

# Homework 9: Control

EECS/BioE/MechE C106A/206A  
Introduction to Robotics

Due: November 13, 2023

# Problem 1: Cat and mouse

Kaylene's cat Zena loves chasing wand toys, but when her roommates are out working on robots she doesn't have anyone to play with her :( Kaylene wants to build a wand toy that can be controlled autonomously when she isn't home. In this problem, we'll derive the Lagrangian of the toy in terms of the angles  $\theta$  and  $\phi$  so that we can calculate the torques that should be applied to make the toy follow a path that's fun for Zena to chase.

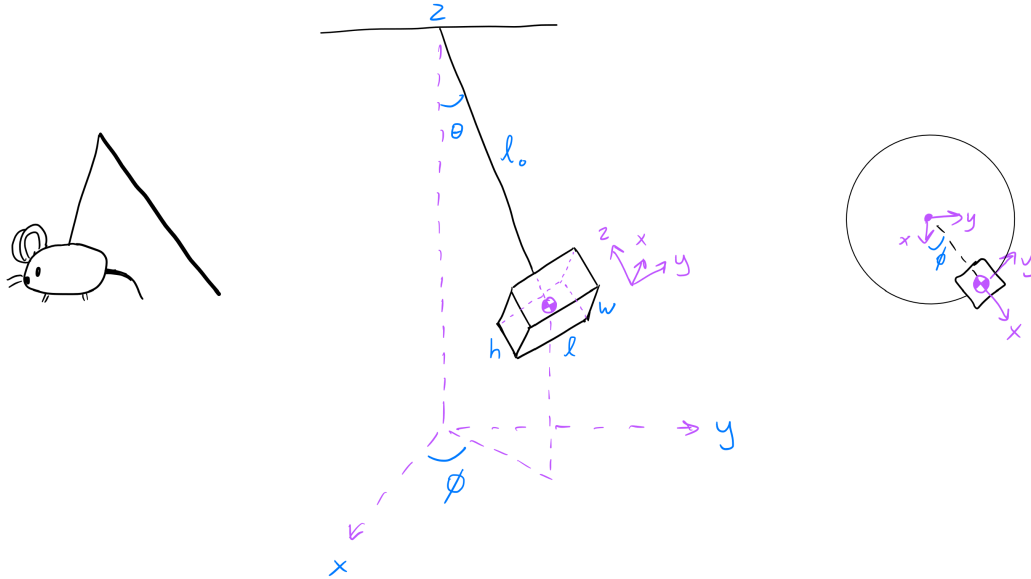


Figure 1: Left: The wand toy with a stuffed mouse. Middle: Model of the toy. Right: top-view diagram of the toy model.

- (a) Find the Lagrangian of the toy if we model the dangling mouse as a point mass of mass  $m$ . Use  $\theta$  and  $\phi$  for the generalized coordinates, and let the height of the top of the string be 0 for the potential energy.
- (b) After implementing a controller based on the Lagrangian for part (a), we find that it doesn't work as well as we hoped. To obtain a more accurate model, we decide to model the mouse as a homogeneous box instead of a point mass. What is the Lagrangian for this system? Note that the projection of the string that anchors the box and the  $x$ -axis of the box onto the  $xy$  plane are aligned (i.e. the  $x$ -axis of the body frame always points away from the anchor point - see the top-view diagram).

## Problem 2: OSIRIS-RExploring

On 24 September, the OSIRIS-REx satellite safely returned samples from the asteroid Bennu to Earth! Before collecting its samples, it spent 2 years orbiting Bennu and mapping its surface. If we model the satellite and asteroid as point masses, we can write the normalized equations of motion of the satellite as

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \quad (1)$$

$$\ddot{\theta} = -2\frac{\dot{\theta}}{r}\dot{r} + \frac{1}{r}u_2 \quad (2)$$

where  $r$  is the distance from the satellite to the asteroid,  $\theta$  is the satellite's angular position,  $k$  is a normalization constant, and  $u_1$  and  $u_2$  are control inputs from the satellite's thrusters.

