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A0206518H: HW2

Info. from the Question:

'N' high-income customers with demand: $P_H = A - BQ_H$

'n' low-income customers with demand: $P_L = a - bQ_L$

where

$A > a > 0$ and

Cost: $C = F + cQ$ where $0 < c < a$

→ We assume that 'B' and 'b' are positive here, so that demand is sloping downwards.

Let $W_H(Q)$ and $W_L(Q)$ be willingness to pay for high and low income respectively for quantity 'Q'.

Then,

$$W_H(Q) = \int_0^Q P_H(x)\,dx = AQ - \frac{BQ^2}{2}$$

$$W_L(Q) = \int_0^Q P_L(x)\,dx = aQ - \frac{bQ^2}{2}$$

(a.)

(i) If the target customers are high-end;

$$\pi_H = N\left[W_H(Q) - (F + cQ)\right] = N\left[AQ - \frac{BQ^2}{2} - F - cQ\right]$$

To maximise profit;

$$\frac{d\pi_H}{dQ} = 0 \Rightarrow N\left[A - BQ - c\right] = 0 \Rightarrow \boxed{Q_H = \frac{A - c}{B}}$$

So, $V_H = W_H\left(\frac{A-C}{B}\right) = A\left(\frac{A-C}{B}\right) - \frac{1}{2B}(A-C)^2$

$$= \left(\frac{A-C}{B}\right)\left(\frac{A+C}{2}\right) = \boxed{\frac{A^2-C^2}{2B}}$$

(ii) If $N << n$; then low income customers have to included in package design.

So,
$$\pi_L = (n+N)\left[W_L(Q) - (F+cQ)\right]$$
$$= (n+N)\left[aQ - \frac{bQ^2}{2} - (F+cQ)\right]$$

To maximise profit again,

$$\frac{d\pi_L}{dQ} = 0 \Rightarrow (n+N)\left[a - bQ - c\right] = 0$$
$$\Rightarrow \boxed{Q_L = \frac{a-c}{b}}$$

$$\therefore V_L = W_L\left(\frac{a-c}{b}\right) = Q\left(a - \frac{bQ}{2}\right)$$
$$= \left(\frac{a-c}{b}\right)\left[a - \left(\frac{a-c}{2}\right)\right]$$
$$= \boxed{\frac{a^2-c^2}{2b}}$$

(b) For 2^{nd} degree price discrimination,

1. $V_L = W_L(Q_L) \Rightarrow$ low-income customers
2. $W_H(Q_H) - V_H = W_H(Q_L) - V_L \Rightarrow$ incentive-compatibility constraint

From ②,

$$V_H = W_H(Q_H) - W_H(Q_L) + V_L$$

$$= AQ_H - \frac{BQ_H^2}{2} - \left[AQ_L - \frac{BQ_L^2}{2}\right] + \underbrace{aQ_L - \frac{bQ_L^2}{2}}_{\text{From ①}}$$

So,

$$\text{Profit}, \pi = N\left[V_H - (F + cQ_H)\right] + n\left[V_L - (F + cQ_L)\right]$$

Solving for quantities of high and low income customers,

$$\rightarrow \frac{\partial \pi}{\partial Q_H} = 0 \Rightarrow N\left(\frac{\partial V_H}{\partial Q_H} - c\right) = 0$$

$$\Rightarrow A - BQ_H - c = 0 \Rightarrow \boxed{Q_H^* = \frac{A-c}{B}}$$

$$\rightarrow \frac{\partial \pi}{\partial Q_L} = 0 \Rightarrow N\left(\frac{\partial V_H}{\partial Q_L}\right) - n\left(\frac{\partial V_L}{\partial Q_L} - c\right) = 0$$

$$\Rightarrow N\left[Q_L(B-b) - (A-a)\right] - n\left[a - bQ_L - c\right] = 0$$

$$\Rightarrow NQ_L(B-b) - N(A-a) - na - nbQ_L - nc = 0$$

$$\Rightarrow Q_L(NB - Nb - nb) = n(a+c) + N(A-a)$$

$$\boxed{\therefore Q_L^* = \frac{n(a+c) + N(A-a)}{N(B-b) - nb}}$$

The 2 packages are : $\left[Q_H^*, V_H^* = W_H(Q_H^*)\right]$ and

$\left[Q_L^*, V_L^* = W_L(Q_L^*)\right]$ iff $Q_L^* > 0$