EE106A Discussion 3: Forward Kinematics

1 Forward kinematics

1.1 Numbering notation for joints and links

We use the following notation for joints and links: Number the links from 0 to n starting from the base. Then, number the connecting joints such that joint i connects links $i-1$ and i. Typically, we attach the *base frame* S to be stationary with respect to link 0, and the *tool frame* T to the robot end-effector.

Definition 1. The *joint space* Q of a manipulator is composed of all possible values of the joint variables of the robot. This is equivalent to the configuration space of the robot. Each joint is parameterized by its *joint angle* θ, even though both angles and displacements are allowed depending on the joint type (revolute or prismatic).

1.2 Forward kinematics problem statement

The forward kinematics of a robot determines the configuration of the end-effector (the gripper or tool mounted on the end of the robot) given the relative configurations of each pair of adjacent links of the robot. Thus, the main objective of forward kinematics is finding the transformation $g_{ST}(\theta_1, \theta_2, ...\theta_n)$ as a function of the joint angles $\theta_1...\theta_n$.

1.3 Product of exponentials formula

Given the initial configuration $g_{ST}(0)$ when all joint angles are at 0, we can use the twist $\widehat{\xi_i}$ associated with each joint i and compose them to get the resulting configuration $g_{ST}(\theta)$ as a function of $\theta_1...\theta_n$ which we call θ here.

$$g_{ST}(\theta) = e^{\widehat{\xi}_1\theta_1} e^{\widehat{\xi}_2\theta_2} ... e^{\widehat{\xi}_n\theta_n} g_{ST}(0) \tag{1}$$

Recall the twist for a revolute joint where ω is a unit vector in the direction of the twist axis, and q is any point on the axis:

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix} \tag{2}$$

and the twist for a primsatic joint, where v is a unit vector in the direction of translation:

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix} \tag{3}$$

Problem 1. *Does the composition work in any other order in general? Why or why not?*
The composition works in another order only if the twists are updated according to how previous joint transformations may change them. If we maintain the order in the product of exponentials formula, we don't have to worry about this dependency.

2 SCARA forward kinematics

Problem 2. *Find the forward kinematics map for the manipulator shown in Fig. 2. This is an "RRRP" chain: 3 revolute joints followed by a single prismatic joint. Assume the orientation of T matches S.*

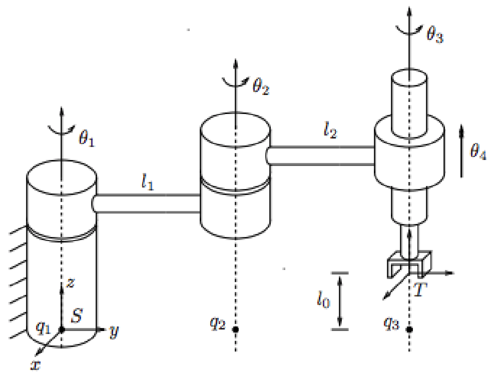

Figure 1: SCARA manipulator in its reference configuration (joint angles all 0).

For the initial configuration, we have

$$g_{ST}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 + l_2 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

The twists of each joint are given by

$$\omega_1 = [0,0,1]^T \qquad v_1 = [0,0,0]^T \tag{5}$$
$$\omega_2 = [0,0,1]^T \qquad v_2 = [l_1,0,0]^T \tag{6}$$
$$\omega_3 = [0,0,1]^T \qquad v_3 = [l_1 + l_2,0,0]^T \tag{7}$$
$$\omega_4 = [0,0,0]^T \qquad v_4 = [0,0,1]^T \tag{8}$$

To get the final forward kinematics, we compose the transformation corresponding to each twist with the initial configuration:

$$g_{ST}(\theta_1,\theta_2,\theta_3,\theta_4) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} g_{ST}(0) \tag{9}$$

Note that we usually call this map $g_{ST}(\theta)$ for brevity. You can now use your code from the previous homeworks to find the configuration of the end effector for any combination of joint angles θ!

Problem 3. *Show that you can arrive at the same result using rigid body transformations in non-exponential coordinates.*

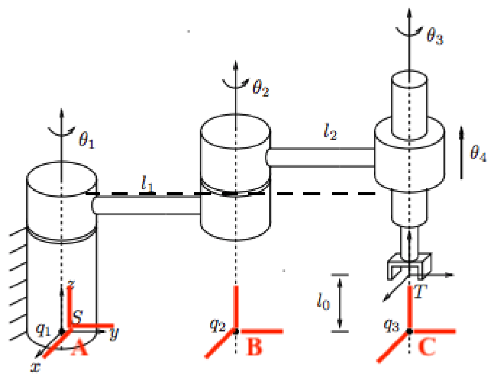

Figure 2: SCARA manipulator in its reference configuration (joint angles all 0).

