C h a p t e r T h r e e

Noise and Distortion in Microwave Systems

The effect of noise is one of the most important considerations when evaluating the performance of wireless systems because noise ultimately determines the threshold for the minimum signal level that can be reliably detected by a receiver. Noise is a random process associated with a variety of sources, including thermal noise generated by RF components and devices, noise generated by the atmosphere and interstellar radiation, and man-made interference. Noise is omnipresent in RF and microwave systems, with noise power being introduced through the receive antenna from the external environment, as well as generated internally by the receiver circuitry. In our later study of modulation methods, we will see that parameters such as signal-to-noise ratio, bit error rates, dynamic range, and the minimum detectable signal level are all directly dependent on noise effects.

Our objective in this chapter is to present a quantitative overview of noise and its characterization in RF and microwave systems. Since noise is a random process, we begin with a review of random variables and associated techniques for the mathematical treatment of noise and its effects. Next we discuss the physical basis and a model for thermal noise sources, followed by an application to basic threshold detection of binary signals in the presence of noise. The noise power generated by passive and active RF components and devices can be characterized equivalently by either noise temperature or noise figure, and these parameters are discussed in Section 3.4, followed by the propagation and accumulation of noise power through a cascade of two-port networks. A more detailed treatment of the noise figure of general passive networks is given in Section 3.5. Finally, we consider the problem of dynamic range and signal distortion in general nonlinear systems. These effects are important for large signal levels in mixers and amplifiers, and can thus be viewed as complementary to the effect of noise, which is an issue for small signal levels.

3.1 REVIEW OF RANDOM PROCESSES

In this section we review some basic principles, definitions, and techniques of random processes that we will need in our study of noise and its effects in wireless communications systems. These include basic probability, random variables, probability density functions, cumulative distribution functions, autocorrelation, power spectral density, and expected values. We assume the reader has had a beginning course in random variables, and so will not require a full exposition of the subject. References [1]–[3] should be useful for a more thorough discussion of the required concepts.

Probability and Random Variables

Probability is the likelihood of the occurrence of a particular event, and is written as $P\{\text{event}\}$. The probability of an event is a numerical value between zero and unity, where zero implies the event will never occur, and unity implies the event will always occur. Probability events may include the occurrence of an equality, such as $P\{x = 5\}$, or events related to a range of values, such as $P\{x \leq 5\}$.

In contrast to the actual terminology, a *random variable* is neither random nor a variable, but is a function that maps sample values from a random event or process into real numbers. Random variables may be used for both discrete and continuous processes. Examples of discrete processes include tossing coins and dice, counting pedestrians crossing a street, and the occurrence of errors in the transmission of data. Continuous random variables can be used for modeling smoothly varying real quantities such as temperature, noise voltage, and received signal amplitude or phase. We will be primarily concerned with continuous random variables.

Consider a continuous random variable X, representing a random process with real continuous sample values x, where $-\infty < x < \infty$. Since the random variable X may assume any one of an uncountably infinite number of values, the probability that X is exactly equal to a specific value, x_0, must be zero. Thus, $P\{X = x_0\} = 0$. On the other hand, the probability that X is less than a specific value of x may be greater than zero: $0 \leq P\{X \leq x_0\} \leq 1$. In the limit as $x_0 \to \infty$, $P\{X \leq x_0\} \to 1$, as the event becomes a certainty.

We will not adopt any particular notation for random variables in this book, as it should be clear from the context which variables are random and which are deterministic. In most cases, the only random variables we will encounter will be associated with noise voltages, and typically denoted as $v_n(t)$, or $n(t)$.

The Cumulative Distribution Function

The *cumulative distribution function* (CDF), $F_X(x)$, of the random variable X is defined as the probability that X is less than or equal to a particular value, x. Thus

$$F_X(x) = P\{X \leq x\}. \tag{3.1}$$

It can be shown that the cumulative distribution function satisfies the following properties:

$$(1)\quad F_X(x) \geq 0 \tag{3.2a}$$

$$(2)\quad F_X(\infty) = 1 \tag{3.2b}$$

$$(3)\quad F_X(-\infty) = 0 \tag{3.2c}$$

$$(4)\quad F_X(x_1) \leq F_X(x_2) \text{ if } x_1 \leq x_2 \tag{3.2d}$$

The last property is a statement that the cumulative distribution function is a monotonic (nondecreasing) function. The definition in (3.1) shows that the result of (3.2d) is equivalent to the statement that

$$P\{x_1 < x \le x_2\} = F_X(x_2) - F_X(x_1).$$

The Probability Density Function

The *probability density function* (PDF), $f_x(x)$, of a random variable X is defined as the derivative of the CDF:

$$f_X(x) = \frac{dF_X(x)}{dx}. \tag{3.3}$$

Since the CDF is monotonically nondecreasing, $f_X(x) \ge 0$ for all x. The PDF may contain delta functions, as in the case of discrete random variables, for which the CDF is a "stair-step" type of function.

By the fundamental theorem of calculus, (3.3) can be inverted to give the following useful result:

$$P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx. \tag{3.4}$$

In addition, since $F(-\infty) = 0$ from (3.2c), (3.4) reduces to the following result that directly relates the CDF to the PDF:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du. \tag{3.5}$$

Finally, because $F(\infty) = 1$ from (3.2b), (3.5) leads to the fact that the total area under a probability density function is unity:

$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1. \tag{3.6}$$

These results can be extended to cases of two random variables. Thus, the joint CDF associated with random variables X and Y is defined as

$$F_{XY}(x, y) = P\{X \le x \text{ and } Y \le y\}. \tag{3.7}$$

Then the joint PDF is calculated as

$$f_{xy}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{xy}(x, y). \tag{3.8}$$

Similar to the result of (3.4), the probability of x and y both occurring in given ranges is found from

$$P\{x_1 < X \le x_2 \text{ and } y_1 < Y \le y_2\} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{xy}(x, y)\,dx\,dy. \tag{3.9}$$

The individual probability density functions for X and Y can be recovered from the joint PDF by integration over one of the variables:

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x, y)\,dx, \tag{3.10a}$$

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x, y)\,dy. \tag{3.10b}$$

For the special case where the random variables X and Y are statistically independent, the joint PDF is the product of the PDFs of X and Y:

$$f_{xy}(x, y) = f_x(x) f_y(y). \tag{3.11}$$

Some Important Probability Density Functions

For reference, we list here some of the probability density functions that we will be using in this book. The most basic is the PDF of a uniform distribution, defined as a constant over a finite range of the independent variable:

$$f_x(x) = \frac{1}{b-a} \quad \text{for} \quad a \le x \le b. \tag{3.12a}$$

The constant 1/(b−a) is chosen to properly normalize the PDF according to (3.6). Many random variables have gaussian statistics, with the general gaussian PDF given by

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} \quad \text{for} \quad -\infty < x < \infty, \tag{3.12b}$$

where m is the mean of the distribution, and σ^2 is the variance.
In our study of fading and digital modulation we will encounter the Rayleigh PDF, given by

$$f_r(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \quad \text{for} \quad 0 \le r < \infty. \tag{3.12c}$$

The reader can verify that these each satisfy the normalization condition of (3.6).

Expected Values

Since random variables are nondeterministic, we cannot predict with certainty the value of a particular sample from a random event or process, but instead must rely on statistical averages such as the mean, variance, and standard deviation. We denote the *expected value* of the random variable X as $\bar{x}$, or $E\{X\}$. The expected value is also sometimes called the mean, or average value. For discrete random variables the expected value is given as the sum of the N possible samples, x_i, weighted by the probabilities of the occurrence of that sample:

$$\bar{x} = E\{X\} = \sum_{i=1}^{N} x_i P\{X = x_i\}. \tag{3.13a}$$

This result directly generalizes to the case of continuous random variables:

$$\bar{x} = E\{X\} = \int_{-\infty}^{\infty} x f_x(x)\,dx. \tag{3.13b}$$

It is easy to show that the process of taking the expected value of a random variable is a linear operation, and that the following two properties therefore apply (assume X and Y are random variables, with c a constant):

$$(1)\quad E\{cX\} = cE\{X\} \tag{3.14a}$$

$$(2)\quad E\{X + Y\} = E\{X\} + E\{Y\}. \tag{3.14b}$$

We will also often be interested in finding the expected value of a function of a random variable. If we have a random variable x, and a function $y = g(x)$ that maps values from x

to a new random variable y, then the expected value of y can be found for the discrete case as

$$\bar{y} = E\{y\} = E\{g(x)\} = \sum_{i=1}^{N} g(x_i)P\{x = x_i\}. \tag{3.15a}$$

For the case of a continuous random variable this becomes

$$\bar{y} = E\{y\} = E\{g(x)\} = \int_{-\infty}^{\infty} g(x) f_x(x)\, dx. \tag{3.15b}$$

The result of (3.15b) can be used to find higher-order statistical averages, such as the *nth moment* of the random variable, X:

$$\overline{x^n} = E\{x^n\} = \int_{-\infty}^{\infty} x^n f_x(x)\, dx. \tag{3.16}$$

The *variance*, σ^2, of the random variable X is found by calculating the second moment of X after subtracting the mean of X:

$$\begin{aligned} \sigma^2 &= E\{(x-\bar{x})^2\} = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_x(x)\, dx \\ &= E\{x^2 - 2x\bar{x} + \bar{x}^2\} = \overline{x^2} - \bar{x}^2. \end{aligned} \tag{3.17}$$

The *root-mean-square* (rms) value of the distribution is σ, the square root of the variance. If a particular zero-mean random voltage is represented by the random variable x, the power delivered to a 1 Ω load by this voltage source will be equal to the variance of x.

The expected value of a function of two random variables involves the joint PDF:

$$\overline{g(x, y)} = E\{g(x, y)\} = \int_{-\infty}^{\infty} g(x, y) f_{xy}(x, y)\, dx dy. \tag{3.18}$$

This result can be applied to the product of two random variables, x and y, by letting the function $g(x, y) = xy$. For the special case of independent random variables the joint PDF is the product of the individual PDFs by (3.11), so (3.18) reduces to

$$\overline{xy} = E\{xy\} = \int_{-\infty}^{\infty} x f_x(x)\, dx \int_{-\infty}^{\infty} y f_y(y)\, dy = E\{x\}E\{y\}. \tag{3.19}$$

Autocorrelation and Power Spectral Density

An important characteristic of both deterministic and random signals is how rapidly their sample values vary with time. This characteristic can be quantified with the *autocorrelation function*, defined for a complex deterministic signal, $x(t)$, as the time average of the product of the conjugate of $x(t)$ and a time-shifted version, $x(t+\tau)$:

$$R(\tau) = \int_{-\infty}^{\infty} x^*(t) x(t+\tau)\, dt. \tag{3.20}$$

It can be shown that $R(0) \geq R(\tau)$, and $R(\tau) = R(-\tau)$. Also, $R(0)$ is the normalized energy of the signal.

For stationary random processes, such as noise processes, the autocorrelation function is defined as

$$R(\tau) = E\{x^*(t)x(t+\tau)\}. \tag{3.21}$$

Because of the relation between the time variation of a signal and its frequency spectrum, we can also characterize the variation of random signals by examining the spectra of the autocorrelation function in the frequency domain. For stationary random processes, the *power spectral density* (PSD), $S(\omega)$, is defined as the Fourier transform of the autocorrelation function:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau}\, d\tau. \tag{3.22a}$$

The inverse transform can be used to find the autocorrelation from a known PSD:

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau}\, d\omega. \tag{3.22b}$$

For a noise voltage, the power spectral density represents the noise power density in the spectral (frequency) domain, assuming a 1 Ω load resistor. If $v(t)$ represents the noise voltage, the power delivered to a 1 Ω load can be found as

$$P_L = \overline{v^2(t)} = E\{v^2(t)\} = R(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_v(\omega)\, d\omega = \int_{-\infty}^{\infty} S_v(2\pi f)\, df \ W, \tag{3.23}$$

where $S_v(\omega)$ is the PSD of $v(t)$. The last equality follows from a change of variable with $\omega = 2\pi f$. Writing this integral in terms of f (in Hz) is convenient because $S_v(\omega)$ has dimension W/Hz, and therefore appears as a power density relative to frequency in Hertz.

EXAMPLE 3.1 OPERATIONS WITH RANDOM VARIABLES

Consider a sinusoidal voltage source, $V_0 \cos \omega_0 t$, which is randomly sampled in time to form a random process $v(t) = V_0 \cos\theta$, where $\theta = \omega_0 t$ is a random variable representing the sample time. Assume θ is uniformly distributed over the interval $0 \leq \theta < 2\pi$, since the cosine function is periodic with period 2π. Find the mean of the sample voltages, the average power delivered to a 1 Ω load, the autocorrelation function of the random process $v(t)$, and the power spectral density.

