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BOUNDARY STIMULATION OF SOCIAL INFLUENCE NETWORKS

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Abstract. Our aim in this article is to tackle formally a simple version of the Friedkin–Johnsen approach to social influence networks, by following the methodology of discrete analysis on graphs, the way it is motivated by the mathematical theory of linear parabolic and elliptic partial differential equations (heat/diffusion and Laplace equations). This analysis may shed some light on the dynamics of opinion propagation inside a social network, which, in this way, can be examined formally under the angle of how structure responds to opinion stimulations triggered at the network boundary in the presence of differentiated intrinsic proclivities and susceptibilities for opinion change or perseverance. Since, in this formal guise, social influence network problems are linear, their solutions can be easily computed in matrix (or linear operator) form. Moreover, one may define a number of matrix or vector measures of network influenceability that attune to known measures of adjacency and degree-closeness centrality in social network analysis. Examples of computation of these measures are given here with regards to artificially generated random multilayer networks possessing randomly assigned or fixed (across layers) influence susceptibilities.

By a social influence network one usually understands a social network, formed by a group of persons, who are assumed to hold an opinion or an attitude, which is subject to sequential changes induced by the force of influence that one person exerts on another person, while they are mutually interacting in the network (French, 1956, Friedkin, 1997). Nevertheless, the extent of revision of a person's opinion is assumed to depend upon the degree of openness or susceptibility that a person exhibits towards the clout of her peers. As a result, in the context of the basic formalization of social influence networks (Friedkin, 1997, Friedkin and Johnsen, 2011), given a social network of persons, in which each person holds an initial opinion and possesses a certain degree of susceptibility to influence, one postulates a specific rule of how exactly persons' opinions are revised and, hence, one expects to be able to predict the subsequently established distribution of opinions on the social network. This is exactly the formal problem raised by the analysis of a social influence network, when one tries to understand the opinion dynamics developing on such a network.

In the basic formulation of this problem (Friedkin, 1997, Friedkin and Johnsen, 2011), one has to solve a certain system of linear equations, i.e., a problem of linear algebra. However, since this problem is actually defined on a distributed network, one of the terms nested inside the governing equations is the (possibly weighted) adjacency matrix of the graph, which is the mathematical abstraction that represents the (empirically given) social network. As a matter of fact, elaborating on the graph–theoretic formulation of this problem, one should fall upon the graph Laplacian matrix. Hence, it is in this guise that the evolutionary problem at hand can be identified to a specific form of the discrete analogue of the diffusion or heat equation (i.e., the parabolic partial differential equations (PDEs) that describe the time variation of density in a material undergoing diffusion or the distribution of heat in a given material over time), whence the relevance of examining the opinion dynamics on a social influence network as a particular type of diffusion process on a social network.

Thus, our aim in this article is to pursue a graph–theoretic and PDE's inspired version of the social influence network problem, which can be directly presented and solved in the form of a discrete PDE on a graph. Finally, we are going to discuss a number of measures of influenceability that naturally arise in the context of such a theorization of social influence multilayer networks.

Let $G = (V, E)$ be a simple undirected graph with n vertices, i.e., $n = |V|$,[1] so that, without any loss of generality, vertices can be denoted as (positive) integers $i, j, \dots \in [n]$; thus, the set of graph vertices is $V = [n]$ and the set of graph edges is $E = \{(i, j) : i \sim j, \text{ for some } i, j \in [n], i \neq j\}$ (with the understanding that edges (i, j) and (j, i) are identical). If vertices i and j are adjacent, we write $i \sim j$, and we denote by $\deg(i)$ the degree of i, i.e., $\deg(i) = |\{j \in [n] : j \sim i\}|$. Again without any loss of generality, we are also assuming that G is connected (which, in particular, would imply that, for any $i \in V, \deg(i) \neq 0$). As usually, $A = \{A_{ij}\}_{i,j\in[n]}$ is the adjacency matrix of G, i.e., A is a binary matrix $n \times n$ (one may write $A \in \{0, 1\}^{V\times V}$) such that $A_{ij} = 1$, whenever $i \sim j$, and $A_{ij} = 0$, otherwise. Furthermore, the **walk matrix** of G is defined as a $n \times n$ matrix $N = \{N_{ij}\}_{i,j\in[n]} \in \mathbb{R}^{V\times V}$ such that

$$N = D^{-1} A, \tag{0.0.1}$$

where D is the diagonal matrix with entries equal to the degree $\deg(i)$ of vertex i, i.e., as[2]

$$N_{ij} = \begin{cases} \frac{1}{\deg(i)}, & \text{whenever } i \sim j, \\ 0, & \text{otherwise,} \end{cases}$$

and the (**combinatorial**) **Laplacian matrix** of G is defined as a $n \times n$ matrix $L = \{L_{ij}\}_{i,j\in[n]} \in \mathbb{R}^{V\times V}$ such that

$$\begin{aligned} L &= D - A \\ &= D(I - N), \end{aligned} \tag{0.0.2}$$

where I is the unit matrix, i.e., as

$$L_{ij} = \begin{cases} \deg(i), & \text{whenever } i = j, \\ -1, & \text{whenever } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

Given a positive integer k such that $1 \leq k \leq n$, let $\mathscr{V}_k = \{V_\alpha\}_{\alpha\in[k]}$ be a family of k nonempty subsets V_α of V. We say that $\mathscr{V}_k$ is a k–**partition** of the set of vertices V,[3] whenever

- the subsets V_α are mutually disjoint and
- $V = \bigcup_{\alpha=1}^{k} V_\alpha$.

Definition: Saying that G is a **multilayer graph** composed of k layers (or a k–**layer graph**),[4] we mean that G admits a k–partition $\mathscr{V}_k = \{V_\alpha\}_{\alpha\in[k]}$. The vertex

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[1]For any finite set X, we denote by $|X|$ the number of elements of X. In addition, for any positive integer x, we denote by $[x]$ the set of integers from 1 to x, i.e., $[x] = \{1, 2, \dots, x\}$.

[2]In the language of random walks, the walk matrix is the transition matrix describing the transitions of a random walk on G.

[3]Apparently, a vertex k–partition of a graph with n vertices corresponds to an integer partition of n into k positive integers $n_1, n_2, \dots, n_k$ such that $n_1 + n_2 + \dots + n_k = n$, where $n_\alpha = |V_\alpha|$, for $\alpha = 1, 2, \dots, k$.

[4]We are going to disregard the two trivial cases: $k = 1$, when the graph G itself may be considered a 1-layer graph through the trivial 1–partition $\mathscr{V}_1 = \{V\}$, and $k = n$, when G may be considered a n-layer graph through the trivial n–partition that makes all V_α's to be singletons.

subsets V_α's (and also the corresponding induced subgraphs $G(V_\alpha)$) are the **layers** of graph G, i.e., for any $\alpha \in [k]$, V_α (and $G(V_\alpha)$) is the α–th layer (and the α–th layer subgraph, resp.) of G.[5]

When V_α and V_β are two (not necessarily distinct) layers of the multilayer graph G, we denote by $E_{\alpha\beta}$ the restriction of the edge set E on $V_\alpha \times V_\beta$. The elements of $E_{\alpha\alpha}$ are called α–**intralayer edges**, while, for $\alpha \neq \beta$, the elements of $E_{\alpha\beta}$ are called $\alpha\beta$–**interlayer edges**. Clearly, the α–th layer subgraph of G is $G_\alpha = G(V_\alpha) = (V_\alpha, E_{\alpha\alpha})$ (for any $\alpha \in [k]$), while the edge–induced subgraph by the interlayer edges $E_{\alpha\beta}$ (for any $\alpha \neq \beta \in [k]$) is a bipartite graph with bipartition (V_α, V_β), which is denoted as $H_{\alpha\beta} = G(E_{\alpha\beta}) = (V_\alpha, V_\beta, E_{\alpha\beta})$. (Notice that $H_{\beta\alpha} = H_{\alpha\beta}$.) Thus, G can be decomposed as the graph union of all the intralayer subgraphs and an interlayer multipartite graph:

$$G = \Big(\bigcup_{\alpha\in[k]} G_\alpha\Big) \cup \Big(\bigcup_{\alpha,\beta\in[k],\alpha\neq\beta} H_{\alpha\beta}\Big).$$

When considering a (real valued) matrix M defined over V (i.e., $M \in \mathbb{R}^{V\times V}$), for any two (nonempty) subsets of vertices $U_1, U_2 \subset V$, we use $M_{U_1U_2}$ to denote the submatrix (or block) of M with rows indexed by U_1 and columns indexed by U_2 (i.e., $M_{U_1U_2} \in \mathbb{R}^{U_1\times U_2}$). In this way, if A is the adjacency matrix of the k–layer graph G, for any $\alpha \in [k]$, $A_{V_\alpha V_\alpha}$ is the adjacency matrix of the intralayer subgraph $G(V_\alpha)$, while for any $\beta \in [k]$, $\beta \neq \alpha$, $A_{V_\alpha V_\beta}$ is the biadjacency matrix of the interlayer bipartite graph $H_{\alpha\beta}$ (Notice that $A_{V_\beta V_\alpha} = (A_{V_\alpha V_\beta})^T$). Moreover, for the sake of convenience, we are going to write $A_{\alpha\beta}$ instead of $A_{V_\alpha V_\beta}$, for any layers $\alpha, \beta \in [k]$. Of course, similar notation applies to all matrices I, D, N and L.

