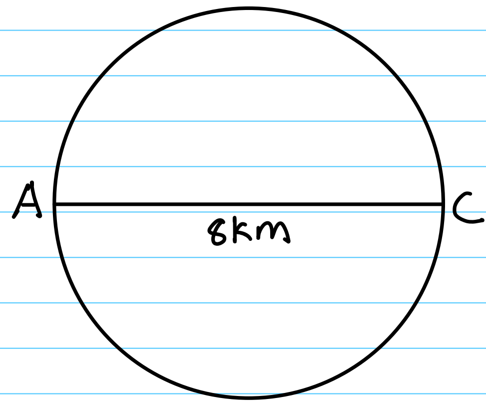
Week 5 & 6 Optimization and Multivariable Differentiation



a) How long? → Time

Distance = Time x speed

$$\text{Time} = \frac{\text{Distance}}{\text{speed}}$$

Flying over the lake across the straight line

radius = 4km

Diameter = 4 X 2 = 8

Diameter → Distance from A to C → 8km

speed = 2 km/h

$$\text{Time} = \frac{8\text{ km}}{2\text{ km/h}} = \overset{4}{\cancel{8\text{km}}} \times \frac{h}{1\cancel{2}\cancel{\text{km}}}$$

= 4 hours

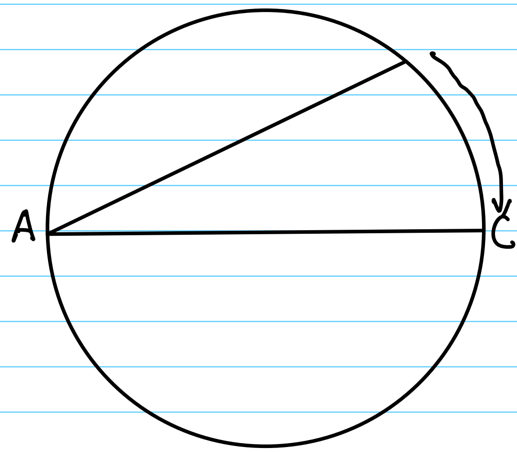
b Distance → circumference of the lake

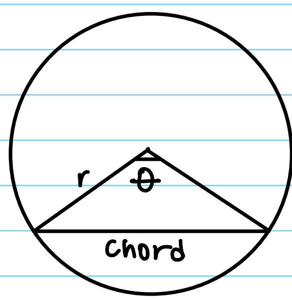
$$= 2 \times \pi r /_{2} \quad = \pi r$$

speed = 3km/h

$$\text{Time} = \frac{\pi r}{3\,km/h}$$

$$= \frac{22}{7} \times 4 \times \frac{1}{3} = \frac{88}{21} = 4.19 \text{ hours}$$



c) 

$$\text{chord length} = 2 \times r \times \sin\frac{(\theta)}{2}$$

$$= \cancel{2} \times 4 \times \frac{\sin(\theta)}{\cancel{2}}$$

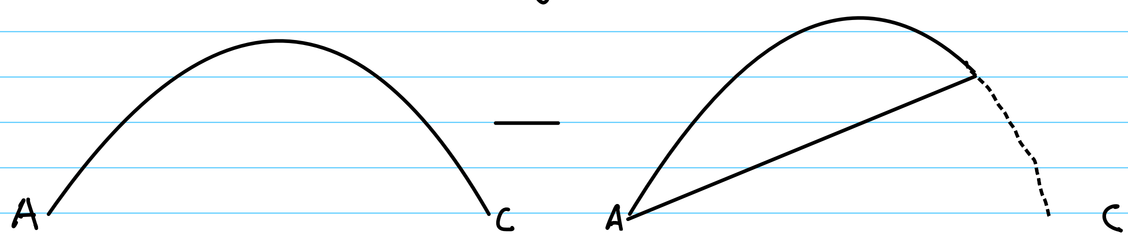
$$= 4\sin\theta$$

How long will it take the drone to traverse the chord?