Solution

The PDF for the random variable θ is $f_\theta(\theta) = \frac{1}{2\pi}$, for $0 \leq \theta < 2\pi$. Then we can calculate the average voltage as

$$\overline{v(t)} = E\{v(t)\} = \int_0^{2\pi} v(t) f_\theta(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, d\theta = 0.$$

The average power delivered to a 1 Ω load is given by the variance of $v(t)$:

$$P_L = \overline{v^2(t)} = E\{v^2(t)\} = \int_0^{2\pi} v^2(t) f_\theta(\theta)\, d\theta = \frac{V_0^2}{2\pi}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{V_0^2}{2} W.$$

The autocorrelation can be calculated using (3.21):

$$\begin{aligned} R_v(\tau) &= E\{v(t)v(t+\tau)\} = V_0^2 E\{\cos\omega_0 t \cos\omega_0(t+\tau)\} \\ &= \frac{V_0^2}{2\pi}\int_0^{2\pi} \cos\theta\cos(\theta+\omega_0\tau)\, d\theta = \frac{V_0^2}{4\pi}\int_0^{2\pi} [\cos\omega_0\tau + \cos(2\theta+\omega_0\tau)]\, d\theta \\ &= \frac{V_0^2}{2}\cos\omega_0\tau \end{aligned}$$

Note that $R_v(0) = V_0^2/2$, which is the variance of $v(t)$. The power spectral density is found using (3.22a):

$$\begin{aligned} S_v(\omega) &= \int_{-\infty}^{\infty} R_v(\tau) e^{-j\omega\tau}\, d\tau = \frac{V_0^2}{4}\int_{-\infty}^{\infty} \left[e^{-j(\omega-\omega_0)\tau} + e^{-j(\omega+\omega_0)\tau}\right] d\tau \\ &= \frac{\pi V_0^2}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \end{aligned}$$

This result shows that power is concentrated at $\omega = \omega_0$ and its image at $-\omega_0$. The total power can also be calculated by integrating the PSD over frequency, as in (3.23):

$$P_L = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_v(\omega)\,d\omega = \frac{V_0^2}{4}\int_{-\infty}^{\infty}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\,d\omega = \frac{V_0^2}{2}.$$

This result agrees with the earlier result obtained as the variance using the PDF. ○

3.2 THERMAL NOISE

Thermal noise, also known as Nyquist, or Johnson, noise, is caused by the random motion of charge carriers, and is the most prevalent type of noise encountered in RF and microwave systems. Thermal noise is generated in any passive circuit element that contains loss, such as resistors, lossy transmission lines, and other lossy components. It can also be generated by atmospheric attenuation and interstellar background radiation, which similarly involve random motion of thermally excited charges. Other sources of noise include *shot noise*, due to the random motion of charge carriers in electron tubes and solid-state devices; *flicker noise*, also occurring in solid-sate devices and vacuum tubes; *plasma noise*, caused by random motions of charged particles in an ionized gas or sparking electrical contacts; and *quantum noise*, resulting from the quantized nature of charge carriers and photons. Although these other types of noise differ from thermal noise in terms of their origin, their characteristics are similar enough that they can generally be treated in the same way as thermal noise.

Noise Voltage and Power

Figure 3.1a shows a resistor of value R at temperature T degrees Kelvin (K). The electrons in the resistor are in random motion, with a kinetic energy that is proportional to the temperature, T. These random motions produce small random voltage fluctuations across the terminals of the resistor, as illustrated in Figure 3.1b. The mean value of this voltage is zero, but its nonzero rms value in a narrow frequency bandwidth B is given by

$$V_n = \sqrt{4kTBR}, \tag{3.24}$$

where

$k = 1.380 \times 10^{-23}$ J/K is Boltzmann's constant
T is the temperature, in degrees Kelvin (K)
B is the bandwidth, in Hz
R is the resistance, in Ω

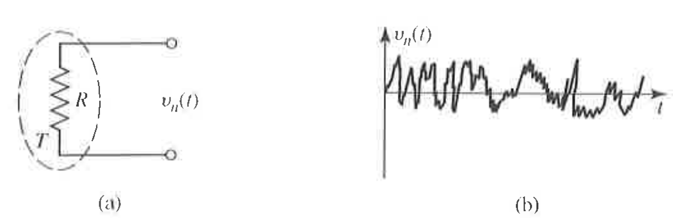

FIGURE 3.1 (a) A resistor at temperature T produces the noise voltage $v_n(t)$. (b) The random noise voltage generated by a resistor at temperature T.

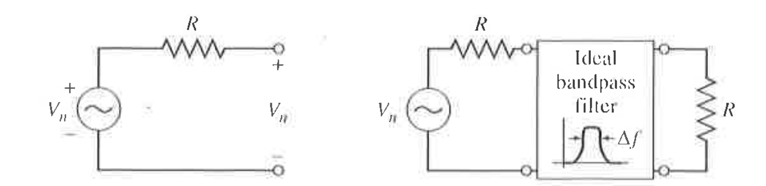

FIGURE 3.2 (a) The Thevenin equivalent circuit for a noisy resistor. (b) Maximum power transfer of noise power from a noisy resistor to a load over a bandwidth B.

The result in (3.24) is known as the Rayleigh–Jeans approximation, and is valid for frequencies up through the microwave band [4].

The noisy resistor can be modeled using a Thevenin equivalent circuit as an ideal (noiseless) resistor with a voltage generator to represent the noise voltage, as shown in Figure 3.2a. The *available noise power* is defined as the maximum power that can be delivered from the noise source to a load resistor. As shown in Figure 3.2b, maximum power transfer occurs when the load is conjugately matched to the source. Then the available noise power can be calculated as

$$P_n = \left(\frac{V_n}{2}\right)^2 \frac{1}{R} = \frac{V_n^2}{4R} = kTB, \tag{3.25}$$

where V_n is the rms noise voltage of the resistor. This is a fundamental result that is useful in a wide variety of problems involving noise. Note that the noise power decreases as the system bandwidth decrease. This implies that systems with smaller bandwidths collect less noise power. Also note that noise power decreases as temperature decreases, which implies that internally generated noise effects can be reduced by cooling a system to low temperatures. Finally, note that the noise power of (3.25) depends on absolute bandwidth, but not on the center frequency of the band. Since thermal noise power is independent of frequency, it is referred to as *white noise*, because of the analogy with white light and its makeup of all other visible light frequencies. It has been found experimentally, and verified by quantum mechanics, that thermal noise is independent of frequency for $0 < f < 1000$ GHz.

The noise power of (3.25) can also be represented in terms of the power spectral density according to (3.23). Since the power given by (3.25) is independent of frequency, the power spectral density must also be independent of frequency, and so we have that,

$$S_n(\omega) = \frac{P_n}{2B} = \frac{kT}{2} = \frac{n_0}{2}. \tag{3.26}$$

This is known as the *two-sided power spectral density* of thermal noise, meaning that the frequency range from $-B$ to B (Hz) is included in the integration of (3.23). This is the conventional definition as used in communication systems work. The notation defined in (3.26), where $n_0/2 = kT/2$ is the two-sided power spectral density for white noise, will be used throughout this book. (Note that n_0 is a constant, with the subscript 'zero'. This should not be confused with the notation $n_o(t)$, which we will often use to denote a noise output signal. The subscript for this latter notation is 'oh', and will always be written as a function of time.)

Since the power spectral density of thermal noise is constant with frequency, its autocorrelation must be a delta function according to (3.22b):

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{n_0}{2}e^{-j\omega\tau}\,d\omega = \frac{n_0}{2}\delta(\tau). \tag{3.27}$$

By the central limit theorem, the probability density function of white noise is gaussian

FIGURE 3.3 Circuit for Example 3.2.

with zero mean:

$$f_n(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-n^2/2\sigma^2}, \tag{3.28}$$

where σ^2 is the variance of the gaussian noise. Thermal noise having a zero mean gaussian PDF is known as *white gaussian noise*. Since the variance of the sum of two independent random variables is the sum of the individual variances (see Problem 3.5), and the variance is equivalent to power delivered to a 1 Ω load, the noise powers generated by two independent noise sources add in a common load. This is in contrast to the case of deterministic sources, where voltages add.

Note that (3.27) is not completely consistent with (3.28), since (3.27) indicates that $R(0)$, the variance of white noise, is infinite while (3.28) implies a finite variance. This problem arises because of the mathematical assumption that white noise has a constant power spectral density, and therefore infinite power. In fact, as we discussed earlier, thermal noise has a constant PSD only over a finite, but very wide, frequency band. We can resolve this issue if we understand our use of the concept of white noise to actually mean a bandlimited PSD having a finite frequency range, but broader than the system bandwidth with which we are working.

EXAMPLE 3.2 CALCULATION OF NOISE POWER

Two noisy resistors, R_1 and R_2, at temperature T, are shown in Figure 3.3. Calculate the available noise power from these sources by considering the individual noise power from each resistor separately. Next, consider the resistors as equivalent to a single resistor of value $R_1 + R_2$, and verify that the same available noise power is obtained. Assume a bandwidth B for the system.

Solution

The equivalent noise voltage from each resistor is found from (3.24):

$$V_{n1} = \sqrt{4kTBR_1}$$
$$V_{n2} = \sqrt{4kTBR_2}.$$

For maximum power transfer, the load resistance should be $R_1 + R_2$. Then the noise power delivered to the load from each noise source is

$$P_{n1} = \left(\frac{V_{n1}}{2}\right)^2 \frac{1}{R_1 + R_2} = \frac{kTBR_1}{R_1 + R_2}$$
$$P_{n2} = \left(\frac{V_{n2}}{2}\right)^2 \frac{1}{R_1 + R_2} = \frac{kTBR_2}{R_1 + R_2}.$$

So the total available noise power is

$$P_n = P_{n1} + P_{n2} = kTB.$$

Considering the two resistors as a single resistor of value $R_1 + R_2$, with a load resistance of $R_1 + R_2$, gives an available noise power of

$$P_n = kTB,$$

in agreement with the first result. ○

3.3 NOISE IN LINEAR SYSTEMS

In a wireless radio receiver, both desired signals and undesired noise pass through various stages, such as RF amplifiers, filters, and mixers. These functions generally alter the statistical properties of the noise, and so it is useful to study these effects by considering the general case of transmission of noise through a linear system. We then consider some important special cases, such as filters and integrators, and the nonlinear situation where noise undergoes frequency conversion by mixing.

Autocorrelation and Power Spectral Density in Linear Systems

In the case of deterministic signals, we can find the response of a linear time-invariant system to an input excitation in the time domain by using convolution with the impulse response of the system, or in the frequency domain by using the transfer function of the system. Similar results apply to wide-sense stationary random processes, in terms of either the autocorrelation function or the power spectral density.

Consider the linear time-invariant system shown in Figure 3.4, where the input random process, $x(t)$, has an autocorrelation $R_x(\tau)$ and power spectral density $S_x(\omega)$, and the output random process, $y(t)$, has an autocorrelation $R_y(\tau)$ and power spectral density $S_y(\omega)$. If the impulse response of the system is $h(t)$, we can calculate the output response as

$$y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)\,du. \tag{3.29a}$$

Similarly, a time-shifted version of $y(t)$ is

$$y(t+\tau) = \int_{-\infty}^{\infty} h(v)x(t+\tau-v)\,dv. \tag{3.29b}$$

So the autocorrelation of $y(t)$ can be found as

$$R_y(\tau) = E\{y(t)y(t+\tau)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)E\{x(t-u)x(t+\tau-v)\}\,du\,dv$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)R_x(\tau+u-v)\,du\,dv. \tag{3.30}$$

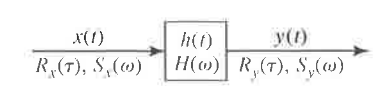

FIGURE 3.4 A linear system with an impulse response $h(t)$ and transfer function $H(\omega)$. The input is a random process $x(t)$, having autocorrelation $R_x(\tau)$ and *PSD* $S_x(\omega)$. The output random process is $y(t)$, having autocorrelation $R_y(\tau)$ and *PSD* $S_y(\omega)$.

FIGURE 3.5 System symbols and frequency responses for low-pass, high-pass, and band-pass filters.

This result shows that the autocorrelation of the output is given by the double convolution of the autocorrelation of the input with the impulse response: $R_y(\tau) = h(\tau) \otimes h(-\tau) \otimes R_x(\tau)$. We can derive the equivalent result in terms of power spectral density by taking the Fourier transform of both sides of (3.30), in view of (3.22a):

$$\int_{-\infty}^{\infty} R_y(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)\int_{-\infty}^{\infty} R_x(\tau+u-v)e^{-j\omega\tau}\,d\tau\,du\,dv.$$

Now perform a change of variable to $\alpha = \tau + u - v$, so that $d\alpha = d\tau$. Then we obtain

$$\int_{-\infty}^{\infty} R_y(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} h(u)e^{j\omega u}\int_{-\infty}^{\infty} h(v)e^{-j\omega v}\int_{-\infty}^{\infty} R_x(\alpha)e^{-j\omega\alpha}\,d\alpha\,du\,dv.$$

Since H(ω) is the Fourier transform of $h(t)$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt, \tag{3.31}$$

the above simplifies to the following important result:

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega). \tag{3.32}$$

We will now demonstrate the utility of these results with several applications.

Gaussian White Noise through an Ideal Low-pass Filter

As we will see in Chapters 5, 9, and 10, filters play an important role in wireless receivers and transmitters. The main function of a filter is to provide *frequency selectivity*, by allowing a certain range of frequencies to pass, while blocking other frequencies. Figure 3.5 shows the symbols and associated idealized frequency responses for low-pass, high-pass, and bandpass filters. Here we examine the effect of an ideal low-pass filter on noise.

Figure 3.6 shows white noise passing through a low-pass filter. The filter has a transfer function, $H(f)$, as shown, with a cutoff frequency of Δf. Note that the transfer function is defined for both positive and negative frequency, since we will be using the two-sided power spectral density. Our usual notation will be to use lowercase letters, such as $n_i(t)$ and

FIGURE 3.6 White noise passing through an ideal low-pass filter, and the transfer function of the filter.

$n_o(t)$, for noise and signal voltages in the time domain, and capital letters, such as N_i and N_o, for average powers of noise and signals.

Since the input noise is white, the two-sided power spectral density of the input noise is constant, as given in (3.26):

$$S_{n_i}(f) = \frac{n_0}{2} \text{ (all f)}. \tag{3.33}$$

Then from (3.32) the output power spectral density is given by

$$S_{n_o}(f) = |H(f)|^2 S_{n_i}(f) = \begin{cases} \frac{n_0}{2} & \text{for } |f| < \Delta f \\ 0 & \text{for } |f| > \Delta f \end{cases}. \tag{3.34}$$

The output noise power is then

$$N_0 = (2\Delta f)S_{n_o}(f) = \Delta f n_0. \tag{3.35}$$

We see that the output noise power is proportional to the filter bandwidth.

Gaussian White Noise through an Ideal Integrator

As we will see in Chapter 9, integrators are critical components for the detection and demodulation of digital signals. Here we derive an expression for the output noise power from an integrator with white noise input; this result will be used later for the derivation of error probabilities for digital modulation in Chapter 9.

Figure 3.7 shows a noise signal, $n_i(t)$, applied to the input of an ideal integrator. The output noise signal is $n_o(t)$. The output of the integrator is the value of the integral, at time $t = T$, of the input signal. We need to find the average power of the output noise.