Now, let us assume that G is the underlying graph of a social network and that the vertices of G represent a group of persons, who maintain their interactions over a pattern of ties represented by the set of edges of G. Thus, from now on, we are going to use interchangeably the terms network–graph, vertices–persons and edges–ties (or interactions). Moreover, we assume that each person possesses an **opinion** (about a certain issue) in such a way that one's opinion might possibly change in time as one interacts with her peers in the social network. In this way, opinion is identified to a real valued function of the person who holds it and the time the opinion is held, i.e., person's i opinion at time t is denoted as $v^t(i) \in \mathbb{R}$, where $i \in V$ and t is the time with values $t = 0, 1, 2, \ldots$ (in the discrete case considered here). Of course, for each t, the mapping $i \mapsto v^t(i)$ (from $V = [n]$ to $\mathbb{R}$) also forms a vector $v^t \in \mathbb{R}$ such that, for any $i \in V$, the i–th component of the vector v^t is the opinion $v^t(i)$ of person i at time t.

According to the **DeGroot–Friedkin** model of social influence in a social network (DeGroot, 1974, Friedkin, 1998), person's i opinion at time t is updated at next instance $t+1$ through a $n \times n$ matrix $W = \{w_{ij}\}_{i,j\in V}$ of interpersonal influences (where $0 \leq w_{ij} \leq 1, \sum_{j\in V} w_{ij} = 1$) in such a way that the influence that person i receives from her peers at time t, denoted as $v^t_{\text{received}}(i)$, is set equal to the weighted average of all the opinions that i receives at time t by all j's having $w_{ij} > 0$ multiplied by the weight w_{ij}, i.e., for any $i \in V$,

$$v^t_{\text{received}}(i) = \frac{\sum_{j\in V} w_{ij}\, v^t(j)}{|\{j \in V : w_{ij} \neq 0\}|}.$$

[5]Apparently, every vertex belongs to a unique layer.

Here, we are going to consider a simplified version of the DeGroot–Friedkin model, according to which (i) any person can be only influenced by her immediate neighbors in the social network and (ii) that, in the expression of $v^t_{\text{received}}(i)$, the influence that i's neighbors exert on i is unweighted. This means that we are considering here the special case that the influence matrix W coincides with the adjacency matrix A of G, i.e., that the influence reaching i is the average of the opinions that i's neighbors hold, which is denoted by $v^t_{\text{neighbors}}(i)$:

$$\begin{aligned} v^t_{\text{neighbors}}(i) &= v^t_{\text{received}}(i) \\ &= \frac{1}{\deg(i)} \sum_{j \sim i} v^t(j) \\ &= D^{-1} A\, v^t(i) \\ &= N\, v^t(i), \end{aligned}$$

where N is the walk matrix on G.

Furthermore, in the version of the **DeGroot–Friedkin** that we are following here (Friedkin and Johnsen, 1990 and 1999), a person's opinion is updated to a new opinion at the next time, after comparing the neighbors' (average) opinion at that time with the initial opinion held by that person. Formally, such a comparison is represented by a convex combination of the average opinion $v^t_{\text{neighbors}}(i)$ of i's neighbors at time t and the initial opinion v^0 of i (at time $t = 0$):

$$\begin{aligned} v^{t+1}(i) &= s_i\, v^t_{\text{neighbors}}(i) + (1 - s_i)\, v^0(i) \\ &= s_i\, N\, v^t(i) + (1 - s_i)\, v^0(i), \end{aligned}$$

where s_i is i's **susceptibility coefficient**, a scalar parameter that is assumed to vary in the interval $[0, 1]$, for all $i \in V$. This means that, if $s_i = 0$, then i's opinion does not change ($v^{t+1}(i) = v^0(i)$), if $s_i = 1$, then i adopts the average opinion of her neighbors $v^t_{\text{neighbors}}(i) = N\, v^t(i)$, while if $0 < s_i < 1$, then i's opinion is inserted in–between $v^t_{\text{neighbors}}(i)$ and $v^0(i)$, where the exact inserted position is weighted by s_i. Moreover, let us denote by $s = \{s_i\}_{i \in [n]} \in [0, 1]^n$ the vector of susceptibility coefficients of all persons (vertices) i in the social network (graph) G and by S is the $n \times n$ diagonal matrix with its diagonal entries equal to the susceptibility coefficients s_i's.

In other words, in the context of the DeGroot–Friedkin model of social influence in a network, in order to find the distribution of the time–dependent opinions of all persons in the network, when the initial distribution of their opinions is known, one has to solve the following system of linear equations on $\mathbb{R}^V$:

$$v^{t+1} = S\, N\, v^t + (I - S)\, v^0, \text{ for } t = 0, 1, 2, \ldots. \tag{0.0.3}$$

In the sequel, we are going to follow a PDEs–based approach by tackling the above system of difference equations (0.0.3) as a (discrete) initial boundary value problem. In this approach, one first needs to specify a boundary constraint on the set of vertices V of graph G, over which (0.0.3) has to be solved. As a matter of fact, in the version of the model of social influence networks that we are considering here, it is through the distribution of susceptibility coefficients that one may obtain an appropriate "natural" boundary inside the graph. For this purpose, one just needs to isolate persons with zero susceptibility coefficient and assume that not everybody has zero susceptibility. Thus, let

$$\begin{aligned}\delta\Omega &= \{i \in V : s_i = 0\},\\ \delta &= |\delta\Omega|, \text{ where } \delta < |V| = n,\\ \Omega &= \{i \in V : s_i > 0\} \text{ and } V = \Omega \cup \delta\Omega,\\ |\Omega| &= n' = n - \delta > 0.\end{aligned}$$

In other words, there exist δ persons located in the boundary $\delta\Omega$, who are "stubborn" in the sense that they are always holding the same opinion, while $n' = n - \delta$ is the number of persons with malleable opinions, who are located inside the remaining part Ω of graph V.

Assuming that each stubborn person is adjacent to at least one person with positive susceptibility coefficient, i.e.,

$$\text{for each } j \in \delta\Omega, \{i : i \sim j\} \cap \Omega \neq \varnothing$$

and taking into account that G was already assumed to be connected, $\delta\Omega$ becomes a proper **boundary** (in the context of spectral graph theory Chung (1997)) of the (discrete) region Ω (over which one needs to solve the system of social influence equations (0.0.3)), i.e.,

$$\delta\Omega = \{j \notin \Omega : i \sim j \text{ for some } i \in \Omega\}.$$

Accordingly, the boundary condition for equation (0.0.3) is that the solution v on the boundary $\delta\Omega$ should be fixed and taken to be equal to 1, i.e.,

$$v_i^t = 1, \text{ for } i \in \delta\Omega. \tag{0.0.4}$$

Finally, to solve system (0.0.3), one needs an **initial condition**:

$$v_i^0 = \phi_i, \text{ for } i \in V = \Omega \cup \delta\Omega,$$

where $\phi \in \mathbb{R}^V$ is a given vector satisfying the boundary condition (0.0.4), i.e., such that $\phi_i = 1$, for $i \in \delta\Omega$.

