CHAPTER

Endogenous Firm Dynamics and Labor Flows via Heterogeneous Agents*

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Human beings, viewed as behaving systems, are quite simple. The apparent complexity of our behavior over time is largely a reflection of the complexity of the environment in which we find ourselves.

Herbert Simon (1996, p. 53)

1 INTRODUCTION

While it is conventional, in a wide variety of economic models, to assume that the economy is in general equilibrium, there is, in fact, substantial dynamism in real economies. Consider the U.S. private sector: over the last decade the workforce has ranged from 115 to 120 million employees annually, with nearly 3 million workers changing employers *each month* on average (Davis et al., 2006). Over this same period there were, each year, 5.7–6.0 million firms with employees of which, on average, nearly 100 thousand went out of business *monthly* while a comparable number started up (Fairlie, 2012). Such high levels of turnover in the American economy—1 in 40 workers changing employers monthly, 1 in 60 firms terminating its operations—portrays a kind of *perpetual economic flux* in the U.S. How are we to interpret such persistent adjustments and reorganizations of productive activities? Conventionally, they are believed to represent the reallocation of human resources to more productive uses (Caves, 1998). But do they generate actual productivity gains at the firm level? Do such fluxes partially result from previous changes, e.g., filling jobs previously opened? Do they cause new fluxes in the next period? Are they produced by *exogenous* shocks, whether aggregate or firm-specific (e.g., technological or productivity-related), or are they due to *endogenous* agent interactions and decisions? If we stipulate that the economy is in general equilibrium then there is no way to realize micro-dynamics except by the imposition of external shocks. Can microeconomic models *endogenously* produce the kinds of dynamics observed empirically when the incentives agents have to change jobs are fully represented?

The main result of the research described here is a microeconomic model capable of producing, *without* exogenous shocks, firm and labor dynamics of the size and type experienced by the U.S. economy prior to the recent financial crisis. In addition to the nearly 3 million people who change jobs in the U.S. each month, about half as many, some 1.5 million workers, separate from their employers monthly without new jobs, becoming unemployed, while a comparable number move off unemployment into new jobs; another 1.5 million people either leave the workforce for a spell or else begin a job after being out of the workforce. These flows sum to approximately 9 million labor market events per month at steady-state (Fallick and Fleischman, 2004). Further, many of the vacancies created by such *inter-firm* flows are filled by *intra-firm* job changes, about which there are few data. All told, perhaps 12 million distinct job change events occur each *month* in the U.S., primarily among the 120 million people in the private sector. Clearly, over the course of a year there is enormous turnover in the matching of people to jobs in the U.S. While conventional explanations for these large labor flows exist (e.g., Krusell et al., 2011, 2017), here I provide a microeconomic explanation *without the need for aggregate shocks*.

This model also reproduces a variety of cross-sectional properties of U.S. businesses. Over the past decade there have appeared increasing amounts of micro-data on U.S. firms, including administratively *comprehensive* (tax record-based) data on firm sizes, ages, growth rates, labor productivity, job tenure, and wages. Extant theories place few restrictions on these data.[1] Lucas (1978) derives Pareto-distributed firm sizes from a postulated Pareto distribution of managerial talent. Luttmer (2007, 2010)

[1]A generation ago Simon noted the inability of the neoclassical theory of the firm to explain the empirical size distribution (Ijiri and Simon, 1977, pp. 7–11, 138–140; Simon, 1997). Transaction cost (e.g.,

obtains Zipf-distributed firm sizes and exponential firm ages (2011) in a variety of general equilibrium settings, driven by exogenous shocks. Rossi-Hansberg and Wright (2007) study establishment growth and exit rates arising in general equilibrium due to industry-specific productivity shocks. Elsby and Michaels (2013) and Arkolakis (2013) simulate heterogeneous firm growth rates due to productivity shocks. However, there are *many* more data on firms and labor to be explained. The model described below reproduces more than three dozen features of the empirical data *without* recourse to exogenous shocks—such shocks are not necessary in a model with worker-level dynamics.

The model draws together threads from various theoretical literatures. It is written at the level of individual agents and incentive problems of the type studied in the principal-agent literature manifest themselves. The agents work in perpetually novel environments, so contracts are incomplete and transaction costs are implicit. Each firm is a coalition of agents making the theory of coalition formation relevant (Ray, 2007). Agent decisions generate firm growth and decline in the spirit of evolutionary economics (Nelson and Winter, 1982).

Specifically, the model consists of a heterogeneous population of agents with preferences for income and leisure. Production takes place under increasing returns to scale, so agents who work together can produce more output per unit effort than by working alone. However, agents act non-cooperatively[2]: they select effort levels that improve their own welfare, and may migrate between firms or start-up new firms when it is advantageous to do so. Analytically, Nash equilibria within a firm can be unstable. Large firms are ultimately unstable because each agent's compensation is imperfectly related to its effort level, making free-riding possible. Highly productive agents eventually leave large firms and such firms eventually decline. All firms have finite lives. The dynamics of firms perpetually forming, growing and perishing are studied. It will be shown that *this non-equilibrium regime provides greater welfare than equilibrium*.

These dynamics mean it is analytically difficult to relate agent level behavior to aggregate outcomes. Therefore, features that emerge at the firm population level are studied using agent-based computing (Holland and Miller, 1991; Vriend, 1995; Axtell, 2000; Tesfatsion, 2002). In agent computing individual software objects represent people and have behavioral rules governing their interactions. Agent models are 'spun' forward in time and regularities emerge from the interactions (e.g., Grimm et al., 2005). The shorthand for this is that macro-structure "grows" from the bottom-up.[3] No equations governing the aggregate level are specified. Nor do agents have either complete information or correct models for how

Williamson, 1985) and game theoretic explanations of the firm (e.g., Hart, 1995; Zame, 2007) make few empirical claims. Sutton (1998) bounds the extent of intra-industry concentration, constraining the shape of size distributions.

[2]For a cooperative game theoretic view of firms see Ichiishi (1993).

[3]'Growing' social phenomena in this way automatically entails sufficient explanation, since a demonstration is provided that the phenomenon to be explained (the explanandum) directly results from agents following certain behavioral rules (explanans); to wit: a social phenomenon has been explained if it has

the economy will unfold. Instead, they glean data inductively from the environment and from their social networks, through direct interactions, and make imperfect forecasts of economic opportunities (Arthur, 1994b). The macroscopic properties of the model *emerge* from the agent interactions. This methodology facilitates modeling agent heterogeneity (Kirman, 1992), non-equilibrium dynamics (Arthur, 2006), local interactions (Kirman, 1997), and bounded rationality (Arthur, 1991; Kirman, 1993).

2 DYNAMICS OF TEAM PRODUCTION

Consider a group of agents A, $|A| = n$, engaged in team production, each agent contributing some amount of effort, generating team output.[4] Specifically, agent i has endowment $\omega_i > 0$ and contributes effort level $e_{i \in A} \in [0, \omega_i]$, to the group. The total effort of the group is then $E \equiv \sum_{i \in A} e_i$. The group produces output, O, as a function of E, according to $O(E) = aE + bE^{\beta}$, $\beta > 1$, without capital as in Hopenhayn (1992).[5] For $b > 0$ there are increasing returns to effort.[6] Increasing returns in production means that agents working together can produce more than they can as individuals.[7] To see this, consider two agents having effort levels e_1 and e_2, with $\beta = 2$. As individuals they produce total output $O_1 + O_2 = a(e_1 + e_2) + b(e_1^2 + e_2^2)$, while working together they make $a(e_1 + e_2) + b(e_1 + e_2)^2$. Clearly this latter quantity is at least as large as the former since $(e_1 + e_2)^2 \geq e_1^2 + e_2^2$. Agents earn according to a compensation rule. For now consider agents sharing total output equally: at the end of each period all output is sold for unit price and each agent receives an O/N

been 'grown' in this sense. By contrast, Epstein (2006) has asserted that if a phenomenon has not been 'grown' then it has not been explained, a much stronger claim. Insofar as Newton, Darwin, and Maxwell did not 'grow' gravity, evolution, or electromagnetism, respectively, from the bottom up, i.e., in terms of lower level principles, yet each provided scientific explanations for the natural phenomena they investigated, Epstein's position is not tenable as philosophy, although perhaps useful as a motto, a kind of rallying cry for the science of emergence (Laughlin and Pines, 2000). Rather, the converse of Epstein's claim—if a phenomenon has not been explained then it has not been grown—is defensible as the contrapositive version of the original claim, that if one has 'grown' a phenomenon then it has been explained.

[4] The model derives from Canning (1995), Huberman and Glance (1998), and Glance et al. (1997).

[5] While $O(E)$ relates inputs to outputs, like a standard production function, E is not the choice of a single decision-maker, since it results from the actions of autonomous agents. Thus, $O(E)$ cannot be made the subject of a math program, as in conventional production theory, yet does describe production possibilities.

[6] Increasing returns at the firm level goes back at least to Marshall (1920) and was the basis of theoretical controversies in the 1920s (Sraffa, 1926; Young, 1928). Recent work on increasing returns is reprinted in Arthur (1994a) and Buchanan and Yoon (1994). Colander and Landreth (1999) give a history of the idea.