The transfer function of the integration operation is

$$H(\omega) = \frac{1}{j\omega}(1 - e^{-j\omega T}), \tag{3.36}$$

where T is the integration interval time. Evaluating the magnitude squared of (3.36) gives

$$\begin{aligned} |H(\omega)|^2 = H(\omega)H^*(\omega) &= \frac{(1-e^{-j\omega T})(1-e^{j\omega T})}{\omega^2} = \frac{2 - 2\cos\omega T}{\omega^2} \\ &= \frac{4\sin^2\omega T/2}{\omega^2} = \frac{\sin^2\pi f T}{(\pi f)^2} = T^2\left(\frac{\sin\pi f T}{\pi f T}\right)^2, \end{aligned} \tag{3.37}$$

since $\omega = 2\pi$ f.

If we assume white noise at the input, with $S_{n_i}(\omega) = n_0/2$, then the output noise power can be calculated using (3.23) and (3.32) to give

$$\begin{aligned} N_o &= \int_{-\infty}^{\infty} \frac{n_0}{2}|H(f)|^2\,df = \frac{n_0 T^2}{2}\int_{-\infty}^{\infty}\left(\frac{\sin\pi f T}{\pi f T}\right)^2 df \\ &= \frac{n_0 T}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{n_0 T}{2} \end{aligned} \tag{3.38}$$

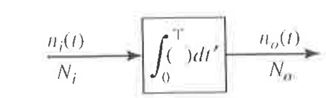

FIGURE 3.7 White noise passing through an ideal integrator.

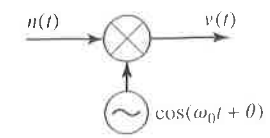

FIGURE 3.8 White noise passing through a mixer with a local oscillator signal, $\cos(\omega_0 t + \theta)$.

The integral was evaluated by using a change of variables, $x = \pi f T$, with $dx = \pi T df$, and a standard integral listed in Appendix B.

Mixing of Noise

One of the common functions of a receiver is to perform *frequency conversion*, by mixing a signal with a local oscillator to shift the original signal spectrum up or down in frequency. When noise coexists with the signal, the noise spectrum will also be shifted in frequency. While we will study mixers in detail in Chapter 7, here we idealize the function of mixing by considering it as a process of multiplication of the input signal by a local oscillator signal, as shown in Figure 3.8. We wish to find the average noise power of the output signal.

We assume that $n(t)$ is a bandlimited white gaussian noise signal with variance, or average power, $\sigma^2 = E\{n^2(t)\}$. The local oscillator signal is given by $\cos(\omega_0 t + \theta)$, where the phase, θ, is a random variable uniformly distributed on the interval $0 < \theta \leq 2\pi$, and is independent of $n(t)$. The output of the idealized mixer is

$$v(t) = n(t)\cos(\omega_0 t + \theta). \tag{3.39}$$

The average output power from the mixer can then be calculated as the variance of $v(t)$:

$$\begin{aligned} N_o &= E\{v^2(t)\} = E\{n^2(t)\cos^2(\omega_0 t + \theta)\} = E\{n^2(t)\}E\{\cos^2(\omega_0 t + \theta)\} \\ &= \frac{\sigma^2}{2\pi}\int_0^{2\pi} \cos^2(\omega_0 t + \theta)\, d\theta = \frac{\sigma^2}{2} \end{aligned} \tag{3.40}$$

This result shows that mixing reduces the average noise power by half. In this case, the factor of one-half is due to the ensemble averaging of the $\cos^2 \omega_0 t$ term over the range of random phase.

If we now consider a deterministic local oscillator signal of the form $\cos \omega_0 t$ (without a random phase), the variance of the output signal becomes

$$E\{v^2(t)\} = E\{n^2(t)\cos^2 \omega_0 t\} = \sigma^2 \cos^2 \omega_0 t. \tag{3.41}$$

The last result follows because the $\cos^2 \omega_0 t$ term is unaffected by the expected value operator, since it is no longer a random variable. (In addition, $v(t)$ is no longer stationary, and therefore does not have an autocorrelation function or power spectral density.) The variance of the output signal is now a function of time, and represents the instantaneous power of the output signal. To find the time-average output power, we must take the time-average of the variance found in (3.41):

$$N_0 = \frac{1}{T}\int_0^T E\{v^2(t)\}\, dt = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \sigma^2 \cos^2 \omega_0 t dt = \frac{\sigma^2}{2}, \tag{3.42}$$

since $T = 1/f = 2\pi/\omega_0$. We see that the same average output power is obtained whether the averaging is over the ensemble phase variation, or over time.

EXAMPLE 3.3 MIXING NOISE

Consider the complex mixing product $x(t) = n(t)e^{j\omega_0 t}$, formed by mixing noise voltage $n(t)$ with a complex exponential. If the autocorrelation and PSD of $n(t)$ are $R_n(\tau)$ and $S_n(\omega)$, find the autocorrelation and PSD of $x(t)$.

Solution

Using the definition of autocorrelation for random processes given in (3.21) we have

$$\begin{aligned} R_x(\tau) &= E\{x^*(t)x(t+\tau)\} = E\{n(t)e^{-j\omega_0 t} n(t+\tau)e^{j\omega_0(t+\tau)}\} \\ &= E\{n(t)n(t+\tau)\}e^{j\omega_0\tau} = R_n(\tau)e^{j\omega_0\tau} \end{aligned}$$

Note that $x(t)$ is still a stationary process, since it has a proper autocorrelation function. From (3.22a) the power spectral density is

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega_0\tau}\, d\tau = \int_{-\infty}^{\infty} R_n(\tau)e^{-j(\omega-\omega_0)\tau}\, d\tau = S_n(\omega - \omega_0),$$

where the last result follows by replacing ω with $\omega - \omega_0$ in $S_n(\omega) = \int_{-\infty}^{\infty} R_n(\tau)e^{-j\omega\tau}d\tau$. ○

Narrowband Representation of Noise

In many receiver circuits, signals and noise are passed through a bandpass filter. In this case, it becomes possible to represent the noise in a form that is more convenient for analysis. This is called the *narrowband representation* of noise, a result that will be very useful in Chapter 9 when analyzing the effect of noise on the demodulation of signals.

Figure 3.9 shows gaussian white noise passing through a bandpass filter with a center frequency ω_0 and bandwidth $\Delta\omega$. If the two-sided power spectral density of the input noise is $n_0/2$, then the PSD of the output noise, $n(t)$, is as shown in the figure. If $\Delta\omega \ll \omega_0$, then $n(t)$ can be represented as

$$n(t) = x(t)\cos \omega_0 t + y(t)\sin \omega_0 t, \tag{3.43}$$

where $x(t)$ and $y(t)$ are random processes, but are slowly varying due to the narrow bandwidth of the filter. To show that the above representation is valid, consider the circuit of Figure 3.10, which can be used to generate $x(t)$ and $y(t)$. Here the noise $n(t)$ is divided and mixed separately with $2\cos \omega_0 t$ and $2\sin \omega_0 t$, which produces the following results:

$$\begin{aligned} 2n(t)\cos \omega_0 t &= 2x(t)\cos^2 \omega_0 t + 2y(t)\sin \omega_0 t \cos \omega_0 t \\ &= x(t) + x(t)\cos 2\omega_0 t + y(t)\sin 2\omega_0 t \\ 2n(t)\sin \omega_0 t &= 2x(t)\cos \omega_0 t \sin \omega_0 t + 2y(t)\sin^2 \omega_0 t \\ &= y(t) - y(t)\cos 2\omega_0 t + x(t)\sin 2\omega_0 t. \end{aligned}$$

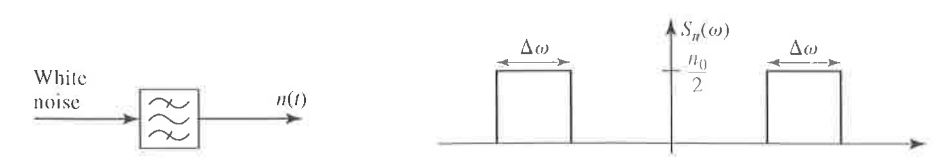

FIGURE 3.9 White noise passing through a bandpass filter, and the power spectral density of the output noise.

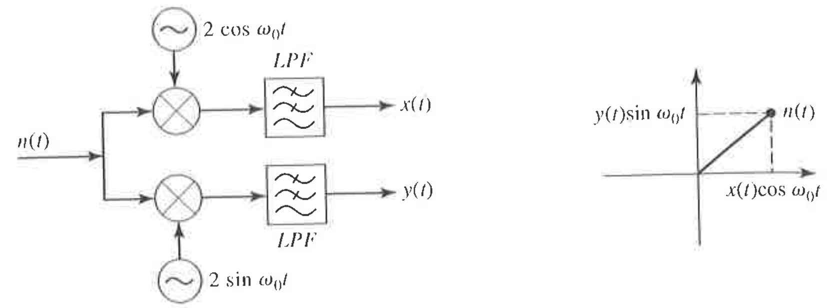

FIGURE 3.10 Circuit to generate low-pass noise $x(t)$ and $y(t)$ from a bandpass noise source $n(t)$.

After low-pass filtering with a low-pass cutoff frequency of $f_c = \Delta\omega/4\pi$, only the $x(t)$ and $y(t)$ terms will remain in the above results. Thus the output noises $x(t)$ and $y(t)$ are limited in bandwidth to $\Delta\omega/2$, and can be viewed as the bandpass noise at ω_0 shifted down to zero frequency. Since $x(t)$ and $y(t)$ represent the *in-phase* and *quadrature* components of $n(t)$, as indicated in the phasor diagram of Figure 3.10, (3.43) is also known as the *quadrature representation* of narrowband noise. We now find the statistics of $x(t)$ and $y(t)$, and show that these gaussian random processes have zero mean, the same variance as $n(t)$, and are uncorrelated. We will also find the power spectral densities of $x(t)$ and $y(t)$.

Since $E\{n(t)\} = 0$, we have from (3.43) that

$$E\{n(t)\} = 0 = E\{x(t)\}\cos\omega_0 t + E\{y(t)\}\sin\omega_0 t.$$

Since $\cos\omega_0 t$ and $\sin\omega_0 t$ vary differently with time, we must have

$$E\{x(t)\} = E\{y(t)\} = 0. \tag{3.44}$$

Next, we evaluate the autocorrelation of $n(t)$ using (3.43):

$$\begin{aligned}
R_n(\tau) &= E\{n(t)n(t+\tau)\} \\
&= E\{[x(t)\cos\omega_0 t + y(t)\sin\omega_0 t][x(t+\tau)\cos\omega_0(t+\tau) + y(t+\tau)\sin\omega_0(t+\tau)]\} \\
&= R_x(\tau)\cos\omega_0 t\cos\omega_0(t+\tau) + R_{xy}(\tau)\cos\omega_0 t\sin\omega_0(t+\tau) \\
&\quad + R_{yx}(\tau)\sin\omega_0 t\cos\omega_0(t+\tau) + R_y(\tau)\sin\omega_0 t\sin\omega_0(t+\tau)
\end{aligned}$$

where $R_{xy}(\tau) = E\{x(t)y(t+\tau)\}$ and $R_{yx}(\tau) = E\{y(t)x(t+\tau)\}$ are the cross-correlation functions of $x(t)$ and $y(t)$. Using standard identities to expand the trigonometric products gives the following:

$$\begin{aligned}
R_n(\tau) = {} & \tfrac{1}{2}R_x(\tau)[\cos\omega_0\tau + \cos\omega_0(2t+\tau)] \\
& + \tfrac{1}{2}R_{xy}(\tau)[\sin\omega_0\tau + \sin\omega_0(2t+\tau)] \\
& + \tfrac{1}{2}R_{yx}(\tau)[-\sin\omega_0\tau + \sin\omega_0(2t+\tau)] \\
& + \tfrac{1}{2}R_y(\tau)[\cos\omega_0\tau - \cos\omega_0(2t+\tau)].
\end{aligned} \tag{3.45}$$

Because $n(t)$ is a stationary process, its autocorrelation must be a function only of τ, and cannot vary with t. Thus the coefficients of $\cos\omega_0(2t+\tau)$ and $\sin\omega_0(2t+\tau)$ must vanish.

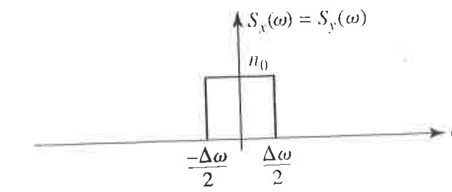

FIGURE 3.11 Power spectral density of $x(t)$ and $y(t)$.

This gives

$$R_x(\tau) = R_y(\tau) \tag{3.46a}$$

$$R_{xy}(\tau) = -R_{yx}(\tau), \tag{3.46b}$$

and then (3.45) reduces to

$$R_n(\tau) = R_x(\tau)\cos\omega_0\tau + R_{xy}(\tau)\sin\omega_0\tau = R_y(\tau)\cos\omega_0\tau - R_{yx}(\tau)\sin\omega_0\tau. \tag{3.47}$$

This result shows that $n(t)$, $x(t)$, and $y(t)$ all have the same variance, since $R_n(0) = R_x(0) = R_y(0)$.

We can also find $R_n(\tau)$ directly by evaluating the inverse Fourier transform of $S_n(\omega)$, which is shown in Figure 3.9. Since $R_n(\tau)$ is real, and symmetric about $\omega = 0$, we have

$$\begin{aligned}
R_n(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_0^{\infty} S_n(\omega)\cos\omega\tau\,d\omega = \frac{n_0}{2\pi}\int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2}\cos\omega\tau\,d\omega \\
&= \frac{n_0}{2\pi\tau}\left[\sin\left(\omega_0 + \frac{\Delta\omega}{2}\right)\tau - \sin\left(\omega_0 - \frac{\Delta\omega}{2}\right)\tau\right] = \frac{n_0}{\pi\tau}\sin\frac{\Delta\omega\tau}{2}\cos\omega_0\tau.
\end{aligned} \tag{3.48}$$

Comparing (3.48) to (3.47) shows that

$$R_x(\tau) = R_y(\tau) = \frac{n_0}{\pi\tau}\sin\frac{\Delta\omega\tau}{2} \tag{3.49a}$$

$$R_{xy}(\tau) = R_{yx}(\tau) = 0. \tag{3.49b}$$

The fact that $R_{xy}(\tau) = 0$ implies that $x(t)$ and $y(t)$ are statistically independent. Finally, we can find the PSD of $x(t)$ and $y(t)$ by taking the Fourier transform of $R_x(\tau)$:

$$S_x(\omega) = S_y(\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau}\,d\tau = \begin{cases} n_0 & \text{for } |\omega| < \Delta\omega/2 \\ 0 & \text{for } |\omega| > \Delta\omega/2 \end{cases} \tag{3.50}$$

where the required Fourier transform may be found in Appendix C. This power spectral density is shown in Figure 3.11, and can be viewed as the bandlimited PSD of $n(t)$ shifted up and down in frequency by the amount ω_0, and low-pass filtered. Note that the peak value of the PSD of $x(t)$ and $y(t)$ is n_0, twice that of the PSD of $n(t)$.

