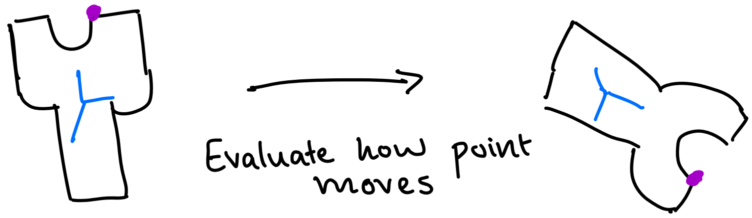
A Summary



- **Rigid body transformations** preserve orientation and direction
- They're affine transformations (Rx + p), rotation then translation
- Points can translate, but vectors simply rotate (since they only represent direction)

- **Homogeneous coordinates** can help us represent this movement

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

- Now we can represent rigid transformations for both points and vectors using a single matrix (convert from affine form to linear form)

$$\bar{q}_a = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix} = g_{ab} \bar{q}_b$$

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- Can stack and invert

$$g_{AC} = g_{AB} g_{BC}$$

$$g_{AC} = g_{CA}^{-1}$$

$$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

- If we want to parametrize our motion by time, then we can use **exponential coordinates** to generate our transformation matrices

- Create rotation matrix:

$$R(t) = e^{\hat{\omega} t}$$

ω = axis of rotation

* Same as the Rodrigues Formula

- Can also create homogeneous transformation matrix
- Use the twist (both linear and angular velocity)

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6 \qquad \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

  - Pure rotation (revolute joint)

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

  - Pure translation (prismatic joint)

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

  - Rotation and translation (screw)

$$\xi = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}$$

Exponential coordinates:

(ξ, θ)

ξ → How we're rotating

θ → How much we have rotated

$$g = e^{\hat{\xi}\theta}$$

$$p(t) = e^{\hat{\xi} t} p(0)$$

$$e^{\hat{\xi}\theta} = \begin{cases} \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} & \omega = 0 \\ \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} & \omega \neq 0,\ ||\omega|| = 1 \end{cases}$$

Discussion 2: Exponential Coordinates

Tarun Amarnath

Announcements:

- **Homework 2** released! Due on Tuesday
  - Much longer than HW 1, start early!
  - About what to expect for the rest of the semester
- **Lab 1** this week, **Lab 2** next week
- **Thursday discussions** now **in-person** because Wi-Fi garbo
  - We'll still post recordings

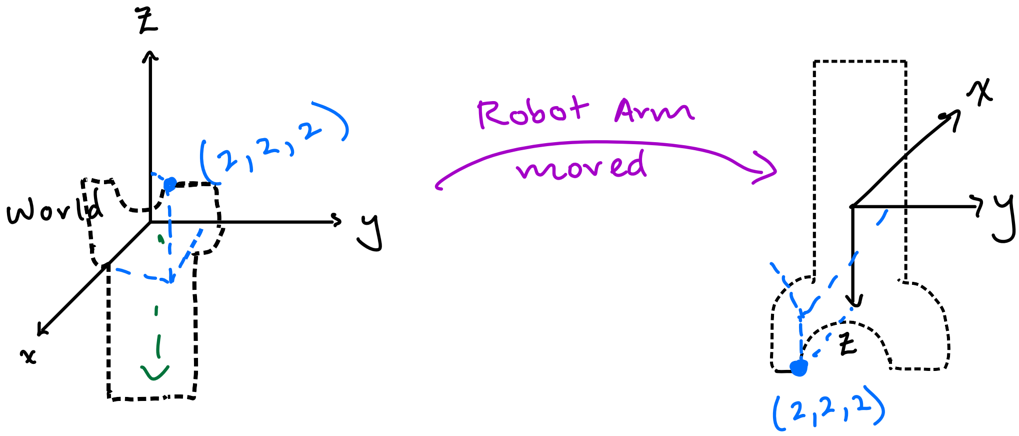
1. Rigid Body Transformations

- **Length-Preserving** $\|p-q\| = \|g(p)-g(q)\|$
  - **All points stay the same distance** from each other
- **Orientation-Preserving**
  - **Points don't switch positions** $g(v \times w) = g(v) \times g(w)$
  - Same angle relative to each other
  - If your camera is on the top of your phone, it stays on the top
- In other words, a rigid body stays rigid. It's a solid solid.
- **Rotations** are rigid body transformations