[7] There are many ways to motivate increasing returns, including 'four hands problems': two people working together are able to perform a task that neither could do alone, like carrying a piano up a flight of stairs.

share of the total output.[8] Agents have Cobb–Douglas preferences for income and leisure, parameterized by θ.[9] All time not spent working is spent in leisure, so agent i's utility can be written as a function of its effort, e_i, and the effort of other agents, $E_{\sim i} \equiv E - e_i$ as

$$U_i\left(e_i\right) = \left(\frac{a\left(e_i + E_{\sim i}\right) + b\left(e_i + E_{\sim i}\right)^{\beta}}{n}\right)^{\theta_i} \left(\omega_i - e_i\right)^{1-\theta_i}. \tag{1}$$

2.1 EQUILIBRIUM OF THE TEAM PRODUCTION GAME

Consider the individual efforts of agents as unobservable. From team output, O, each agent i determines E and, from its contribution to production, e_i, can figure out $E_{\sim i}$. Agent i then selects the effort level that maximizes its utility, i.e.,

$$e_i^*\left(\theta_i, \omega_i, E_{\sim i}, a, b\right) = \arg\max_{e_i} U_i\left(e_i; \theta_i, \omega_i, E_{\sim i}, n, a, b, \beta\right).$$

For $\beta = 2$, in symbols,

$$e_i^* = \max\left\{0, \frac{2b\left(\theta_i\omega_i - E_{\sim i}\right) - a + \sqrt{4b\theta_i^2\left(\omega_i + E_{\sim i}\right)\left[a + b\left(\omega_i + E_{\sim i}\right)\right] + a^2}}{2b\left(1 + \theta_i\right)}\right\}. \tag{2}$$

Note that e_i^* does not depend on n but does depend on $E_{\sim i}$—the effort put in by the other agents. To develop intuition for the general dependence of e_i^* on its parameters, Fig. 1 plots it for $a = b = 1$ and $\omega_i = 10$, as functions of $E_{\sim i}$ and θ_i. Optimal effort decreases monotonically as 'other agent effort,' $E_{\sim i}$, increases. For each θ_i there exists some $E_{\sim i}$ beyond which it is rational for agent i to put in no effort. For constant returns, i.e. $b = 0$, e_i^* falls linearly with $E_{\sim i}$ with slope $\theta_i - 1$.

Singleton Firms

The $E_{\sim i} = 0$ solution of (2) corresponds to agents working alone in single agent firms. For this case the expression for the optimal effort level is

$$e^* = \frac{2b\theta\omega - a + \sqrt{4b\theta^2\omega\left[a + b\omega\right] + a^2}}{2b\left(1 + \theta\right)}. \tag{3}$$

For $\theta = 0$, $e^* = 0$ while for $\theta = 1$, $e^* = \omega$.

[8] The model yields roughly constant total output, so in a competitive market the price of output would be nearly constant. Since there are no fixed costs, agent shares sum to total cost, which equals total revenue. The shares can be thought of as either uniform wages in pure competition or profit shares in a partnership.
[9] Appendix A gives a more general model of preferences, yielding qualitatively identical results.

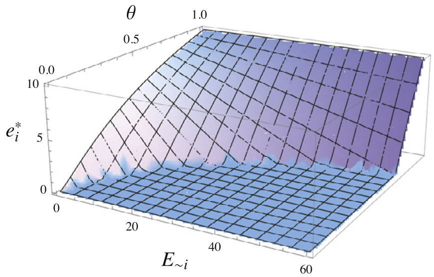

FIGURE 1

Dependence of e_i^* on $E_{\sim i}$ and θ for $a = 1$, $b = 1$, $\omega_i = 10$.

Nash Equilibrium

Equilibrium in a team corresponds to each agent working with effort e_i^* from (2), using $E_{\sim i}^*$ in place of $E_{\sim i}$ such that $E_{\sim i}^* = \sum_{j \neq i} e_j^*$. This leads to:

Proposition 1. *Nash equilibrium exists and is unique (Rosen, 1965; Watts, 2002).*

Proof. From the continuity of the *RHS* of (2) and the convexity and compactness of the space of effort levels, a fixed point exists by the Brouwer theorem. Each fixed point is a Nash equilibrium, since once it is established no agent can make itself better off by working at some other effort level. □

Proposition 2. *There exists a set of efforts that Pareto dominate Nash equilibrium (Hölmstrom, 1982), a subset of which are Pareto optimal. These (a) involve larger effort levels than the Nash equilibrium, and (b) are not individually rational.*

Proof. To see (a) note that

$$dU_i(e_i^*) = \frac{\partial U_i}{\partial e_i} de_i + \frac{\partial U_i}{\partial E_{\sim i}} dE_{\sim i} > 0,$$

since the first term on the *RHS* vanishes at the Nash equilibrium and

$$\frac{\partial U_i}{\partial E_{\sim i}} = \frac{\theta_i \left(a + 2b\left(e_i + E_{\sim i}\right)\right)\left(\omega_i - e_i\right)^{1-\theta_i}}{n^{\theta_i}\left[\left(e_i + E_{\sim i}\right)\left(a + b\left(e_i + E_{\sim i}\right)\right)\right]^{1-\theta_i}} > 0.$$

For (b), each agent's utility is monotone increasing on the interval $[0, e_i^*)$, and monotone decreasing on $(e_i^*, \omega_i]$. Therefore, $\partial U_i / \partial e_i < 0 \; \forall e_i > e_i^*, E_{\sim i} > E_{\sim i}^*$. □

The effort region that Pareto-dominates Nash equilibrium is the space where individuals who are part of the firm can achieve higher welfare than they do either working alone or at Nash equilibrium within the team.

Example 1 (*Nash equilibrium in a team with free agent entry and exit*)**.** Four agents having θs of $\{0.6, 0.7, 0.8, 0.9\}$ work together in a team in which $a =$

$b = 1$ and the agents have equal endowments, $\omega_i = 1$. Equilibrium, from (2), has agents working with efforts $\{0.15, 0.45, 0.68, 0.86\}$, respectively, producing 6.74 units of output. The corresponding utilities are $\{1.28, 1.20, 1.21, 1.32\}$. If these agents worked alone they would, by (3), put in efforts $\{0.68, 0.77, 0.85, 0.93\}$, generating outputs of $\{1.14, 1.36, 1.58, 1.80\}$ and total output of 6.07. Their utilities would be $\{0.69, 0.80, 0.98, 1.30\}$. Working together they put in less effort and receive greater reward. This is the essence of team production. Now say a $\theta = 0.75$ agent joins the team. The four original members adjust their effort to $\{0.05, 0.39, 0.64, 0.84\}$—i.e., all workless—while total output rises to 8.41. Their utilities increase to $\{1.34, 1.24, 1.23, 1.33\}$. The new agent works with effort 0.52, receiving utility of 1.23, above its singleton utility of 0.80. If another agent having $\theta = 0.75$ joins the team the new equilibrium efforts of the original group members are $\{0.00, 0.33, 0.61, 0.83\}$, while the two newest agents contribute 0.48. The total output rises to 10.09 with utilities $\{1.37, 1.28, 1.26, 1.34\}$ for the original agents and 1.26 for each of the twins. Overall, even though the new agent induces one co-worker to free ride, the net effect is a Pareto improvement. Next, an agent with $\theta = 0.55$ (or less) joins. Such an agent will free ride and not affect the effort or output levels, so efforts of the extant group members will not change. However, since output must be shared with one additional agent, all utilities fall. For the 4 originals these become $\{1.25, 1.15, 1.11, 1.17\}$. For the twins their utility falls to 1.12 and that of the $\theta = 0.9$ agent is now below what it can get working alone (1.17 vs. 1.30). Since agents may exit the group freely, it is rational for this agent to do so, causing further adjustment: the three original agents work with efforts $\{0.10, 0.42, 0.66\}$, while the twins add 0.55 and the newest agent free rides. Output is 7.52, yielding utility of $\{1.10, 0.99, 0.96\}$ for the original three, 0.97 for the twins, and 1.13 for the free rider. Unfortunately for the group, the $\theta = 0.8$ agent now can do better by working alone—utility of 0.98 versus 0.96, inducing further adjustments: the original two work with efforts 0.21 and 0.49, respectively, the twins put in effort of 0.61, and the $\theta = 0.55$ agent rises out of free-riding to work at the 0.04 level; output drops to 5.80. The utilities of the originals are now 0.99 and 0.90, 0.88 for the twins, and 1.07 for the newest agent. Now the $\theta = 0.75$ agents are indifferent between staying or starting new singleton teams.

Homogeneous Teams

Consider a team composed of agents of the same type (identical θ and ω). In a homogeneous group each agent works with the same effort in equilibrium, determined from (2) by substituting $(n-1)\, e_i^*$ for $E_{\sim i}$, and solving for e^*, yielding:

$$e^* = \frac{2bn\theta\omega - a\left(\theta + n\left(1-\theta\right)\right) + \sqrt{4bn\theta^2\omega\left(a + bn\omega\right) + a^2\left(\theta + n\left(1-\theta\right)\right)^2}}{2bn\left(2\theta + n\left(1-\theta\right)\right)}. \tag{4}$$

It is easy to see that (4) specializes to (3) in the case of $n = 1$. These efforts are shown in Fig. 2 as a function of θ with $a = b = 1$ and various n, for two values of ω. Clearly, effort in monotonically increasing in preference for income, θ. Note that agents with

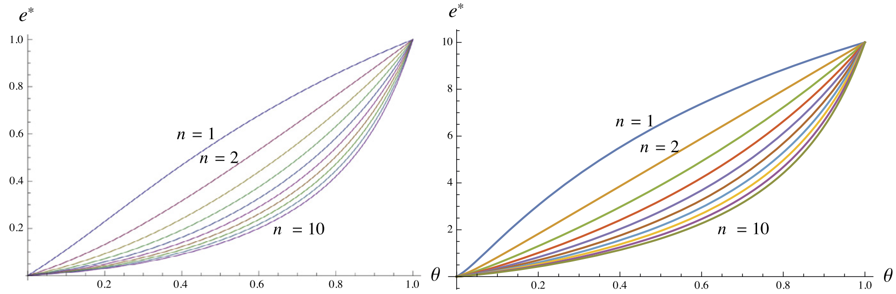

FIGURE 2

Optimal agent effort in homogeneous teams as a function of n for various θ, with $a = b = 1$ and $\omega = 1$ (left) and $\omega = 10$ (right).

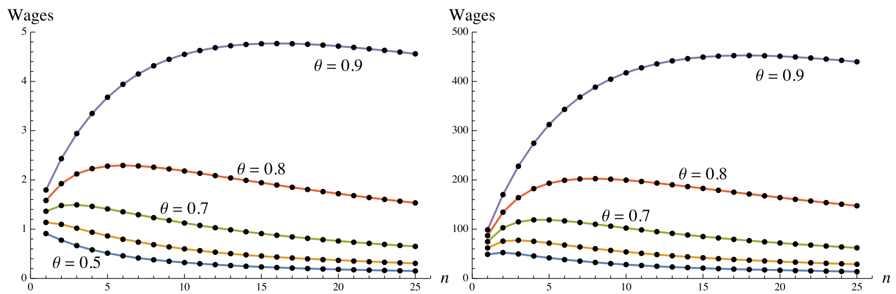

FIGURE 3

Agent wages in homogeneous teams as a function of n for various θ, with $a = b = 1$ and $\omega = 1$ (left) and $\omega = 10$ (right).

