

EE106A Discussion 8: Jacobians

1 Review: Spatial and body twists

Last week, we learned about spatial and body velocity twists between two frames A and B . These velocity twists are useful because they allow us to find the instantaneous velocity of the B frame expressed in both spatial and body coordinates.

$$v_{q_a}(t) := \dot{q}_a(t) = \dot{g}_{ab}(t)q_b = \underbrace{\dot{g}_{ab}(t)g_{ab}^{-1}(t)}_{:=\widehat{V}_{ab}^s} q_a = \widehat{V}_{ab}^s q_a \quad (1)$$

$$v_{q_b}(t) := g_{ab}^{-1}(t)v_{q_a}(t) = \underbrace{g_{ab}^{-1}(t)\dot{g}_{ab}(t)}_{:=\widehat{V}_{ab}^b} q_b = \widehat{V}_{ab}^b q_b \quad (2)$$

Today, we will be thinking of velocities in the context of robotic manipulators. We will be finding the velocities between the fixed frame S and the end effector frame T , \widehat{V}_{st}^s and \widehat{V}_{st}^b .

To do so, we will introduce the notion of spatial and body manipulator Jacobians. Then, we will see how these manipulator Jacobians help us detect singular configurations.

1.1 Adjoint for Twist Coordinate Change

When working with twists, we can transform a twist matrix $\widehat{\xi}$ into a different coordinate system defined by g , so that it becomes $\widehat{\xi}'$

$$\widehat{\xi}' = g\widehat{\xi}g^{-1} \quad (3)$$

In twist coordinates,

$$\xi' = Ad_g \xi \quad (4)$$

2 Jacobians

2.1 Spatial Jacobian Definition

As before, we have the expression for \widehat{V}_{st}^s as a function of the transformation between S and T :

$$\widehat{V}_{st}^s = \dot{g}_{st}(\theta)g_{st}^{-1}(\theta) \quad (5)$$

In twist coordinates,

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta} \quad (6)$$

where the *spatial manipulator Jacobian* $J_{st}^s(\theta)$ is defined as

$$J_{st}^s(\theta) = \begin{bmatrix} (\frac{\partial g_{st}}{\partial \theta_1})^\vee & \dots & (\frac{\partial g_{st}}{\partial \theta_n})^\vee \end{bmatrix} \quad (7)$$

$$= [\xi_1 \quad \xi'_2 \quad \dots \quad \xi'_n] \quad (8)$$

$$\xi'_i = Ad_{(e^{\widehat{\xi}_1\theta_1}\dots e^{\widehat{\xi}_{i-1}\theta_{i-1}})}\xi_i \quad (9)$$

2.2 Interpretation

The spatial Jacobian allows us to calculate the velocity of the robot end effector as a function of the current state of the individual robot joints, θ , and the velocities of each joint, $\dot{\theta}$.

The i^{th} column of the spatial Jacobian ξ'_i is equal to the i^{th} joint twist transformed to the current manipulator configuration and written in spatial coordinates.

Problem 1. *Explain how this physical interpretation is true.*

2.3 How it's used

We can use the spatial Jacobian to compute the instantaneous velocity of a point q attached to the end-effector relative to the spatial frame. This velocity is

$$v_{q_s} = \widehat{V}_{st}^s q_s = (J_{st}^s(\theta)\dot{\theta})^\wedge q_s \quad (10)$$

where q_s is the coordinates of q in the spatial frame.

2.4 Body Jacobian definition

The body Jacobian is very similar to the spatial one - it allows us to calculate the instantaneous velocity of the robot end effector (and any point attached to it) as a function of the joint velocities. The only difference is that the velocity is with respect to the body frame instead of the spatial frame.

$$\widehat{V}_{st}^b = g_{st}^{-1}(\theta) \dot{g}_{st}(\theta) \quad (11)$$

In twist coordinates,

$$V_{st}^b = J_{st}^b(\theta) \dot{\theta} \quad (12)$$

where the *body manipulator Jacobian* $J_{st}^b(\theta)$ is defined as

$$J_{st}^b(\theta) = [\xi_1^\dagger \quad \xi_2^\dagger \quad \dots \quad \xi_n^\dagger] \quad (13)$$

$$\xi_i^\dagger = Ad_{(e^{\widehat{\xi}_{i+1}\theta_{i+1}} \dots e^{\widehat{\xi}_n\theta_n} g_{st}(0))}^{-1} \xi_i \quad (14)$$

2.5 Interpretation

For some configuration θ , the body manipulator Jacobian maps the joint velocity vector $\dot{\theta}$ into the body velocity twist coordinates of the end-effector.