3.4 BASIC THRESHOLD DETECTION

We now have enough background in the topics of noise and systems to discuss an application to basic threshold detection. Threshold detection is relevant to most digital modulation schemes, and so we will see this topic again in more detail in Chapter 9. Here we evaluate the probability of error for a simple binary communications channel.

FIGURE 3.12 Input signal and noise voltage for a basic threshold detection system.

Consider a communications system where binary signals are transmitted in the presence of bandlimited white gaussian noise. Thus, the received signal, $r(t)$, can be written as the transmitted signal voltage, $s(t)$, plus a noise voltage, $n(t)$:

$$r(t) = s(t) + n(t) \tag{3.51}$$

where $n(t)$ has zero mean and variance σ^2. A sketch of a possible received voltage is shown in Figure 3.12.

When a binary "1" is transmitted the signaling voltage will be $s(t) = v_0$, and when a binary "0" is transmitted we will have $s(t) = 0$. The receiver must be designed to process the received voltage, and detect whether a "1" or a "0" has been transmitted. In the absence of noise we can simply sample the receive voltage and determine whether it is above or below a *threshold* level. In this case, the logical choice for a threshold voltage would be $v_0/2$, so that if $r(t) > v_0/2$ the receiver would detect a "1," and if $r(t) < v_0/2$ the receiver would detect a "0." This detection process can be implemented using a simple sampler and comparator circuit. In practical receivers threshold detection could incorporate matched filters or integrators to minimize the effect of noise, but here we consider only the sampling of the received signal at its maximum or minimum point.

Because the possible noise voltage amplitude ranges from $-\infty$ to ∞, the received signal may sometimes be less than the threshold when a "1" has been sent, and may be greater than the threshold when a "0" has been sent. Either of these cases will result in a detection error. In fact, there are four detection possibilities, as listed in Table 3.1.

Probability of Error

We can now find the *probability of error* for threshold detection. We define $P_e^{(1)}$ as the probability of an error in detection when a binary "1" has been transmitted, and $P_e^{(0)}$ as the probability of an error when a binary "0" has been sent. Knowing these two probabilities then defines the likelihood of all the outcomes in Table 3.1, since the probability of a correct outcome is 1 − (probability of an error).

TABLE 3.1 Possible Outcomes of Threshold Detection

Transmitted Binary Data	$s(t)$	$r(t) > v_0/2$?	Detection Outcome	Correct Detection
0	0	no	0	yes!
0	0	yes	1	error
1	v_0	yes	1	yes!
1	v_0	no	0	error

When a binary "1" is transmitted, a detection error will occur if the received signal and noise is less than the threshold level at the sampling time. For a threshold of $v_0/2$, the probability of this event is

$$\begin{aligned} P_e^{(1)} &= P\{r(t) = v_0 + n(t) < v_0/2\} \\ &= \int_{-\infty}^{v_0/2} f_r(r)\,dr = \int_{-\infty}^{v_0/2} \frac{e^{-(r-v_0)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dr, \end{aligned} \tag{3.52}$$

where we have used (3.1), (3.5), and the gaussian probability density function given in (3.12b). Since $n(t)$ is gaussian with zero mean, the receive signal $r(t)$ is also gaussian, but with a mean value of v_0 when a binary "1" is being transmitted. The expression in (3.52) can be reduced to a standard form by using the change of variable $x = (v_0 - r)/\sqrt{2\sigma^2}$. Then we have

$$P_e^{(1)} = \frac{1}{\sqrt{\pi}} \int_{x_0}^{\infty} e^{-x^2}\,dx, \tag{3.53}$$

where the lower limit is

$$x_0 = \frac{v_0}{2\sqrt{2\sigma^2}}. \tag{3.54}$$

The integral occurring in (3.53) is related to the *complementary error function*, written as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2}\,du. \tag{3.55}$$

Details on properties of the complementary error function, including an algorithm for calculating $erfc(x)$, can be found in Appendix D. Using the definition of (3.55) allows (3.53) to be written as

$$P_e^{(1)} = \frac{1}{2} erfc(x_0) = \frac{1}{2} erfc\left(\frac{v_0}{2\sqrt{2\sigma^2}}\right), \tag{3.56}$$

which is our final expression for $P_e^{(1)}$. By a similar analysis we can find $P_e^{(0)}$, the probability of error when a binary "0" is sent. It is left as a problem to show that $P_e^{(0)} = P_e^{(1)}$, as might be expected from the symmetry resulting from a threshold of $v_0/2$. The result of (3.56) is dependent on the ratio v_0/σ, which can be considered a *signal-to-noise ratio* (SNR), since v_0 is the maximum signal voltage, and σ is the rms value of the noise voltage. Since $erfc(x)$ decreases monotonically with x, large SNR results in lower probability of error.

A graphical interpretation of threshold detection is shown in Figure 3.13. The probability density functions are shown for the received signal and noise for the two cases of sending a binary "0" or a "1." The former has a PDF centered at $r = 0$, while the latter has

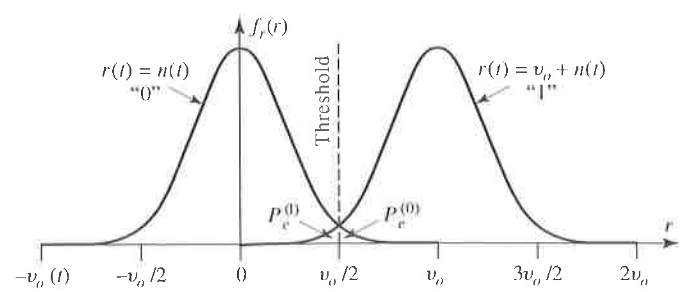

FIGURE 3.13 Graphical interpretation of the probability of error for threshold detection.

a PDF centered at $r = v_0$. The threshold of $v_0/2$ is located midway between these values. The probabilities of error are the areas of the tails of the two PDFs either above or below the threshold value.

EXAMPLE 3.4 PROBABILITY OF ERROR FOR THRESHOLD DETECTION

Calculate and plot the probability of error for threshold detection versus the signal-to-noise ratio, v_0/σ, in dB. Use a logarithmic scale for the probability of error.

Solution

Since we are dealing with voltages, the signal-to-noise ratio in dB is calculated as

$$\frac{v_0}{\sigma}(dB) = 20 \log \frac{v_0}{\sigma}.$$

Then (3.56) can be used to evaluate $P_e^{(1)}$. The algorithm of Appendix D can be used to calculate values of the complementary error function. A sample calculation follows for $v_0/\sigma = 6$ dB:

For $v_0/\sigma = 6$ dB we have a numerical value of

$$\frac{v_0}{\sigma} = 10^{6/20} = 2.0.$$

Then the argument of the complementary error function is, from (3.54)

$$x_0 = \frac{v_0}{2\sqrt{2}\sigma} = \frac{2.0}{2\sqrt{2}} = 0.707.$$

Equation (3.56) gives

$$P_e^{(1)} = \tfrac{1}{2} erfc(x_0) = \tfrac{1}{2} erfc(0.707) = \tfrac{1}{2}(0.317) = 0.159.$$

The same method can be used for other values of v_0/σ, and the result is plotted in Figure 3.14. Note that for large values of *S/N* the probability of error becomes very small. Error probabilities in the range of 10^{-5} to 10^{-8} are often desired in practice. ○

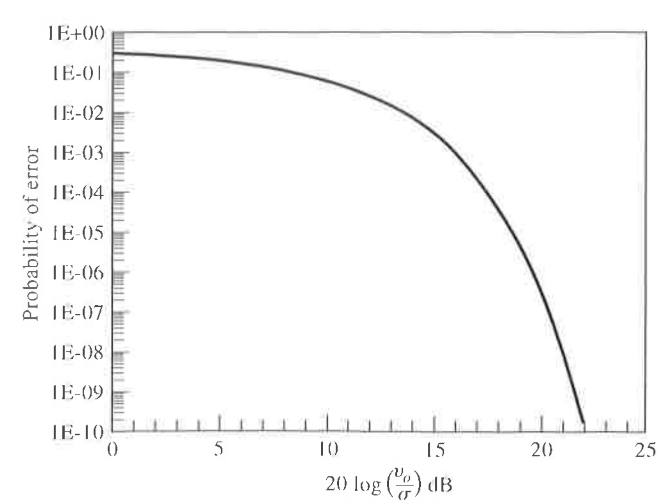

FIGURE 3.14 Probability of error versus signal-to-noise ratio for threshold detection.

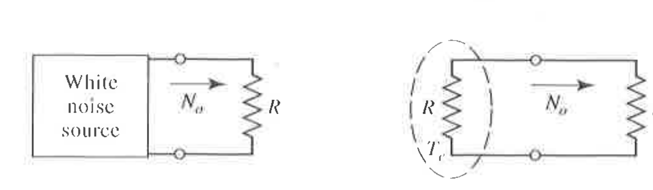

FIGURE 3.15 Equivalent noise temperature of an arbitrary white noise source.

3.5 NOISE TEMPERATURE AND NOISE FIGURE

Besides being received from the external environment by the antenna, noise is also generated by passive and active components of a wireless receiver system. In this section we will study ways of characterizing the noise properties of components such as amplifiers, mixers, couplers, and filters, and the transmission of noise through a multistage system.

Equivalent Noise Temperature

If an arbitrary noise source is white, so that its power spectral density is not a function of frequency (at least over the frequency range of interest), it can be modeled as an equivalent thermal noise source, and characterized by an *equivalent noise temperature*. This situation is illustrated in Figure 3.15, where an arbitrary white noise source of driving point impedance R delivers noise power N_o to a load resistor R. This noise source can be replaced with a noisy resistor of value R, at temperature T_e, where T_e is an equivalent temperature selected so that the same noise power is delivered to the load. Thus

$$T_e = \frac{N_o}{kB}. \tag{3.57}$$

Wireless components and receiver systems can then be characterized in terms of their equivalent noise temperature, T_e, expressed in degrees Kelvin (K). Note that $T_e \geq 0$, and may be greater or less than $T_0 = 290$ K. In addition, note that the result in (3.57) implies some fixed bandwidth, B, which is generally the bandwidth of the component or system. As an example, consider a noisy amplifier having bandwidth B and power gain G. Let the amplifier be matched to noiseless source and load resistors, as shown in Figure 3.16a. If the source resistor of Figure 3.16a is at a (hypothetical) temperature of $T_s = 0$ K, then the input noise power to the amplifier will be $N_i = 0$, and the output noise power N_o will be due only to the noise generated by the amplifier itself. We can obtain the same output noise power

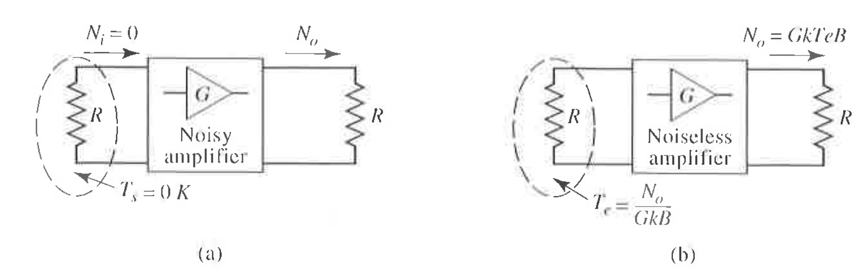

FIGURE 3.16 Equivalent noise temperature of a noisy amplifier. (a) Noisy amplifier. (b) Equivalent noiseless amplifier.

by driving an ideal noiseless amplifier with a resistor at a temperature

$$T_e = \frac{N_o}{GkB}, \tag{3.58}$$

so that the output noise power in both cases is $N_o = GkT_eB$, as illustrated in Figure 3.16b. Then T_e is the equivalent noise temperature of the amplifier. Note the important point that the equivalent noise source is applied at the *input* to the device, and that the input noise power, $N_i = kT_eB$, must be multiplied by the gain of the amplifier to obtain the output noise power. This is the convention that we will use throughout most of this book, but be aware that it is also possible to reference the equivalent noise source at the output of the component.

Measurement of Noise Temperature

A direct way to measure equivalent noise temperature is to drive the component with a known noise source, and measure the increase in output noise power. Because the thermal noise power generated by a resistor is so small, active noise sources are available for this purpose. Active noise sources use a diode or an electron tube to provide a calibrated noise power output, and are useful for laboratory tests and measurements. Active noise generators can be characterized by their equivalent noise temperature, but a more common measure of noise power for such components is the *excess noise ratio* (ENR), defined as

$$ENR(dB) = 10\log\frac{N_g - N_0}{N_0} = 10\log\frac{T_g - T_0}{T_0}, \tag{3.59}$$

where N_g and T_g are the noise power and equivalent temperature of the noise generator, and $N_0 = kT_0B$ and $T_0 = 290$ K are the noise power and temperature associated with a passive source (a matched load) at room temperature. Solid-state noise generators typically have ENRs ranging from 20 to 40 dB, meaning that their noise power output is 100 to 10,000 times greater than the thermal noise power from a matched load at room temperature.

