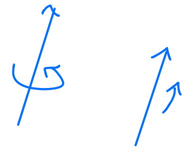
Discussion 4: Inverse Kinematics

Recall our problem set up. We are given a desired end effector configuration g_d (input) $\in SE(3)$, and we need to find $(\theta_1, ..., \theta_n)$ (output) such that

$$\underbrace{e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_n\theta_n}\, g_{st}(0)}_{g_{st}(\theta)} = g_d \quad (1)$$

known: $\hat{\xi}_1 \ldots \hat{\xi}_n$; solve for: $\theta_1 \ldots \theta_n$; known: $g_{st}(0)$; input: g_d



$$\underbrace{e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_n\theta_n}} = g_d\, g_{st}^{-1}(0) := \underbrace{g}_{\text{known}}$$

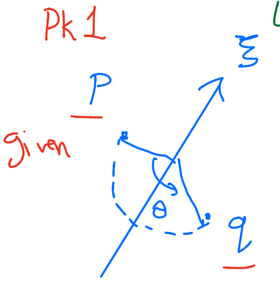
Paden-Kahan Subproblems

Can be used to simplify inverse kinematics for revolute joints when n <= 6.

Need to specify:

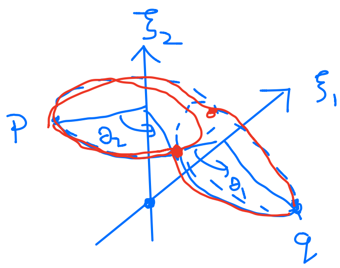
Subproblem 1:	$e^{\hat{\xi}\theta}p = q$	rotate one point onto another	(p, q)
Subproblem 2:	$\boxed{e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p = q}$	rotate about two intersecting axes	(p, q)
Subproblem 3:	$\boxed{\left\lVert e^{\hat{\xi}\theta}p - q\right\rVert = \delta}$	move one point to a specified distance from another	(p, q, δ)

PK1



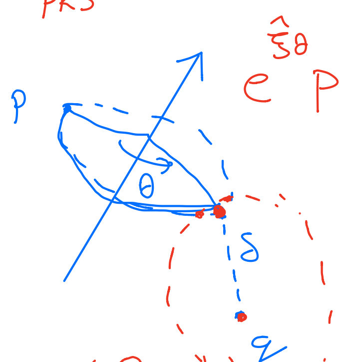
≤ 1 solution

PK2



≤ 2 solutions

PK3



≤ 2 solutions

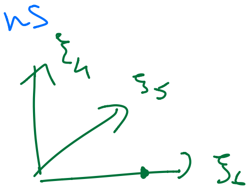
$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \cdots \underbrace{e^{\hat{\xi}_6\theta_6}} = g$$

Trick 1: to eliminate variables from the Right hand side of PoE

eg: $e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \underbrace{e^{\hat{\xi}_3\theta_3} p} = gp$

→ pick a point p on axis of joint 3: $e^{\hat{\xi}_3\theta_3}p = p$

$$\underbrace{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = gp}_{\text{pk2 with } q = gp} \leftarrow \text{only 2 unknowns}$$



Trick 2: Eliminate variables from left handside:

eg: $$e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g$$

pick a point q on axis of joint 2.

also pick p not on the axis of joint 3

$$e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p - q = gp - q$$

$$\|gp\| = \|p\|$$

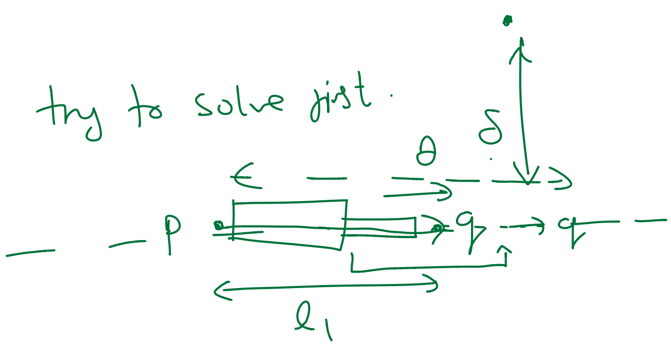
$$\Big\| \underbrace{e^{\hat{\xi}_2\theta_2}}_{\text{rigid body transform}} \big(\underbrace{e^{\hat{\xi}_3\theta_3} p - q}_{\text{vector}} \big) \Big\| = \|gp - q\|$$

$$\underbrace{\| e^{\hat{\xi}_3\theta_3} p - q \| = \| gp - q \|}_{\delta = \|gp - q\|}$$

Deal with prismatic joints?