Rigid Transformation of a Point

- We can move and rotate a coordinate frame
- Points on that frame move and rotate with it

- Ex. Robot arm: flips upside down and moves by 1 unit in the y-direction



- Moved by +1 in y-direction
- z-axis flipped

z = -3

$$p_{AB} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad R_{AB} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$g_{ab}\, q = p_{ab} + R_{ab}\, q_b$$

$$g_{ab} \cdot q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

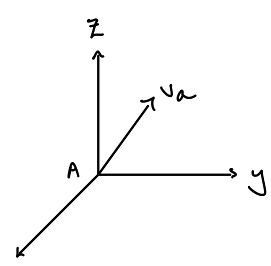
$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$$

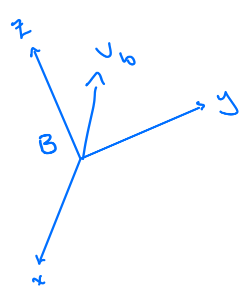
$$= \boxed{\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}}$$

- Affine transformation: f(x) = Mx + b
  - M is linear ~> not dependent on x
  - b is in the space of Y

Rigid Transformation of a Vector

- Just a rotation
- Vectors only have direction, no positional information





$$g(v) = g(s-r) = g(s) - g(r)$$

$$= \cancel{p_{ab}} + R_{ab} \cdot s - \cancel{p_{ab}} - R_{ab} \cdot r$$

$$= R_{ab} \cdot s - R_{ab} \cdot r$$

$$= R_{ab} \cdot (s - r)$$

$$= R_{ab} \cdot \vec{v}$$

Homogeneous Coordinates

- Can be used with both points and vectors
  - 4-dimensional array

Point: $q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix}$ Vector: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$

- Combine rotation and translation

$$g_{ab} = \left[\begin{array}{c|c} R_{ab} \in \mathbb{R}^{3\times3} & p_{ab} \in \mathbb{R}^{3\times1} \\ \hline 0 & 1 \end{array}\right] \in \mathbb{R}^{4\times4}$$

$p_{ab} + R_{ab} \cdot q_b = g_{ab} \cdot q_b$

- Ex. Flip z-axis and move in the +x direction by 1 unit

$$g_{ab} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad q_b = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_b = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

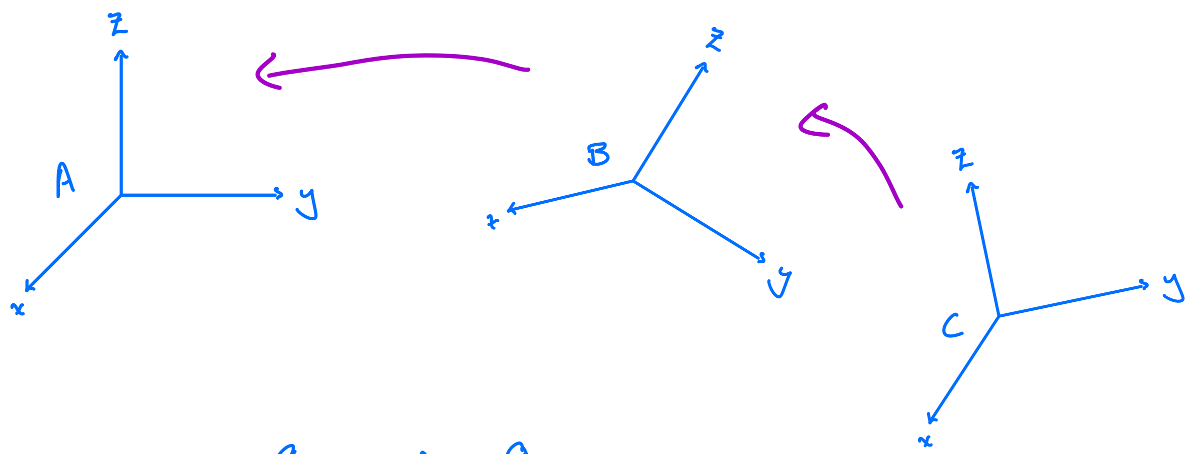
Point: $g_{ab}\, q_b = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}$ $= 2 \cdot -1 + 1 \cdot 1$ → Has moved 1 unit