$$= \frac{\cancel{4}\sin\theta}{\cancel{2}} = 2\sin\theta \text{ hrs}$$

d) The length of remaining path :

Perimeter of lake − length of arc



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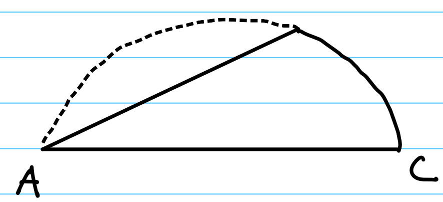


Perimeter of lake $= \pi r$
Length of arc $= \theta \times r$

$$\pi r - \theta r$$
$$\frac{22}{7} \times 4 - 4\theta$$
$$= 4\left(\frac{22}{7} - \theta\right)$$
$$= 4(\pi - \theta)$$

How long it will take the drone to complete the journey?

It is a circular path

$\therefore$ speed $= 3 km/h$

$$\text{Time} = \frac{4(\pi-\theta)}{3} = 1.3(\pi - \theta)$$

$= 1{\cdot}3(22/7 - \theta)$

$= (4{\cdot}189 - 1{\cdot}3\theta)$ hrs

e) Combining both results:

$T = 2\sin\theta + 4{\cdot}189 - 1{\cdot}3\theta$
$= 2\sin\theta - 1{\cdot}3\theta + 4{\cdot}189$

f) To find minimum /maximum value of θ:
Differentiate T wrt θ

$\frac{dT}{d\theta} = (2 \times \cos\theta + \sin\theta) - 1{\cdot}3$

$= 2\cos\theta + \sin\theta - 1{\cdot}3$

The critical point:

$2\cos\theta + \sin\theta - 1{\cdot}3 = 0$

$2\cos\theta + \sin\theta = 1{\cdot}3$

$2\cos\theta + \sqrt{(1-\cos^2\theta)} = 1{\cdot}3$

Let $\cos\theta = x$

$2x + \sqrt{1-x^2} = 1{\cdot}3$

$\sqrt{1-x^2} = 1{\cdot}3 - 2x$

Square both sides

$$1 - x^2 = (1.3 - 2x)^2$$

$$1 - x^2 = (1.3 - 2x)(1.3 - 2x)$$

$$1 - x^2 = 1.69 - 2.6x - 2.6x + 4x^2$$

$$1 - x^2 = 1.69 - 5.2x + 4x^2$$

$$4x^2 + x^2 - 5.2x + 1.69 - 1 = 0$$

$$5x^2 - 5.2x + 0.69 = 0$$

Solving Using Tiger algebra

$$x_1 = \frac{13}{25} + \frac{\sqrt{331}}{50} = 0.884$$

$$x_2 = \frac{13}{25} - \frac{\sqrt{331}}{50} = 0.156$$

But $\cos\theta = x$

$$\theta = \cos^{-1}(x)$$

$$\theta_1 = \cos^{-1}(0.884) = 27.87$$

$$\theta_2 = \cos^{-1}(0.156) = 81.025$$

To find the minimum and maximum values, plug in θ in the Original equation

$2\sin\theta - 1.3\theta + 4.189$

$2\sin(27.87) - 1.3(27.87) + 4.189$

$= -31.255$

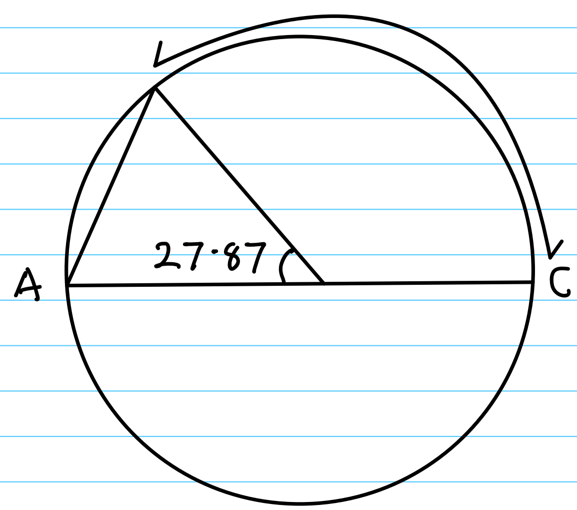
$2\sin(81.025) - 1.3(81.025) + 4.189$

$= -102.364$

$\theta = 81.025$ would minimize time

$\theta = 27.87$ would maximize time

g) I would want to minimize time, so, I'll use $\theta = 27.87$



Then from the end of the chord, I'll pilot the drone round the circumference to point C