$\theta = 0$ do work not at all, i.e., $e^*(0, \omega, n) = 0$, while those with $\theta = 1$ contribute everything to production, $e^*(1, \omega, n) = \omega$, and nothing to leisure, independent of the size of the team, n. Also note that $e^*(\theta, \omega, i) > e^*(\theta, \omega, i + 1)$, that is, effort levels decrease as team size increases, other things being equal.

From (4) wages as a function of agent preference, θ, endowment, ω, and team size, n, can be determined explicitly for homogeneous teams. There results a long expression that is not terribly revealing so it is omitted here. Instead, wages as a function of n for $a = b = 1$ are plotted in Fig. 3, for $\theta \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$, and for two values of endowments, ω. Note that teams with high endowments earn *much* more than low endowment teams. For agents with relatively low preference for income and low endowment, wages are monotonically decreasing as team size increases. For sufficiently large preference for income or endowment there is a team size that maximizes wages. For instance, in the left plot of Fig. 4, for $\theta = 0.9$ the team size that maximizes wages is 16.

But wages are only part of what motivates our agents. They also care about leisure, i.e., the non-contributions to production: $\omega - e^*$. Fig. 4 plots utilities as a function of

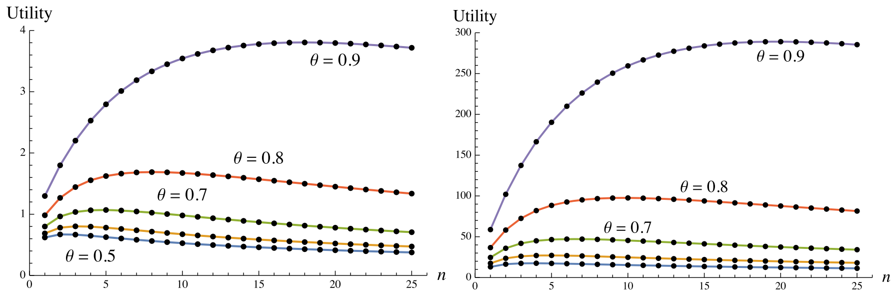

FIGURE 4

Agent utility in homogeneous teams as a function of n for various θ, with $a = b = 1$ and $\omega = 1$ (left) and $\omega = 10$ (right).

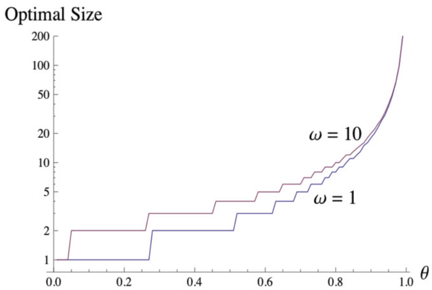

FIGURE 5

Optimal sizes of homogeneous teams as a function of θ, with $a = b = 1$ and $\omega = 1, 10$.

n for $a = b = 1$ and $\theta \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$. Now note that each curve in Fig. 5 is single-peaked, so there is an optimal team size for every θ. Since utility involves both wages and leisure, the team size that maximizes utility will not generally be the same as team size that maximizes wages. For example, in the left plot of Fig. 5, for $\theta = 0.9$ the team size that maximizes utility is 18, versus 16 for maximizing wages. In a team of size 18 each agent has almost the same income as in a 16 person team and somewhat more leisure time. Utility maximizing team sizes are shown in Fig. 5 as a function of θ for two values of ω. For high θ agents the curves are approximately coincident. When homogeneous agents are arranged into optimal teams, how much utility do they receive? Once again, an analytical expression can be written down but it is long and its overall shape not obvious. Se we plot it numerically in Fig. 6, as a function of θ, and compare it to singleton utility, for two values of endowments, ω. Optimal team sizes rise quickly with θ (note log scale in the higher endowment case). Gains from being in a team are greater for high θ agents.[10]

[10]For analytical characterization of an equal share (partnership) model with perfect exclusionary power see Farrell and Scotchmer (1988); an extension to heterogeneous skills is given by Sherstyuk (1998).

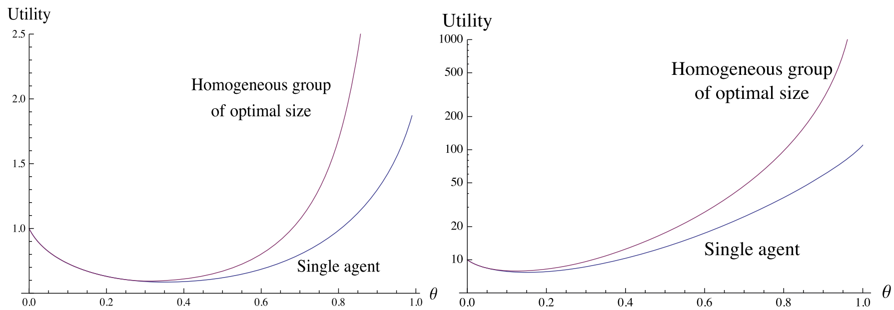

FIGURE 6

Agent utility in optimally-sized homogeneous teams as a function of θ, with $a = b = 1$ and $\omega = 1$ (left) and $\omega = 10$ (right).

For homogeneous teams it is instructive to study the effort that would be contributed at the (symmetric) Pareto solution. This can be explicitly determined by substituting $(n - 1)e_i$ for $E_{\sim i}$ in (1) then differentiating with respect to e_i, setting the resulting expression equal to 0 and solving for e^*. Doing this yields

$$e^*_{Pareto} = \frac{2bn\theta\omega - a + \sqrt{4bn\theta^2\omega\,(a + bn\omega) + a^2}}{2bn\,(1 + \theta)}. \tag{5}$$

Conceptually, it must be the case that the Pareto effort level exceeds the Nash level, i.e., (5) always greater than (4). To demonstrate this involves lengthy algebra so instead, to build up some intuition for what the difference between the two terms looks like, we simply plot it in Fig. 7 for a specific parameterization. Clearly this surface is everywhere greater than 0, numerically confirming the general result that agents under-supply effort in Nash equilibrium.

Example 2 (*Graphical depiction of the solution space 2 two identical agents*)**.** Consider two agents with $\theta = 0.5$ and $\omega = 1$. Solving (2) for e^* with $E_{\sim i} = e^*$ and $a = b = 1$ yields $e^* = 0.4215$, corresponding to utility level 0.6704. Effort deviations by either agent alone are Pareto dominated by the Nash equilibrium, e.g., decreasing the first agent's effort to $e_1 = 0.4000$, with e_2 at the Nash level yields utility levels of 0.6700 and 0.6579, respectively. An effort increase to $e_1 = 0.4400$ with e_2 unchanged produces utility levels of 0.6701 and 0.6811, respectively, a loss for the first agent while the second gains. If both agents decrease their effort from the Nash level their utilities fall, while joint increases in effort are welfare-improving. There exist symmetric Pareto optimal efforts of 0.6080 and utility of 0.7267. However, efforts exceeding Nash levels are not individually rational—each agent gains by putting in less effort. Fig. 8 plots iso-utility contours for these agents as a function of effort. The U-shaped lines are for the first agent, utility increasing upwards. The C-shaped curves refer to the second agent, utility larger to the right. Point 'N' is the Nash equilibrium.

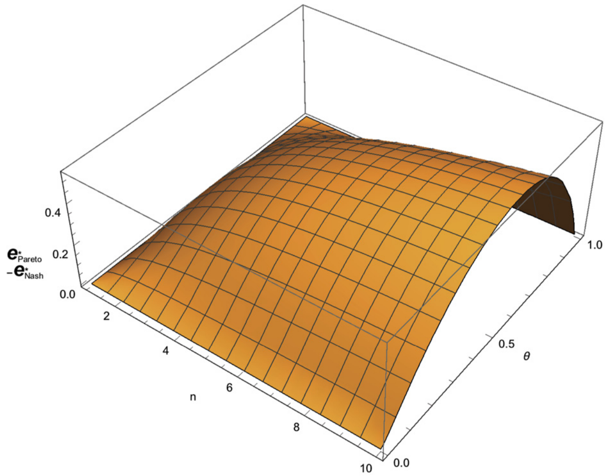

FIGURE 7

Difference in effort levels between Pareto and Nash solutions in team production as a function of n and θ, with $a = b = 1$ and $\omega = 1$.

The ‘core’ shaped region extending above and to the right of ‘N’ is the set of efforts that Pareto-dominate Nash. The set of efforts from ‘P’ to ‘P’ are Pareto optimal, with the subset from ‘D’ to ‘D’ being Nash dominant.

For two agents with different θs the qualitative structure of the effort space shown in Fig. 8 is preserved, but the symmetry is lost. Increasing returns insures the existence of effort levels that Pareto-dominate the Nash equilibrium.

2.2 STABILITY OF NASH EQUILIBRIUM, DEPENDENCE ON TEAM SIZE

A unique Nash equilibrium always exists but for sufficiently large group size it is unstable. To see this, consider a team operating away from equilibrium, each agent adjusting its effort. As long as the adjustment functions are decreasing in $E_{\sim i}$ then one expects the Nash levels to obtain. Because aggregate effort is a linear combination of individual efforts, the adjustment dynamics can be conceived of in aggregate terms. In particular, the total effort level at time $t + 1$, $E(t + 1)$, is a decreasing function of $E(t)$, as depicted notionally in Fig. 9 for a five agent firm, with the dependence of $E(t + 1)$ on $E(t)$ shown as piecewise linear. The intersection of this function with the 45° line is the equilibrium total effort. However, if the slope at the intersection is less than -1, the equilibrium will be unstable. Thus, every team has a maximum stable size, dependent on agent θs.