$2 \to -1$

Vector: $g_{ab}\vec{v}_b = \begin{bmatrix} -2 \\ 2 \\ -2 \\ 0 \end{bmatrix} \rightarrow$ Has not moved
0 in the 4th position

Composition Rule

- Product of 2 rigid body transforms performs both of them
- Go from right to left
- Same as rotation matrices basically, but this also includes translation



$$g_{AC} = g_{AB} \cdot g_{BC}$$

Invertibility

- They're invertible
- Can go from one place to another and back

$$g_{AB}^{-1} = g_{BA}$$

$$g = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} \qquad g^{-1} = \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix}$$

2. Exponential Coordinates

Matrix Exponential

- Recall from homework 0 some definitions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow \text{Taylor Series}$$

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \longrightarrow \text{With a matrix!}$$

$$= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

Takeaway: Matrix exponential behaves the same as normal exponential

- Differential equation

$$\frac{dx}{dt} = \dot{x} = Ax \qquad x(0) = x_0$$

$$\boxed{x(t) = e^{At} \cdot x(0)}$$

- Some exercises

1. By differentiating the series representation, show that if $Y(t) = e^{At}$ then $\dot{Y}(t) = Ae^{At} = e^{At}A$.

* Check Solutions

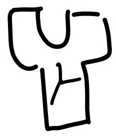
2. By differentiating the function $y(t) = e^{-At}x(t)$, show that $x(t) = e^{At}x_0$ is the unique solution to $\dot{x} = Ax$ with initial condition $x(0) = x_0$.

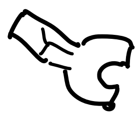
Motivation

$[R]$ Rotation

Homogeneous $\begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$

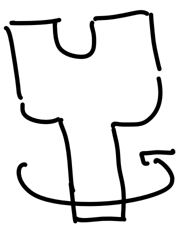
- We want to construct a **transformation matrix**
- Understand how some point moves with coordinate axes
  - Ex. Where in the world frame does some point on a robot arm end up

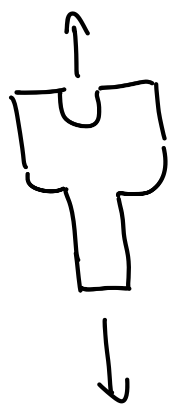




→ Make sure it doesn't hit a table!

- But the thing with robots is that they have continuous motion
- A joint can spin around or move forward and back





- **Our transformation matrix changes with movement**
- This means we need the matrix to be a **function of theta** (how much the arm has moved)

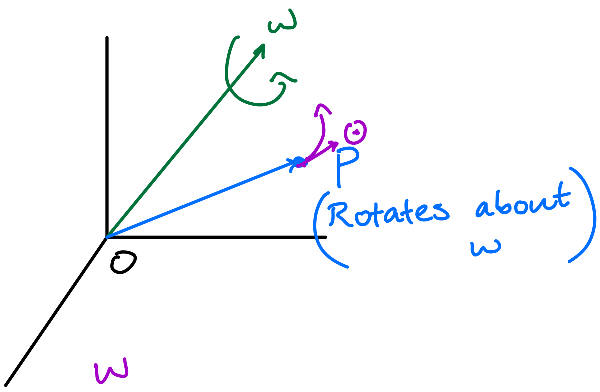
- How do we do that?
- We look at **how the joint moves** (i.e. linear and angular velocities)
- Then integrate!
  - (But this is a DE as we'll see, so it's really an exponential