Therefore, the initial boundary value problem (IBVP) corresponding to the system of equations (0.0.3) for the DeGroot–Friedkin model of social influence on $V = \Omega \cup \delta\Omega$ is the problem of finding $v^t \in \mathbb{R}^V$ such that, given $\phi \in \mathbb{R}^\Omega$, v^t satisfies the following system of equations, for any $t = 0, 1, 2, \ldots,$

$$v^{t+1} = S\,N\,v^t + (I - S)\,\phi, \text{ in } \Omega, \tag{0.0.5}$$

$$v^t = 1, \text{ on } \delta\Omega, \tag{0.0.6}$$

$$v^0 = \begin{cases} \phi, & \text{in } \Omega,\\ 1, & \text{on } \delta\Omega. \end{cases} \tag{0.0.7}$$

Actually, in (0.0.5), v^t, S, N, I should have been denoted as $v^t_\Omega, S_{\Omega\Omega}, N_{\Omega\Omega}, I_{\Omega\Omega}$ (resp.) and, in (0.0.6), v^t as $v^t_{\delta\Omega}$, according to the notation introduced above. However, for the sake of simplicity, we are going to keep disregarding the subscript "Ω" or "$\delta\Omega$" from anything that is explicitly prescribed to be valid "in Ω" or " on $\delta\Omega$."

Denoting by u^t the rate of opinion change at the unit time step, i.e., $u^t = v^t - v^{t-1}$, apparently, u^t statisfies the folllowing system of equations,[6] for any

[6]After having solved for the rate of opinion change u^t, the vector of opinions can be retrieved as $v^t = \sum_{\tau=1}^{t} u^\tau + \phi$.

$t = 1, 2, \ldots,$

$$\begin{aligned} u^{t+1} &= S\,(I - D^{-1}\,L)\,u^t, \text{ in } \Omega, \\ u^t &= 0, \text{ on } \delta\Omega, \\ u^1 &= \begin{cases} S\,(N - I)\,\phi, & \text{in } \Omega, \\ 0, & \text{on } \delta\Omega, \end{cases} \end{aligned}$$

having used the expression $N = I - D^{-1}\,L$ (because of (0.0.1) and (0.0.2)), where $D^{-1}L$ is the **random walk normalized Laplacian** G. Thus, setting $u^t = D^{-1/2}\,w^t$ and denoting by $\mathscr{L} = D^{-1/2}\,L\,D^{-1/2}$ the **symmetric normalized Laplacian**, we obtain the following system of equations for w^t

$$\begin{aligned} w^{t+1} &= S\,(I - \mathscr{L})\,w^t, \text{ in } \Omega, \\ w^t &= 0, \text{ on } \delta\Omega, \\ w^1 &= \begin{cases} D^{1/2}\,S\,(N - I)\,\phi, & \text{in } \Omega, \\ 0, & \text{on } \delta\Omega, \end{cases} \end{aligned}$$

which is a **parabolic** discrete (in time and space variables) IBVP equipped with a Dirichlet boundary condition, i.e., it is similar to the discrete IBVP for the **diffusion** (or **heat**) equation. (In fact, the case $S = I$ on Ω reduces the system of social influence equations to the ordinary diffusion or heat IBVP in Ω.) Therefore (for the required technical details on continuous time parabolic equations on graphs, see F.R.K. Chung, 1977, and S.–Y. Chung et al., 2007), the unique solution of the previous problem is given by

$$w^t = \mathscr{H}_t^s\,w^1, \text{ for any } t = 1, 2, \ldots,$$

where $\mathscr{H}_t^s$ is an s–dependent **Dirichlet heat kernel** matrix

$$\mathscr{H}_t^s = (S\,(I - \mathscr{L}))^{t-1}.$$

Moreover, it is known, in a similar way with the continuous time case Chung (1997), that, for a given connected subgraph Ω of graph G and for any $t = 1, 2, \ldots$, $\mathscr{H}_t^s$ is defined by

$$\mathscr{H}_t^s(i,j) = \sum_{k=1}^{n'} \left(s_k\,(1 - \lambda_k)\right)^{t-1}\,\phi_k(i)\phi_k(j),$$

where, for $k = 1, 2, \ldots, n'$, the λ_k's are the eigenvalues of the symmetric normalized Laplacian matrix $\mathscr{L}$ on Ω and ϕ_k are the corresponding orthonormal eigenvectors. Notice that, in the case $S = I$, $\mathscr{H}_t^s$ is reduced to the heat kernel of the ordinary heat/diffusion equation for the symmetric normalized Laplacian matrix $\mathscr{L}$ on Ω.

To find the corresponding **steady state** (or **equilibrium**) system of equations for the social influence IBVP (0.0.5), (0.0.6) and (0.0.7), one first needs to homogenize the initial condition (0.0.7) by subtracting v^0 from v^t so that the corresponding IBVP with homogeneous initial (and boundary) condition becomes, for any $t = 0, 1, 2, \ldots,$

$$\begin{aligned} v^{t+1} &= S\,N\,v^t + S\,(N - I)\,\phi, \text{ in } \Omega, \\ v^t &= 0, \text{ on } \delta\Omega, \\ v^0 &= 0, \text{ in } \Omega \cup \delta\Omega. \end{aligned}$$

Thus, at the asymptotic limit $\overline{v} = \lim_{t\to+\infty} v^t$, one gets the following boundary value problem (BVP)

$$\big(I - S\,(I - \mathscr{L})\big)\,\overline{v} = S\,(N - I)\,\phi, \text{ in } \Omega, \tag{0.0.8}$$

$$\overline{v} = 0, \text{ on } \delta\Omega. \tag{0.0.9}$$

As previously, to solve these equations in the context of spectral graph theory, after setting $\overline{v} = D^{-1/2}\,\overline{w}$, we get

$$\begin{aligned}\big(I - S\,(I - \mathscr{L})\big)\,\overline{w} &= b, \text{ in } \Omega,\\ \overline{w} &= 0, \text{ on } \delta\Omega.\end{aligned}$$

where

$$b = S\,D^{-1/2}\,(N - I)\,\phi.$$

Again, this is a nonhomogeneous **elliptic** discrete (in space variables) BVP equipped with a Dirichlet boundary condition and, in the context of spectral graph theory (Chung, 1997, Chung and Simpson, 2015), it admits a unique solution in the form

$$\overline{w} = \mathscr{G}^s\,b,$$

where $\mathscr{G}^s$ is the corresponding s–dependent **Green function** (matrix), i.e., the following matrix left inverse

$$\mathscr{G}^s = \big(I - S\,(I - \mathscr{L})\big)^{-1},$$

which can be computed explicitly as in Chung (1997) in terms of the eigenvalues and the eigenvectors of the symmetric normalized Laplacian matrix $\mathscr{L}$ on Ω. (In the particular case $S = I$, $\mathscr{G}^s$ is reduced to the Green function of the symmetric normalized Laplacian matrix $\mathscr{L}$ on Ω.)

Consequently, the methodology of spectral graph theory for the solution of parabolic and elliptic PDEs on graphs is applicable to the problem of solving either the nonstationary or the stationary problem of social influence on graphs. Although the linear operators used in this theory should be applied to normalized variables, without any loss of generality, we are going to use here the same notation for the expressions of solutions, even when the arguments in the problem at hand are un–normalized. Thus, given the vector s of susceptibility coefficients of the vertices (persons) of the graph (social network) G and the vector ϕ of initial opinions of these vertices (persons), the vector of opinions v^t at time t solving the IBVP (0.0.5), (0.0.6) and (0.0.7) will be denoted in terms of the s–dependent heat kernel $\mathscr{H}_t^s$ as

$$v^t = v^t(s,\phi) = \mathscr{H}_t^s\,\phi, \text{ for } t = 1, 2, \ldots.$$

Similarly, the vector of asymptotic opinions $\overline{v}$ solving the BVP (0.0.8) and (0.0.9) will be denoted in terms of the s–dependent Green function $\mathscr{G}^s$ as

$$\overline{v} = \overline{v}(s,\phi) = \mathscr{G}^s\,f(s,\phi),$$

where the vector f is

$$f(s,\phi) = S\,(N - I)\,\phi.$$

In this framework, social influence on a network can be studied as an "incomplete" diffusion process, in the sense that the deterministic social influence process (of the DeGroot–Friedkin model) satisfies a certain s–perturbed IBVP and BVP, which depend regularly on s and, in the limit $s \to 1$ (where 1 is the vector with all its components equal to 1), the corresponding solutions settle in the solutions of ordinary diffusion on a graph. Therefore, for any i, t, s and ϕ, $\left(\mathscr{H}_t^s\,\phi\right)_i \le \left(\mathscr{H}_t^1\,\phi\right)_i$

and $\left(\mathscr{G}^s f(s,\phi)\right)_i \leq \left(\mathscr{G}^1 f(s,\phi)\right)_i$. In other words, the solutions of the nonstationary and stationary social influence problems are pointwise bounded above by the solutions of the corresponding diffusion problems.

