Practice Midterm

Problem 2

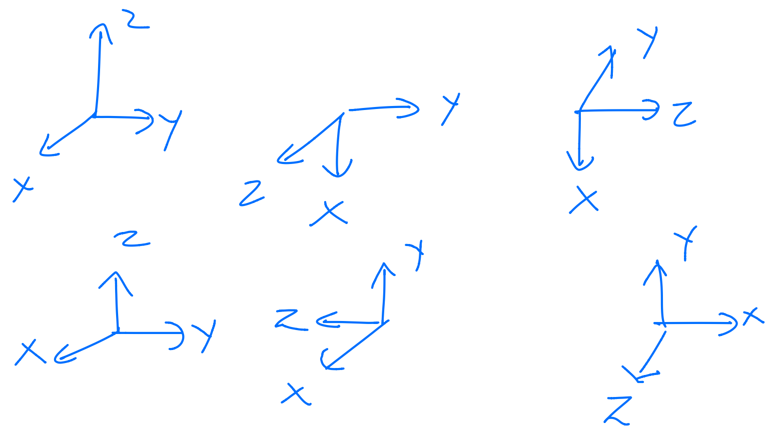
a) which operations are always commutative.

A, B are commutative $\iff AB = BA$

① R_1, R_2 about orthogonal axes. X

$R_x(\theta_1)\, R_y(\theta_2) \neq R_y(\theta_2)\, R_x(\theta_1)$

$R_x(\pi/2)\, R_y(\pi/2)$



② $R_\omega(\theta_1)\, R_\omega(\theta_2)$

$R_\omega(\theta_2)\, R_\omega(\theta_1)$

$\rightarrow R_\omega(\theta_1 + \theta_2)$ ✓

$R_1 = e^{\hat{\omega}\theta_1}$

$R_2 = e^{\hat{\omega}\theta_2}$

$R_1 R_2 = e^{\hat{\omega}(\theta_1+\theta_2)}$

③ $g_1 = \begin{bmatrix} I & p_1 \\ 0 & 1 \end{bmatrix}$ $g_2 = \begin{bmatrix} I & p_2 \\ 0 & 1 \end{bmatrix}$ ✓

$g_1 g_2 = g_2 g_1 = \begin{bmatrix} I & p_1 + p_2 \\ 0 & 1 \end{bmatrix}$

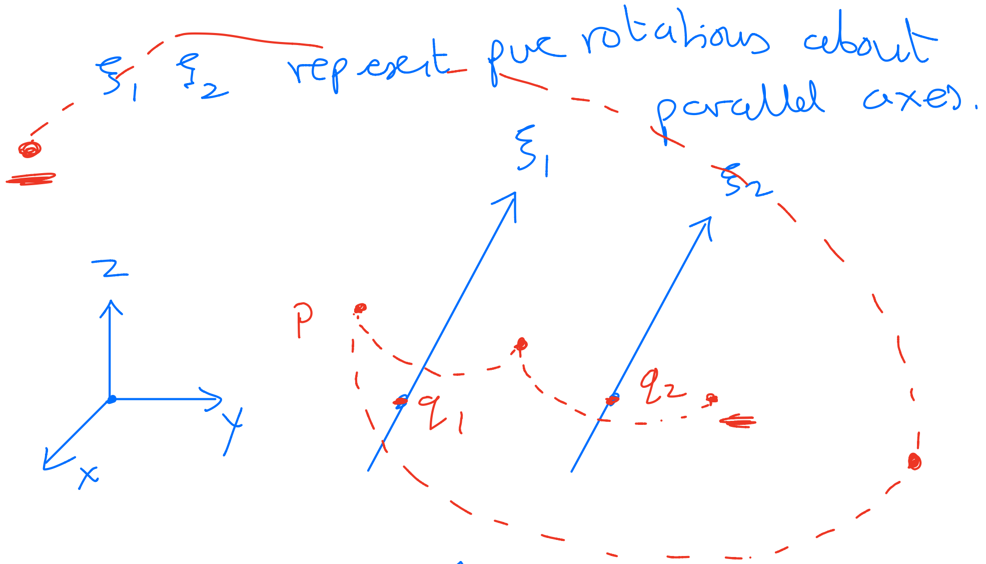
④ X $g_1 = \begin{bmatrix} R & p_1 \\ 0 & 1 \end{bmatrix}$ $g_2 = \begin{bmatrix} R & p_2 \\ 0 & 1 \end{bmatrix}$ $R = R_x(\pi/4)$

$$g_1 g_2 = \begin{bmatrix} \underline{R} & \underline{p_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{R} & \underline{p_2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^2 & Rp_2 + p_1 \\ 0 & 1 \end{bmatrix}$$