Exponential Coordinates for Rotation

- Basically, we're constructing the **rotation matrix** using this technique
- (We'll get to the full homogeneous matrix next)

Problem 1. *Find the rotation matrix $R(\omega, \theta)$ for a rotation about some axis ω by amount θ. How is Rodrigues' formula related?*

★ Assume unit angular velocity $\|\omega\| = 1$



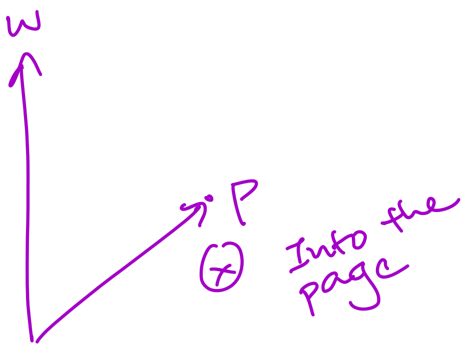
Velocity of Particle:

$\dot{p}(t) = \omega \times p(t)$ → Linear motion Tangential to rotation

$\dot{p}(t) = \hat{\omega}\, p(t)$

$\frac{dp}{dt} = \hat{\omega}\, p(t)$

$$\boxed{p(t) = e^{\hat{\omega} t} \cdot \underline{p(0)}}$$



$\underline{\text{Linear}}$ velocity : $\omega \times p$

$e^{\hat{\omega} t}$ → Rotation matrix parameterized by time

$= e^{\hat{\omega}\theta}$

$$\boxed{R(\omega, \theta) = e^{\hat{\omega}\theta}}$$

= Rodrigues Formula

$= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$

2.2.1 Exercise

(Axis of rotation, θ) $= (\omega, \theta)$

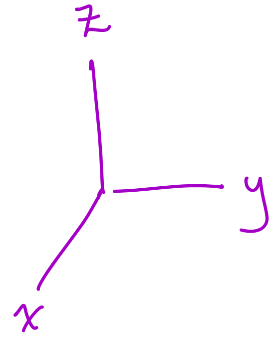
Find the exponential coordinates of the following rotation matrices:

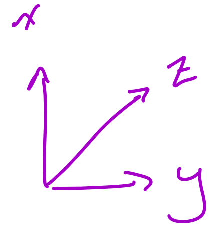
1. $R_x(\pi/2)$, the Euler x rotation matrix.
2. $R_y(-\pi/2)$
3. $R = R_x(\pi/2) R_y(-\pi)$

world x

$e^{\hat{\omega}_1\theta_1} \cdot e^{\hat{\omega}_2\theta_2} \rightarrow e^{\hat{\omega}_1\theta_1 + \hat{\omega}_2\theta_2}$

