrix}$$

Warning: $\omega_3 \neq \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$: coordinate in frame 2.

All R_3, ω_3, p_3, q_3 are defined in frame 0.

$$R_3 = \begin{bmatrix} \frac{\sin 2\theta}{2} & (\sin\theta)^2 & -\cos\theta \\ (\sin\theta)^2 - 1 & -\frac{\sin 2\theta}{2} & -\sin\theta \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}, \left(\omega_3 = \frac{1}{\sqrt{3}} R_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \theta_3 = \frac{2}{3}\pi\right), q_3 = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix}.$$ Then, $p_3 = (I - R_3)q_3$.

To find R_3, first find ω_3, θ_3.