Consider the n agent group in some state other than equilibrium at time t, with effort levels, $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))$. At $t + 1$ let each agent adjust its effort

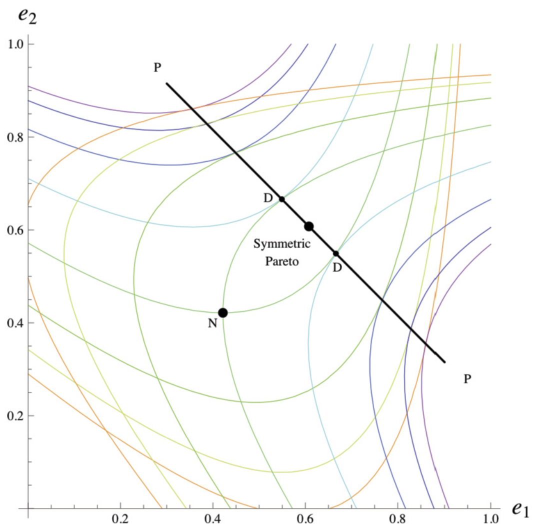

FIGURE 8

Effort level space for two agents with $\theta = 0.5$ and $a = b = \omega = 1$; colored lines are iso-utility contours, 'N' designates the Nash equilibrium, the heavy line from P–P are the Pareto optima, and the segment D–D represents the Pareto optima that dominate the Nash equilibrium.

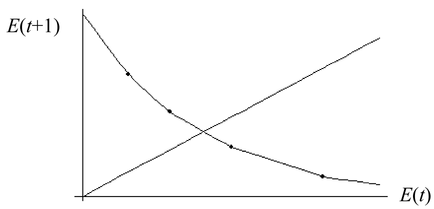

FIGURE 9

Phase space of effort level adjustment, $n = 5$.

using (2), a 'best reply' to the previous period's value of $E_{\sim i}$,[11]

$$e_i(t+1) = \max\left\{0, \frac{2b\left[\theta_i\omega_i - E_{\sim i}(t)\right] - a + \sqrt{4b\theta_i^2\left(\omega_i + E_{\sim i}(t)\right)\left[a + b\left(\omega_i + E_{\sim i}(t)\right)\right] + a^2}}{2b\left(1+\theta_i\right)}\right\}. \tag{6}$$

[11]Effort adjustment functions that are decreasing in $E_{\sim i}$ and increasing in θ_i yield qualitatively similar results; see Appendix A. While this is a dynamic strategic environment, agents make no attempt to deduce optimal multi-period strategies. Rather, at each period they myopically 'best respond'. This simple behavior is sufficient to produce very complex dynamics, suggesting sub-game perfection is implausible.

This results in an n-dimensional dynamical system, for which it can be shown:

Proposition 3. *All teams are unstable for sufficiently large group size.*

Proof. Start by assessing the eigenvalues of the Jacobian matrix[12]:

$$J_{ij} \equiv \frac{\partial e_i}{\partial e_j} = \frac{1}{1+\theta_i}\left\{\theta_i^2 \frac{a + 2b(\omega_i + E^*_{\sim i})}{\sqrt{a^2 + 4b\theta_i^2(\omega_i + E^*_{\sim i})[a + 2b(\omega_i + E^*_{\sim i})]}} - 1\right\} \tag{7}$$

with $J_{ii} = 0$. Since each $\theta_i \in [0, 1]$ it can be shown that $J_{ij} \in [-1, 0]$, and J_{ij} is monotone increasing with θ_i. The *RHS* of (7) is independent of j, so each row of the Jacobian has the same value off the diagonal, i.e., $J_{ij} \equiv k_i$ for all $j \neq i$. Overall,

$$J = \begin{bmatrix} 0 & k_1 & \cdots & k_1 \\ k_2 & 0 & \cdots & k_2 \\ \vdots & & \ddots & \vdots \\ k_n & \cdots & k_n & 0 \end{bmatrix},$$

with each of the $k_i \leq 0$. Stability of equilibrium requires that this matrix's dominant eigenvalue, λ_0, have modulus strictly inside the unit circle. It will now be shown that this condition holds only for sufficiently small group sizes. Call ρ_i the row sum of the ith row of J. It is well-known (Luenberger, 1979, pp. 194–195) that $\min_i \rho_i \leq \lambda_0 \leq \max_i \rho_i$. Since the rows of J are comprised of identical entries

$$(n-1)\min_i k_i \leq \lambda_0 \leq (n-1)\max_i k_i. \tag{8}$$

Consider the upper bound: since the largest $k_i < 0$ there is some value of n beyond which $\lambda_0 < -1$ and the solution is unstable. Furthermore, since large k_i corresponds to agents with high θ_i, it is these agents who determine group stability. From (8), compute the maximum stable group size, $N^{\max}$, by setting $\lambda_0 = -1$ and rearranging:

$$n^{\max} \leq \left\lfloor \frac{\max_i k_i - 1}{\max_i k_i} \right\rfloor, \tag{9}$$

where $\lfloor z \rfloor$ refers to the largest integer less than or equal to z. Groups larger than $n^{\max}$ will never be stable, that is, (9) is an upper bound on group size. □

For any of b, $E_{\sim i}$ or $\omega_i \gg a$, such as when $a \sim 0$, $k_i \approx (\theta_i - 1)/(\theta_i + 1)$. Using this together with (9) we obtain an expression for $n^{\max}$ in terms of preferences

$$n^{\max} \leq \left\lfloor \frac{2}{1 - \max_i \theta_i} \right\rfloor. \tag{10}$$

[12] Technically, agents who put in no effort do not contribute to the dynamics, so the effective dimension of the system will be strictly less than n when such agents are present.

Table 1 Onset of instability in a group having $\theta = 0.7$; Nash equilibrium in groups larger than 6 are unstable

n	e^*	$U(e^*)$	k	$\lambda_0 = (n-1)k$
1	0.770	0.799	not applicable	not applicable
2	0.646	0.964	−0.188	−0.188
3	0.558	1.036	−0.184	−0.368
4	0.492	1.065	−0.182	−0.547
5	0.441	1.069	−0.181	−0.726
6	0.399	1.061	−0.181	−0.904
7	0.364	1.045	−0.180	−1.082

The agent with *highest* income preference thus determines the maximum stable group size. Other bounds on λ_0 can be obtained via column sums of J. Noting the ith column sum by γ_i, we have $\min_i \gamma_i \le \lambda_0 \le \max_i \gamma_i$, which means that

$$\sum_{i=1}^{n} k_i - \min_i k_i \le \lambda_0 \le \sum_{i=1}^{n} k_i - \max_i k_i . \tag{11}$$

These bounds on λ_0 can be written in terms of the group size by substituting $n\,\overline{k}$ for the sums. Then an expression for $n^{\max}$ can be obtained by substituting $\lambda_0 = -1$ in the upper bound of (11) and solving for the maximum group size, yielding

$$n^{\max} \le \left\lfloor \frac{\max_i k_i - 1}{\overline{k}} \right\rfloor . \tag{12}$$

The bounds given by (9) and (12) are the same (tight) for homogeneous groups, since the denominators are identical in this case.

Example 3 (*Onset of instability with increasing team size*)**.** Consider a homogeneous group of agents having $\theta = 0.7$, with $a = b = \omega = 1$. From (8) the maximum stable group size is 6. Here we investigate how instability arises as the group grows. For an agent working alone the optimal effort, from (3), is 0.770, utility is 0.799. Now imagine two agents working together. From (4) the Nash efforts are 0.646 and utility increases to 0.964. Each element of the Jacobian (6) is identical; call this k. For $n = 2$, $k = -0.188 = \lambda_0$. For $n = 3$ the utility is higher and $\lambda_0 = -0.368$. The same qualitative results hold for group sizes 4 and 5, with λ_0 approaching -1. At $n = 6$ efforts again decline and now each agent's utility is lower. Adding one more agent to the group ($n = 7$) causes λ_0 to fall to -1.082: the group is *unstable*—any perturbation of the Nash equilibrium creates dynamics that do not settle down. All of this is summarized in Table 1. Groups of greater size are also unstable in this sense. For lesser θ instability occurs at smaller sizes, while groups having higher θ can support larger numbers. Fig. 10 shows the maximum stable firm size (in green) for all θ with $a = b = 1$ and $\omega = 1$, with the smallest size at which instability occurs (red). The

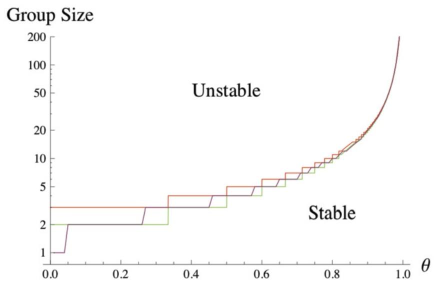

FIGURE 10

Unstable Nash equilibria in homogeneous teams as a function of income preference θ.

lower (magenta) line is the optimal firm size (Fig. 5), which is very near the stability boundary, sometimes in the unstable region. This is reminiscent of the 'edge of chaos' literature, for systems poised at the boundary between order and disorder (Levitan et al., 2002).

Unstable Equilibria and Pattern Formation Far from Agent Level Equilibria

Unstable equilibria may be viewed as problematical if one assumes agent level equilibria are *necessary* for social regularity. Games in which optimal strategies are cycles have long been known (e.g., Shapley, 1964; Shubik, 1997). Solution concepts can be defined to include such possibilities (Gilboa and Matsui, 1991). While agent level equilibria are *sufficient* for macro-regularity, they are not *necessary*. When agents are learning or in combinatorially rich environments, as they are here, fixed points are unlikely to be realized. Non-equilibrium models in economics include Papageorgiou and Smith (1983) and Krugman (1996).[13]

Real firms are inherently dynamic: workers leave, new ones arrive, everyone adjusts.[14] Indeed, there is vast turnover of jobs and firms, as already indicated. Of the largest 5000 U.S. firms in 1982, in excess of 65% of them no longer existed as independent entities by 1996 (Blair et al., 2000)! 'Turbulence' well describes such volatility (Beesley and Hamilton, 1984; Ericson and Pakes, 1995).

