EECS/BioE C106A/206A Introduction to Robotics

Lost Section 7

Nov 13 Fri 7 – 9 PM

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Summary

- Vision
- Velocities (twist X magnitude), twists, and Jacobians
- Singularity
- Dynamics using Lagrangian (Kinetic energy – Potential energy)
- Control: computed torque method

Linear Algebra and Adjoint Matrix

- $\xi = (v, \omega)^{\mathrm{T}} \in \mathbb{R}^6$
- $\hat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$

- $\mathrm{Ad}_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$
- $g\hat{\xi}g^{-1} = \left(\mathrm{Ad}_g \xi\right)^{\wedge}$
- $\left(g\hat{\xi}g^{-1}\right)^{\vee} = \mathrm{Ad}_g \xi$
- $\mathrm{Ad}_{g^{-1}} = \left(\mathrm{Ad}_g\right)^{-1}$
- $\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1}\,\mathrm{Ad}_{g_2}$
- $V^s = \mathrm{Ad}_g V^b$
- $J^s = \mathrm{Ad}_g J^b$

1. Vision

Homogeneous coordinate: $x = [u, v, 1]^{\mathrm{T}} \in \mathbb{R}^3$

Spatial coordinate of the point: $X \in \mathbb{R}^3, \lambda \in \mathbb{R}$,

$$\lambda x = KX,$$

where K is the camera matrix. If the camera is calibrated and normalized,

$$K = I, \qquad \text{and} \qquad \lambda x = X.$$

Least square solution: $A \in \mathbb{R}^{m \times n}, \bar{\lambda} \in \mathbb{R}^n$,

$$A\bar{\lambda} = b$$

Case 1. $m = n$ and A is invertible:

there is a unique solution.

Case 2. $m < n$:

there are infinite solutions.

Case 3. $m > n$:

The problem is over-constrained: the number of solutions could be 0, 1, or ∞.

For Case 3, we can use the least square method: find $\bar{\lambda}$ such that

$$\min_{\bar{\lambda}} \left(A\bar{\lambda} - b\right)^2 .$$

The solution for $\bar{\lambda}$ is

$$\bar{\lambda} = \left(A^{\mathrm{T}}A\right)^{-1} A^T b.$$

1. Vision

The essential matrix and epipolar constraint

Consider two cameras:

$x_1, x_2 \in \mathbb{R}^3$: the homogeneous coordinates of the projection of the same point p onto the two cameras.

$X_1, X_2 \in \mathbb{R}^3$: the coordinates of the same point p relative to the two camera frames.

$g = (R, T)$: the rigid-body transformation from frame 1 to 2

Suppose the two cameras are calibrated and normalized.

$$\lambda_1 x_1 = X_1 \text{ and } \lambda_2 x_2 = X_2.$$

$$X_2 = RX_1 + T.$$

Essential matrix: $E = \hat{T}R$

Epipolar constraint $(0 = x_2^{\mathrm{T}} E x_1)$: to find a constraint without λ_1, λ_2

Proof.

$$\lambda_1 x_1 = X_1 \text{ and } \lambda_2 x_2 = X_2.$$
$$X_2 = RX_1 + T.$$

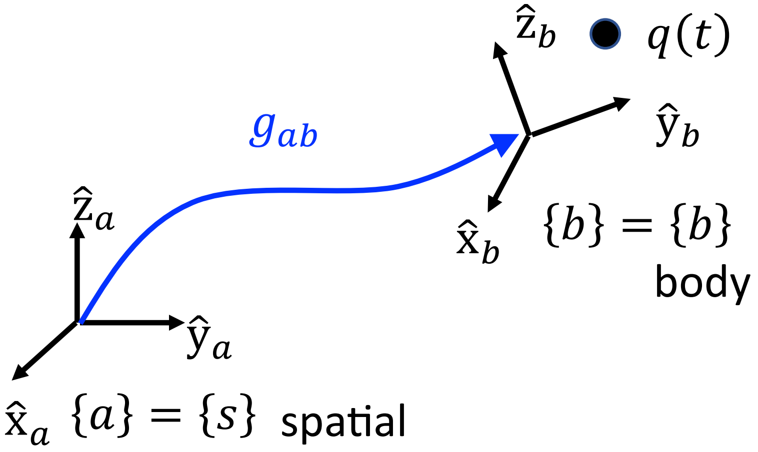
Then, we have $\lambda_1 x_1 = R\lambda_2 x_2 + T$. By multiplying $\hat{T}$ on the both side,

$$\lambda_1 \hat{T} x_1 = \hat{T} R \lambda_2 x_2.$$

Since $x_1 \perp \hat{T} x_1$, by multiplying x_1^{T} on the both side,

$$0 = x_2^{\mathrm{T}} E x_1.$$

2. Velocities (twist X magnitude), Twists, and Jacobians



$q(t)$: moving point

$q_a(t)$: coordinate of the point w.r.t. $\{a\}$
q_b: coordinate of the fixed point w.r.t. $\{b\}$