In order to be able to understand more thoroughly the structure of solutions to the problems of social influence that we are investigating here, we are going to restrict our attention on a particular type of vectors of susceptibility coefficients and initial opinions. As a matter of fact, from now on, we will be concerned only with vectors s_0 of susceptibility coefficients such that, corresponding to the vector s_0, there exists a single (unique) $i(s_0) \in V$, the susceptibility coefficient of which is 0, while, for any other $j \neq i(s_0)$, we have $0 < s_j \leq 1$. The vertex (person) $i(s_0)$ in the social network (graph) G will be called **source**, for reasons that will become clear in what follows. Apparently, such a vector s_0 induces a partition of the set of vertices (persons) V into a singleton–boundary $\delta\Omega = \{i(s_0)\}$ and an interior $\Omega = V \setminus \{i(s_0)\}$. In addition, from now on, we will be considering only vectors ϕ_0 of initial opinions, which are zero everywhere in Ω, while they take the value 1 on the source $i(s_0)$, i.e., on the boundary $\delta\Omega$. In other words,

$$(\phi_0)_i = \begin{cases} 0, & \text{for } i \neq i(s_0), \text{ i.e., in } \Omega, \\ 1, & \text{for } i = i(s_0), \text{ i.e., on } \delta\Omega. \end{cases}$$

Apparently, such an initial condition ϕ_0 is going to trigger the emergence of nonzero opinions in Ω sustained by the incessant influx of opinion from the pertinacious boundary $\delta\Omega$ that keeps on being diffused all–over the graph (that is why the boundary vertex $i(s_0)$ was called *source*). Moreover, by solving the s_0–perturbed IBVP and BVP for social influence with initial opinion vector ϕ_0, we can compute the following two characteristic times and two characteristic opinions for each person (vertex) $i \in V$:

- The **opinion kick–off time** $T_i^0 = T_i^0(s_0, \phi_0)$, at which i's opinion becomes nonzero for the first time, i.e.,
$$T_i^0 = \inf_{t=0,1,2,\ldots} \{v_i^t(s_0,\phi_0) \neq 0\}.$$
- The **germinated opinion** $v_i^{T_i^0} = v_i^{T_i^0}(s_0, \phi_0)$, i.e., i's opinion at the kick–off time T_i^0.
- The (approximate) **opinion stabilization time** $T_i^\infty(\epsilon) = T_i^\infty(\epsilon; s_0, \phi_0)$, at which i's opinion tends to be stabilized at the corresponding opinion steady state, i.e.,
$$T_i^\infty(\epsilon) = \inf_{t=0,1,2,\ldots} \{|v_i^t(s_0,\phi_0) - \overline{v}_i(s_0,\phi_0)| < \epsilon\},$$
for certain **precision of approximation** $0 < \varepsilon \ll 1$.
- The **steady state opinion** $\overline{v}_i = \overline{v}_i(s_0,\phi_0)$, to which the temporal solution from s_0 and ϕ_0 is stabilized.

It is not difficult to show that the above quantities satisfy the following monotonicity properties, denoting by $d(i,j)$ the geodesic distance between vertices i and j and by d the diameter of G:

- $T_i^0 = 0$ if and only if $i \in \delta\Omega$, i.e., $i = i(s_0)$,
- $T_i^0 = d(i, i(s_0))$, for any $i \in \Omega$,
- $0 < T_i^0 \leq d \leq T_i^\infty(\epsilon)$, for any $i \in \Omega$ and for some $0 < \varepsilon \ll 1$,
- $v_i^{T_i^0} \leq \overline{v}_i$.

- $T_i^0 \le T_j^0$ if and only if $d(i, i(s_0)) \le d(j, i(s_0))$,
- $T_i^0 \le T_j^0$ if and only if $v_i^{T_i^0} \le v_j^{T_j^0}$,
- $T_i^0 \le T_j^0$ if and only if $T_i^\infty(\epsilon) \le T_j^\infty(\epsilon)$, for some $0 < \varepsilon \ll 1$,
- $T_i^0 \le T_j^0$ if and only if $\overline{v}_i \le \overline{v}_j$.

In the previous definitions, the source $i(s_0)$ was fixed (i.e., a specific vertex of the graph was set to be the source). Next, let us allow the source to range all-over the graph vertices. To do so, we are going to assume that we are starting with a given vector of susceptibility coefficients s, the components of which are *all* nonzero, and, for each vertex $i \in V$, we are going to consider a modified vector of susceptibility coefficients $s[i]$ such that, for any $j \in V$, $s[i]_j = (1 - \delta_{ij})\, s_j$ (where δ_{ij} is the Kronecker delta), i.e.,

$$s[i]_j = \begin{cases} 0, & \text{for } j = i, \\ s_j, & \text{for } j \neq i. \end{cases}$$

In correspondence to the vector of susceptilibity coefficients $s[i]$, we are going to consider a vector of initial opinions $\phi[i]$ such that $\phi[i]_j = \delta_{ij}$. Then, for each $i \in V$, after solving the social influence IBVPs with vector of susceptibility coefficients $s[i]$ and vector of initial opinions $\phi[i]$, we would get (for each $t = 0, 1, 2, \ldots$) an opinion matrix $\mathscr{U}^t = \{\mathscr{U}^t_{i,j}\}_{i,j\in[n]} \in \mathbb{R}^{V\times V}$ such that, for each $i, j \in \Omega$,

$$\mathscr{U}^t_{i,j} = v^t_j(s[i], \phi[i]). \tag{0.0.10}$$

In other words, the ij–entry of $\mathscr{U}^t$ gives j's opinion at time t generated by the social influence IBVP, when starting with vector of susceptilibity coefficients $s[i]$ and vector of initial opinions $\phi[i]$, i.e., when having the source located at i, where $s_i = 0$ and $\phi_i = 1$, while $s_j > 0$ and $\phi_j = 0$, for any other $j \neq i$.

In this way, we may also compute four $n \times n$ matrices on the social influence graph G, denoted as $\mathscr{U}^\infty = \{\mathscr{U}^\infty_{i,j}\}_{i,j\in[n]}$, $\mathscr{T} = \{\mathscr{T}_{i,j}\}_{i,j\in[n]}$, $\mathscr{E} = \{\mathscr{E}_{i,j}\}_{i,j\in[n]}$ and $\mathscr{S}(\epsilon) = \{\mathscr{S}(\epsilon)_{i,j}\}_{i,j\in[n]}$, which are defined as:

$$\begin{aligned} \mathscr{U}^\infty_{i,j} &= \overline{v}_j(s[i], \phi[i]), \\ \mathscr{T}_{i,j} &= T^0_j(s[i], \phi[i]), \\ \mathscr{E}_{i,j} &= v_j^{T_i^0}(s[i], \phi[i]), \\ \mathscr{S}(\epsilon)_{i,j} &= T^\infty_j(\epsilon, s[i], \phi[i]). \end{aligned}$$

Of course, among these four matrices, $\mathscr{T}$ is the only symmetric one (in general, for any graph G and any vector of susceptibility coefficients s). Notice that the sum of the i–th row (or column) of $\mathscr{T}$ gives the inverse of the un–normalized closeness index c_c of vertex i, i.e., for each $i \in V$,

$$\sum_{j\in[n]} \mathscr{T}_{i,j} = \sum_{j\in[n]} \mathscr{T}_{j,i} = \frac{n-1}{c_{\text{closeness}}(i)}.$$

On the other side, in what concerns the nonsymmetric matrix $\mathscr{U}^\infty$, the sums of its rows and columns generate two interesting measures that quantify the contribution of each vertex in the dynamics of opinion propagation inside the graph:

- $\mathrm{u}_i^{\infty,\text{out}} = \sum_{j\in[n], j\neq i} \mathscr{U}^\infty_{i,j}$, which will be called i's **volume of opinion broadcasting**, determines the cumulative amount of steady state opinion that i can broadcast to all other vertices (persons), when i is the sole (exclusive) source of this opinion transmission. Apparently, for a given graph

G, the volume of opinion broadcasting of i depends only on the vector of susceptibility coefficients s in such a way that $0 = \mathfrak{u}_i^{\infty,\mathrm{out}}|_{s=0} \leq \mathfrak{u}_i^{\infty,\mathrm{out}} \leq \mathfrak{u}_i^{\infty,\mathrm{out}}|_{s=1} = n-1$. Also note that the normalized index of i's volume of opinion broadcasting $\frac{\mathfrak{u}_i^{\infty,\mathrm{out}}}{n-1}$ signifies i's density of opinion broadcasting.