Analysis of multiple variable functions

$$f(x,y) = \frac{x^2-3x}{1+x^2y^2}$$

$$= (x^2-3x)(1+x^2y^2)^{-1}$$

1st Diff wrt x

$$(x^2-3x) \times -1(1+x^2y^2)^{-2} \times y^2(2x) +$$
$$(1+x^2y^2)^{-1} \times (2x-3)$$

$$-1(x^2-3x)(1+x^2y^2)^{-2} \times y^2(2x) + (1+x^2y^2)^{-1}$$
$$\times (2x-3)$$

$$= -y^2(x^2-3x)(1+x^2y^2)^{-2}(2x) + (1+x^2y^2)^{-1}$$
$$\times (2x-3)$$

2nd Diff wrt y

$$(x^2-3x)(1+x^2y^2)^{-1}$$

$$(x^2-3x) \times -1(1+x^2y^2)^{-2} \times x^2(2y) +$$
$$(1+x^2y^2)^{-1}(0)$$

$$-x^2(x^2-3x)(1+x^2y^2)^{-2}(2y)$$

Second derivatives

$f_{xx}, f_{yy}, f_{xy}, f_{yx}$

$$f_{xy} = f_{yx}$$

$$f_{xx} = \frac{d}{dx}(f_x)$$

$$f_x = \frac{3y^2x^2+2x-3}{(1+y^2x^2)^2}$$

$a = 3x^2y^2 + 2x - 3$

$b = (1+y^2x^2)^2$

$$f_{xx} = \frac{a'b - b'a}{b^2}$$

$$a' = \frac{d}{dx}(3y^2x^2+2x-3)$$

Using the power and sum rule,

$a' = 6y^2x + 2$

$$b' = \frac{d}{dx}(1+y^2x^2)^2$$

Using chain rule

$$b' = 2 \times 2xy^2 \times (1+y^2x^2)$$
$$= 4xy^2(1+y^2x^2)$$

$$f_{xx} = \frac{(6y^2x+2)(1+y^2x^2)^2 - 4xy^2(1+y^2x^2)(3x^2y^2+2x-3)}{(1+y^2x^2)^{2\times 2}}$$

$$= \frac{(6y^2x+2)(1+y^2x) - (4xy^2)(3x^2y^2+2x-3)}{(1+y^2x^2)^3}$$

f_{yy}

$$f_{yy} = \frac{d}{dy}(f_y)$$

$$f_y = \frac{2x^3y(3-x)}{(1+x^2y^2)^2}$$

$$f_y = \frac{6x^3y - 2x^4y}{(1+x^2y^2)^2}$$

$$a = 6x^3y - 2x^4y$$

$$b = (1 + x^2y^2)^2$$

$$f_{yy} = \frac{d}{dy}\left(\frac{a}{b}\right)$$

Using quotient rule

$$a' = \frac{d}{dy}(6x^3y - 2x^4y)$$

$$a' = 6x^3 - 2x^4$$

$$b' = \frac{d}{dy}(1+x^2y^2)^2$$

$$b' = 2 \times 2x^2y \times (1+x^2y^2)$$

$$b' = 4x^2y(1 + x^2y^2)$$

$$f_{yy} = \frac{(6x^3-2x^4)(1+x^2y^2)^2 - 4x^2y(1+x^2y^2)(6x^3y-2x^4y)}{(1+x^2y^2)^4}$$