$\xi \leftarrow$ prismatic joint try to solve first.

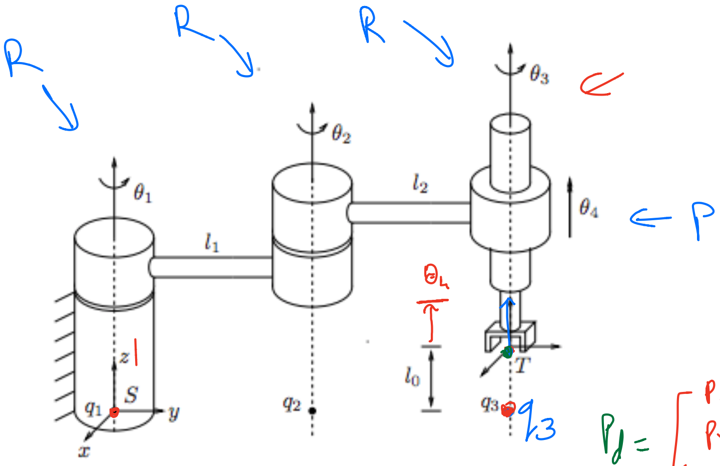
$$\| e^{\hat{\xi}\theta} p - q \| = \delta$$



unique solution by physics.

$$l_1 + \theta = \delta$$

$$\Rightarrow \theta = \delta - l_1$$



$$u^T v = \|v\|\|u\| \cos\theta$$

$$\|u \times v\| = \|v\|\|u\| \sin\theta$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\|u\times v\|}{u^T v}$$

$$P_d = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \qquad \begin{bmatrix} R_d & P_d \\ 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} = g_d\, g_{st}^{-1}(0) := g$$

① start with θ_4: $\underbrace{P_z = l_0 + \theta_4}$ $\Rightarrow$ $\theta_4 = P_z - l_0$ ✓

θ_4 uniquely determines Z component

How many θ_4?

$\times$ 1 sol.

$$g_4 = e^{\hat{\xi}_4\theta_4} \leftarrow \text{known}$$

$$\underbrace{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g\, g_4^{-1} := g'}$$

② pick q_3

$$\underbrace{e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} q_3} = g' q_3$$

Trick 2: pick q_1

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} q_3 - q_1 = g' q_3 - q_1$$

$$\rightarrow \quad e^{\hat{\xi}_1\theta_1}\left(e^{\hat{\xi}_2\theta_2} q_3 - q_1\right) = g' q_3 - q_1$$

$$\rightarrow \quad \|e^{\hat{\xi}_2\theta_2} q_3 - q_1\| = \|g' q_3 - q_1\|$$

PK3 with $\delta = \| g' q_3 - q_1 \|$

$p = q_3$

$q = q_1$

found θ_2 using PK3 ✓ $\times 2$ sol.

we know $g_2 = e^{\hat{\xi}_2 \theta_2}$

$$e^{\hat{\xi}_1 \theta_1} g_2 \, e^{\hat{\xi}_3 \theta_3} = g'$$

(3) pick q_3 again

$$\underbrace{e^{\hat{\xi}_1 \theta_1} \boxed{(g_2 q_3)} = (g' q_3)}$$

PK1 $p = g_2 q_3$ $\times 1$ sol

$q = g' q_3$

use PK1 to find θ_1

Now, $g_1 = e^{\hat{\xi}_1 \theta_1}$ is known.

$$\underbrace{g_1 g_2} e^{\hat{\xi}_3 \theta_3} = g'$$

$$e^{\hat{\xi}_3 \theta_3} = g_2^{-1} g_1^{-1} g' := g''$$

$$e^{\hat{\xi}_3\theta_3} = g''$$

(4) Let p be any point <u>not</u> on joint 3.

$$\underbrace{e^{\hat{\xi}_3\theta_3} \underbrace{p}_{} = \underbrace{g'' p}_{}}_{}$$

$\times 1$ sol.

PK 1. use to solve θ_3

DONE: θ_1 θ_2 θ_3 θ_4

$1 \times 2 \times 1 \times 1 = 2$ solutions

<u>at most</u>.