- $\mathfrak{u}_i^{\infty,\mathrm{in}} = \sum_{j\in[n],j\neq i} \mathscr{U}_{j,i}^{\infty}$, which will be called i's **volume of opinion reception**, determines the cumulative amount of steady state opinion that can be delivered to i from all other vertices (persons) being the opinion transmitters (aggregating all the opinion releases received by i each time with a different $j \neq i$ being the sole (exclusive) source of the opinion transmission). Again, for a given graph G, the volume of opinion reception of i depends only on the vector of susceptibility coefficients s in such a way that $0 = \mathfrak{u}_i^{\infty,\mathrm{in}}|_{s=0} \leq \mathfrak{u}_i^{\infty,\mathrm{in}} \leq \mathfrak{u}_i^{\infty,\mathrm{in}}|_{s=1} = n-1$. Also now, the normalized index of i's volume of opinion reception $\frac{\mathfrak{u}_i^{\infty,\mathrm{in}}}{n-1}$ signifies i's density of opinion reception.

Clearly, the following balance relationship holds in a social influence network:

$$\sum_{i\in[n]} \mathfrak{u}_i^{\infty,\mathrm{out}} = \sum_{i\in[n]} \mathfrak{u}_i^{\infty,\mathrm{in}} = \mathscr{I},$$

where the quantity $\mathscr{I}$, called **graph influenceability** is such that $0 = \mathscr{I}|_{s=0} \leq \mathscr{I} \leq \mathscr{I}|_{s=1} = (n-1)n$.

Furthermore, $\mathscr{U}^{\infty}$ might generate two local (at the level of vertices) influence–related centrality indices, for any vertex i:

- The **opinion broadcasting degree centrality** index, $c_{\mathrm{OBdegree}}(i)$, defined as:
$$c_{\mathrm{OBdegree}}(i) = \frac{1}{n-1} \sum_{j\in[n],j\sim i} \mathscr{U}_{i,j}^{\infty}.$$
- The **opinion broadcasting closeness centrality** index, $c_{\mathrm{OBcloseness}}(i)$,
$$c_{\mathrm{OBcloseness}}(i) = \frac{(n-1)\,\upsilon}{\sum_{j\in[n]} \mathscr{U}_{i,j}^{\infty}\, d(i,j)},$$
where $\upsilon = \min_{i,j\in[n]} \mathscr{U}_{i,j}^{\infty}$.

Of course, these indices are normalized in $[0,1]$ and they are reduced to the indices of degree centrality and degree closeness (respectively), when $s=1$, while, otherwise, they remain bounded by the latter. Moreover, the previous indices correspond to the following two global measures (at the level of graph G), which are also bounded by the corresponding degree and closeness centralization indices (respectively):

- The **opinion broadcasting degree centralization** index, $C_{\mathrm{OBdegree}}(G)$, defined as:
$$C_{\mathrm{OBdegree}}(G) = \sum_{i\in[n]} \Big[\Big(\max_{i\in[n]} c_{\mathrm{OBdegree}}(i)\Big) - c_{\mathrm{OBdegree}}(i)\Big].$$
- The **opinion broadcasting closeness centralization** index, $C_{\mathrm{OBcloseness}}(G)$,
$$C_{\mathrm{OBcloseness}}(G) = \sum_{i\in[n]} \Big[\Big(\max_{i\in[n]} c_{\mathrm{OBcloseness}}(i)\Big) - c_{\mathrm{OBcloseness}}(i)\Big].$$

Example 1:
Let G be a 3–layer graph composed of 60 vertices (with each layer containing 20 vertices), which is plotted below. This is an artificially generated Erdos–Renyi random graph with probabilities 0.1 inside each layer and 0.03 across layers. The set of vertices of this graph is $V = \{0, 1, 2, \ldots 59\}$ and it is partitioned in Layer 1 containing vertices $\{0, 1, \ldots, 19\}$, Layer 2 with vertices $\{20, 21, \ldots, 39\}$ and Layer 3 with vertices $\{40, 41$

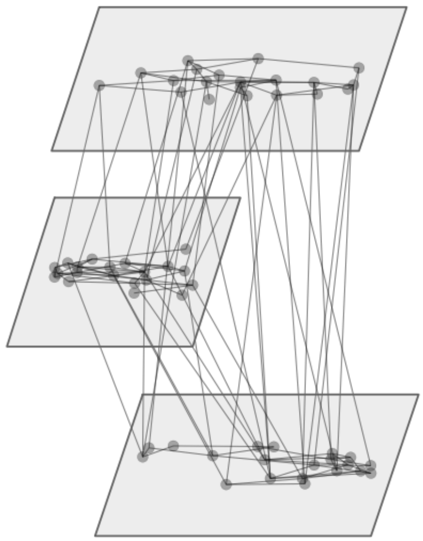


FIGURE 1. The 3–layer random graph of Example 1.

These graph vertices are equipped with certain randomly chosen susceptibility coefficients, which are distributed according to the following histogram:

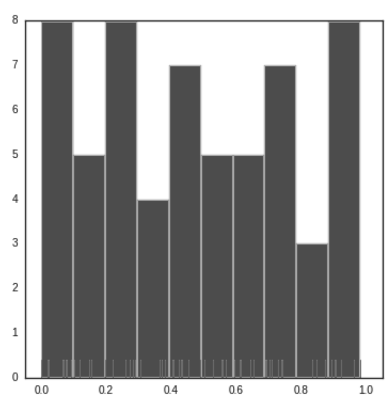
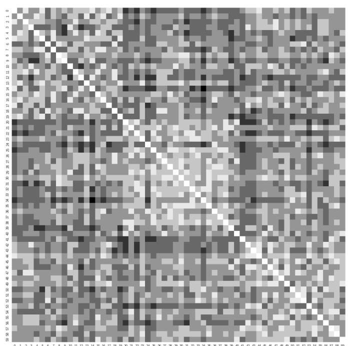


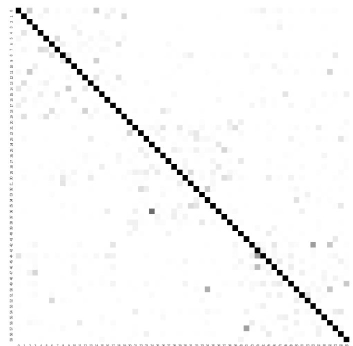
FIGURE 2. The histogram of susceptibility coefficients of the 3–layer random graph of Example 1.

The following figures show the heat maps of the four matrices, i.e, the opinion kick–off times $\mathscr{T}$, the germinated opinions $\mathscr{E}$, the opinion stabilization times $\mathscr{S}(\epsilon)$

(for $\epsilon = 0.01$) and the steady state opinions $\mathscr{U}^\infty$ corresponding to the graph of this example.



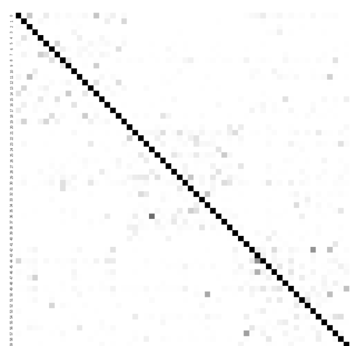
(A)



(B)

FIGURE 3. The heat maps of the matrices (a) $\mathscr{T}$ and (b) $\mathscr{E}$ of the 3–layer random graph of Example 1.