$$f_{yy} = \frac{(6x^3-2x^4)(1-3x^2y^2)}{(1+x^2y^2)^3}$$

$$f_{yx} = \frac{d}{dx}(f_y)$$

$$f_y = \frac{(6x^3 - 2x^4)(1 - 3x^2y^2)}{(1 + x^2y^2)^3}$$

can be written as

$$f_y = \frac{6x^3 - 18x^5y^2 - 2x^4 + 6x^6y^2}{(1 + x^2y^2)^3}$$

$$f_{yx} = \frac{d}{dx}\left(\frac{6x^3 - 18x^5y^2 - 2x^4 + 6x^6y^2}{(1 + x^2y^2)^3}\right)$$

$$a = 6x^3 - 18x^5y^2 - 2x^4 + 6x^6y^2$$

$$b = (1 + x^2y^2)^3$$

$$f_{yx} = \frac{d}{dy}\left(\frac{a}{b}\right)$$

Quotient rule,

$$f_{yx} = \frac{a' \cdot b - b' \cdot a}{b^2}$$

$$a' = \frac{d}{dx}(a)$$

$$a' = 18x^2 - 90x^4y^2 - 8x^3 + 36x^5y^2$$

$$b' = \frac{d}{dx}(b)$$

Using the chain rule

$$b' = 3 \times 2xy^2 \times (1+x^2y^2)^2$$
$$= 6xy^2(1+x^2y^2)^2$$

$$f_{yx} = \frac{2x^2(9y^4x^4 + 20y^2x^3 - 54y^2x^2 - 4x + 9)}{(y^2x^2+1)^4}$$

b Equation of tangent

$$z - z_0 = m(x - x_0) + n(y - y_0)$$

$$z_0 = f(2,3)$$

$$z_0 = \frac{(2)^2 - 3(2)}{1 + (2)^2 \cdot (3)^2}$$

$$= \frac{4 - 6}{1 + 4 \times 9}$$

$$= -2/37$$

m will be $f_x(2,3)$

$$m = \frac{3y^2x^2 + 2x - 3}{(1+y^2x^2)^2}$$

$$m = \frac{3(3)^2(2)^2 + 2(2) - 3}{(1 + 3^2 \times 2^2)^2}$$

$$m = \frac{108 + 4 - 3}{(1+36)^2}$$

$$m = \frac{109}{1369}$$

n is $f_y(2,3)$

$$n = \frac{6x^3y - 2x^4y}{(1+x^2y^2)^2}$$

$$n = \frac{6(2)^3(3) - 2(2)^4(3)}{(1+(2)^2(3)^2)^2}$$

$$n = \frac{144 - 96}{1369}$$

$$n = \frac{48}{1369}$$

$$z - \left(\frac{-2}{37}\right) = \frac{109}{1369}(x-2) + \frac{24}{1369}(y-3)$$

$$z + \frac{2}{37} = \frac{109(x-2) + 48(y-3)}{1369}$$

$$z = \frac{109x - 218 + 48y - 144}{1369} - 2/37$$

$$z = \frac{109x + 48y - 362 - 74}{1369}$$

$$z = \frac{109x + 48y - 436}{1369}$$

Using the above formula to estimate $f(2.1, 2.95)$

$$f(2.1, 2.95) = \frac{109(2.1) + 48(2.95) - 436}{1369}$$

$$= -65.5/1369 \quad = -0.0478$$

Lets evaluate actual value

$$f(2.1, 2.95) = \frac{(2.1)^2 - 3(2.1)}{1 + (2.1)^2 (2.95)^2}$$

$$= \frac{-1.89}{39.378025}$$

$$= -0.047996$$

$$100 - \left(\frac{-0.000196}{-0.047996}\right) \times 100$$

$$= 99.59\%$$

This is the accuracy.