3 FROM ONE TEAM TO SIX MILLION FIRMS, COMPUTATIONALLY

The U.S. private sector consists of some 120 million employees who work in 6 million firms. How can we relate the abstract team production model described in the

[13] Non-equilibrium models are better known and well-established in other sciences, e.g., in mathematical biology the instabilities of certain PDE systems are the basis for pattern formation (Murray, 1993).
[14] Arguments against firm equilibrium include Kaldor (1972, 1985), Moss (1981), and Lazonick (1991).

previous section to a real economy? Analytically, there would appear to be little hope of linking the model to data without thinking in terms of to some kind of 'representative firm,' unless heroic assumptions were made about employee agents being substantially homogeneous, for otherwise how could large numbers of equations be solved? Alternatively, since there are millions of workers and firms, perhaps probabilistic or statistical reasoning could be brought to bear on the analytical model to connect it to the data, just as statistical mechanics relates the microscopic behavior of atoms or molecules to aggregate properties of a gas or liquid. While it might be possible to make progress using these approaches, we have employed, instead, an emerging computational technique known as agent-based modeling (*ABM*). In this approach, agents are created in software as objects, each having internal data and purposive behavioral rules. The agents interact and over time patterns and regularities emerge at the population level. With sufficient computational resources it is possible to instantiate very large populations of agents—indeed, below we report results for an *ABM* at full-scale with the U.S. economy, some 120 million worker agents. The question then becomes what specific rules of agent behavior are sufficient to produce teams that quantitatively resemble U.S. business firms, in terms sizes, ages, productivities, growth rates, entry and exit rates, and so on? Perhaps surprisingly, it turns out that basing agent behavior on the analytical model of the previous section is one way to accomplish this, when supplemented with rules for changing jobs and starting up new firms. Specifically, consider agents arranged in teams, with each agent perpetually adjusting its effort level according to (6), based on the adjustments of others, with new people being hired and co-workers leaving for new employment elsewhere. If a specific agent's team becomes unstable or populated by other agents who are not working very hard we permit it to look for employment in other teams and to consider forming a new team as well. What happens overall? Do lots of little teams form or a few big ones? Is a static equilibrium of specific agents in particular teams reached if we wait long enough? Are patterns produced in the population of teams that are recognizable vis-á-vis real firms? Parameterizing the firm formation model of the previous section appropriately, adding rules for how individuals seek new employment, yields an *ABM* that is capable of producing patterns and regularities that can be made to closely resemble the data on U.S. firms.

3.1 SET-UP OF THE COMPUTATIONAL MODEL USING AGENTS

To study the formation of teams within a population using software agents the *ABM* follows the analytical model. Total output of a firm consists of both constant and increasing returns, requiring specification of three parameters for each firm, a, b, and β; probability distributions for each will be used so the realized firms will be heterogeneous. Preferences and endowments, θ and ω respectively, are also specified probabilistically, so agents are heterogeneous. When agent i acts it searches over $[0, \omega_i]$ for the effort maximizing its next period utility. It makes no explicit forecast about future utility, knowing that the composition of its team will likely change—in essence, it knows it will adapt to changing circumstances so its 'default forecast'

Table 2 'Base case' configuration of the computational model

Model attribute	Value
number of agents	120,000,000
constant returns coefficient, a	*uniform* on [0, 1/2]
increasing returns coefficient, b	*uniform* on [3/4, 5/4]
increasing returns exponent, β	*uniform* on [3/2, 2]
distribution of preferences, θ	*uniform* on (0, 1)
endowments, ω	1
compensation rule	equal shares
number of neighbors, v	*uniform* on [2, 6]
activation regime	uniform (all agents active each period)
probability of agent activation/period	4% of total agents (4,800,000)
time calibration: one model period	one month of calendar time
initial condition	all agents in singleton firms

is that thing will remain about the same and when things begin to go downhill it will move on. Many firms arise in the *ABM* and each agent, when activated, always evaluates the utility it could receive from working elsewhere. Thus it is necessary to specify how agents search other firms for employment opportunities. Here each agent is given an exogenous social network consisting of v_i other agents, connected at random with uniform probability, effectively an Erdös–Renyi random graph. When activated, each agent considers (a) staying in its current firm, (b) joining v_i other firms—in essence an on-the-job search over its social network (Granovetter, 1973; Montgomery, 1991)—and (c) starting up a new firm. It chooses the option that yields greatest utility. Since agents evaluate only a small number of firms their information is very limited. We utilize 120 million agents, roughly the size of the U.S. private sector.[15] Specifically, about 5 million agents are activated each period, corresponding to one calendar month, in rough accord with job search frequency (Fallick and Fleischman, 2001). The 'base case' parameterization of the model is shown in Table 2. This was developed by heuristically seeking good fits to the many empirical data described in the next several subsections.[16]

[15]For many years we labored under the constraint of not having enough computing power to work at full-scale, instead using 10 million agents, or 1 million, or even 10,000 in our earliest efforts (Axtell, 1999). So the pros and cons of small and large models are well-known to us. While it may appear to not be parsimonious to work with full-scale models involving hundreds of millions of agents, the one great advantage of doing so is that the many model outputs do *not* have to be re-scaled in order to be compared to data emanating from the actual economy. In practice, each output measure from a sub-scale model has to be at least be interpreted in order to be compared to reality, such as in explaining a cut-off in the upper tail of a distribution, as when a 1 million agent model fails to produce a firm bigger than 10,000. But more problematically, many outputs have to be quantitatively adjusted for the different variability, skewness, and so on produced by the smaller model. The bottom line is that a 1 million agent economy is not simply a (1/100)th scale replica of a 100 million agent economy.

[16]For model attributes with random values, each agent or firm is given a single value when it is instantiated.

- **INSTANTIATE and INITIALIZE time, agent, firm, and data objects;**
- **REPEAT:**
  - **FOR each agent, activate it probabilistically; if active:**
    - **Compute e^* and $U(e^*)$ in current firm;**
    - **Compute e^* and $U(e^*)$ for starting up a new firm;**
    - **FOR each firm in the agent's social network:**
      - **Compute e^* and $U(e^*)$;**
    - **IF current firm is not best choice THEN leave:**
      - **IF start-up firm is best THEN form start-up;**
      - **IF another firm is best THEN join other firm;**
  - **FOR each firm:**
    - **Sum agent inputs and then do production;**
    - **Distribute output according to compensation rule;**
  - **COLLECT monthly and annual statistics;**
  - **INCREMENT time and reset data objects;**

ALGORITHM 1

High-level representation of the ABM code.

Execution of the model is summarized in Algorithm 1, presented as pseudo-code, with the parameters of Table 2 calibrated such that each pass through the main REPEAT loop represents one month of calendar time in the real world. Each worker is represented as an agent in this model, and both agents and firms are software objects. It is important to emphasize that this is *not* a numerical model: there are no (explicit) equations governing the aggregate level; each agent does some calculations to figure out *how hard* to work in its own firm and in other firms it evaluates, and then compares the various alternatives in order to come to a decision about *where* to work. "Solving" an *ABM* means marching it forward in time to see what patterns emerge (cf. Axtell, 2000).

The ways individual worker agents form firms are described in Section 3.2. The aggregate steady-state that eventually emerges in the model is the subject of Section 3.3. Then in Section 3.4 the population of *firms* in the steady-state is studied—distributions of firm sizes, ages, growth rates, and so on. In Section 3.5 the population of *employees* in the steady-state is characterized, including distributions of income, job tenure, etc. Section 3.6 looks at the movement of workers between firms. The final Section 3.7 investigates the welfare advantages of an economy organized into firms, from the perspective of individual workers as well as from an aggregate/social planner point of view.

3.2 A TYPICAL REALIZATION OF THE MODEL: AGENTS FORM FIRMS

As specified in Table 2, we start the model with all agents working for themselves, in one person firms.[17] However, this configuration of the economy immediately breaks

[17]This is not strictly necessary, and the code base supports starting agents in teams, but beginning as singletons serves two purposes. First, running from the most extreme 'atomized' initial condition and

down as each agent, when activated, discovers it can do better working with another agent to jointly produce output, taking advantage of increasing returns to effort. Over time some teams expand as certain agents find it welfare-improving to join those teams, while other teams contract as their employee agents discover better opportunities elsewhere. New firms are started-up by agents who do not find better opportunities.[18] Overall, once an initial transient passes,[19] an approximately stationary macrostate emerges.[20] In this macro steady-state agents continue to adjust their efforts and change jobs, causing firms to evolve, and so there is no static equilibrium at the agent level, at least not in the entire population—there can be 'pockets' of agents who are in temporary equilibrium but this never lasts. Nor can the agent-level behavior be characterized by some form of mixed strategy Nash equilibria, since in even modestly-sized populations the specific social situation each agent finds itself in at any time is idiosyncratic and essentially never repeats. In the next subsection we shall attempt to characterize this steady-state, but first we investigate the formation, evolution, and eventual death of a 'typical' firm.

During a realization of the model, when a firm starts operating it adds agents who contribute effort to production. Generally this is good for the firm since through the increasing returns mechanism current employees of the firm will either receive more income when new employees add effort or, by reducing their effort levels, current employees gain utility. However, there is no guarantee that the new employee will work as hard as the current employees. That is, at the margin it is always hard to find people who will contribute as much effort, on average, as current workers, and it particularly difficult to find people who will *raise* firm productivity. So after a period of growth, in which the firm attracts some number of productive new employees, it inevitably enters a phase in which agents continue to adjust their effort levels downward and it soon becomes preferable—typically for those having the largest preference for income including those who founded the firm—to leave, either to take a job elsewhere or to start-up a new firm. By withdrawing their efforts from the firm the exiting agents impact the rest of the firm in two ways, first by reducing

targeting empirically-credible results is a high-bar which, if accomplished, is strong evidence that the behavioral rules employed are reasonably correct, that they are doing all the work. By contrast, if agents were initially arranged in groups that closely resembled the real world then it would be unclear whether the final result was due to the model or the initial condition. Second, it turns out to be quite hard to initialize agents into teams that are even temporarily stable, where the agents are willing to stay, even briefly. That is, for most group initial conditions large numbers of agents change jobs right away, leading to unrealistically high job turnover initially, which may take a long time to settle down.

[18]To give some intuition for how this works, movies of this process in a small population of agents are available at css.gmu.edu/~axtell/Rob/Research/Pages/Firms.html#6.

[19]The duration of the transient depends on model parameters.

[20]As is typical of all *ABMs*, while each run of the model is deterministic—distinct runs made with the same stream of pseudo-random numbers always produce the same output—there can be substantial run-to-run variation in the evolution of the economy at the agent level due to the underlying stochasticity of both the agent population (set by the parameters of Table 2) and the behavioral rules. However, for a given parameterization of the model the aggregate steady-state eventually produced is statistically the same across individual runs.