(A)



(B)

FIGURE 4. The heat maps of the matrices (a) $\mathscr{S}(0.01)$ and (b) $\mathscr{U}^\infty$ of the 3–layer random graph of Example 1.

In particular, from the matrix $\mathscr{U}^\infty$ of steady state opinions (where $\mathscr{U}^\infty_{i,j}$ is the steady state opinion of j triggered by a persistent opinion source at node i), we obtain the following information about the dynamics of opinion propagation inside G:

- The normalized graph influenceability is equal to 0.034, which is very low.

- The vertex with the highest volume of opinion broadcasting is vertex 49 on Layer 3 (with maximum density of opinion broadcasting equal to 0.034).
- The vertex with the lowest volume of opinion broadcasting is vertex 2 on Layer 1 (with minimum density of opinion broadcasting equal to 0.001).
- The vertex with the highest volume of opinion reception is vertex 43 on Layer 3 (with maximum density of opinion broadcasting equal to 0.002).
- The vertex with the lowest volume of opinion reception is vertex 27 on Layer 2 (with minimum density of opinion broadcasting equal to 0.001).

If we compute the intralayer and interlayer influenceabilities (from the submatrices inside each layer or across layers), we find the results of the next table. Apparently, the highest (lowest) intralayer influenceability is inside Layer 2 (resp., Layer 1), while the highest (lowest) interlayer influenceability is from Layer 2 to Layer 3 (resp., from Layer 2 to Layer 1). (Notice that the values of intralayer/interlayer influenceabilities are not symmetric, since the matrix $\mathscr{U}^\infty$ is not.)

Table 1. Intralayer and Interlayer Influenceabilities of the Graph of Example 1.

	Layer 1	Layer 2	Layer 3
Layer 1	0.188	0.029	0.064
Layer 2	0.056	0.228	0.080
Layer 3	0.077	0.055	0.216

We have also computed the opinion broadcasting degree centrality index $c_{\text{OBdegree}}(i)$, the opinion broadcasting closeness centrality index $c_{\text{OBcloseness}}(i)$, the opinion broadcasting degree centralization index for the graph $C_{\text{OBdegree}}(G)$, the opinion broadcasting closeness centralization index for the graph $C_{\text{OBcloseness}}(G)$ and we found:

- $\max c_{\text{OBdegree}} = 0.023$ is attained at vertex 44 on Layer 3.
- $\min c_{\text{OBdegree}} = 0.001$ is attained at vertex 2 on Layer 1.
- $\max c_{\text{OBcloseness}} = 0.013$ is attained at vertices 2 and 4 on Layer 1.
- $\min c_{\text{OBcloseness}} = 0.001$ is attained at vertex 49 on Layer 3.
- $C_{\text{OBdegree}}(G) = 0.014$.
- $C_{\text{OBcloseness}}(G) = 0.012$.

Notice that, for this graph, the range of values of the degree centrality index $c_{\text{degree}}(i)$, the closeness centrality index $c_{\text{OBdegree}}(i)$ and the values of the degree centralization index for the graph $C_{\text{degree}}(G)$ and the closeness centralization index for the graph $C_{\text{closeness}}(G)$ are:

- $\max c_{\text{degree}} = 0.169$ is attained at vertex 30 on Layer 2.
- $\min c_{\text{degree}} = 0.016$ is attained at vertex 41 on Layer 3.
- $\max c_{\text{closeness}} = 0.446$ is attained at vertex 30 on Layer 2.
- $\min c_{\text{closeness}} = 0.001$ is attained at vertex 24 on Layer 2.
- $C_{\text{degree}}(G) = 0.095$.
- $C_{\text{closeness}}(G) = 0.102$.

Furthermore, we have also computed the opinion broadcasting degree centrality index $c_{\text{OBdegree}}(i)$ and the opinion broadcasting closeness centrality index $c_{\text{OBcloseness}}(i)$, for each vertex i of this graph, and we show the corresponding pairplots in the next three figures. We found the following Pearson correlation coefficients for the network of Example 1:

- $r_{c_{\text{OBdegree}},c_{\text{OBcloseness}}} = -0.707.$
- $r_{c_{\text{OBdegree}},c_{\text{degree}}} = 0.755.$
- $r_{c_{\text{OBdegree}},c_{\text{closeness}}} = 0.511.$
- $r_{c_{\text{OBcloseness}},c_{\text{degree}}} = -0.624.$
- $r_{c_{\text{OBcloseness}},c_{\text{closeness}}} = -0.505.$
- $r_{c_{\text{degree}},c_{\text{closeness}}} = 0.874.$

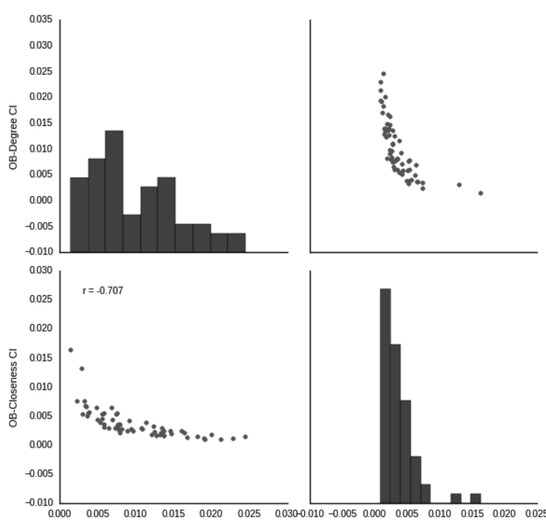


FIGURE 5. The pairplot between the indices of opinion broadcasting degree centrality and opinion broadcasting closeness centrality for the graph of Example 1.

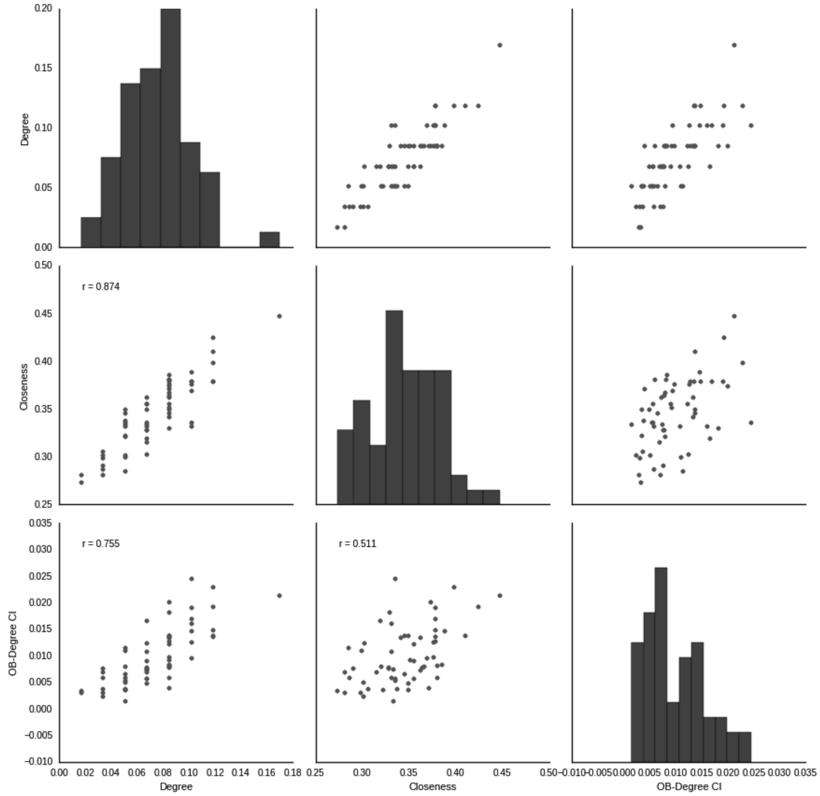


FIGURE 6. The pairplot between the indices of opinion broadcasting degree centrality, degree centrality and closeness centrality for the graph of Example 1.