The problem with direct measurement of noise temperature is that most components generate only small levels of noise, which are difficult to measure reliably. Instead, a ratio technique, called the *Y-factor method*, is often used in practice. This method is illustrated in Figure 3.17, where the device under test is connected to one of two different matched loads at different temperatures. Let T_1 be the temperature of the hotter load, and T_2 the temperature of the cooler load, and let the respective output noise powers be N_1 and N_2. Since the source noise is uncorrelated with the noise of the device under test, the total output noise powers for the two cases can be written as

$$N_1 = GkT_1B + GkT_eB \tag{3.60a}$$

$$N_2 = GkT_2B + GkT_eB, \tag{3.60b}$$

where the first term is due to the input noise power, and the second term is due to the noise generated by the device under test. This set of equations has two unknowns: T_e (the desired

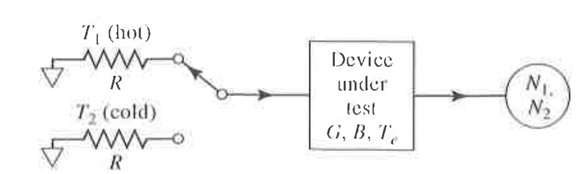

FIGURE 3.17 The Y-factor method for measuring equivalent noise temperature.

noise temperature of the device under test), and GB (the gain-bandwidth product of the amplifier). Define the Y factor as

$$Y = \frac{N_1}{N_2} = \frac{T_1 + T_e}{T_2 + T_e} \geq 1, \tag{3.61}$$

which is a ratio determined via power measurements, and does not depend on GB. Then (3.60) can be solved for the equivalent noise temperature of the device under test:

$$T_e = \frac{T_1 - YT_2}{Y - 1}. \tag{3.62}$$

To obtain accurate results with this method, the two source temperatures should not be too close together, so that Y is not close to unity. In practice, one noise source may be a resistor at room temperature, while the other is either hotter or colder, depending on whether T_e is greater or lesser than T_0. An active noise source can be used as a "hot" source, while a "cold" source can be obtained by immersing a load resistor in liquid nitrogen ($T = 77$ K), or liquid helium ($T = 4$ K).

Noise Figure

We have seen that noisy RF and microwave components can be characterized by an equivalent noise temperature. An alternative characterization is the *noise figure*, which can be viewed as a measure of the degradation in the signal-to-noise ratio between the input and output of the component. When noise and a desired signal are applied to the input of a noiseless network, both noise and signal will be attenuated or amplified by the same factor, so that the SNR will be unchanged. But if the network is noisy, the output noise power will be increased to a greater degree than the output signal power, so that the output SNR will be reduced. The noise figure, F, is a measure of this reduction in SNR, and is defined as

$$F = \frac{S_i/N_i}{S_o/N_o} \geq 1, \tag{3.63}$$

where S_i and N_i are the input signal and noise powers, and S_o and N_o are the output signal and noise powers. By definition, the input noise power must be the noise power from a matched load at $T_0 = 290$ K; that is, $N_i = kT_0B$. Noise figure is usually expressed in dB, obtained as $F(\text{dB}) = 10\log F$. (Note: some authors define the numerical value of F as the *noise factor*, and the corresponding value in dB as the noise figure, but we will not make this distinction.)

We can establish the relation between noise figure and equivalent noise temperature by referring to Figure 3.18, which shows noise power N_i and signal power S_i being fed into a noisy two-port network. The network is characterized by a power gain G, a bandwidth B, and an equivalent noise temperature T_e. Note that the input noise power is $N_i = kT_0B$, as required by the definition of noise figure. The output signal power is $S_o = GS_i$, while

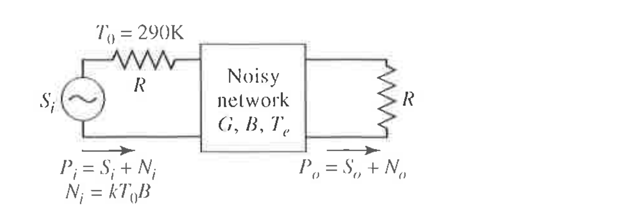

FIGURE 3.18 Relating the noise figure of a noisy network to its equivalent noise temperature.

the output noise power is the sum of the amplified input noise power and the internally generated noise:

$$N_o = kGB(T_0 + T_e).$$

Using these results in (3.63) gives the noise figure as

$$F = \frac{S_i}{kT_0B}\frac{kGB(T_0 + T_e)}{GS_i} = 1 + \frac{T_e}{T_0} \geq 1. \tag{3.64}$$

This result can be solved for T_e in terms of F to give

$$T_e = (F - 1)T_0. \tag{3.65}$$

If the network were perfectly noiseless, its equivalent noise temperature would be zero, and its noise figure would be unity, or 0 dB. These results show that equivalent noise temperature and noise figure are interchangeable ways of characterizing the noise properties of RF and microwave components. In practice, mixers and amplifiers are usually specified in terms of noise figure, while antennas and receivers are often specified in terms of noise temperature.

Again referring to the two-port network of Figure 3.18, if we define N_{added} as the noise power added by the network, then the output noise power can be expressed as

$$N_o = G(N_i + N_{added}),$$

assuming that N_{added} is applied to the input of the network. Then, using (3.63) and the fact that $S_o = GS_i$, allows the noise figure to be written as

$$F = \frac{S_i/N_i}{GS_i/G(N_i + N_{added})} = 1 + \frac{N_{added}}{N_i}. \tag{3.66}$$

Since N_{added} is independent of N_i it cannot be proportional to N_i, and so the noise figure depends on the particular value chosen for the input noise power. For this reason N_i must be defined according to a fixed standard if the noise figure is to be meaningful in a general sense. The standard chosen is $N_i = kT_0B$.

Noise Figure of a Lossy Line

We can now determine the noise figure of an important practical component—the lossy transmission line (or attenuator). Figure 3.19 shows a lossy transmission line of characteristic impedance $Z_0 = R$, and held at a physical temperature T. The power gain, G, of a lossy network is less than unity; the *power loss factor*, L, can be defined as $L = 1/G > 1$.

If the input of the line is terminated with a matched load at temperature T, then the entire system is in thermal equilibrium, and the output of the line will appear as a resistor of value R, at temperature T. Thus the available noise power at the output must be $N_o = kTB$. But we can also view the output noise power as a sum of the input noise power attenuated through the lossy line, and the noise power added by the lossy line itself. In this case we

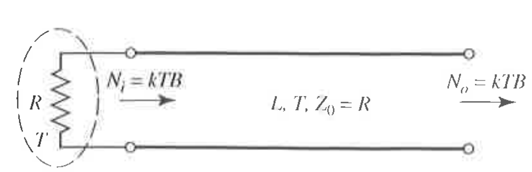

FIGURE 3.19 Determining the noise figure of a lossy line or attenuator with loss L and temperature T.

would write

$$N_o = kTB = G(kTB + N_{added}), \tag{3.67}$$

where N_{added} is the noise generated by the line, as if it appeared at the input terminals of the line. Solving (3.67) for this power gives

$$N_{added} = \frac{1 - G}{G}kTB = (L - 1)kTB. \tag{3.68}$$

Then (3.57) shows that the equivalent noise temperature of the lossy line (as referred to the input) is

$$T_e = \frac{N_{added}}{kB} = (L - 1)T. \tag{3.69}$$

The noise figure of the lossy line can then be found using (3.64) to be

$$F = 1 + \frac{T_e}{T_0} = 1 + (L - 1)\frac{T}{T_0}. \tag{3.70}$$

Note that in the limiting case of a lossless line, with $L = 1$, the above results reduce to $T_e = 0$ and $F = 1$ (0 dB) as expected, since a lossless component does not generate thermal noise. Another special case occurs when the line or attenuator is at room temperature. Then $T = T_0$, and the noise figure reduces to $F = L$. For instance, a 6 dB attenuator at room temperature has a noise figure of 6 dB. At higher temperatures, however, the noise figure will be higher.

Noise Figure of Cascaded Components

In a typical wireless receiver the input signal travels through a cascade of several different components such as filters, amplifiers, mixers, and transmission lines. Each of these stages will progressively degrade the signal-to-noise ratio, so it is important to quantify this effect to evaluate the overall performance of the receiver. If we know the noise figure (or noise temperature) of the individual stages, we can determine the noise figure of the cascade connection of stages. We will see that the most critical stage is usually the first, and that later stages generally have a progressively reduced effect on the overall noise figure. This is an important consideration for the design and layout of receiver circuitry.

Consider a cascade of two components having power gains G_1 and G_2, noise figures F_1 and F_2, and noise temperatures T_{e1} and T_{e2}, as shown in Figure 3.20a. We wish to find the overall noise figure, F, and noise temperature, T_e, of the cascade as if it were the single component of Figure 3.20b. Note that we set the input noise power to be $N_i = kT_0B$.

Using noise temperatures, the noise power at the output of the first stage is

$$N_1 = G_1kT_0B + G_1kT_{e1}B. \tag{3.71}$$

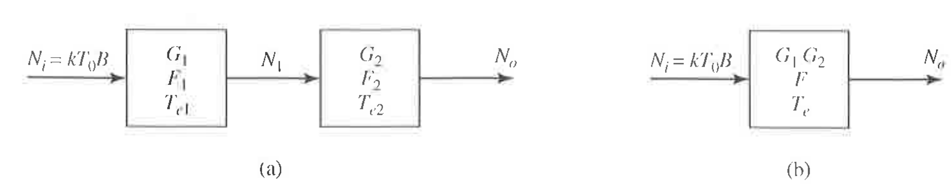

FIGURE 3.20 Noise figure and equivalent noise temperature of a cascaded system. (a) Two cascaded networks. (b) Equivalent network.

Then the noise power at the output of the second stage is

$$N_o = G_2 N_1 + G_2 k T_{e2} B$$
$$= G_1 G_2 k B\left(T_0 + T_{e1} + \frac{T_{e2}}{G_1}\right). \tag{3.72}$$

For the equivalent system of Figure 3.20b the output noise power can be written as

$$N_o = G_1 G_2 k B (T_e + T_0), \tag{3.73}$$

so comparison with (3.72) gives the noise temperature of the cascade system:

$$T_e = T_{e1} + \frac{T_{e2}}{G_1}. \tag{3.74}$$

Using (3.64) to convert noise temperature to noise figure gives the noise figure of the cascade system:

$$F = F_1 + \frac{F_2 - 1}{G_1}. \tag{3.75}$$

The above results are for two cascaded networks, but can be generalized to an arbitrary number of stages as follows:

$$T_e = T_{e1} + \frac{T_{e2}}{G_1} + \frac{T_{e3}}{G_1 G_2} + \cdots, \tag{3.76}$$

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \cdots. \tag{3.77}$$

These results show that the noise characteristics of a cascaded system are dominated by the first few stages, since the effect of later stages is reduced by the product of the gains of the preceding stages. Thus, for best overall system noise performance, the first stage of a receiver should have a low noise figure and at least moderate gain. Expense and effort are most rewarded when applied to improving the noise characteristics of the first or second stage, as opposed to later stages, since later stages have a diminished impact on overall noise performance.

EXAMPLE 3.5 ANALYSIS OF A WIRELESS RECEIVER

The block diagram of a wireless receiver front end is shown in Figure 3.21. Compute the overall noise figure of this subsystem. If the input noise power from a feeding antenna is $N_i = kT_a B$, where $T_a = 15$ K, find the output noise power in dBm. What is the two-sided power spectral density of the output noise? If we require a minimum SNR of 20 dB at the output of the receiver, what is the minimum signal voltage that can be applied at the receiver input? Assume the system is at temperature T_0, with a characteristic impedance of 50 Ω, and an IF bandwidth of 10 MHz.

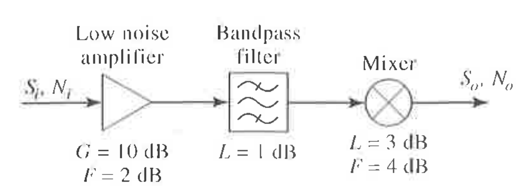

FIGURE 3.21 Block diagram of a wireless receiver front-end circuit.

Solution

We first carry out the required conversions from dB to numerical values:

$G = 10$ dB $= 10$	$G = -1.0$ dB $= 0.79$	$G = -3.0$ dB $= 0.5$
$F = 2$ dB $= 1.58$	$F = 1$ dB $= 1.26$	$F = 4$ dB $= 2.51$

Then we can use (3.77) to find the overall noise figure of the system:

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} = 1.58 + \frac{(1.26 - 1)}{10} + \frac{(2.51 - 1)}{(10)(0.79)}$$
$$= 1.80 = 2.55 \text{ dB}$$

The best way to compute the output noise power is to use noise temperatures. From (3.65), the equivalent noise temperature of the overall system is

$$T_e = (F - 1)T_0 = (1.80 - 1)(290) = 232 \text{ K}.$$

The overall gain of the system is $G = (10)(0.79)(0.5) = 3.95$. Then we can find the output noise power as

$$N_o = k(T_a + T_e)BG = (1.38 \times 10^{-23})(15 + 232)(10 \times 10^6)(3.95)$$
$$= 1.35 \times 10^{-13} \text{ W} = -98.7 \text{ dBm}$$

From (3.26), the power spectral density of the output noise over the IF bandwidth is

$$S_n(\omega) = \frac{N_o}{2B} = \frac{1.35 \times 10^{-13} \text{ W}}{2(10 \times 10^6)} = 6.8 \times 10^{-21} \text{ W/Hz}.$$

Finally, for an output SNR of 20 dB $= 100$, the input signal power must be

$$S_i = \frac{S_o}{G} = \frac{S_o}{N_o}\frac{N_o}{G} = 100\,\frac{1.35 \times 10^{-13}}{3.95} = 3.42 \times 10^{-12} \text{ W} = -84.7 \text{ dBm}.$$

For a 50 Ω system impedance, this corresponds to an input signal voltage of

$$V_i = \sqrt{Z_0 S_i} = \sqrt{(50)(3.42 \times 10^{-12})} = 1.31 \times 10^{-5} \text{ v} = 13.1\ \mu\text{V (rms)}.$$

Note: It may be tempting to compute the output noise power from the definition of the noise figure as

$$N_o = N_i F\left(\frac{S_o}{S_i}\right) = N_i F G = kT_a BFG$$
$$= (1.38 \times 10^{-23})(15)(10 \times 10^6)(1.8)(3.95) = 1.47 \times 10^{-14} \text{ W}.$$

This is an *incorrect* result! The reason for the disparity with the earlier result is because the definition of noise figure assumes an input noise level of kT_0B, while this problem involves an input noise of kT_aB, with $T_a = 15$ K. This is a common error and suggests that when computing absolute noise powers it is often safer to use noise temperatures to avoid this confusion. ○

3.6 NOISE FIGURE OF PASSIVE NETWORKS

As we have seen in the previous section, the noise figure of an RF or microwave system can be evaluated if we know the noise figures of the individual components. In the previous section we derived the noise figure for a matched lossy line or attenuator by

FIGURE 3.22 A passive two-port network with impedance mismatches. The network is at physical temperature T.

using a thermodynamic argument, but that method is not useful for very many circuits. Here we extend the thermodynamic method to evaluate the noise figure of general passive RF and microwave networks (networks that do not contain active devices such as diodes or transistors, which generate nonthermal noise). In addition, this method will account for the change in noise figure that occurs when a component is impedance mismatched at either its input or output port.