$v_{q_a}(t)$: coordinate of the velocity of the point w.r.t. $\{a\}$
$v_{q_b}(t)$: coordinate of the velocity of the point w.r.t. $\{b\}$

Definition of the spatial velocity:

$$v_{q_a}(t) = \hat{V}_{ab}^s q_a(t)$$

In the book, we have $\hat{V}_{ab}^s = \dot{g}_{ab} g_{ab}^{-1}$.

Definition of the body velocity:

$$v_{q_b}(t) = \hat{V}_{ab}^b q_b$$

In the book, we have $\hat{V}_{ab}^b = g_{ab}^{-1} \dot{g}_{ab}$.

$$V_{ab}^s = \mathrm{Ad}_{g_{ab}} V_{ab}^b$$

2. Velocities (twist X magnitude), Twists, and Jacobians

HW 6. Problem 3

Problem 3. Velocities as Twists

Consider a smooth rotational trajectory $R(t) \in SO(3)$ where $t \in [0, \infty)$ denotes time. In this problem, we will derive the notion of angular and rigid body velocities directly from our knowledge of exponential coordinates.

(a) Let $t \in [0, \infty)$ and a small $\Delta t > 0$ be given. Argue that there exists $\hat{\omega} \in \mathfrak{so}(3)$ such that

$$R(t + \Delta t) = e^{\hat{\omega}\Delta t} \cdot R(t) \tag{10}$$

Note that ξ is a function of both t and Δt.

(b) Now take the limit as $\Delta t \to 0$. Show that in this limit, $\hat{\omega}$ approaches $\dot{R}R^T = \hat{\omega}^s(t)$. That is, in the limit, this infinitessimal rotation approaches the spatial angular velocity of R. *Hint: It may help to recall that for small* Δt *we have* $e^{A\Delta t} \approx I + A\Delta t$. *It may also help to recall the limit definition of the derivative.*

(c) Conclude that the spatial angular velocity of R is simply the *instantaneous rotation axis* of the body, with magnitude equal to the instantaneous angular speed.

(d) Repeat the exercise in parts (a)-(c) except with a smooth *rigid-body motion* trajectory $g(t) \in SE(3)$. Interpret the spatial velocity $V^s(t)$ in terms of the twist associated with the *instantaneous screw motion* that the body is undergoing at time t.

This ω is not normalized one, and it already has its magnitude.

$$\hat{\omega} \to \dot{R}R^T$$

$$\hat{\xi} \to \dot{g}g^{-1}$$

The twist can be easily found from the screw motion.

$$\xi = (v, \omega), \qquad v = -\omega \times q$$

2. Velocities (twist X magnitude), Twists, and Jacobians

The spatial velocity and Jacobian: $V^s(=\dot{g}g^{-1}) = J^s\dot{\theta}$

The body velocity and Jacobian: $V^b(=g^{-1}\dot{g}) = J^b\dot{\theta}$

$$V_{ab}^{s} = \mathrm{Ad}_{g_{ab}} V_{ab}^{b}$$

$$J^{s} = \mathrm{Ad}_{g_{ab}} J^{b}$$

2. Velocities (twist X magnitude), Twists, and Jacobians

The spatial velocity and Jacobian: $V^s(=\dot{g}g^{-1}) = J^s\dot{\theta}$

The body velocity and Jacobian: $V^b(=g^{-1}\dot{g}) = J^b\dot{\theta}$

Open chain manipulator $\quad g = e^{\hat{\xi}_1\theta_1}\dots e^{\hat{\xi}_n\theta_n}g(0)$

$$J^s = [\xi_1', \dots, \xi_n']$$

General theory:

$$\hat{V}^s = \dot{g}g^{-1} = \sum_{i=1}^{n}\left(\frac{\partial g}{\partial\theta_i}\dot{\theta}_i\right)g^{-1}$$

$$= \begin{bmatrix}\frac{\partial g}{\partial\theta_1}g^{-1} & \frac{\partial g}{\partial\theta_2}g^{-1} & \cdots & \frac{\partial g}{\partial\theta_n}g^{-1}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \vdots\\ \dot{\theta}_n\end{bmatrix}$$

$$V^s = \underbrace{\begin{bmatrix}\left(\frac{\partial g}{\partial\theta_1}g^{-1}\right)^{\vee} & \left(\frac{\partial g}{\partial\theta_2}g^{-1}\right)^{\vee} & \cdots & \left(\frac{\partial g}{\partial\theta_n}g^{-1}\right)^{\vee}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \vdots\\ \dot{\theta}_n\end{bmatrix}}_{J^s = [\xi_1' \quad \xi_2' \quad \cdots \quad \xi_n']}$$

$$\xi_1' = \xi_1, \qquad \xi_2' = \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}}\xi_2, \qquad \dots, \qquad \xi_n' = \mathrm{Ad}_{e^{\hat{\xi}_1\theta_1}\dots e^{\hat{\xi}_{n-1}\theta_{n-1}}}\xi_n$$