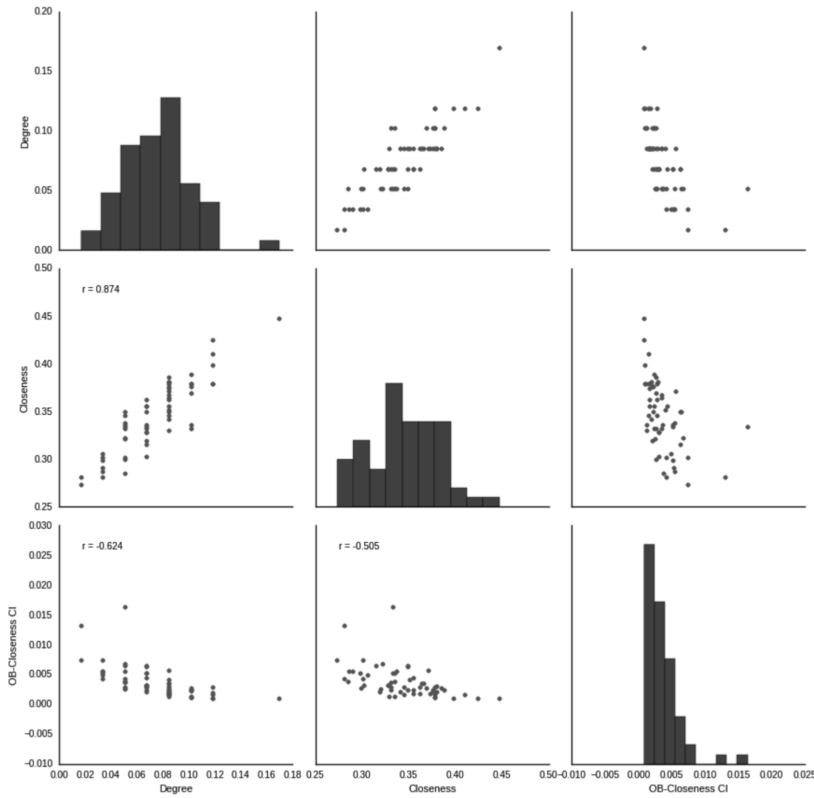
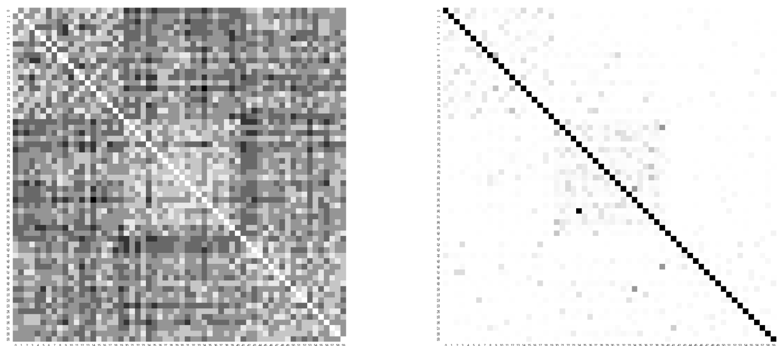


FIGURE 7. The pairplot between the indices of opinion broadcasting closeness centrality, degree centrality and closeness centrality for the graph of Example 1.

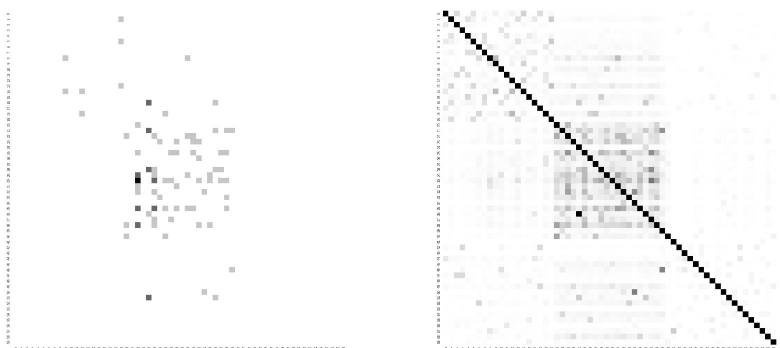
Example 2:
Let G be the 3–layer random graph of Example 1 but now equipped with susceptibility coefficients, which are constant in each layer. In particular, all vertices in Layer 1 have susceptibility coefficient 0.7, in Layer 2 they have susceptibility coefficient 1 and in Layer 3 the susceptibility coefficient is 0.3.

For this example, the heat maps of the four matrices, i.e., the opinion kick–off times $\mathscr{T}$, the germinated opinions $\mathscr{E}$, the opinion stabilization times $\mathscr{S}(\epsilon)$ (for $\epsilon = 0.01$) and the steady state opinions $\mathscr{U}^{\infty}$ are shown in the following two figures.



(A) (B)

Figure 8. The heat maps of the matrices (a) $\mathscr{T}$ and (b) $\mathscr{E}$ of the 3–layer random graph of Example 2.



(A) (B)

Figure 9. The heat maps of the matrices (a) $\mathscr{S}(0.01)$ and (b) $\mathscr{U}^\infty$ of the 3–layer random graph of Example 2.

In particular, from the matrix $\mathscr{U}^\infty$ of steady state opinions, we obtain the following information about the dynamics of opinion propagation inside this graph:

- The normalized graph influenceability is equal to 0.074, which is relatively low.
- The vertex with the highest volume of opinion broadcasting is vertex 30 on Layer 2 (with maximum density of opinion broadcasting equal to 0.548).
- The vertex with the lowest volume of opinion broadcasting is vertex 48 on Layer 3 (with minimum density of opinion broadcasting equal to 0.119).

- The vertex with the highest volume of opinion reception is vertex 37 on Layer 2 (with maximum density of opinion broadcasting equal to 0.475).
- The vertex with the lowest volume of opinion reception is vertex 53 on Layer 3 (with minimum density of opinion broadcasting equal to 0.096).

Comparing the above results with the corresponding ones obtained in Example 1, we observe that by fixing the susceptibility coefficients in constant values on each layer ($0.7, 1.0$ and 0.3), the graph influenceability and the volumes/densities of opinion broadcasting and reception increase significantly with regards to the corresponding values they had attained, when the susceptibility coefficients were assigned to vertices in a random way.

Computing the intralayer and interlayer influenceabilities (from the submatrices inside each layer or across layers), we find the results of the next table. Apparently, the highest (lowest) intralayer influenceability is inside Layer 1 (resp., Layer 2), while the highest (lowest) interlayer influenceability is from Layer 1 to Layer 2 (resp., from Layer 2 to Layer 3). (Notice that the values of intralayer/interlayer influenceabilities are not symmetric the matrix $\mathscr{U}^\infty$ is not.)

TABLE 2. Intralayer and Interlayer Influenceabilities of the Graph of Example 2.

	Layer 1	Layer 2	Layer 3
Layer 1	0.346	0.272	0.045
Layer 2	0.183	1.104	0.073
Layer 3	0.134	0.324	0.105

Again for this case too, we have computed the opinion broadcasting degree centrality index $c_{\text{OBdegree}}(i)$, the opinion broadcasting closeness centrality index $c_{\text{OBcloseness}}(i)$, the opinion broadcasting degree centralization index for the graph $C_{\text{OBdegree}}(G)$, the opinion broadcasting closeness centralization index for the graph $C_{\text{OBcloseness}}(G)$ and we found:

- $\max c_{\text{OBdegree}} = 0.042$ is attained at vertex 25 on Layer 2.
- $\min c_{\text{OBdegree}} = 0.001$ is attained at vertex 41 on Layer 3.
- $\max c_{\text{OBcloseness}} = 0.737$ is attained at vertex 17 on Layer 1.
- $\min c_{\text{OBcloseness}} = 0.054$ is attained at vertex 30 on Layer 2.
- $C_{\text{OBdegree}}(G) = 0.037$.
- $C_{\text{OBcloseness}}(G) = 0.257$.

Furthermore, we have also computed the opinion broadcasting degree centrality index $c_{\text{OBdegree}}(i)$ and the opinion broadcasting closeness centrality index $c_{\text{OBcloseness}}(i)$, for each vertex i of this graph, and we show the corresponding pairplots in the next three figures. We found the following Pearson correlation coefficients for the network of Example 2:

- $r_{c_{\text{OBdegree}}, c_{\text{OBcloseness}}} = -0.444$.
- $r_{c_{\text{OBdegree}}, c_{\text{degree}}} = 0.771$.
- $r_{c_{\text{OBdegree}}, c_{\text{closeness}}} = 0.551$.
- $r_{c_{\text{OBcloseness}}, c_{\text{degree}}} = -0.502$.
- $r_{c_{\text{OBcloseness}}, c_{\text{closeness}}} = -0.444$.
- $r_{c_{\text{degree}}, c_{\text{closeness}}} = 0.874$.

