Introduction

For High-income customers

- No of customers: N
- Demand function:

$$P_H = A - BQ$$

- Willingness to pay:

$$W_H(Q) = \int_0^Q P_H(x)dx = \int_0^Q (A - Bx)dx = AQ - \frac{BQ^2}{2}$$

For Low-income customers

- No of customers: n
- Demand function:

$$P_L = a - bQ$$

- Willingness to pay:

$$W_L(Q) = \int_0^Q P_L(x)dx = \int_0^Q (a - bx)dx = aQ - \frac{bQ^2}{2}$$

- Given the following:

$$A > a > 0$$

Cost is same for both products

Total cost function is given as,

$$C = cQ + F \text{ with } 0 < c < a$$

a) If only one package is allowed to be sold.

i) Only selling to High income customers

$$\pi_H = Total\ Revenue - Total\ Cost = NQ_H V_H - cNQ_H - F$$

$$=> \pi_H = NW_H(Q) - cNQ - F = N(AQ - \frac{BQ^2}{2} - cQ) - F$$

To maximize profits:

$$\frac{d\pi_H}{dQ} = 0$$

$$=> A - BQ - c = 0$$

$$=> \boldsymbol{Q^* = \frac{A - c}{B}}$$

$$=> \boldsymbol{V_H^* = W_H(Q) = \frac{A^2 - c^2}{2B}}$$

$$=> \boldsymbol{\pi_H^* = N[\frac{(A - c)^2}{2B}] - F}$$

ii) Selling to both High and Low customers

$$\pi_L = n[W_L(Q) - cQ] - F = n(aQ - \frac{bQ^2}{2} - cQ) - F$$

To Maximize profits:

$$\frac{d\pi_l}{dQ} = 0$$

$$=> \boldsymbol{Q^* = \frac{a - c}{b}}$$

$$=> \boldsymbol{V_L^* = W_l(Q) = \frac{a^2 - c^2}{2b}}$$

$$=> \boldsymbol{\pi_L^* = (N + n)[\frac{(a - c)^2}{2b}] - F}$$

When to sell only to High Income customers and when to sell to everyone

When $\pi_H > \pi_L$ we will sell only to high-income customers, but this is not true when $n \gg N$ so in that case we sell to both high-income and low-income customers.

b) Two packages are allowed to be sold

Calculating the 2nd Degree Prices and Quantities for the 2 types of customers.

For 2nd degree price discrimination to be possible these following conditions need to be satisfied:

- $V_L \leq W_L(Q_L)$
- $W_L(Q_L) - V_L \geq W_L(Q_H) - V_H$
- $V_H \leq W_H(Q_H)$
- $W_H(Q_H) - V_H \geq W_H(Q_L) - V_L$

For profit maximization, constraints 1 and 4 need to be binding. So,

$$V_L = W_L(Q_L)$$

$$V_H = W_H(Q_H) - W_H(Q_L) + V_L = AQ_H - \frac{BQ_H^2}{2} - \left(AQ_L - \frac{BQ_L^2}{2}\right) + (aQ_L - \frac{bQ_L^2}{2})$$

Therefore, the profit function becomes:

$$\pi = N(V_H - cQ_H) + n(V_L - cQ_L) - F$$

$$=> \pi = N\left[AQ_H - \frac{BQ_H^2}{2} - \left(AQ_L - \frac{BQ_L^2}{2}\right) + \left(aQ_L - \frac{bQ_L^2}{2}\right) - cQ_H\right] + n\left[aQ_L - \frac{bQ_L^2}{2} - cQ_L\right] - F$$

Solving for FOC for both $\frac{\delta\pi}{\delta Q_H} = 0$ and $\frac{\delta\pi}{\delta Q_L} = 0$ we will get:

$$N(A - BQ_H - c) = 0 => \boldsymbol{Q_H^* = \frac{A-c}{B}}$$

$$N(-A + BQ_L) + N(a - bQ_L) + n(a - bQ_L - c) = 0 => \boldsymbol{Q_L^* = \frac{n(a-c) - N(A-a)}{N(b-B) + nb}}$$

$$\boldsymbol{V_L^* = aQ_L^* - \frac{bQ_L^{*2}}{2}}$$

$$\boldsymbol{V_H^* = AQ_H^* - \frac{BQ_H^*}{2} - \left(AQ_L^* - \frac{BQ_L^{*2}}{2}\right) + V_L^*}$$

Putting this value of Q_H and Q_L in V_H and V_L functions we get an optimal package of (V_H^*, Q_H^*) and (V_L^*, Q_L^*).

Optimal Pricing Strategy

The pricing is only optimal when,

$$\boldsymbol{\frac{n(a-c) - N(A-a)}{N(b-B) + nb} > 0}$$

In other words, when $n(a-c) - N(A-a)$ and $N(b-B) + nb$ are of the same sign.

If the two are of the opposite signs, then the optimal strategy is to sell only to high income customers that is sell only 1 package which of $(V_H, Q_H) = \left(\frac{(A^2-c^2)}{2B}, \frac{A-c}{B}\right)$