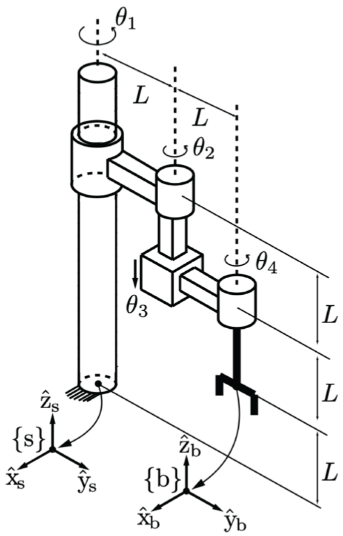
i-th column of Jacobian is the contribution of i-th joint (the twist of i-th joint).

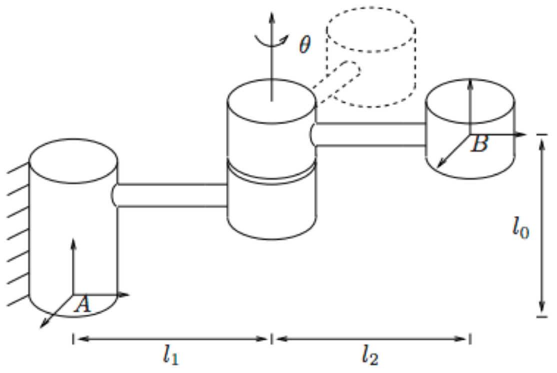
Find a twist using a screw motion:

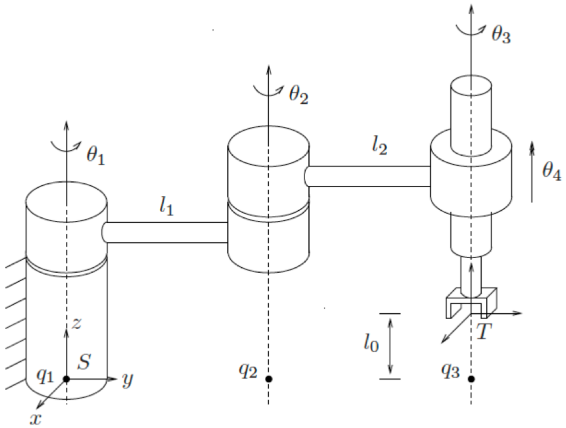
for a revolute joint i, find the rotation direction $\omega_i' = \omega_i'(\theta_1, \dots, \theta_{i-1})$ and the center of the rotation $q_i' = q_i'(\theta_1, \dots, \theta_{i-1})$.
for a prismatic joint i, find the translational direction $v_i' = v_i'(\theta_1, \dots, \theta_{i-1})$.

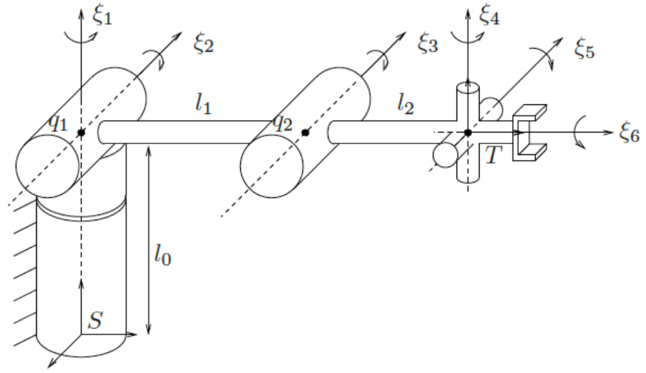
2. Velocities (twist X magnitude), Twists, and Jacobians

How to find Jacobian for the following examples?