Generally it is easier and more accurate to find the noise characteristics of an active device, such as a diode or transistor, by direct measurement than by calculation from first principles. Once the noise parameters of a device are known, the overall noise figure of a circuit containing that device can be evaluated. This is demonstrated in Chapter 6 for the design of low-noise amplifiers.

This section requires knowledge of S-parameters and available gain, topics which are briefly treated in Chapter 2, and in more detail in reference [4]. If the reader does not have this background, he or she may skip this section without any loss of continuity for the rest of the book.

Noise Figure of a Passive Two-port Network

Figure 3.22 show an arbitrary passive two-port network, with a generator at port 1 and a load at port 2. The network is characterized by its S-parameter matrix, $[S]$. In the general case, impedance mismatches may exist at each port, and we define these mismatches in terms of the following reflection coefficients:

Γ_s = reflection coefficient looking toward generator

Γ_{in} = reflection coefficient looking toward port 1 of network

Γ_{out} = reflection coefficient looking toward port 2 of network

Γ_L = reflection coefficient looking toward load

If we assume the network is at temperature T, and an input noise power of $N_1 = kTB$ is applied to the input of the network, the available output noise power at port 2 can be written as

$$N_2 = G_{21}kTB + G_{21}N_{added} \tag{3.78}$$

where N_{added} is the noise power generated internally by the network (referenced to port 1), and G_{21} is the *available gain* of the network from port 1 to port 2. As derived in reference [4], and later in Chapter 6, the available gain can be expressed in terms of the S-parameters

of the network and the port mismatches as

$$G_{21} = \frac{\text{power available from network}}{\text{power available from source}} = \frac{|S_{21}|^2(1-|\Gamma_S|^2)}{|1-S_{11}\Gamma_S|^2(1-|\Gamma_{out}|^2)}. \tag{3.79}$$

Also, the output mismatch can be expressed as

$$\Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1-S_{11}\Gamma_S}. \tag{3.80}$$

Observe that when the network is matched to its external circuitry, so that $\Gamma_s = 0$ and $S_{22} = 0$, we have $\Gamma_{out} = 0$ and $G_{21} = |S_{21}|^2$, which is the gain of the network when it is matched. Also observe that the available gain of the network does not depend on the load mismatch, Γ_L. This is because available gain is defined in terms of the maximum power that is available from the network, which occurs when the load impedance is conjugately matched to the output impedance of the network.

Since the input noise is kTB, and the network is at temperature T, the network is in thermodynamic equilibrium, and so the available output noise power must be $N_2 = kTB$. Then we can solve for N_{added} from (3.78) to give

$$N_{added} = \frac{1-G_{21}}{G_{21}}kTB. \tag{3.81}$$

Then the equivalent noise temperature of the network is

$$T_e = \frac{N_{added}}{kB} = \frac{1-G_{21}}{G_{21}}T, \tag{3.82}$$

and the noise figure of the network is

$$F = 1 + \frac{T_e}{T_0} = 1 + \frac{1-G_{21}}{G_{21}}\frac{T}{T_0}. \tag{3.83}$$

Note the similarity of (3.81)–(3.83) to the results in (3.68)–(3.70) for the lossy line—the essential difference is that here we are using the available gain of the network, which accounts for impedance mismatches between the network and the external circuit. We will now illustrate the use of this result with some applications to problems of practical interest.

Application to a Mismatched Lossy Line

In Section 3.5 we found the noise figure of a lossy transmission line under the assumption that it was matched to its input and output circuits. Now we consider the case where the line is mismatched to its input circuit.

Figure 3.23 shows a transmission line of length ℓ at temperature T, with a power loss factor $L = 1/G$, and an impedance mismatch between the line and the generator. Thus,

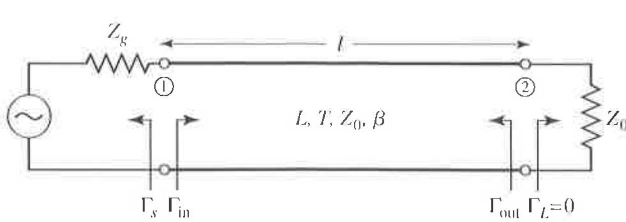

FIGURE 3.23 A lossy transmission line at temperature T with an impedance mismatch at its input port.

$Z_g \neq Z_0$, and the reflection coefficient looking toward the generator can be written as

$$\Gamma_s = \frac{Z_g - Z_0}{Z_g + Z_0} \neq 0.$$

The scattering matrix of the lossy line of characteristic impedance Z_0 can be written as

$$[S] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{e^{-j\beta\ell}}{\sqrt{L}}, \tag{3.84}$$

where β is the propagation constant of the line. Using (3.80) gives the reflection coefficient looking into port 2 of the line as

$$\Gamma_{\text{out}} = S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s} = \frac{\Gamma_s}{L} e^{-2j\beta\ell}. \tag{3.85}$$

Then the available gain, from (3.79), is

$$G_{21} = \frac{\frac{1}{L}(1 - |\Gamma_s|^2)}{1 - |\Gamma_{\text{out}}|^2} = \frac{L(1 - |\Gamma_s|^2)}{L^2 - |\Gamma_s|^2}. \tag{3.86}$$

We can verify two limiting cases of (3.86): when $L = 1$ we have $G_{21} = 1$, and when $\Gamma_s = 0$ we have $G_{21} = 1/L$.

Using (3.82) gives the equivalent noise temperature of the mismatched lossy line as

$$T_e = \frac{1 - G_{21}}{G_{21}} T = \frac{(L - 1)(L + |\Gamma_s|^2)}{L(1 - |\Gamma_s|^2)} T. \tag{3.87}$$

The corresponding noise figure can then be evaluated using (3.64). Observe that when the line is matched, $\Gamma_s = 0$ and (3.87) reduces to $T_e = (L - 1)T$, in agreement with the result for the matched lossy line given by (3.69). If the line is lossless, then $L = 1$ and (3.87) reduces to $T_e = 0$ regardless of mismatch, as expected. But when the line is lossy and mismatched, so that $L > 1$ and $|\Gamma_s| > 0$, then the noise temperature given by (3.87) is greater than $T_e = (L - 1)T$, the noise temperature of the matched lossy line. The reason for this increase is that the lossy line actually delivers noise power out of both its ports, but when the input port is mismatched some of the available noise power at port 1 is reflected from the source back into port 1, and appears at port 2. When the generator is matched to port 1, none of the available power from port 1 is reflected back into the line, so the noise power available at port 2 is a minimum.

Application to a Wilkinson Power Divider

Here we evaluate the noise figure of a power divider, which is another common component found in wireless systems. Figure 3.24 shows a Wilkinson power divider. A detailed analysis and description of this circuit are given in reference [4], but for our purposes knowledge of the scattering matrix is sufficient:

$$[S] = \frac{-j}{\sqrt{2L}} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{3.88}$$

where L is the dissipative insertion power loss from port 1 to port 2 or 3, due to the loss of the quarter-wave transmission lines connecting those ports. The scattering matrix shows that the divider is matched at all ports, and divides input power at port 1 evenly to ports 2 and 3, when those ports are matched. The shunt resistor across ports 2 and 3 provides

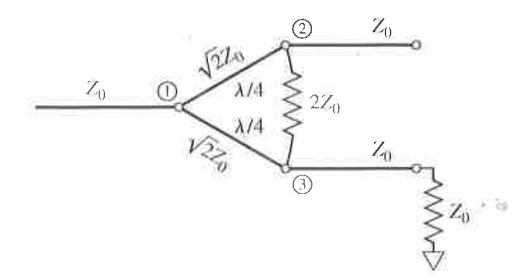

FIGURE 3.24 A Wilkinson power divider with port 3 terminated in a matched load.

isolation between those ports ($S_{23} = S_{32} = 0$), but does not dissipate input power when ports 2 and 3 are terminated with matched loads.

To evaluate the noise figure of the Wilkinson divider, we first terminate port 3 with a matched load; this converts the 3-port device to a 2-port device. If we assume a matched source at port 1, we have $\Gamma_s = 0$. Equation (3.80) then gives $\Gamma_{\text{out}} = S_{22} = 0$, and so the available gain can be calculated from (3.79) as

$$G_{21} = |S_{21}|^2 = \frac{1}{2L}. \tag{3.89}$$

Then the equivalent noise temperature of the Wilkinson divider is, from (3.82),

$$T_e = \frac{1 - G_{21}}{G_{21}} T = (2L - 1)T, \tag{3.90}$$

where T is the physical temperature of the divider. Using (3.64) gives the noise figure as

$$F = 1 + \frac{T_e}{T_0} = 1 + (2L - 1)\frac{T}{T_0}. \tag{3.91}$$

Observe that if the divider is at room temperature, then $T = T_0$ and (3.91) reduces to $F = 2L$. If the divider is at room temperature and lossless, (3.91) reduces to $F = 2 = 3$ dB. In this case the source of the noise power is the isolation resistor.

Because the divider circuit of Figure 3.24 is matched at its input and output, it is easy to obtain these same results using thermodynamic arguments. Thus, if we apply an input noise power of kTB to port 1 of the matched Wilkinson divider at temperature T, the system will be in thermal equilibrium and the output noise power must therefore be kTB. We can also express the output noise power as the sum of the input power times the gain of the divider, and N_{added}, the noise power added by the divider itself (referenced at the input to the divider):

$$kTB = \frac{kTB}{2L} + \frac{N_{\text{added}}}{2L}. \tag{3.92}$$

Solving for N_{added} gives

$$N_{\text{added}} = kTB(2L - 1), \tag{3.93}$$

so the equivalent noise temperature is

$$T_e = \frac{N_{\text{added}}}{kB} = (2L - 1)T,$$

in agreement with (3.90).

3.7 DYNAMIC RANGE AND INTERMODULATION DISTORTION

Since thermal noise is generated by any lossy component, and all realistic components have at least a small loss, the ideal linear component or network does not exist in the sense that its output response is always exactly proportional to its input excitation. Thus, all realistic devices are nonlinear at very low power levels due to noise effects. In addition, all practical components also become nonlinear at high power levels. This may ultimately be the result of catastrophic destruction of the device at very high powers or, in the case of active devices such as diodes and transistors, due to effects such as gain compression or the generation of spurious frequency components due to device nonlinearities. In either case these effects set a minimum and maximum realistic power range, or *dynamic range*, over which a given component or network will operate as desired. In this section we will study dynamic range, and the response of nonlinear devices in general. These results will be useful for our later discussions of amplifiers (Chapter 6), mixers (Chapter 7), and wireless receiver design (Chapter 10).

Devices such as diodes and transistors are nonlinear components, and it is this nonlinearity that is of great utility for functions such as amplification, detection, and frequency conversion [5]. Nonlinear device characteristics, however, can also lead to undesired responses such as gain compression and the generation of spurious frequency components. These effects may produce increased losses, signal distortion, and possible interference with other radio channels or services.

Figure 3.25 shows a general nonlinear network, having an input voltage v_i and an output voltage v_o. In the most general sense, the output response of a nonlinear circuit can be modeled as a Taylor series in terms of the input signal voltage:

$$v_o = a_0 + a_1 v_i + a_2 v_i^2 + a_3 v_i^3 + \cdots, \tag{3.94}$$

where the Taylor coefficients are defined as

$$a_0 = v_o(0) \quad \text{(DC output)} \tag{3.95a}$$

$$a_1 = \left.\frac{dv_o}{dv_i}\right|_{v_i=0} \quad \text{(linear output)} \tag{3.95b}$$

$$a_2 = \left.\frac{d^2v_o}{dv_i^2}\right|_{v_i=0} \quad \text{(squared output)} \tag{3.95c}$$

and higher order terms. Thus, different functions can be obtained from the nonlinear network depending on the dominance of particular terms in the expansion. If a_0 is the only nonzero coefficient in (3.94), the network functions as a rectifier, converting an AC signal to DC. If a_1 is the only nonzero coefficient, we have a linear attenuator ($a_1 < 1$) or amplifier ($a_1 > 1$). If a_2 is the only nonzero coefficient, we can achieve mixing and other frequency conversion functions. Usually, however, practical devices have a series expansion containing many nonzero terms, and a combination of several of these effects will occur. We consider some important special cases below.

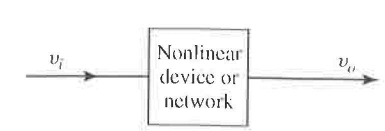

FIGURE 3.25 A general nonlinear device or network.

Gain Compression

First consider the case where a single frequency sinusoid is applied to the input of a general nonlinear network, such as an amplifier:

$$v_i = V_0 \cos \omega_0 t. \tag{3.96}$$

Then (3.94) gives the output voltage as

$$\begin{aligned} v_o &= a_0 + a_1 V_0 \cos \omega_0 t + a_2 V_0^2 \cos^2 \omega_0 t + a_3 V_0^3 \cos^3 \omega_0 t + \cdots \\ &= \left(a_0 + \tfrac{1}{2} a_2 V_0^2\right) + \left(a_1 V_0 + \tfrac{3}{4} a_3 V_0^3\right) \cos \omega_0 t \\ &\quad + \tfrac{1}{2} a_2 V_0^2 \cos 2\omega_0 t + \tfrac{1}{4} a_3 V_0^3 \cos 3\omega_0 t + \cdots. \end{aligned} \tag{3.97}$$

This result leads to the voltage gain of the signal component at frequency ω_0:

$$G_v = \frac{v_o^{(\omega_0)}}{v_i^{(\omega_0)}} = \frac{a_1 V_0 + \frac{3}{4} a_3 V_0^3}{V_0} = a_1 + \tfrac{3}{4} a_3 V_0^2, \tag{3.98}$$

where we have retained only terms through the third order.