c We will find values for x and y for which $f_x = 0$ and $f_y = 0$

$$f_x = \frac{3y^2x^2 + 2x - 3}{(1+y^2x^2)^2} = 0$$

$$3y^2x^2 + 2x - 3 = 0$$

$$3y^2x^2 = 3 - 2x$$

$$y^2 = \frac{3-2x}{3x^2}$$

$$y = \sqrt{\frac{3-2x}{3x^2}}$$

$$f_y = \frac{6x^3y - 2x^4y}{(1+x^2y^2)^2} = 0$$

$$6x^3y - 2x^4y = 0$$

$$y(6x^3 - 2x^4) = 0$$

but $y = \sqrt{\frac{3-2x}{3x}}$

$$(6x^3 - 2x^4) \times \sqrt{\frac{3-2x}{3x}}$$

$$\sqrt{\frac{3-2x}{3x}} = 0$$

$$3 - 2x = 0$$
$$3 = 2x$$
$$x = \frac{3}{2}$$

OR//

$$6x^3 - 2x^4 = 0$$
$$2x^3(3 - x) = 0$$
$$2x^3 = 0 \quad \text{OR} \quad 3 - x = 0$$
$$x = 0 \qquad\qquad 3 = x$$

critical values are

$x^* = 0, 3$ or $3/2$

Lets find y

$x = 0$

$$y = \sqrt{\frac{3 - 2(0)}{3(0)}}$$
$$= \sqrt{\frac{3}{0}} \longrightarrow \text{Undefined}$$

$x = 3$

$$y = \sqrt{\frac{3 - 2(3)}{3(3)}}$$
$$y = \pm\sqrt{\frac{-3}{9}}$$

$$y = +\sqrt{-\frac{1}{3}}$$

This is a non-imaginery number, we won't include this

$x = 3/2$

$$y = \sqrt{\frac{3 - 3/2 \times 2}{3 \times 3/2}}$$

$$y = \sqrt{\frac{0}{9/2}}$$

$$y = 0$$

Critical point is $(3/2, 0)$

Lets classify

$$D = f_{xx}(3/2, 0) \times f_{yy}(3/2, 0) - (f_{xy}(3/2, 0))^2$$

After evaluating using sage:

$f_{xx}(3/2, 0) = 2$

$f_{yy}(3/2, 0) = 81/8$

$f_{xy}(3/2, 0) = 0$

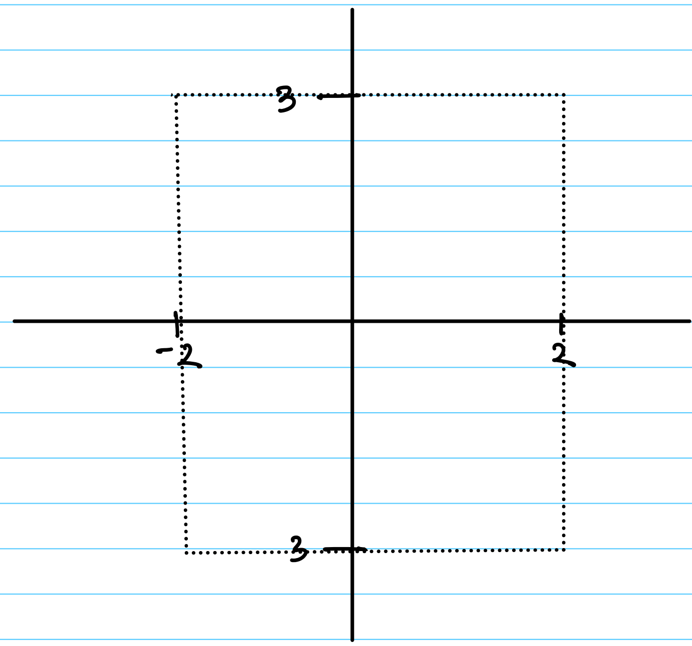
$D = 2 \times 81/8 - (0)^2$

$D = 81/4$

D is positive
$f_{xx}(3/2, 0)$ is positive as well

Therefore $(3/2, 0)$ is a minimum point

d



we already know $(3/2, 0)$ is a local extrema

To find global extrema, we evaluate the critical points within that domain.