The i^{th} column of the body Jacobian ξ_i^\dagger is equal to the i^{th} joint twist transformed to the current manipulator configuration and written in body coordinates.

Problem 2. *Explain how this physical interpretation is true.*

2.6 How it's used

We can use the body Jacobian to compute the instantaneous velocity of a point q attached to the end-effector relative to the body frame. This velocity is

$$v_{qb} = \widehat{V}_{st}^b q_b = (J_{st}^b(\theta) \dot{\theta})^\wedge q_b \quad (15)$$

where q_b is the coordinates of q in the tool frame.

2.7 Converting between Spatial and Body Jacobians

Just as the adjoint map takes us between spatial and body twists, it also takes us between spatial and body Jacobians. This shouldn't be surprising since the columns of the Jacobian are twists!

$$J_{st}^s(\theta) = Ad_{g_{st}(\theta)} J_{st}^b(\theta) \quad (16)$$

2.8 Finding the Jacobian

Problem 3. Find the spatial and body manipulator Jacobians for the Stanford manipulator.

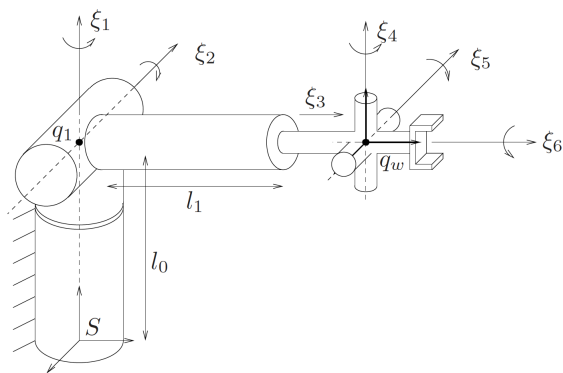
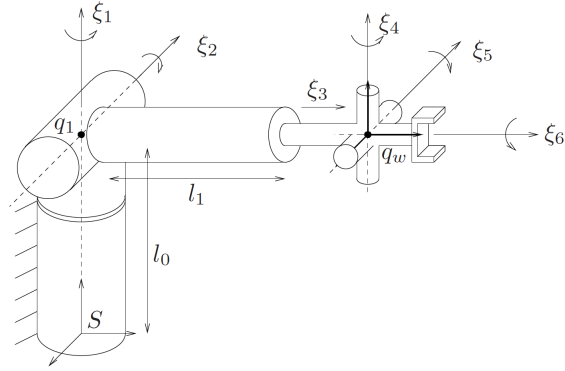


Figure 1: Stanford manipulator

When we want to find the manipulator Jacobians for some specific configuration θ_d , it's easier to do it by inspection rather than having to first find the manipulator Jacobians for general θ , then plugging in θ_d . To find cross products, it may be helpful to draw out circles to visualize direction.

Problem 4. Find the spatial and body manipulator Jacobians for the Stanford manipulator in its initial configuration. In this case, $\theta_d = 0$.



3 Singularities

At some configuration θ_s , it may be possible for $J_{st}^s(\theta_s)$ to *not* have full rank. This corresponds to the manipulator not being able to achieve motion in certain directions (since $J_{st}^s(\theta_s)$ doesn't fully span the space of possible velocities). We call θ_s a *singular configuration*. Since being in singular configurations is not desirable, it's important to figure out what they are for a particular manipulator so they can be avoided.

Problem 5. *Show that a manipulator Jacobian is singular if it has four revolute joint axes that intersect at a single point.*

Problem 6. *When is the elbow manipulator in a singular configuration?*

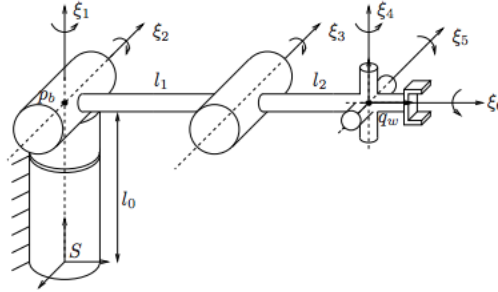


Figure 2: Elbow manipulator