The result of (3.98) shows that the voltage gain is equal to the a_1 coefficient, as expected, but with an additional term proportional to the square of the input voltage amplitude. In most practical amplifiers a_3 is typically negative, so that the gain of the amplifier tends to decrease for large values of V_0. This effect is called *gain compression*, or *saturation*. Physically, this is usually due to the fact that the instantaneous output voltage of an amplifier is limited by the power supply voltage used to bias the active device. Smaller values of a_3 will lead to higher output voltages.

A typical amplifier response is shown in Figure 3.26. For an ideal linear amplifier a plot of the output power versus input power is a straight line with a slope of unity, and the gain of the amplifier is given by the ratio of the output power to the input power. The amplifier response of Figure 3.26 tracks the ideal response over a limited range, then begins to saturate, resulting in reduced gain. To quantify the linear operating range of the amplifier, we define the *1 dB compression point* as the power level for which the output power has decreased by 1 dB from the ideal characteristic. This power level is usually denoted by P_1, and can be stated in terms of either input power or output power. For amplifiers P_1 is

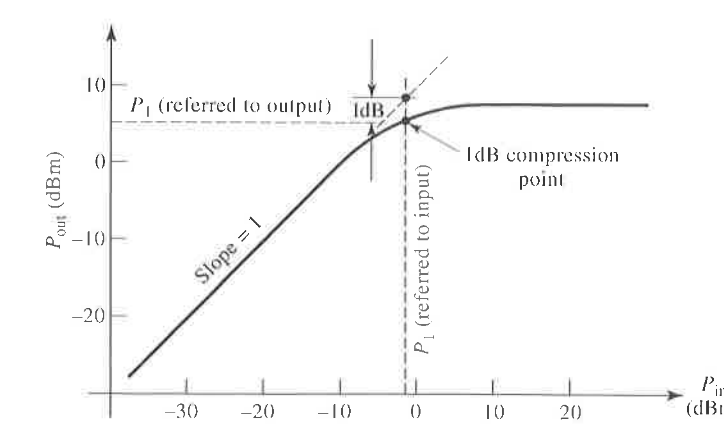

FIGURE 3.26 Definition of the 1 dB compression point for a nonlinear amplifier.

usually specified as an output power, while for mixers P_1 is usually specified in terms of input power.

Intermodulation Distortion

Observe from the expansion of (3.97) that a portion of the input signal at frequency ω_0 is converted to other frequency components. For example, the first term of (3.97) represents a DC voltage, which would be a useful response in a rectifier application. The voltage components at frequencies $2\omega_0$ or $3\omega_0$ might be useful for frequency multiplier circuits. In amplifiers, however, the presence of other frequency components will lead to signal distortion if those components are in the passband of the amplifier.

For a single input frequency, or *tone*, ω_0, the output will in general consist of harmonics of the input frequency of the form $n\omega_0$, for $n = 0, 1, 2, \ldots$. Usually these harmonics lie outside the passband of the amplifier, and so do not interfere with the desired signal at frequency ω_0. The situation is different, however, when the input signal consists of two closely spaced frequencies.

Consider a *two-tone* input voltage, consisting of two closely spaced frequencies, ω_1 and ω_2:

$$v_i = V_0(\cos \omega_1 t + \cos \omega_2 t) \tag{3.99}$$

From (3.94) the output is

$$\begin{aligned} v_o &= a_0 + a_1 V_0(\cos \omega_1 t + \cos \omega_2 t) + a_2 V_0^2(\cos \omega_1 t + \cos \omega_2 t)^2 \\ &\quad + a_3 V_0^3(\cos \omega_1 t + \cos \omega_2 t)^3 + \cdots \\ &= a_0 + a_1 V_0 \cos \omega_1 t + a_1 V_0 \cos \omega_2 t + \tfrac{1}{2} a_2 V_0^2 (1 + \cos 2\omega_1 t) \\ &\quad + \tfrac{1}{2} a_2 V_0^2 (1 + \cos 2\omega_2 t) + a_2 V_0^2 \cos(\omega_1 - \omega_2)t + a_2 V_0^2 \cos(\omega_1 + \omega_2)t \\ &\quad + a_3 V_0^3 \left(\tfrac{3}{4} \cos \omega_1 t + \tfrac{1}{4} \cos 3\omega_1 t\right) + a_3 V_0^3 \left(\tfrac{3}{4} \cos \omega_2 t + \tfrac{1}{4} \cos 3\omega_2 t\right) \\ &\quad + a_3 V_0^3 \left[\tfrac{3}{2} \cos \omega_2 t + \tfrac{3}{4} \cos(2\omega_1 - \omega_2)t + \tfrac{3}{4} \cos(2\omega_1 + \omega_2)t\right] \\ &\quad + a_3 V_0^3 \left[\tfrac{3}{2} \cos \omega_1 t + \tfrac{3}{4} \cos(2\omega_2 - \omega_1)t + \tfrac{3}{4} \cos(2\omega_2 + \omega_1)t\right] + \cdots, \end{aligned} \tag{3.100}$$

where standard trigonometric identities have been used to expand the initial expression. We see that the output spectrum consists of harmonics of the form

$$m\omega_1 + n\omega_2, \tag{3.101}$$

with $m, n = 0, \pm 1, \pm 2, \pm 3, \ldots$. These combinations of the two input frequencies are called *intermodulation products*, and the *order* of a given product is defined as $|m| + |n|$. For example, the squared term of (3.100) gives rise to the following four intermodulation products of second order:

$2\omega_1$	(second harmonic of ω_1)	$m = 2$	$n = 0$	order $= 2$
$2\omega_2$	(second harmonic of ω_2)	$m = 0$	$n = 2$	order $= 2$
$\omega_1 - \omega_2$	(difference frequency)	$m = 1$	$n = -1$	order $= 2$
$\omega_1 + \omega_2$	(sum frequency)	$m = 1$	$n = 1$	order $= 2$

All of these second-order products are undesired in an amplifier, but in a mixer the sum or

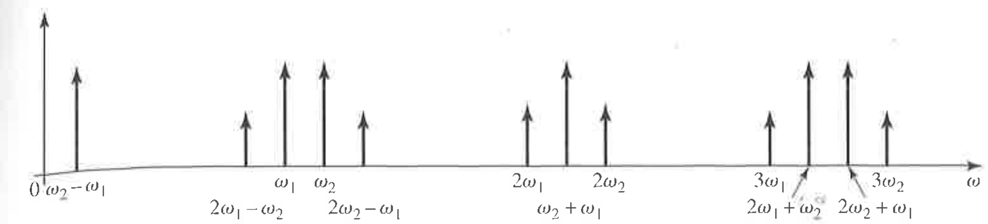

FIGURE 3.27 Output spectrum of second and third-order two-tone intermodulation products, assuming $\omega_1 < \omega_2$.

difference frequencies form the desired outputs. In either case, if ω_1 and ω_2 are close, all the second-order products will be far from ω_1 or ω_2, and can easily be filtered (either passed or rejected) from the output of the component.

The cubed term of (3.100) leads to six third-order intermodulation products: $3\omega_1$, $3\omega_2$, $2\omega_1 + \omega_2$, $2\omega_2 + \omega_1$, $2\omega_1 - \omega_2$, and $2\omega_2 - \omega_1$. The first four of these will again be located far from ω_1 and ω_2, and will typically be outside the passband of the component. But the two difference terms produce products located near the original input signals at ω_1 and ω_2, and so cannot be easily filtered from the passband of an amplifier. Figure 3.27 shows a typical spectrum of the second- and third-order two-tone intermodulation products. For an arbitrary input signal consisting of many frequencies of varying amplitude and phase, the resulting in-band intermodulation products will cause distortion of the output signal. This effect is called *third-order intermodulation distortion*.

Third-Order Intercept Point

Equation (3.100) shows that as the input voltage V_0 increases, the voltage associated with the third-order products increases as V_0^3. Since power is proportional to the square of voltage, we can also say that the output power of third-order products must increase as the cube of the input power. So for small input powers the third-order intermodulation products must be very small, but will increase quickly as input power increases. We can view this effect graphically by plotting the output power for the first- and third-order products versus input power on log-log scales (or in dB), as shown in Figure 3.28.

The output power of the first order, or linear, product is proportional to the input power, and so the line describing this response has a slope of unity (before the onset of compression). The line describing the response of the third-order products has a slope of 3. (The second-order products would have a slope of 2, but since these products are generally not in the passband of the component, we have not plotted their response in Figure 3.28.) Both the linear- and third-order responses will exhibit compression at high input powers, so we show the extension of their idealized responses with dotted lines. Since these two lines have different slopes, they will intersect, typically at a point above the onset of compression, as shown in the figure. This hypothetical intersection point, where the first-order and third-order powers are equal, is called the *third-order intercept point*, denoted P_3, and specified as either an input or an output power. Usually P_3 is referenced at the output for amplifiers, and at the input for mixers.

As depicted in Figure 3.28, P_3 generally occurs at a higher power level than P_1, the 1 dB compression point. Many practical components follow the approximate rule that P_3 is 12 to 15 dB greater than P_1, assuming these powers are referenced at the same point.

We can express P_3 in terms of the Taylor coefficients of the expansion of (3.100) as follows. Define P_{ω_1} as the output power of the desired signal at frequency ω_1. Then from

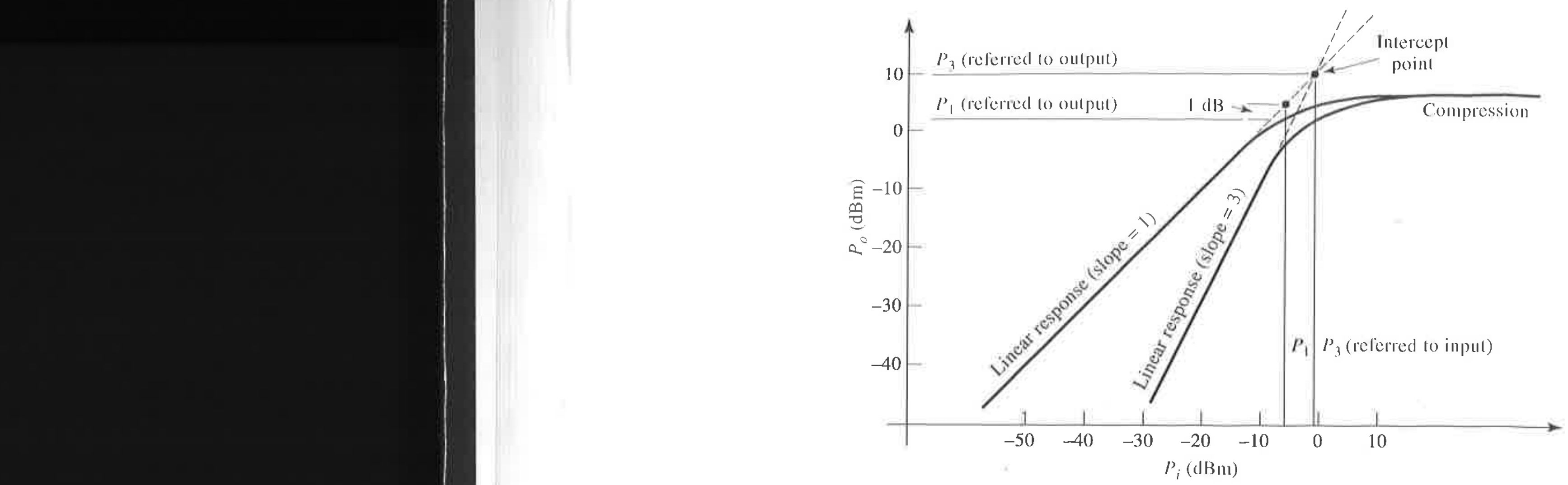

FIGURE 3.28 Third-order intercept diagram for a nonlinear component.

(3.100) we have

$$P_{\omega_1} = \tfrac{1}{2}a_1^2 V_0^2. \tag{3.102}$$

Similarly, define $P_{2\omega_1-\omega_2}$ as the output power of the intermodulation product of frequency $2\omega_1 - \omega_2$. Then from (3.100) we have

$$P_{2\omega_1-\omega_2} = \tfrac{1}{2}\left(\tfrac{3}{4}a_3 V_0^3\right)^2 = \tfrac{9}{32}a_3^2 V_0^6. \tag{3.103}$$

By definition, these two powers are equal at the third-order intercept point. If we define the input signal voltage at the intercept point as V_{IP}, then equating (3.102) and (3.103) gives

$$\tfrac{1}{2}a_1^2 V_{IP}^2 = \tfrac{9}{32}a_3^2 V_{IP}^6.$$

Solving for V_{IP} yields

$$V_{IP} = \sqrt{\frac{4a_1}{3a_3}}. \tag{3.104}$$

Since P_3 is equal to the linear response of P_{ω_1} at the intercept point, we have from (3.102) and (3.104) that

$$P_3 = P_{\omega_1}\Big|_{V_0=V_{IP}} = \frac{1}{2}a_1^2 V_{IP}^2 = \frac{2a_1^3}{3a_3}, \tag{3.105}$$

where P_3 in this case is referred to the output port. This expression will be useful in the following section.

