Neighborhood weighting in N-dim Space

Nathan Zhang

February 6, 2015

1 Restriction

1.1 Problem

For any restriction weighting algorithm, the uniform matrix must be a fixed point value, reducing to a smaller matrix of the same values. Furthermore, the sum of weights around a point must be 1 to prevent value inflation/deflation.

Let P be the N-dimensional coordinate of a non-boundary restriction point. For each neighbor i at position p_i with value v_i in P's neighborhood, let δ_i be the vector $p_i - P$. We then set the weight of this neighbor to be $2^{-(||\delta_i||_1 - N)}$. That is, the value of P post-restriction is

$$\Sigma_{i \in neighborhood(P)} 2^{-(||p_i - P||_1 - N)} * v_i$$

1.2 Weight Calculation

In N-dimensional space the cumulative weight is $\Sigma_{i \in neighborhood(P)} 2^{-(||p_i - P||_1 - N)}$. Each δ falls in $\{-1, 0, 1\}^n$. There are $\binom{N}{k} \cdot 2^k$ δs of 1-norm k, and therefore the sum of all weights at the $||\delta||_1 = k$ level is $\binom{N}{k} \cdot 2^k \cdot 2^{-(k+N)}$ or $\binom{N}{k} 2^{-N}$. Since $\Sigma_{k=0}^{N} \binom{N}{k} = 2^N$, we arrive at the cumulative weight of $2^N \cdot 2^{-N} = 1$

2 Interpolation

2.1 Problem

When interpolating in an N-dimensional space we need to find an algorithm to interpolate into spaces.

Let P be a point we wish to interpolate to in this case. The value of this point is equal to

$$\Sigma_{n \in neighborhood(P)} \begin{cases} 2^{-||n-P||_1} \cdot value(n) & \text{if } defined(n) \\ 0 & \text{otherwise} \end{cases}$$

In order for an algorithm to be valid, the sum of total weights on this point must also be 1.

2.2 Weight Calculation

By construction of our interpolation target matrix, we know that all known points in the neighborhood of every point in our target matrix have the same norm. If the neighborhood of a point consists defined points with 1-norm k, then each point has the weight 2^{-k} We can prove that the sum of contributions is 1 by induction.

In the base case, where $N = 0, k = 0$, we have exactly 1 point (the point itself, so it must be defined), and has a weight of $2^0 = 1$.

We have two inductive cases, $N+1$-dimension, $||\delta||_1 = k+1$; $N+1$-dimension, $||\delta||_1 = 0$

In the first case, we can generate the $||\delta||_1 = k+1$ vectors from appending either -1 or 1 onto each existing $||\delta||_1 = k$ vector, giving twice as many points, or 2^{k+1} points in $N+1$ dimensions.

In the second case, we know that there is exactly one point with $||\delta||_1 = 0$, the point itself. This has a weight of $2^0 = 1$.