We can also find the resulting FK map (ie. the transformation betweeen the world and tool frame) by expressing it as a composition of intermediate transformations. So, we define some intermediate frames attached to the robot (choice of these frames is not unique!)

Define new frames A, B, and C that have the same orientation as S and T initially, but are attached to joints 1, 2, and 3 respectively. Now we can define transformations between each subsequent frame as a function of the joint angles.

$$g_{SA} = \begin{bmatrix} R_z(\theta_1) & [0 \quad 0 \quad l_0]^T \\ 0 & 1 \end{bmatrix}$$

$$g_{AB} = \begin{bmatrix} R_z(\theta_2) & [0 \quad l_1 \quad 0]^T \\ 01 & \end{bmatrix}$$

$$g_{BC} = \begin{bmatrix} R_z(\theta_3) & [0 \quad l_2 \quad 0]^T \\ 0 & 1 \end{bmatrix}$$

$$g_{CT} = \begin{bmatrix} I & [0 \quad 0 \quad \theta_4]^T \\ 0 & 1 \end{bmatrix}$$

Multiply these all together, such that

$$g_{ST} = g_{SA} g_{AB} g_{BC} g_{CT}$$

And the solution should be the same as what we found with exponential coordinates.

3 Elbow manipulator forward kinematics

Problem 4. *Find the forward kinematics map for the elbow manipulator in Fig. 3. Assume that the end effector is positioned at the ξ_4 axis (i.e. the y-position of the end effector is $l_1 + l_2$).*

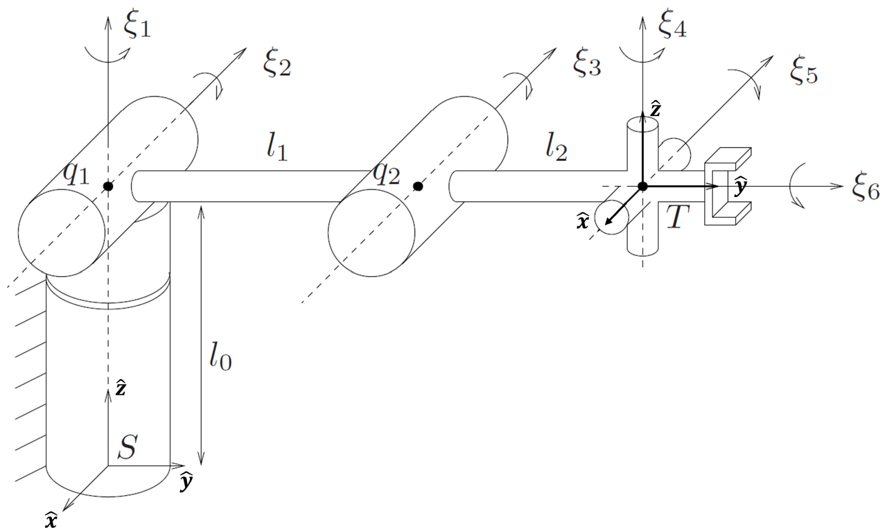

Figure 3: Elbow manipulator in its reference configuration (joint angles all 0).

For the initial configuration, we have

$$g_{ST}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 + l_2 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

The twists of each joint are given by

$$\omega_1 = [0, 0, 1]^T \qquad v_1 = [0, 0, 0]^T \tag{11}$$
$$\omega_2 = [-1, 0, 0]^T \qquad v_2 = [0, -l_0, 0]^T \tag{12}$$
$$\omega_3 = [-1, 0, 0]^T \qquad v_3 = [0, -l_0, l_1]^T \tag{13}$$
$$\omega_4 = [0, 0, 1]^T \qquad v_4 = [l_1 + l_2, 0, 0]^T \tag{14}$$
$$\omega_5 = [-1, 0, 0]^T \qquad v_5 = [0, -l_0, l_1 + l_2]^T \tag{15}$$
$$\omega_6 = [0, 1, 0]^T \qquad v_6 = [-l_0, 0, 0]^T \tag{16}$$

To get the final forward kinematics, we compose the transformation corresponding to each twist with the initial configuration:

$$g_{ST}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{ST}(0) \tag{17}$$