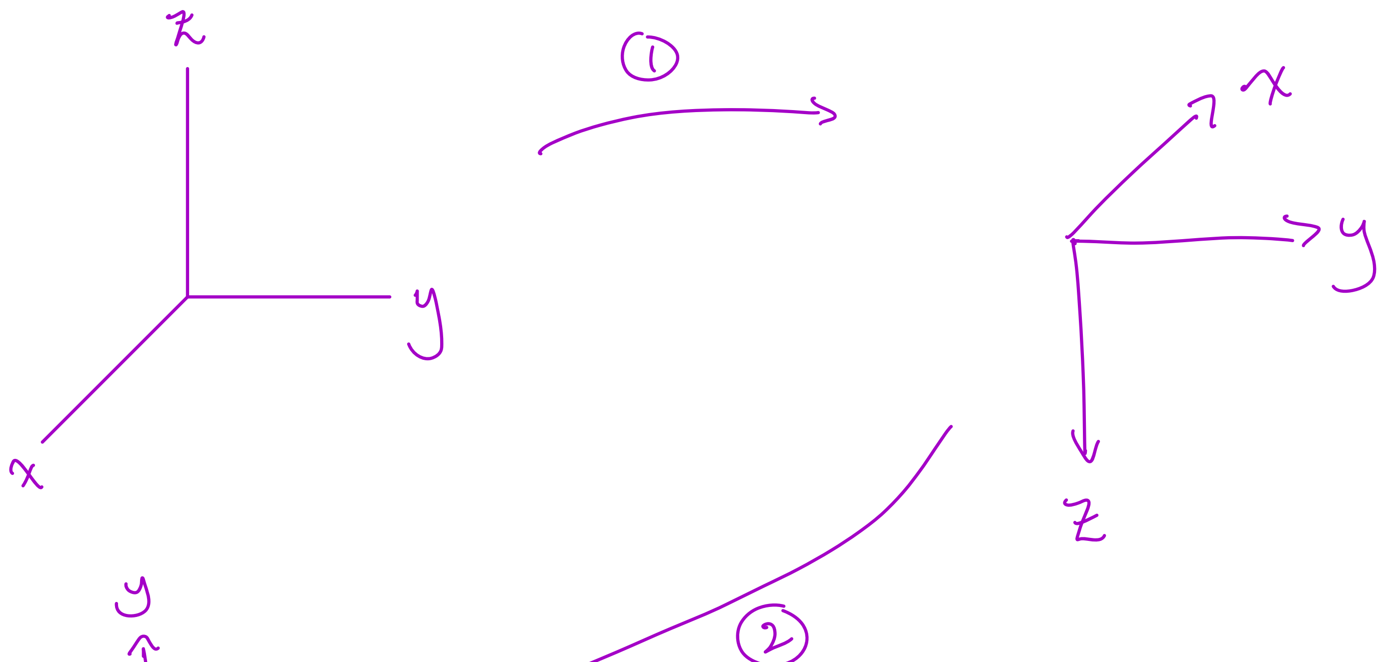
2.

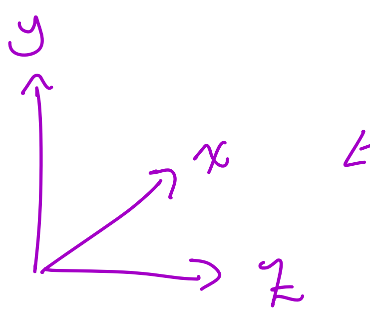




$$\boxed{\omega = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \theta = -\pi/2}$$

3.





- x has been flipped
- y & z have switched

Rotation by π by axis in y-z

Rotation around the middle (rotated by same amt in y-z plane)

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \xrightarrow[\|w\|=1]{\text{norm}} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\boxed{w = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \Theta = \pi}$$

3. Exponential Coordinates for All Rigid Motion

Twists

- Usually we want to find more than just the rotation matrix
- See how position changes too
- We want the **full homogeneous transformation** $[R]$
- We can use **twists** to capture this idea
  - Use **both linear and angular velocities**

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^6$$

$v \rightarrow$ Linear component of velocity (not dependent on $p(t)$)

$\omega \rightarrow$ Axis of rotation, $\|\omega\| = 1$

$$\rightarrow \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Problem 2. *Write the expressions for the velocity of the point p (ie. $\dot{p}(t)$) when attached to both the revolute and prismatic joints in Fig. 2. Assume that $\omega \in \mathbb{R}^3$, $||\omega|| = 1$, and $q \in \mathbb{R}^3$ is some point along the axis of ω.*