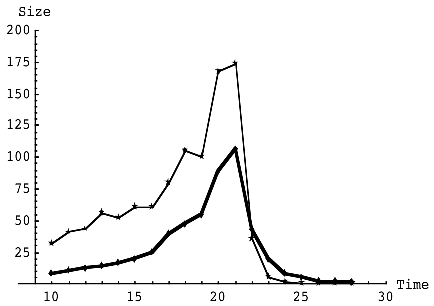

FIGURE 11

Evolution of the number of workers (heavy line) and output (thin line) over the lifetime of a firm.

the superlinear gains produced by increasing returns, and second because their effort contributions are typically above average, often considerably so. Taken together these effects may subsequently induce other employees to work harder (shirk less), partially stabilizing the firm, as was illustrated in Example 1. Then, if the loss of hard-working agents can be partially compensated for by the addition of new employees the demise of the firm may be at least delayed for some time, if not stopped altogether. Alternatively, the loss of highly productive agents may lead to rapid decline. Ultimately, every firm declines in this model, as in the real world—no firm lives forever—despite the fact that the individual agents do live forever, unlike the real world. That is, each firm has a *lifecycle*.

To illustrate the basic firm lifecycle it is instructive to look at the history of a particular firm. In Fig. 11 the evolution of the number of employees and the output of a specific firm are displayed over the course of its life from its birth at time 9 through its demise at time 28. This firm grew nearly exponentially to the point of having some 100 workers around time 21, at which time its production peaked at approximately 175 units, but soon after it went into decline with both its production and its workforce declining monotonically. To see the type of agents who joined this firm and to understand their behavior over time we plot in Fig. 12 the preferences for income, θ, and effort levels among the employees, averages represented by dark lines with ranges in each period also shown. The firm begins life with agents having high θ but over time agents having lower preference for income progressively populate the firm (left figure), although some high θ individuals stay through to the late stages. From the start there is almost monotone decreasing effort contributed (right figure), on average, over time, although right from the start there is a wide range of effort levels, with some agents contributing little and others working hard. As the workers leave this firm they migrate to other firms where they can earn more or have more leisure, gaining utility in the process.

In summary, a typical firm is founded by high θ agents who work hard. Subsequent hires work less hard but the firm prospers due to increasing returns. Eventually the most productive agents leave and the firm goes into decline.

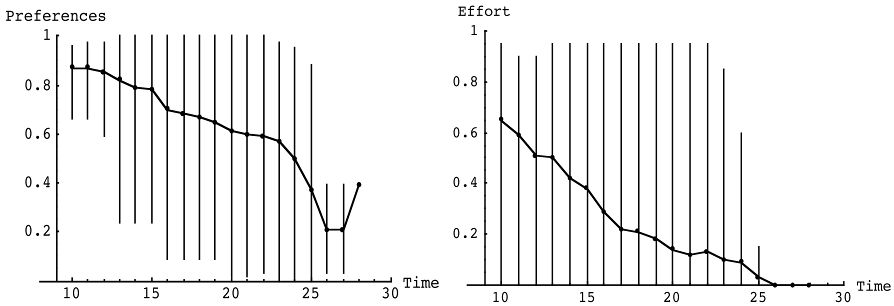

FIGURE 12

Evolution of average preference for income (left figure) and average effort levels (right figure) over the lifetime of a firm, showing minimum and maximum values as well.

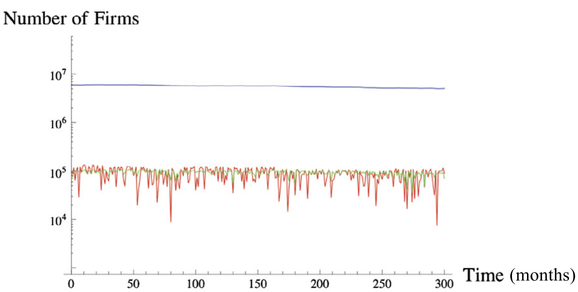

FIGURE 13

Typical time series for the total number of firms (blue), new firms (green), and exiting firms (red) over 25 years (300 months); note higher volatility in exits.

3.3 AN AGGREGATE STEADY-STATE EMERGES: PROPERTIES

While agents perpetually adjust their efforts and change jobs in this model, causing individual firms to evolve, a steady-state emerges at the aggregate level.

Number of Firms, Entrance and Exit, and Average Firm Size

The number of firms varies over time, due both to entry—agents leaving extant firms for start-ups—and the demise of failing firms. In the U.S. about 6 million firms have employees. Fig. 13 shows the number of firms (blue) once a steady-state has been achieved, nearly unchanging over 300 months (25 years) and in good agreement with the data. Before the Financial Crisis of 2008–2009 there were nearly 100K startups with employees in the U.S. monthly (Fairlie, 2012), quite close to the number produced by the model as shown in Fig. 14 (green). Counts of firm exits shown in Fig. 14 (red) are comparable but more volatile. Note that this plot has a logarithmic ordinate, so despite the volatility of entrants and exits the total number of firms is relatively constant.

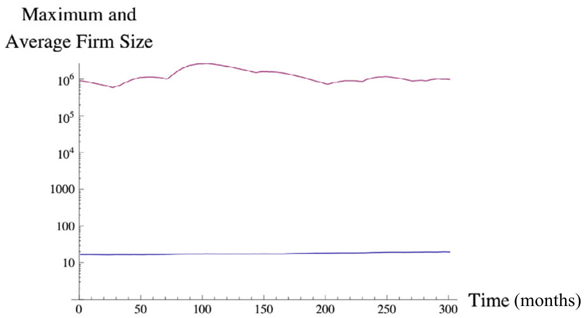

FIGURE 14

Typical time series for average firm size (blue) and maximum firm size (magenta).

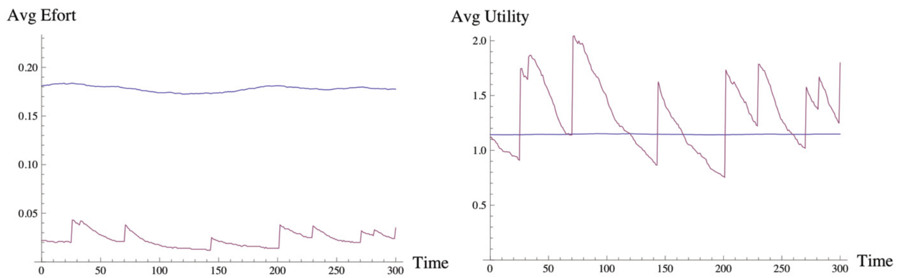

FIGURE 15

Typical time series for average effort level (left) in the population (blue) and in the largest firm (magenta), and (right) average utility (blue) and in the largest firm (magenta).

Mean firm size in the U.S. is about 20 workers/firm (Axtell, 2001). Since there are 120 million agents in the model and the number of firms that emerges is approximately 6 million, mean firm size, as shown in Fig. 14 (blue), is very close to 20. Also shown in Fig. 14 is the size of largest firm (red), which fluctuates around a million. The largest firm in the U.S. (Wal-Mart) employs some 1.4 million Americans today. The several abrupt changes in the size of the largest firm in the figure represent distinct firms, each temporarily having the largest size in the artificial economy.

Typical Effort and Utility Levels

Agents who work together improve upon their singleton utility levels through reduced effort, as shown in Fig. 15. This is the *raison d'être* of firms. While efforts in large firms fluctuate, average effort overall is quite stable (Fig. 15, left). Much of the dynamism in the 'large firm' time series is due to the identity of the largest firm changing. Fig. 15 (right) shows the average agent utility (blue) is usually less than that in the largest firm (red). Occasionally utility in large firms falls below average, signaling that the large firm is in decline.

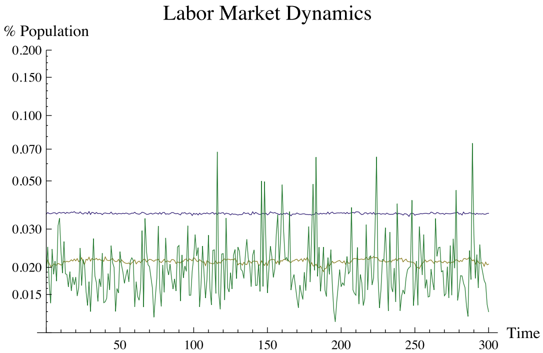

FIGURE 16

Typical monthly job-to-job changes (blue), job creation (yellow), and destruction (green).

Labor Flows

In the U.S. economy people change jobs with, what is to some, "astonishingly high" frequency (Hall, 1999, p. 1151). Job-to-job switching (also known as employer-to-employer flow) represents 30–40% of labor turnover, and is larger than unemployment flows (Fallick and Fleischman, 2001; Faberman and Nagypál, 2008; Nagypál, 2008; Davis et al., 2012). Moving between jobs is intrinsic to the agent-based model. In Fig. 16 the level of monthly job changing at steady-state is shown (blue)—just over 3 million/month—along with measures of jobs created (red) and jobs destroyed (green). Job creation occurs in firms with net monthly hiring, while job destruction means firms lose workers (net). Job destruction is more volatile than job creation, as is the case in the U.S. data (Davis et al., 1996).

Overall, Figs. 13–16 develop intuition about typical dynamics of firm formation, growth and dissolution. They are a 'longitudinal' picture of typical micro-dynamics of agents and firms. We now turn to cross-sectional properties.

3.4 THE STEADY-STATE POPULATION OF FIRMS: SIZES, PRODUCTIVITIES, AGES, SURVIVAL RATES, LIFETIMES, AND GROWTH RATES

Watching firms form, grow, and die in the model movies (see footnote 18), one readily sees the coexistence of big firms, medium-sized ones, and small ones.