Find a twist using a screw motion:

for a revolute joint i, find the rotation direction $\omega_i' = \omega_i'(\theta_1, \dots, \theta_{i-1})$ and the center of the rotation $q_i' = q_i'(\theta_1, \dots, \theta_{i-1})$.

for a prismatic joint i, find the translational direction $v_i' = v_i'(\theta_1, \dots, \theta_{i-1})$.

$$J^s = \begin{bmatrix} \boxed{\begin{matrix} -\omega_1' \times q_1' \\ \omega_1' \end{matrix}} & \boxed{\begin{matrix} v_2' \\ 0 \end{matrix}} & \cdots & \begin{matrix} -\omega_n' \times q_n' \\ \omega_n' \end{matrix} \end{bmatrix}$$

revolute joint prismatic joint

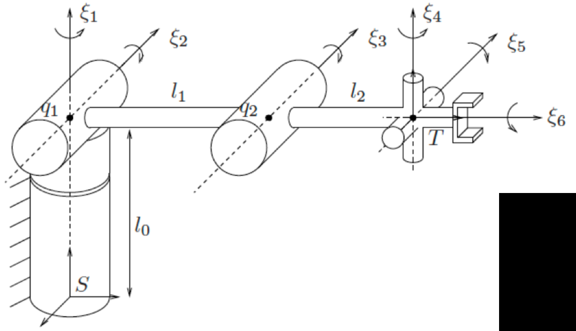
3. Singularity

The Jacobian J^s (or J^b) is singular if J^s is not full rank.

$$\mathrm{Rank}(J^s) = \mathrm{Rank}(J^b)$$

This is because $J^s = \mathrm{Ad}_{g_{ab}} J^b$ and $\mathrm{Ad}_{g_{ab}} \in \mathbb{R}^{6\times 6}$ is always invertible.

$$\mathrm{Rank}(J^s) = \mathrm{Rank}(\bar{J})$$



$\bar{J}$: Jacobian for the velocity of the end effector

$$\begin{bmatrix} v^s(\theta) \\ \omega^s(\theta) \end{bmatrix} = \bar{J}(\theta)\dot{\theta}$$

Find a set of θ such that two axes are on the same line

At the end, I suggest you to prove the columns of the Jacobian is linearly dependent.

4. Dynamics using Lagrangian

$$L = T - V$$

Kinetic energy

$$T = \frac{1}{2} m \|\dot{x}\|^2 + \frac{1}{2} I \dot{\theta}^2$$

Note that $I = 0$ for point mass

Potential energy

$$V = mgx + \frac{1}{2} k (x - x_0)^2$$

$$\Upsilon = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}$$

For angle x_i, Υ_i is a joint torque acting on the i-th axis or body.

For position x_i, Υ_i is a joint force acting on the i-th axis.

5. Control: computed torque method

Dynamics: $M(q)\ddot{q} + C(\dot{q}, q)\dot{q} + N(q) = u$

Control goal: make system to track the desired motion. $q(t) \to q_d(t)$

Computed torque method:

$$u = M(\theta)\left(\ddot{q}_d - K_v\dot{e} - K_p e\right) + C(\dot{q}, q)\dot{q} + N(q),$$

where $e \coloneqq q - q_d$.

Dynamics for the error (e): $\ddot{e} + K_v\dot{e} + K_p e = 0$.

5. Control: computed torque method

Dynamics for the error ($e \in \mathbb{R}^n$): $\ddot{e} + K_v\dot{e} + K_p e = 0.$

Stability analysis for diagonal K_v and K_p.

Consider the Characteristic equation ($\lambda \in \mathbb{R}$)

$$\lambda^2 + K_v\lambda + K_p = 0 \ (K_v, K_p \geq 0)$$

If $\mathrm{Re}(\lambda_{1,2}) < 0$, the system is stable.

$$e(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Case 1. $K_v = 0$

$\lambda_{1,2} = \pm\sqrt{K_p}j$: unstable

Note that $\sqrt{K_p} = 2\pi f$ where f is the frequency and $f = \frac{1}{T}, T\text{: period.}$

Case 2. $K_p = 0$

$\lambda_1 = 0, \lambda_2 = -K_v < 0$: unstable

Case 3. $K_v, K_p > 0$

$\mathrm{Re}(\lambda_{1,2}) < 0$: always true (proved in the textbook)

5. Control: computed torque method

Dynamics for the error ($e \in \mathbb{R}^n$): $\ddot{e} + K_v \dot{e} + K_p e = 0.$

Stability analysis for diagonal K_v and K_p.

Consider the Characteristic equation ($\lambda \in \mathbb{R}$)

$$\lambda^2 + K_v \lambda + K_p = 0 \; (K_v, K_p \geq 0)$$

If $\mathrm{Re}\left(\lambda_{1,2}\right) < 0$, the system is stable.

$$e(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

If λ_1 contains an imaginary number: $\lambda_1 = a + bj \; (a, b \in \mathbb{R})$,

$$\lambda_2 = \bar{\lambda}_1 = a - bj$$

If $b = 0$ (iff $K_v^2 - 4K_p \geq 0$), there is no oscillation.

If $b \neq 0$ (iff $K_v^2 - 4K_p < 0$), there is oscillation.