$$f(-2,y) = \frac{(-2)^2 - 3(-2)}{1 + (-2)^2 y^2}$$

$$= \frac{4+6}{1+4y^2}$$

$$= \frac{10}{1+4y^2}$$

Lets find values for which $y=0$ in $f'(-2,y)$

$$f'(-2,y) = \frac{-80y}{(4y^2+1)^2}$$

$$-80y = 0$$
$$y = 0$$

$(-2,0)$ is a critical point

$$f(-2,0) = 10$$

$$f(2,y) = \frac{(2)^2 - 3(2)}{1 + (2)^2 y^2}$$

$$= \frac{4-6}{1+4y^2}$$

$$f(2,y) = \frac{-2}{1+4y^2}$$

$$= \frac{-2}{1+4y^2}$$

$$f'(2,y) = \frac{16y}{(4y^2+1)^2}$$

$$\frac{16y}{(4y^2+1)^2} = 0$$

$$16y = 0$$
$$y = 0$$

$(2,0)$ is a critical point

$$f(2,0) = -2$$

$$f(x,3) = \frac{x^2-3x}{1+x^2 \times 3^2}$$

$$= \frac{x^2-3x}{1+9x^2}$$

$$f'(x,3) = \frac{2x-3}{9x^2+1} - \frac{18x(x^2-3x)}{(9x^2+1)^2}$$

Calculated using sage Math

At $f'(x,3) = 0$

$$x = \frac{\sqrt{82}-1}{27} \quad \text{or} \quad x = \frac{-\sqrt{82}-1}{27}$$

$x = 0.2983$ or $x = -0.3724$

Both points are within the boundary

$(0.2983, 3)$ $(-0.3724, 3)$

$f(0.2983, 3) = -0.4475$

$f(-0.3724, 3) = 0.5586$

$$f(x, -3) = \frac{x^2 - 3x}{1 + x^2(-3)^2}$$

$$= \frac{x^2 - 3x}{1 + 9x^2}$$

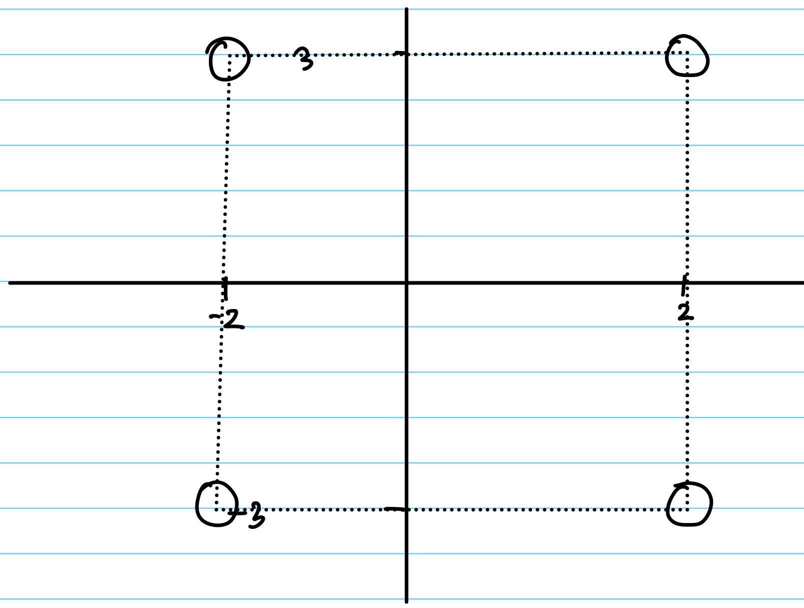
$f(x, -3) = f(x, 3)$

critical points are same

Lets check the endpoints

End points are

$f(-2, 3)$, $f(-2, -3)$, $f(2, 3)$ & $f(2, -3)$



$$f(-2,-3) = \frac{(-2)^2 - 3(-2)}{1 + (-2)^2(-3)^2}$$

$$= 0.2703$$

$$f(-2,3) = \frac{(-2)^2 - 3(-2)}{1 + (-2)^2(3)^2}$$

$$= 0.2703$$

$$f(2,3) = \frac{(2)^2 - 3(2)}{1 + (2)^2(3)^2}$$

$$= -0.05405$$

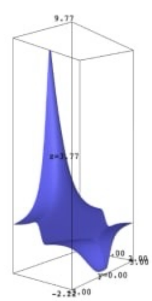
$$f(2,-3) = \frac{(2)^2 - 3(2)}{1 + (2)^2(-3)^2}$$

$$= -0.05405$$

Comparing the values evaluated at the critical points & endpoints, the highest value is at (2,0) and lowest value is at (3/2, 0)

