LECTURE NOTES

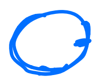
INVERSE KINEMATICS

$\theta \longrightarrow SE(3)$

n torus

$\theta \in S^1 \times S^1 \times \dots S^1 \rightarrow SE(3)$

$\theta = \theta_1 \dots \theta_n$

$\theta_1 \in [0, 2\pi]$ S^1  $0, 2\pi$ $S^1 \times S^1$ 

Torus

$g_{st}(\theta) : T^n \rightarrow SE(3)$

Find θ if they exist

$g = \begin{bmatrix} R & p \\ 0\,0\,0 & 1 \end{bmatrix}$

$R \in SO(3)$

$\|p\| \in [0, L_{max}]$

$0 \leq \|p\| \leq L_{max}$

$$e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} = g_{st}$$

How many solutions exist?

Total no. of Solutions 2^n, 2^{n-1}

$n = 6$ d.o.f.

$SE(3)$ dimension 6

$R \in SO(3)$ dimension 3

$p \in R^3$ dimension 3

$2^6 = 64$ ✗

32 ✗

RULING CONJECTURE

MAX. # of SOLUTIONS ≤ 16 $\boxed{\text{TRUE}}$

$N=6$ Third Geometric Inverse Kinematics $\longleftarrow$ V. Kahan, B. Paden

[Releaux] Design of Mechanism..

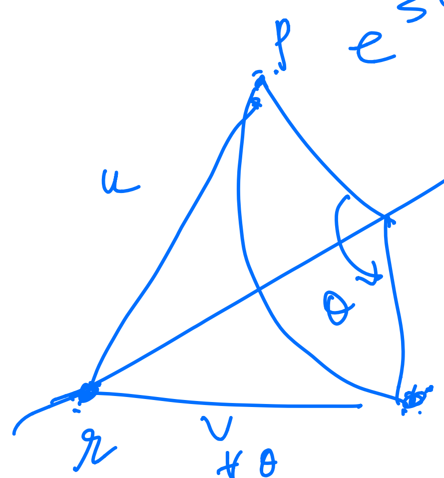
Paden Kahan Subproblems

(1) Given a screw ξ with zero pitch and two points $p, q \in \mathbb{R}^3$

Find, if possible, $\theta \in [0, 2\pi]$

$$e^{\hat{\xi}\theta} p = q$$

θ, $2\pi - \theta$ dont count as two



axis of $\xi = \omega$, $h = 0$

$\forall \theta$

$$e^{\hat{\xi}\theta} r = r$$

Choose $r \in$ axis of ξ

$p - r := u$

$q - r := v$.

$$e^{\hat{\xi}\theta} p - r = q - r$$

$$e^{\hat{\xi}\theta} p - \underbrace{e^{\hat{\xi}\theta} r} = q - r$$

$$e^{\hat{\xi}\theta} (p - r) = q - r$$

$(I - e^{\hat{\omega}\theta}) \omega \times v + \omega\omega^T v \theta$

$$\left[\begin{array}{c|c} e^{\hat{\omega}\theta} & x \\ \hline 0\ 0\ 0 & 0 \end{array}\right] \begin{bmatrix} p - r \\ 0 \end{bmatrix} = \begin{bmatrix} q - r \\ 0 \end{bmatrix}$$

$u = p - s$

$v = q - s$

$$\left[e^{\hat{\omega}\theta} u = v \right] \Rightarrow \omega^T u = \omega^T v$$

$$\omega^T e^{\hat{\omega}\theta} u = \omega^T v$$

$$\omega^T \left(I + \hat{\omega}\theta + \frac{\hat{\omega}^2 \theta^2}{2!} + \dots \right) u = \omega^T v$$

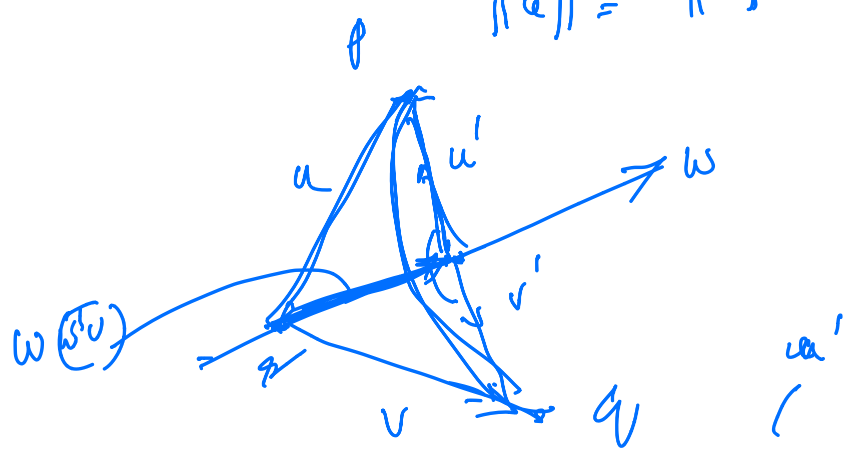
$$\underline{\omega^T u} = \underline{\omega^T v}$$ NEC. CONDITION (1)

$$\| e^{\hat{\omega}\theta} u \|^2 = \| v \|^2$$

$$u^T e^{\hat{\omega}^T \theta} e^{\hat{\omega}\theta} u = v^T v$$

$$u^T e^{-\hat{\omega}\theta} e^{\hat{\omega}\theta} u = v^T v$$

$$\| u \|^2 = \| v \|^2$$ NEC. CONDITION (2)