Firm Sizes (by Employees and Output)

At any instant there exists a distribution of firm sizes in the model. In the steady-state firm sizes are skew, with a few big firms and larger numbers of progressively smaller ones. Typical model output is shown in Fig. 17 for firm size measured by employees (left) and output (right; arbitrary units). The modal firm size is 1 employee with the median between 3 and 4, in agreement with the data on U.S. firms. Firm sizes, S, are

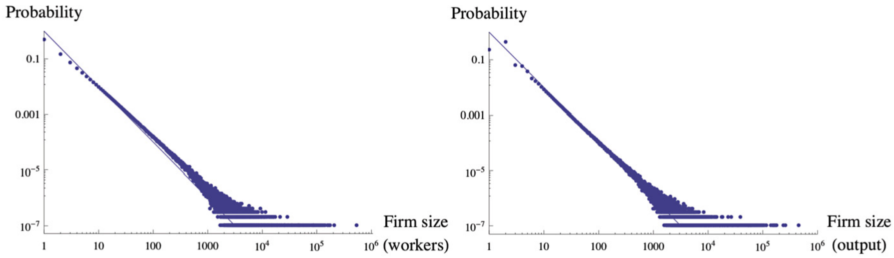

FIGURE 17

Stationary firm size distributions (PMFs) by employees (left) and output (right).

approximately Pareto distributed, the complementary *CDF* of which, $\overline{F}_S(s)$, is

$$Pr[S \geq s] \equiv \overline{F}_S(s; \alpha, s_0) = \left(\frac{s_0}{s}\right)^{\alpha}, \quad s \geq s_0, \ \alpha \geq 0$$

where s_0 is the minimum size, unity for size measured by employees. The U.S. data are well fit by $\alpha \approx -1.06$ (Axtell, 2001), the line in Fig. 17 (left), a *PMF*. The Pareto distribution is a power law and for $\alpha = 1$ is known as Zipf's law. Note that the power law fits almost the *entire distribution* of firm sizes. A variety of explanations for power laws have been proposed.[21] Common to these is the idea that such systems are far from (static) equilibrium at the microscopic (agent) level. Our model is clearly non-equilibrium with agents regularly changing jobs.

Labor Productivity

Firm output per employee is labor productivity. Fig. 18 plots average firm output as a function of firm size. Fitting a line by several methods indicates that $\ln(O)$ scales linearly with $\ln(S)$ with slope close to 1. This represents nearly constant returns to scale, also a feature of U.S. output data; see Basu and Fernald (1997). That nearly *constant returns* occur at the *aggregate* level despite *increasing returns* at the *micro*-level suggests the difficulties of making inferences across levels, i.e., the dual fallacies of division and composition. An explanation of why this occurs is apparent. High productivity firms grow by adding agents who work less hard than incumbents, thus such firms are driven toward the average productivity. That is, firms whose position in the (output, worker) space of Fig. 18 put them above the 45 degree line find that, over time, they evolve toward that line. In essence, when agents change jobs they push their new firm toward the average labor productivity.[22]

It is well known that there is large heterogeneity in labor productivity across firms (e.g., Dosi, 2007). Shown in Fig. 19 (left) are data on gross output per worker for

[21] Bak (1996, pp. 62–64), Marsili and Zhang (1998), Gabaix (1999), Reed (2001), and Saichev et al. (2010); for a review see Mitzenmacher (2004).

[22] As output per worker represents wages in our model, there is only a small wage–size effect (Brown and Medoff, 1989; Even and Macpherson, 2012).

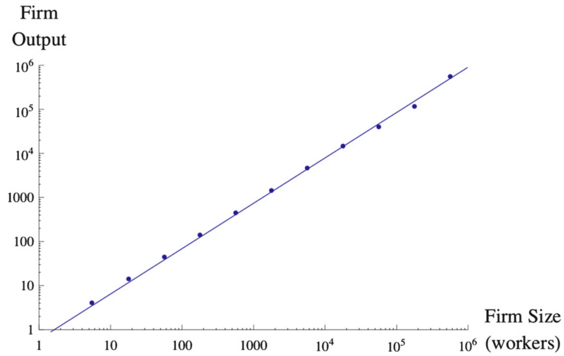

FIGURE 18

Constant returns at the aggregate level despite increasing returns at the micro-level.

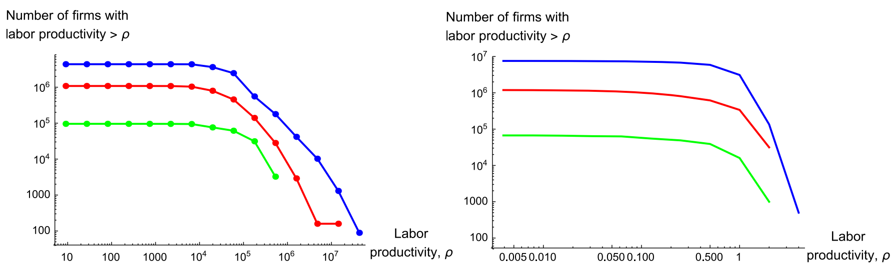

FIGURE 19

Complementary distribution functions of gross labor productivity in the U.S. (left) and in model output (right), by firm size.

all U.S. companies for three size classes: from 1 to 99 employees (blue), between 100 and 9999 (red) and 10,000 and larger (green). Note that log–log coordinates are being used in Fig. 19, meaning the right tail is very nearly a power law. Souma et al. (2009) have studied the productivity of Japanese firms and find similar results. Fig. 19 (right) is model output for the same sizes with productivity measured in arbitrary units. Note its qualitative similarity to the U.S. data, although fewer medium-sized firms having high productivity arise in our model, and the slope of the distribution for small firms appears to be significantly steeper in the model than in the data.

Firm Ages, Survival Rates, and Lifetimes

Using data from the BLS Business Employment Dynamics program, Fig. 20 gives the age distribution (*PMF*) of U.S. firms, in semi-log coordinates, with each colored line representing the distribution reported in a recent year. Model output is overlaid on the raw data as points and agrees reasonably well. Average firm lifetime is about 14 years

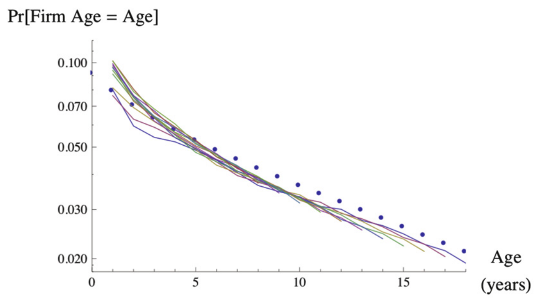

FIGURE 20

Firm age distributions (PMFs), U.S. data 2000–2011 (12 lines) and model (points).

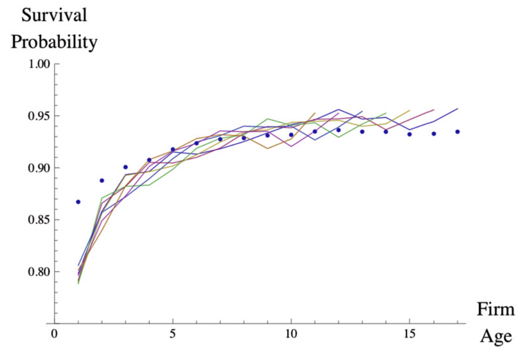

FIGURE 21

Firm survival probability increases with firm age, U.S. data 1994–2000 (7 lines) and model (points).

here, ranging from 12 to 15 over the several years shown. The curvature in the data implies that firm ages are better fit by the Weibull distribution than the exponential, the latter more commonly employed in the literature on firm ages (Coad, 2010; West, 2017).

If firm ages were exactly exponentially distributed then the survival probability would be constant, independent of age (Barlow and Proschan, 1965; Kalbfleisch and Prentice, 1980; Klein and Moeschberger, 1997). The curvature in Fig. 18 indicates that survival probability depends on age. Empirically, survival probability *increases* with age (Evans, 1987a; Hall, 1987; Haltiwanger et al., 2011). This is shown in Fig. 21 for U.S. companies in recent years (lines) along with model output (points). The model over-predicts the survival probabilities of the youngest firms.

Data on U.S. firm ages is right censored in age, thus little systematic information is known about long-lived firms, except that they are rare (de Geus, 1997).

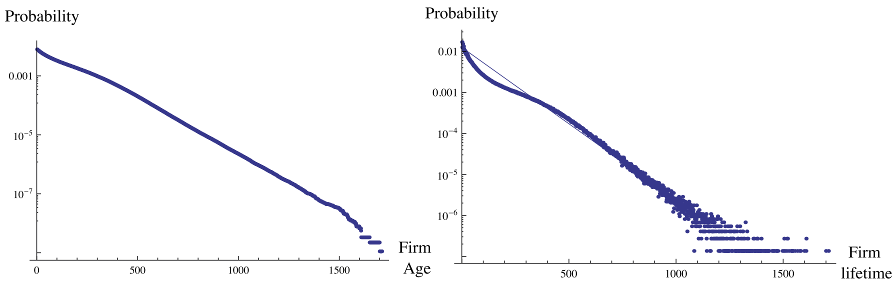

FIGURE 22

Firm age distributions and lifetime distributions (PMFs) in the long run (months).

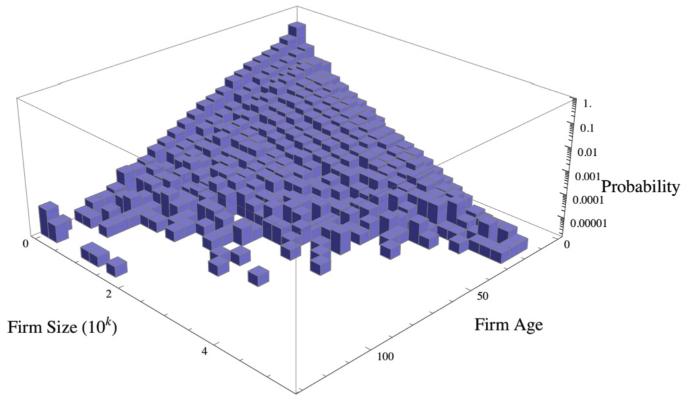

FIGURE 23

Histogram of the steady-state distribution of firms by log (size) and age in the model.

Further, the role of mergers and acquisitions (M&A) makes the lifetime of very long-lived firms ambiguous, as when a younger firm buys an older one. When this model has run for a sufficiently long time in the steady-state approximately stationary firm age and lifetime distributions emerges, as shown in Fig. 22. The durations (ages, lifetimes) shown here approach 150 years. As such, they represent predictions, since we lack appropriate data at present, for how firm lives play out over the long run.

Joint Distribution of Firms by Size and Age

The joint distribution of size and age is shown in Fig. 23, a normalized histogram in log probabilities. Note that log probabilities decline approximately linearly as a function of age and $\log(S)$. Many of the largest firms in the model are relatively young ones that grow rapidly, much like in the U.S. economy (e.g., Luttmer, 2011, Fig. 1).