Global maximum ⟶ (2,0)
Global minimum ⟶ (3/2, 0)

e



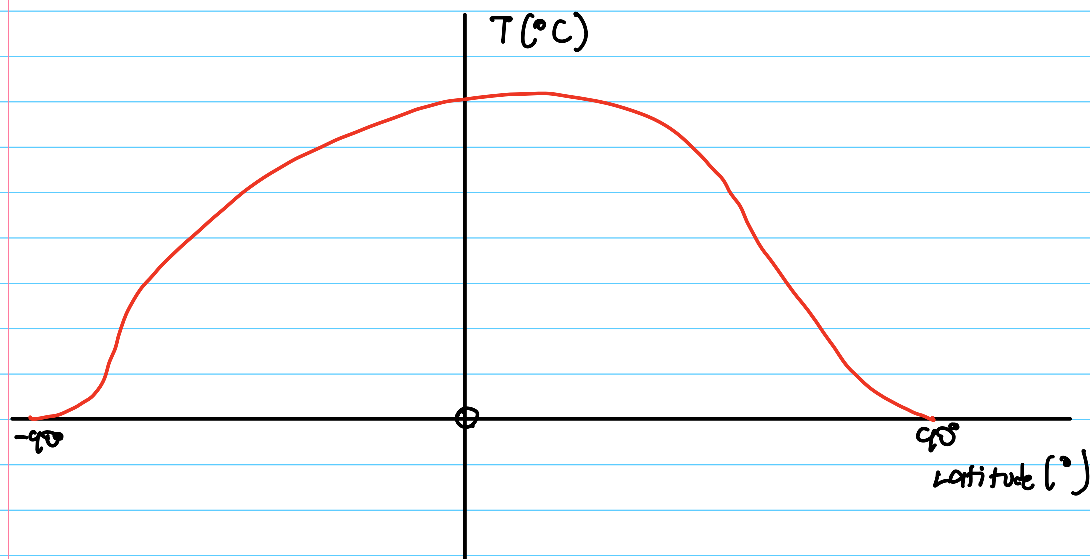
The plot agrees that the maximum occurs at $y=0$ and $x=-2$ and the minimum also occurs at (3/2, 0)

Estimating derivatives

The Latitude & Longitude of my location is 37.775° and -122.419°

Temperature of location varies with latitude but not with longitude.

The graph of Temperature and latitude looks like



Time also varies with temperature in a similar way as latitude. For example, Temperature increases from the morning and reaches it's peak around noon. Then it decreases till night.

Because Time does not change with longitude
If x = longitude
then
$$\frac{dT}{dx} = 0$$

The graph of time & latitude looks similar to the cosine graph. We can use it for an estimate.

$$T_m = T_{(0,0)} \times \cos(y) \times a \quad \dots\dots \quad ①$$

$T_{(0,0)}$ is the temperature at the null point i.e (lat = 0, long = 0), y is the latitude, T_m is the mean location of the place, a is proportionality constant

$$T = T_m \times \cos(12-t) \times b \quad \dots\dots \quad ②$$

t is the 24 hour time, b is the proportionality constant

Substituting 1 in 2, we have

$$T = a \times b \times T_{(0,0)} \times \cos(y) \times \cos(12-t)$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial T}{\partial y} =$$

keeping everything else constant

$$a \times b \times T(0,0) \times \cos(12-t) \times -\sin(y)$$

$$-abT(0,0)\cos(12-t)\sin(y)$$

$$\frac{\partial T}{\partial t} =$$

$$a \times b \times T(0,0) \times \cos(y) \times -\sin(12-t) \times -1$$

$$= ab\, T(0,0) \cos(y) \sin(12-t)$$