Dynamic Range

We can define *dynamic range* in a general sense as the operating range for which a component or system has desirable characteristics. For a power amplifier this may be the

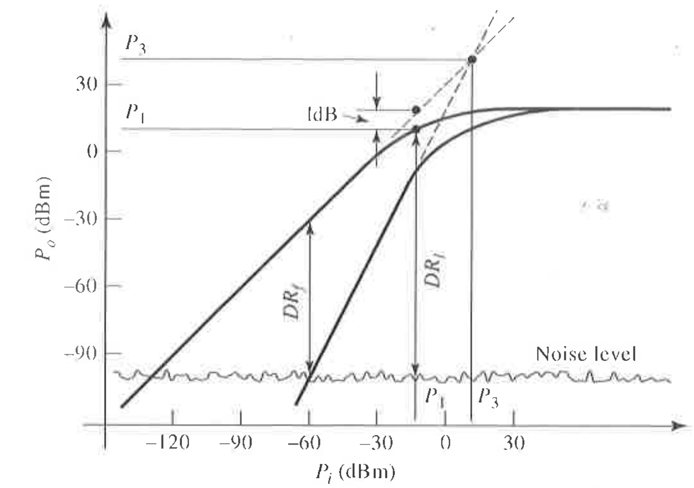

FIGURE 3.29 Illustrating linear dynamic range and spurious free dynamic range.

power range that is limited at the low end by noise and at the high end by the compression point. This is essentially the linear operating range for the amplifier, and is called the *linear dynamic range* (DR_ℓ). For low-noise amplifiers or mixers, operation may be limited by noise at the low end and the maximum power level for which intermodulation distortion becomes unacceptable. This is effectively the operating range for which spurious responses are minimal, and is called the *spurious-free dynamic range* (DR_f).

We thus compute the linear dynamic range DR_ℓ as the ratio of P_1, the 1 dB compression point, to the noise level of the component, as shown in Figure 3.29. These powers can be referenced at either the input or the output of the device. Note that some authors [6] prefer to define the linear dynamic range in terms of a minimum detectable power level. This definition is more appropriate for a receiver system, rather than an individual component, as it depends on factors external to the component itself, such as the type of modulation used, the recommended system SNR, effects of coding, and related factors.

The spurious free dynamic range is defined as the maximum output signal power for which the power of the third-order intermodulation product is equal to the noise level of the component. This situation is shown in Figure 3.29. If P_{ω_1} is the output power of the desired signal at frequency ω_1, and $P_{2\omega_1-\omega_2}$ is the output power of the third-order intermodulation product, then the spurious free dynamic range can be expressed as

$$DR_f = \frac{P_{\omega_1}}{P_{2\omega_1-\omega_2}}, \tag{3.106}$$

with $P_{2\omega_1-\omega_2}$ taken equal to the noise level of the component. $P_{2\omega_1-\omega_2}$ can be written in terms of P_3 and P_{ω_1} as follows:

$$P_{2\omega_1-\omega_2} = \frac{9a_3^2V_0^6}{32} = \frac{\frac{1}{8}a_1^6V_0^6}{\frac{4a_1^6}{9a_3^2}} = \frac{(P_{\omega_1})^3}{(P_3)^2}, \tag{3.107}$$

where (3.102) and (3.105) have been used. Observe that this result clearly shows that the third-order intermodulation power increases as the cube of the input signal power. Solving (3.107) for P_{ω_1}, and applying the result to (3.106) gives the spurious free dynamic range in terms of P_3 and N_o, the output noise power of the component:

$$DR_f = \frac{P_{\omega_1}}{P_{2\omega_1-\omega_2}}\bigg|_{P_{2\omega_1-\omega_2}=N_o} = \left(\frac{P_3}{N_o}\right)^{2/3}. \tag{3.108}$$

This result can be written in terms of dB as

$$DR_f(dB) = \tfrac{2}{3}(P_3 - N_o), \tag{3.109}$$

for P_3 and N_o expressed in dB or dBm. If the output SNR is specified, this can be added to N_o to give the spurious free dynamic range in terms of the minimum detectable signal level. Finally, although we derived this result for the $2\omega_1 - \omega_2$ product, the same result applies for the $2\omega_2 - \omega_1$ product.

EXAMPLE 3.6 DYNAMIC RANGES

A receiver has a noise figure of 7 dB, a 1 dB compression point of 25 dBm (referenced to output), a gain of 40 dB, and a third-order intercept point of 35 dBm (referenced to output). If the receiver is fed with an antenna having a noise temperature of $T_A = 150$ K, and the desired output SNR is 10 dB, find the linear and spurious free dynamic ranges. Assume a receiver bandwidth of 100 MHz.

Solution

The noise power at the receiver output can be calculated as

$$\begin{aligned} N_o &= GkB[T_A + (F-1)T_o] = 10^4(1.38\times 10^{-23})(10^8)[150 + (4.01)(290)] \\ &= 1.8\times 10^{-8}\ \text{W} = -47.4\ \text{dBm}. \end{aligned}$$

Then the linear dynamic range is, in dB

$$DR_\ell = P_1 - N_o = 25\ \text{dBm} + 47.4\ \text{dBm} = 72\ \text{dB}.$$

Equation (3.109) gives the spurious free dynamic range as

$$DR_f = \tfrac{2}{3}(P_3 - N_o - SNR) = \tfrac{2}{3}(35 + 47.4 - 10) = 48.3\ \text{dB}.$$

Observe that $DR_f \ll DR_\ell$. ○

Intercept Point of Cascaded Components

As in the case of noise figure, the cascade connection of components has the effect of degrading (lowering) the third-order intercept point. Unlike the case of a cascade of noisy components, however, the intermodulation products in a cascaded system are deterministic (coherent), so we cannot simply add powers, but must deal with voltages.

With reference to Figure 3.30, let G_1 and P_3' be the power gain and third-order intercept point for the first stage, and G_2 and P_3'' be the corresponding values for the second stage. From (3.107) the third-order distortion power at the output of the first stage is

$$P'_{2\omega_1-\omega_2} = \frac{(P'_{\omega_1})^3}{(P_3')^2}, \tag{3.110}$$

where P'_{ω_1} is the desired signal power at frequency ω_1 at the output of the first stage. The

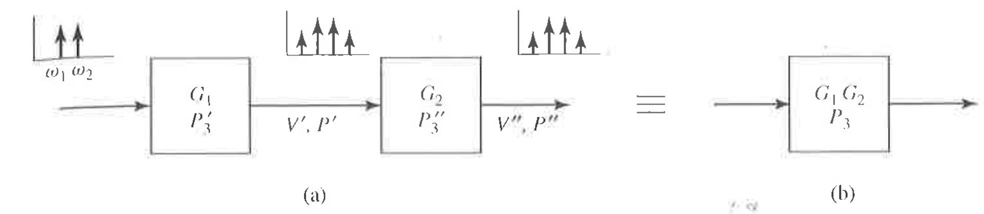

FIGURE 3.30 Third-order intercept point for a cascaded system. (a) Two cascaded networks. (b) Equivalent network.

voltage associated with this power is

$$V'_{2\omega_1-\omega_2} = \sqrt{P'_{2\omega_1-\omega_2}Z_0} = \frac{\sqrt{(P'_{\omega_1})^3 Z_0}}{P_3'}, \tag{3.111}$$

where Z_0 is the system impedance.

The total third-order distortion voltage at the output of the second stage is the sum of this voltage times the voltage gain of the second stage, and the distortion voltage generated by the second stage. This is because these voltages are deterministic and phase-related, unlike the uncorrelated noise powers that occur in cascaded components. Adding these voltages gives the worst-case result for the distortion level, because there may be phase delays within the stages that could cause partial cancellation. Thus we can write the worst-case total distortion voltage at the output of the second stage as

$$V''_{2\omega_1-\omega_2} = \frac{\sqrt{G_2(P'_{\omega_1})^3 Z_0}}{P_3'} + \frac{\sqrt{(P''_{\omega_1})^3 Z_0}}{P_3''}.$$

Since $P''_{\omega_1} = G_2 P'_{\omega_1}$, we have

$$V''_{2\omega_1-\omega_2} = \left(\frac{1}{G_2P_3'} + \frac{1}{P_3''}\right)\sqrt{(P''_{\omega_1})^3 Z_0}. \tag{3.112}$$

Then the output distortion power is

$$P''_{2\omega_1-\omega_2} = \frac{\left(V''_{2\omega_1-\omega_2}\right)^2}{Z_0} = \left(\frac{1}{G_2P_3'} + \frac{1}{P_3''}\right)^2 (P''_{\omega_1})^3 = \frac{(P''_{\omega_1})^3}{P_3^2}. \tag{3.113}$$

Thus the third-order intercept point of the cascaded system is

$$P_3 = \left(\frac{1}{G_2P_3'} + \frac{1}{P_3''}\right)^{-1}. \tag{3.114}$$

Note that $P_3 = G_2P_3'$ for $P_3'' \to \infty$, which is the limiting case when the second stage has no third-order distortion. This result is also useful for transferring P_3 between input and output reference points.

EXAMPLE 3.7 CALCULATION OF CASCADE INTERCEPT POINT

A low-noise amplifier and mixer are shown in Figure 3.31. The amplifier has a gain of 20 dB and a third-order intercept point of 22 dBm (referenced at output), and the mixer has a conversion loss of 6 dB and a third-order intercept point of 13 dBm (referenced at input). Find the intercept point of the cascade network.

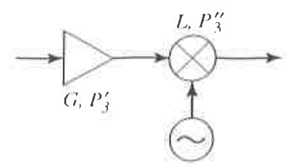

FIGURE 3.31 System for Example 3.7.

Solution
First we transfer the reference of P_3 for the mixer from its input to its output:

$$P''_3 = 13 \text{ dBm} - 6 \text{ dB} = 7 \text{ dBm (referenced at output)}$$

Converting the necessary dB values to numerical values:

$$\begin{aligned} P'_3 &= 22 \text{ dBm} = 158 \text{ mW} && \text{(for amplifier)} \\ P''_3 &= 7 \text{ dBm} = 5 \text{ mW} && \text{(for mixer)} \\ G_2 &= -6 \text{ dB} = 0.25 && \text{(for mixer)} \end{aligned}$$

Then using (3.114) gives the intercept point of the cascade as

$$P_3 = \left(\frac{1}{G_2 P'_3} + \frac{1}{P''_3}\right)^{-1} = \left(\frac{1}{(0.25)(158)} + \frac{1}{5}\right)^{-1} = 4.4 \text{ mW} = 6.4 \text{ dBm},$$

which is seen to be much lower than the P_3 of the individual components. ○

Passive Intermodulation

The above discussion of intermodulation distortion was in the context of active circuits involving diodes and transistors, but it is also possible for intermodulation products to be generated by passive nonlinear effects in connectors, cables, antennas, or almost any component where there is a metal-to-metal contact. This effect is called *passive intermodulation* (PIM) and, as in the case of intermodulation in amplifiers and mixers, occurs when signals at two or more closely spaced frequencies mix to produce spurious products.

Passive intermodulation can be caused by a number of factors, such as poor mechanical contact, oxidation of junctions between ferrous-based metals, contamination of conducting surfaces at RF junctions, or the use of nonlinear materials such as carbon fiber composites or ferromagnetic materials. In addition, when high powers are involved, thermal effects may contribute to the overall nonlinearity of a junction. It is very difficult to predict PIM levels from first principles, so measurement techniques must usually be used.

Because of the third-power dependence of the third-order intermodulation products with input power, passive intermodulation is usually only significant when input signal powers are relatively large. This is frequently the case in cellular telephone base station transmitters, which may operate with powers of 30–40 dBm, with many closely spaced RF channels. It is often desired to maintain the PIM level below −125 dBm, with two 40 dBm transmit signals. This is a very wide dynamic range, and requires careful selection of components used in the high-power portions of the transmitter, including cables, connectors, and antenna components. Because these components are often exposed to the weather, deterioration due to oxidation, vibration, and sunlight must be offset by a careful maintenance program. Passive intermodulation is generally not a problem in receiver systems, due to the much lower power levels.

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PROBLEMS

3.1 Prove the four properties of cumulative distribution functions as given in (3.2a)–(3.2d).

3.2 Use the definition of expected value to prove the linearity properties of (3.14).

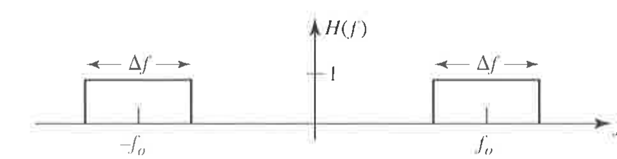
3.3 Evaluate the mean and variance for the uniform, gaussian, and Rayleigh probability density functions given in (3.12).

3.4 Evaluate the nth moment of a random process having a gaussian PDF with zero mean ($m = 0$). Show that $E\{x^n\} = 0$ for odd n.

3.5 Consider the random variable $z = x + y$, where x, y, and z are zero-mean gaussian random variables having variances σ_x^2, σ_y^2, and σ_z^2. If x and y are independent, show that $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$.

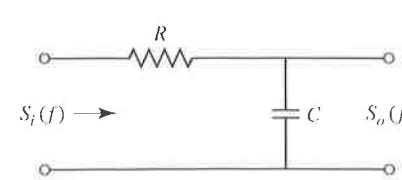
3.6 A gaussian white noise process, X, has zero mean and variance σ^2. Evaluate the probability $P\{-\sigma < x < \sigma\}$, that is, the probability that a particular sample, x, of the process satisfies the inequality $-\sigma < x < \sigma$.

3.7 The autocorrelation function for a white noise source is $R(\tau) = \frac{1}{2}n_0\delta(\tau)$. If this source is applied to an ideal bandpass filter with the frequency response shown below, find the total noise power output.



3.8 A noisy resistor of value R, at temperature T, is connected to a load resistor of value R_L. Calculate and plot the average power dissipated in the load, normalized to kTB, for $0 \le R_L < 10R$. Prove that maximum power transfer occurs for $R_L = R$.

3.9 A gaussian white noise source with a two-sided power spectral density $S_i(f) = n_0/2$ is applied to the RC low-pass filter circuit shown below. Find the output noise power, N_o, in terms of n_0 and $f_c = 1/2\pi RC$ (the cutoff frequency of the filter).



3.10 Derive the result of (3.38) for white noise passing through an ideal integrator by using the autocorrelation function, instead of the power spectral density. Find the output noise power from $N_o = E\{n_o{}^2(t)\}$, and evaluate $n_o(t)$ by integrating $n_i(t)$ according to the integrator operation.