$\omega^T u = \omega^T v$

$\| u \|^2 = \| v \|^2$

$u' \perp \omega$

$\omega^T u'$

$\omega^T (I - \omega\omega^T) u$

$= (\omega^T - \omega^T \omega \omega^T) u$

$= (\omega^T - \omega^T) u$

$= 0$

$v' \perp \omega$

$$u = \omega \omega^T u + \underbrace{(I - \omega\omega^T) u}_{u'}$$

$$\| u \|^2 = (\omega^T u)^2 + \| u' \|^2$$

$$v = \omega \omega^T v + \overbrace{(I - \omega\omega^T) v}^{v'}$$

$$\| v \|^2 = (\omega^T v)^2 + \| v' \|^2$$

$\perp w$

$u = \_\_\, w + \_\_$

$= (w^T u) w + (I - w w^T) u$

$= \underbrace{w w^T u}_{\parallel w} + \underbrace{(I - w w^T) u}_{\perp w} \to u'$

$\|u\|^2 = (w^T u)^2 + \|u'\|^2$

$\|v\|^2 = (w^T v)^2 + \|v'\|^2$

$\|u\|^2 = \|v\|^2 = \cancel{(w^T u)^2} + \|u'\|^2 = \cancel{(w^T v)^2} + \|v'\|^2$

$\{ \|u\| = \|v\|, \ \|u'\| = \|v'\|, \ w^T u = w^T v \}$

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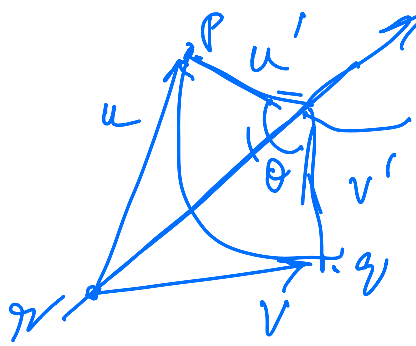


$w(w^T u) = w(w^T v)$

$\cos^{-1}\left(\frac{u'^T v'}{\|u'\|\|v'\|}\right)$

p, q

Choose on ξ axis

$u' \times v' = \|u'\|\|v'\| \sin\theta \, \underline{w}$

$u'^T v' = \|u'\|\|v'\| \cos\theta$

$\underline{w}^T (u' \times v') = \|u'\|\|v'\| \sin\theta \, w^T w$

$u'^T v' = \|u'\|\|v'\| \cos\theta$

Def

$u = p - r$

$v = q - r$

$\theta = \arctan 2\left(\frac{w^T (u' \times v')}{u'^T v'}\right)$

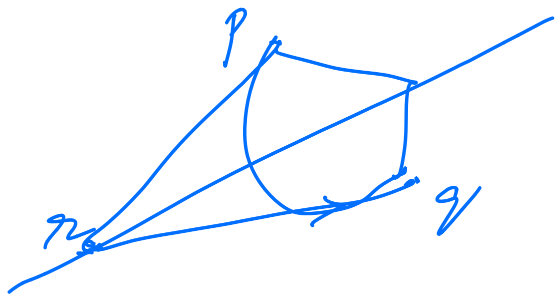
Check

$\left. \begin{matrix} w^T u = w^T v \\ \|u\|^2 = \|v\|^2 \end{matrix} \right\} \overset{\text{Iff}}{\Longrightarrow}$ Iff

$u' = u - w w^T u$

$v' = v - w w^T v$

" " $\Rightarrow$ no solution exists

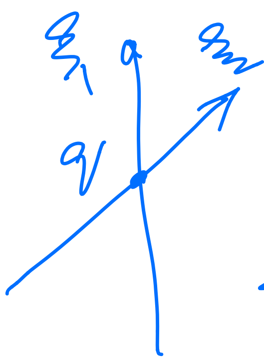


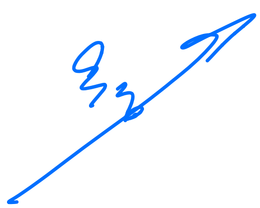
Cleverly!

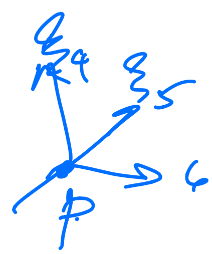
$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots \underbrace{e^{\hat{\xi}_6\theta_6} p} = g p$$

$p \in$ axis of ξ_6

$$e^{\hat{\xi}_6\theta_6} p = p$$







$p \in$ axis $\xi_4 \cap$ axis ξ_5 $\cap$ axis ξ_6

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \underbrace{e^{\hat{\xi}_4\theta_4} \cdots e^{\hat{\xi}_6\theta_6} p}_{p} = g p$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p = g p$$

$q \in$ axis $\xi_1 \cap$ axis ξ_2

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p - q = gp - q$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \left[e^{\hat{\xi}_3\theta_3} p - \underbrace{e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} q}_{} \right]$$

$$= gp - q$$

$$\underbrace{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2}}_{} \left[e^{\hat{\xi}_3\theta_3} p - q \right] = gp - q$$

$$\| e^{\hat{\xi}_3\theta_3} p - q \| = \underbrace{\| gp - q \|}_{} = d$$

SINGLE VARIABLE