- (a) Without any thruster input ( $u_1 = u_2 = 0$ ), the satellite achieves a circular orbit with  $r(t) = p$  and  $\theta(t) = \omega t$ . Linearize the system about this orbit, noting that for this orbit we can replace  $k$  with  $p^3\omega^2$  (from the first equation).
- (b) Characterize the stability of the linearized system when the thrusters are not firing. What does this mean for the satellite's orbit? *Hint: you may use the `eigenvals()` function in SymPy.*
- (c) Is the linearized system completely controllable?

### Problem 3: SpaceMax

Max, who is also passionate about rocketeering, has decided to start his own, rival company, SpaceMax. He designs the rocket depicted in Figure 2.

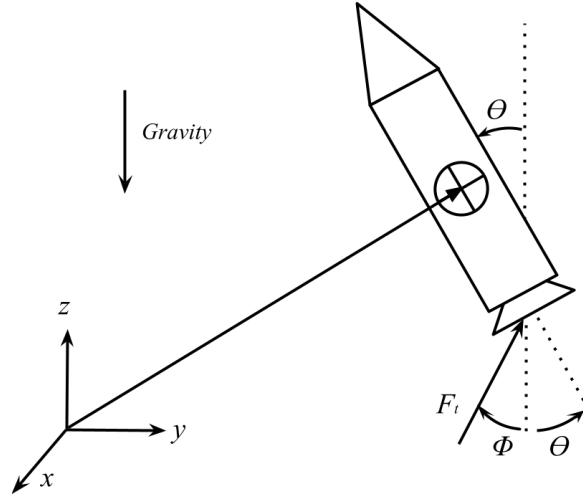


Figure 2: Gimbaled Rocket

The rocket is powered by a gimbaled engine, which rotates by an angle  $\Phi$  with respect to the rocket orientation to control the angle at which the thrust force,  $F_t$  is applied. The rocket has constant mass  $m$ , orientation  $\theta$ , and inertia  $I$  about the center of mass. Note that since the rocket is planar,  $x(t) = 0$  for all  $t$ .

- Write down a vector of generalized coordinates for the system. (*Hint: this should be a vector of length 3.*)
- Find the Lagrangian  $L = T - V$  of the system.
- Find the rocket's equations of motion in terms of a vector of generalized forces  $\Upsilon$ .
- Calculate  $\Upsilon_y$ , the component of the external force in the y-direction. Assume that the thrust force  $F_t$  is the only external force applied to the system.
- Calculate  $\Upsilon_z$ , the component of the external force in the z-direction. Assume that the thrust force  $F_t$  is the only external force applied to the system.
- Max wants to keep his rocket stable about  $z = z_d$ . This suggests trying to drive the error  $e = z_d - z$  to 0. Prove that for any system with  $e \in \mathbb{R}^{n \times n}$ , if  $e$  evolves according to the following ODE,

$$0 = K_p e + K_d \dot{e} + c \ddot{e} \quad (3)$$

where  $c \in \mathbb{R}$  is a constant, we can choose  $K_p, K_d \in \mathbb{R}^{n \times n}$  such that  $\lim_{t \rightarrow \infty} e(t) = 0$ . *Hint: Choose convenient  $K_p$  and  $K_d$  matrices and examine an arbitrary row of the ODE.*

- (g) Assuming a fixed gimbal angle  $\Phi$ , find an expression for the thrust force  $F_t$  that gives error dynamics of:

$$0 = K_p e + K_d \dot{e} + \ddot{e} \quad (4)$$

Where  $e = z_d - z$ . You may leave your answer in terms of gains  $K_p, K_d$ , and constant gimbal angle  $\Phi$ . You may assume that the thrust vector has a positive  $z$  component. Using the result of part (f), we can conclude that the rocket will stably hover at  $z_d$ !