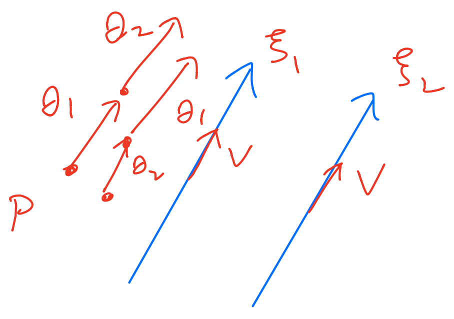
$$g_2 g_1 \qquad = \begin{bmatrix} R^2 & Rp_1 + p_2 \\ 0 & 1 \end{bmatrix}$$

5 ✗ $g_1 = e^{\hat{\xi}_1 \theta_1}$ $\quad g_2 = e^{\hat{\xi}_2 \theta_2}$ $\quad \begin{bmatrix} e^{\hat{\omega}\theta_1} & (I - e^{\hat{\omega}\theta_1}) q_1 \\ 0 & 1 \end{bmatrix}$

ξ_1, ξ_2 represent pure rotations about parallel axes.



6 ✓ $g_1 = e^{\hat{\xi}_1 \theta_1}$ $\quad g_2 = e^{\hat{\xi}_2 \theta_2}$ $\quad \xi_1, \xi_2$ are pure translations about parallel axes $\quad \omega = 0$



$$p' = p + v(\theta_1 + \theta_2)$$

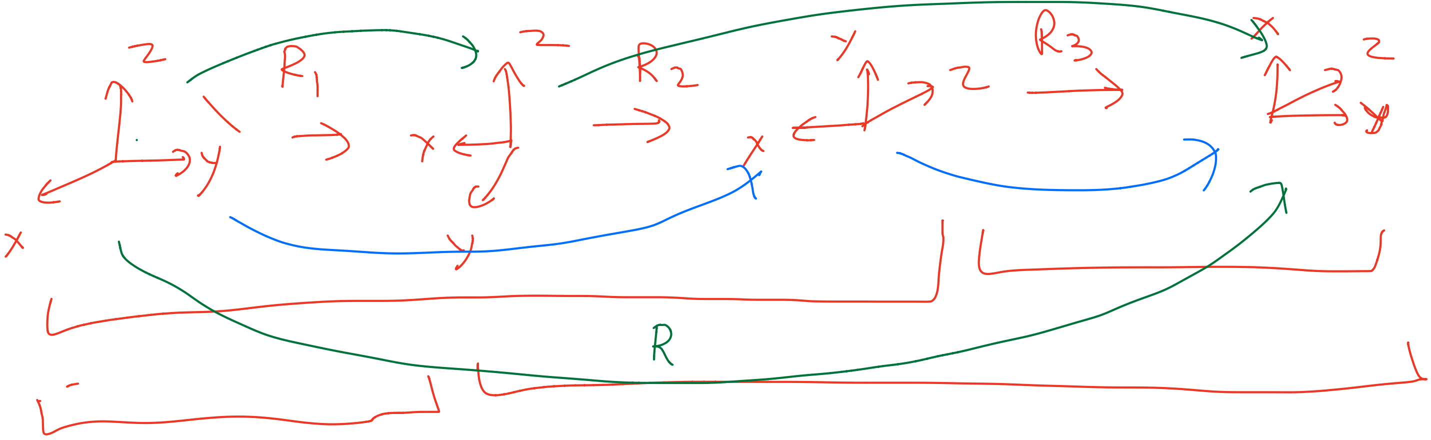
---

b) Are the operations associative

a composition of operations is associative $\Longleftrightarrow$

$$(AB)C = A(BC)$$

$\rightarrow$ all operations compose associatively.