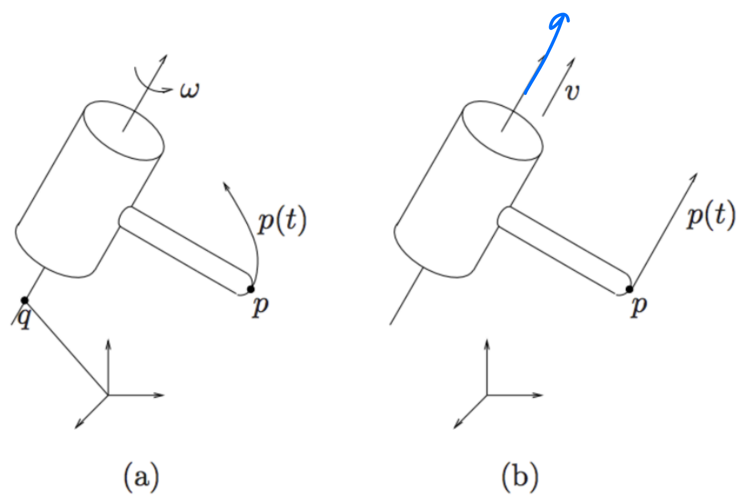


Figure 2: a) A revolute joint and b) a prismatic joint.

Revolute:

$$\dot{p}(t) = \omega \times (p(t) - q)$$

$\hookrightarrow$ Linear velocity

Prismatic:

$$\dot{p}(t) = v$$

$$\omega \times p(t) - \omega \times q$$

$$= \hat{\omega} p(t) - \hat{\omega} q$$

Twist of a Revolute Joint (Rotational Motion)

- Now, let's make the velocity into a DE in homogeneous coordinates

$$\boxed{\dot{p} = A \cdot p(t)}$$

$$\underbrace{\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix}}_{\text{velocity vector}} = \underbrace{\begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \underbrace{\begin{bmatrix} p \\ 1 \end{bmatrix}}_{\text{Point}}$$

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

Twist of a Prismatic Joint (Linear Motion)

$\dot{p}(t) = v$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

More on Twists

Wedge: $\begin{bmatrix} v \\ \omega \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} = \hat{\xi}$

Vee: $\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} v \\ \omega \end{bmatrix} = \xi$

3.4 Solution to differential equation gives us the exponential map

Problem 5. *Write the general solution to the differential equation $\dot{\bar{p}} = \hat{\xi}\bar{p}$. Then, make use of the fact that $||\omega|| = 1$ to reparameterize t to be θ. Specifically, find the expression for $p(\theta)$ in terms of $p(0)$.*

$\dot{p} = \hat{\xi} \cdot p(t)$

$$\boxed{p(t) = e^{\hat{\xi}t} p(0)}$$

* p is in homogeneous coords

$e^{\hat{\xi}t} \longrightarrow$ Homogeneous Transformation Matrix

$= g_{ab}$

- It's a mapping of points from initial coordinates to final coordinates after motion with parameter
- Not a mapping between coordinate frames

$\theta = t$

$$e^{\hat{\xi}\theta} = \begin{cases} \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} & \omega = 0 \\ \\ \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} & \omega \neq 0, \ ||\omega|| = 1 \end{cases}$$

Exponential Coordinates:

$$(\xi, \quad \theta)$$

$$= \left([v, \quad \omega]^T, \quad \theta \right)$$

4. Screw Motion

- Any rigid body translation can be simplified
- Instead of having a rotation and then a translation
- Finite rotation about some axis and then translation about that axis
  - Axis *l*
  - Magnitude M (like theta)
  - Pitch *h* - ratio of translation : rotation
    - h = 0: pure rotation
    - h infinite: pure translation
- Rotation by M (theta)
- Translation by hM (apply ratio)

The transformation g corresponding to S has the following effect on a point p:

$$gp = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega \tag{11}$$

Problem 6. *Convert this transformation to homogeneous coordinates. What do you notice between this expression and the one in Eq. 10?*

$$g\begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\omega \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p \\ 1 \end{bmatrix}$$