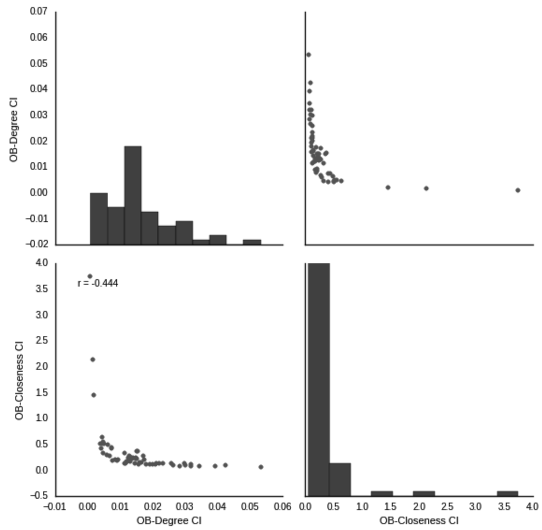


FIGURE 10. The pairplot between the indices of opinion broadcasting degree centrality and opinion broadcasting closeness centrality for the graph of Example 2.

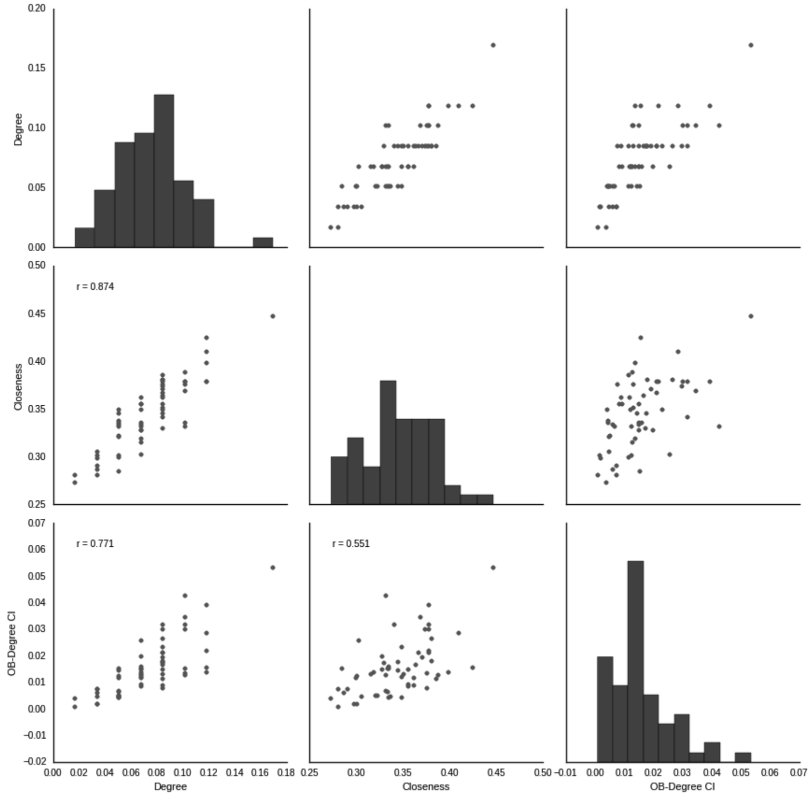


FIGURE 11. The pairplot between the indices of opinion broadcasting degree centrality, degree centrality and closeness centrality for the graph of Example 2.

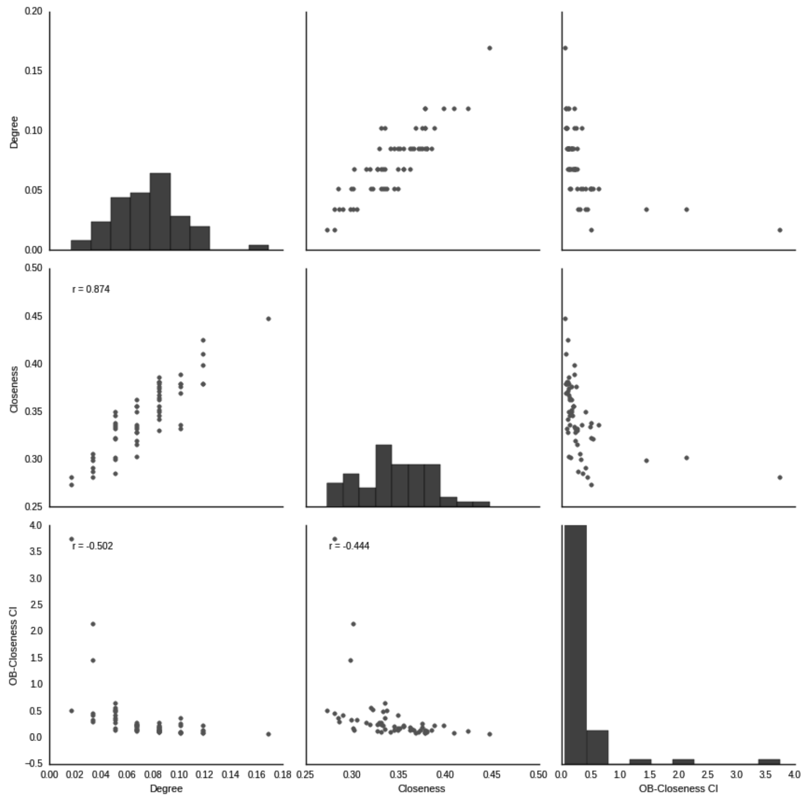


FIGURE 12. The pairplot between the indices of opinion broadcasting closeness centrality, degree centrality and closeness centrality for the graph of Example 2.

Example 3:
Let G be the 2–layer graph, which is a two–dimensional grid composed of 2×10 vertices (with each layer containing 10 vertices) plotted below.

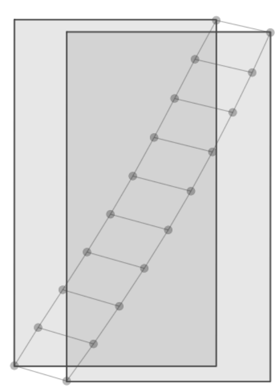


FIGURE 13. The 2–layer grid of Example 3.

We assume that all 10 vertices of Layer 1 have the same susceptibility coefficient equal to s_1 and all 10 vertices of Layer 2 have the same susceptibility coefficient equal to s_2. In this case, we have computed how the opinion broadcasting degree and closeness centralization indices vary in terms of the two–dimensional vector of susceptibility coefficients $s = (s_1, s_2)$ and our findings are plotted in the next two figures.

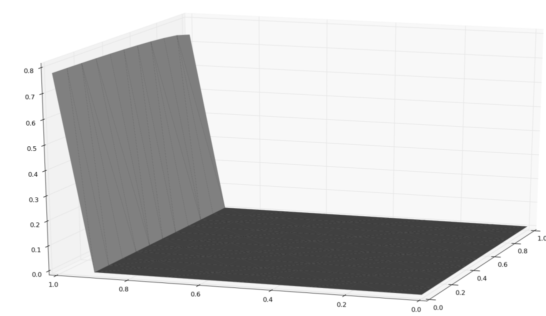


FIGURE 14. The variation of the opinion broadcasting degree centralization index on the two–dimensional vector of susceptibility coefficients.

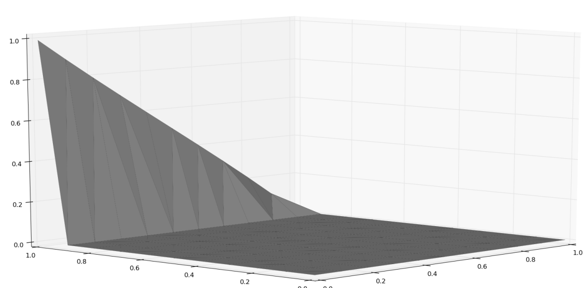


FIGURE 15. The variation of the opinion broadcasting closeness centralization index on the two–dimensional vector of susceptibility coefficients.

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