$$R = R_1 R_2 R_3$$

$$(R_1 R_2) R_3 = R_1 (R_2 R_3)$$

c) $R_{AC} = R_{\vec{AB}} \, R_{\vec{BC}}$

$R_{AB} = R_{BA}^T \leftarrow \quad R_{AB} = R_{BA}^{-1}$

(i) $R_{CA} = R_{CB} R_{BA}$ $\qquad R_{AB}, R_{CB}$

$= R_{CB} R_{AB}^T$

(ii) $R_{BC} = R_{BA} \; R_{AC}$ $\qquad R_{AB}, R_{CA}$

$= R_{AB}^T \; R_{AC}^T$

(iii) $R_{AA} = I$

(iv) $R_{AC} = R_{AB} R_{BC}$ $\qquad R_{AB}^{-1}, R_{BC}^T$

$= (R_{AB}^{-1})^T \, (R_{BC}^T)^T$

Problem 7

$R = I + 2\hat{u}^2 \qquad u \in \mathbb{R}^3 \qquad \|u\| = 1$

a) $R^T R = I$

$$(I + 2\hat{u}^2)^T (I + 2\hat{u}^2)$$

$$= (I + 2\hat{u}^2)^2 = I + 4\hat{u}^2 + 4\hat{u}^4$$

$$= I + 4\hat{u}^2 + 4\hat{u}\,(\hat{u}^3) \qquad \underbrace{\hat{u}^3 = -\hat{u}}$$

$$= I + 4\hat{u}^2 - 4\hat{u}^2 = I \qquad \square$$

b) $\det(R) = 1$

$$f(u) = \underbrace{\det(I + 2\hat{u}^2)}$$

1. $f: \mathbb{R}^3 \to \mathbb{R}$
2. for all u, $f(u) = \pm 1$
3. f is continuous

$f(u) = \det(R)$

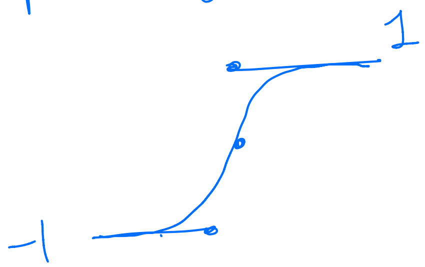
$R^T R = I$

$\Rightarrow \det(R) = \pm 1$

$\det(R^T R) = \det(I) = 1$

$\det(R)^2 = 1 \Rightarrow \det(R) = \pm 1$

Is it possible for f to take on both $+1$ and -1.



$\Rightarrow f$ must be constant: either always $+1$ or always -1.

$u = (1, 0, 0)$

$$f((1,0,0)^T) = \det(I + 2\hat{u}^2)$$

$$= \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 1 \Rightarrow (I+2\hat{u}^2) \text{ is}$$

always a rotation matrix.

c) Exponential coordinates of R? $\doteq (u, \underline{\pi})$

Hint: What does R look like from a new reference frame where u is the x-axis.

$R: P_A \rightarrow P_A'$

$R': P_B \rightarrow P_B'$ B has $\underline{u}$ for an x-axis

$$\boxed{R' = I + 2\hat{u}^2} \quad \text{where} \quad u = \underline{(1,0,0)^T}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \leftarrow R_x(\pi)$$

$$R = R(u, \pi)$$

exp - coords are (u, π)

A ← original ref frame

B ← is any new frame where u is chosen as x axis.

$$R_{BA}u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$R: P_A \mapsto P_A'$

$R': P_B \mapsto P_B'$

$$R' p_B = \underbrace{R_{BA} \underbrace{R \underbrace{R_{AB}\, p_B}_{p_A}}_{p'_A}}_{p'_B} \quad \Longrightarrow \quad R' = R_{BA}\, R\, R_{AB}$$

$$R' = R_{BA} \left(I + 2\hat{u}^2 \right) R_{AB}$$

$$= \cancel{R_{BA} R_{AB}}^{\,I} + 2\, R_{BA}\, \hat{u}^2\, R_{AB}$$

$$= I + 2 \left(\overbrace{R_{BA}\, \hat{u}\, R_{AB}} \right)^2$$

$$= I + 2 \left(\left(\underbrace{R_{BA} u} \right)^{\wedge} \right)^2$$

$$= I + 2 \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{\wedge} \right)^2$$

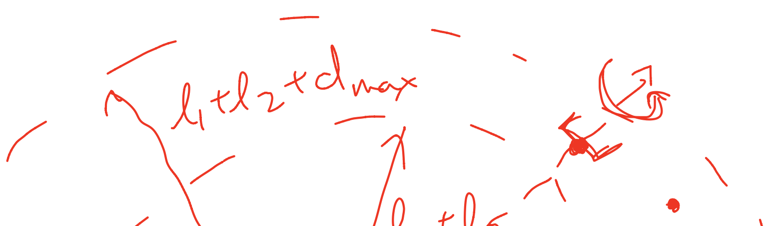
invertible g:

$$g A^2 g^{-1} = \left(g A g^{-1} \right)^2$$

$$R\, \underbrace{\hat{\omega}}\, R^T = \underbrace{(R\omega)}^{\wedge}$$

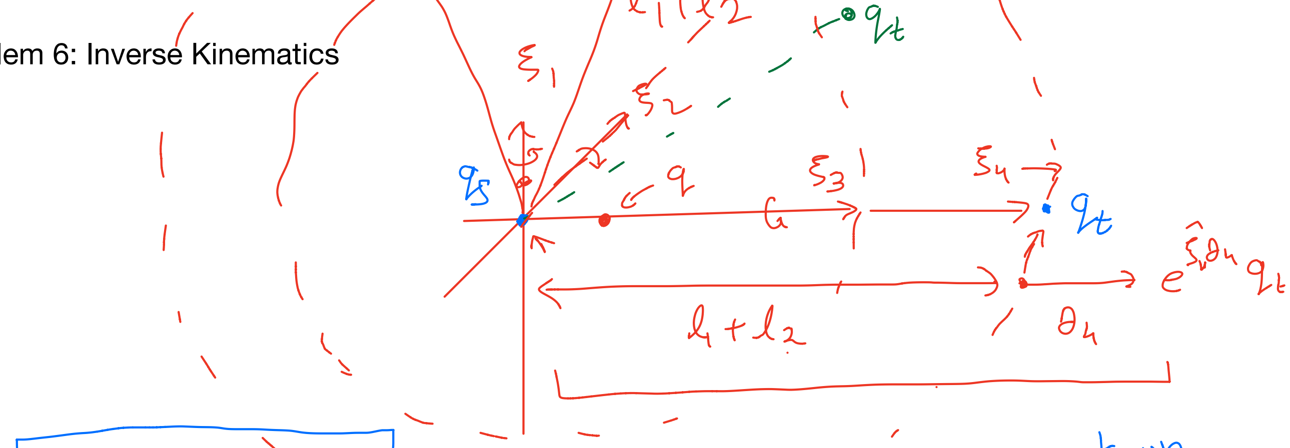
$$= I + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}^2 = I + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = R_x(\pi) \qquad \square$$



$d_4 \le d_{max}$

Problem 6: Inverse Kinematics



known

$$e^{\hat\xi_1\theta_1} e^{\hat\xi_2\theta_2} e^{\hat\xi_3\theta_3} e^{\hat\xi_4\theta_4} = g_d\, g_{st}^{-1}(0) := g \qquad \|gp\| = \|p\|$$

(1)
$$\Big\| \underbrace{e^{\hat\xi_1\theta_1} e^{\hat\xi_2\theta_2} e^{\hat\xi_3\theta_3}}_{\text{rigid body trasform}} \big(\underbrace{e^{\hat\xi_4\theta_4} q_t - q_s}_{\text{vector}}\big) \Big\| = \|g q_t - q_s\|$$

$$\|e^{\hat\xi_4\theta_4} q_t - q_s\| = \|g q_t - q_s\| := \delta$$

$$l_1 + l_2 + \theta_4 = \|g q_t - q_s\|$$

$$\Rightarrow \quad \theta_4 = \|g q_t - q_s\| - (l_1 + l_2)$$

unique solution! ≤ 1

we also know $g_4 = e^{\hat\xi_4\theta_4}$

$$e^{\hat\xi_1\theta_1} e^{\hat\xi_2\theta_2} e^{\hat\xi_3\theta_3} = g g_4^{-1} := g'$$

(2) pick a point q on ξ_3, NOT on axis 1,2.

intersecting axes.

$$\underbrace{e^{\hat\xi_1\theta_1} e^{\hat\xi_2\theta_2} q = g' q}_{\text{PK subproblem 2}}$$

Solve for (θ_1, θ_2) using PK2

≤ 2 solutions

also know $g_1 = e^{\hat{\xi}_1 \theta_1}$ $g_2 = e^{\hat{\xi}_2 \theta_2}$

(θ_1, θ_2)
$\updownarrow$ 2 sols
(θ_1', θ_2')

$$g_1 g_2 e^{\hat{\xi}_3 \theta_3} = g' \Rightarrow \underbrace{e^{\hat{\xi}_3 \theta_3}} = g_2^{-1} g_1^{-1} g' := \underbrace{g''}$$

③ pick any q <u>not</u> on ξ_3

$$\underbrace{e^{\hat{\xi}_3 \theta_3} q = g'' q}_{\text{PK subproblem 1}} \rightarrow \leq 1 \text{ solution}$$

<u>DONE</u>: $\theta_1 \theta_2 \theta_3 \theta_4$

a) reachable space: ~~spherical~~ annulus centered at q_s
of inner radius $l_1 + l_2$
outer radius $l_1 + l_2 + d_{max}$

dextrous space: ϕ

# of solutions $\leq 1 \times 2 \times 1 = 2$