$\rightarrow$ Very similar in form to equation above

Every twist $\Longleftrightarrow$ Equivalent screw

5. Twists from Screw Motion

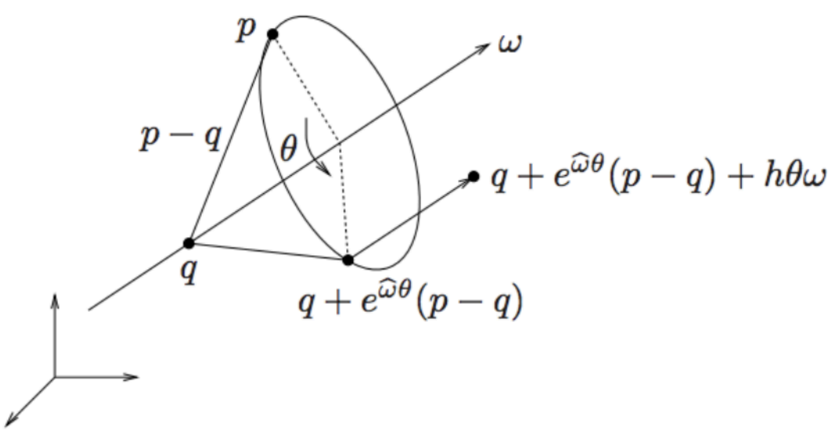
- Screws correspond to rotation and translation
- Can convert them into twists
- 2 cases: pure translation and nonzero rotation + translation

A) Pure Translation (*h* infinite)

$$\hat{\xi} = \begin{bmatrix} v \\ 0 \end{bmatrix}^{\wedge} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$

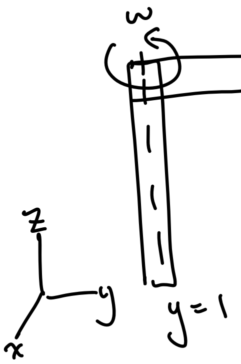
B) Nonzero rotation (*h* finite)

- Rotation by theta
- Axis *w*
  - Passes through point q
- Translation by hθ units



$$\hat{\xi} = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\omega} & -\omega \times q + h\omega \\ 0 & 0 \end{bmatrix}$$

Exercise: Find the twist for the following revolute joint:



$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

$q \rightarrow$ any pt. on axis of rotation

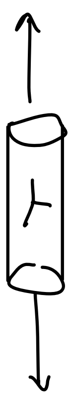
$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{Rotating about } z$$

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

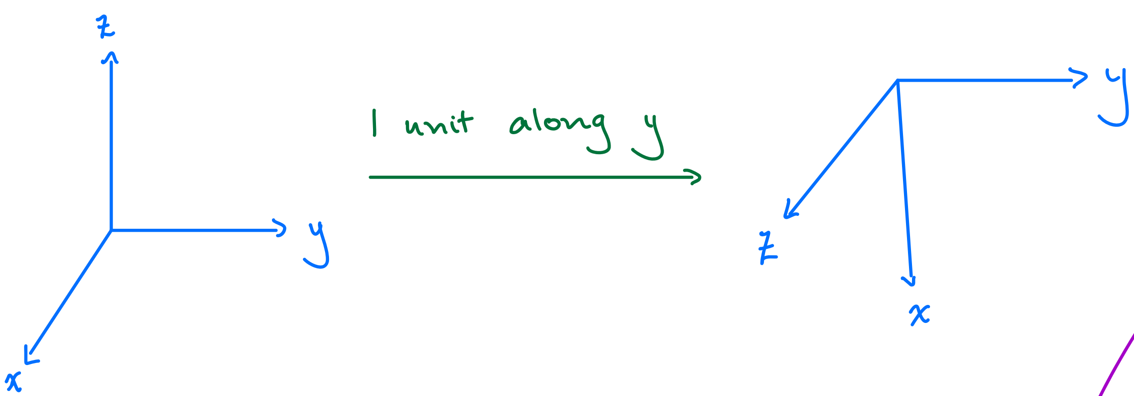
$$\xi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercise: Find the twist for this prismatic joint:



$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise: Find the exponential coordinates for this rigid body transform using the equivalent screw motion.



$$\text{Axis} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$M = \theta = \pi/2$$

$$h = \frac{1}{\pi/2} = \frac{2}{\pi}$$

$$\xi = \begin{bmatrix} -\omega \times q + h\omega \\ \omega \end{bmatrix}$$

tinyurl.com/106a-disc-2