Firm Growth Rates

Call S_t a firm's size at time t. Its one period growth rate is $G \equiv S_{t+1}/S_t \in \boldsymbol{R}_+$.[23] In a population of firms consider G to be a stationary random variable. Gibrat's (1931) law of proportional growth implies that if all firms have the same G then $S_{t+1} = GS_t$ is lognormally distributed for large t while the mean and variance of S grows with time (Sutton, 1997, p. 40), i.e., S is not stationary. Adding firm birth and death processes can lead to stationary firm size distributions (e.g., Simon, 1955; Ijiri and Simon, 1977; Ijiri, 1987; Mitzenmacher, 2004; or de Wit, 2005).

Historically, determination of the overall structure of G was limited by relatively small samples of firm data (e.g., Hart and Prais, 1956). Beginning with Stanley et al. (1996), who analyzed data on publicly-traded U.S. manufacturing firms (Compustat), there has emerged a consensus that $g \equiv \ln(G) \in \boldsymbol{R}$ is well-fit by the Subbotin or exponential power distribution.[24] This distribution embeds the Gaussian and Laplace distributions and has *PDF*

$$\frac{\eta}{2\sigma_g \Gamma(1/\eta)} \exp\left[-\left(\frac{|g-\overline{g}|}{\sigma_g}\right)^{\eta}\right],$$

where $\overline{g}$ is the average log growth rate, σ_g is proportional to the standard deviation, Γ is the gamma function, and η is a parameter; $\eta = 2$ corresponds to the normal distribution, $\eta = 1$ the Laplace or double exponential.[25]

Data on g for all U.S. establishments[26] has been analyzed by Perline et al. (2006), shown as a histogram in Fig. 24 for 1998–1999, decomposed into seven logarithmic size classes. Note the vertical axis is ln (frequency). In comparison to later years, e.g., 1999–2000, 2000–2001, these data are very nearly stationary. Perline et al. (2006) find that $\eta \sim 0.60$ for the size 32–63 size class, lesser for smaller firms, larger for bigger ones. The gross statistical features of g are:

i. Growth rates *depend* on firm size—small and large firms have different g. This means that *Gibrat's law is false*: all firms do *not* have the same G.
ii. The mode of $g \sim 0$, so mode$(G) \sim 1$, i.e., many firms do not grow.
iii. There is more variance for firm decline ($g < 0$) than for growth ($g > 0$), i.e., there is more variability in job destruction than job creation (Davis et al., 1996), requiring an asymmetric Subbotin distribution (Perline et al., 2006).

[23] An alternative definition of G is $2(S_{t+1} - S_t)/(S_t + S_{t+1})$, making $G \in [-2, 2]$ (Davis et al., 1996). Although advantageous because it keeps exiting and entering firms in datasets for one additional period, it obscures differences in growth rate tails by artificially truncating them. Because part of our focus will be tail behavior (high and low growth-rate firms), we will not use this alternative definition here.

[24] Subsequent work includes European pharmaceuticals (Bottazzi et al., 2001) and Italian and French manufacturers (Bottazzi et al., 2007, 2011). Bottazzi and Secchi (2006) give theoretical reasons why g should have $\eta \sim 1$, having to do with the central limit theorem for the number of summands geometrically distributed (Kotz et al., 2001). Schwarzkopf (2010, 2011) argues that g is Levy-stable.

[25] For g Laplace-distributed, G follows the log-Laplace distribution, a kind of double-sided Pareto distribution (Reed, 2001), a combination of the power function distribution on (0, 1) and the Pareto on $(1, \infty)$.

[26] Due to details of tracking firms (enterprises) longitudinally, only establishment data are available.

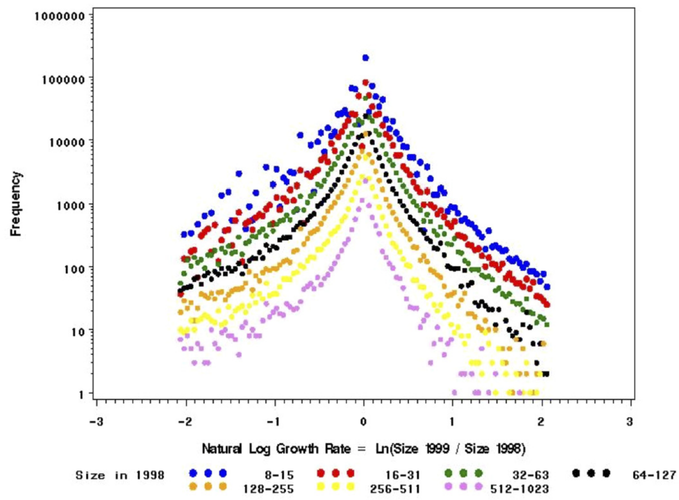

FIGURE 24

Histogram of annual g for all U.S. establishments, by size class. Source: Census.

iv. Growth rate variance declines with firm size (Hymer and Pashigian, 1962; Mansfield, 1962; Evans, 1987a, 1987b; Hall, 1987; Stanley et al., 1996).

There are at least five other well-known regularities concerning firm growth rates that are *not* illustrated by the previous figure:

v. Mean growth is positive, slightly above 0;
vi. Mean grow rate declines with firm size, and is positive for small firms, negative for large firms (Mansfield, 1962; Birch, 1981; Evans, 1987a, 1987b; Hall, 1987; Davis et al., 1996; Neumark et al., 2011);
vii. Mean growth *declines* with age (Evans, 1987a, 1987b; Haltiwanger et al., 2008);
viii. Mean growth *rises* with size, controlling for age (Haltiwanger et al., 2011);
ix. Growth rate variance *declines* with firm age (Evans, 1987a, 1987b).

With these empirical features of firm growth rates as background, Fig. 25 shows distributions of g produced by the model for seven classes of firm sizes, from small (blue) to large (purple) ones. In this plot we can see at least half of the empirical properties of firm growth: g clearly depends on firm size (i), with mode$(g) = 0$ (ii) and $\overline{g} \sim 0.0$ (v). It is harder to see that there is more variance in firm decline than growth (iii) but it is the case numerically. Clearly, variance declines with firm size (iv). Fig. 26 shows mean growth rates as a function of firm (left) size and (right) age. It is clear from these figures that $\overline{g}$ declines with size (vi) and similarly for age (vii). For more than 30 years, since the work of Birch (1981, 1987), economists have debated the meaning of figures like Fig. 26 (left). Specifically, it is not clear whether

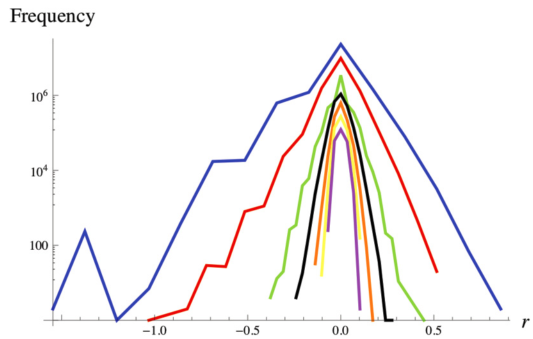

FIGURE 25

Distribution of annual g by firm size: 8–15 (blue), 16–31 (red), 32–63 (green), 64–127 (black), 128–255 (orange), 256–511 (yellow), and 512–1023 (purple).

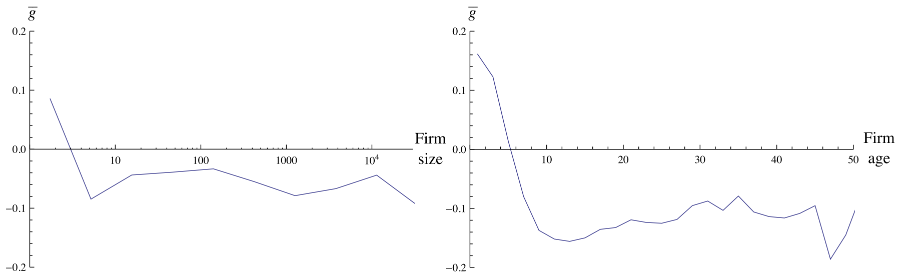

FIGURE 26

Dependence of $\overline{g}$ on (left) firm size and (right) firm age, model output.

size or age plays the larger role in determining positive growth rates. Haltiwanger and co-workers (2008, 2009, 2011) control for age and argue that it is not small firms that create jobs but rather young ones. The problem with such 'controls' for non-monotonic relationships is that they mix effects across distinct (size, age) classes. A different way to understand the distinct effects of size and age is to show how they each effect $\overline{g}$. This is done in Fig. 27, where each firm is placed into a (size, age) bin and the average g computed locally. To see precisely whether size or age matters most, a no growth ($\overline{g} = 0$) plane is superimposed on the model's $\overline{g}$ (size, age). From this we can see that *young* and *small* firms grow the most in our model.

Firm growth rate variability falls with size (iv) and age (ix). Fig. 28 shows these unconditionally for the model. Specifically, the standard deviation of g falls with size in the left plot of Fig. 28. Based on central limit arguments one expects this to be proportional to $S^{-\kappa}$, $\kappa = 1/2$ meaning the fluctuations are independent while $\kappa < 1/2$ implies they are correlated. Stanley et al. (1996) find $\kappa \sim 0.16 \pm 0.03$ for publicly-traded firms (Compustat data) while Perline et al. (2006) estimate $\kappa \sim 0.06$ for all U.S. establishments. From the model output $\kappa = 0.054 \pm 0.010$. A variety of

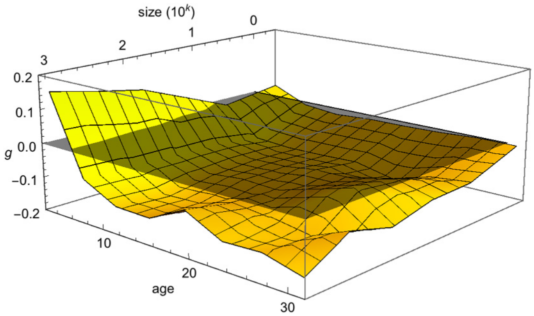

FIGURE 27

Dependence of $\overline{g}$ on firm size and age.

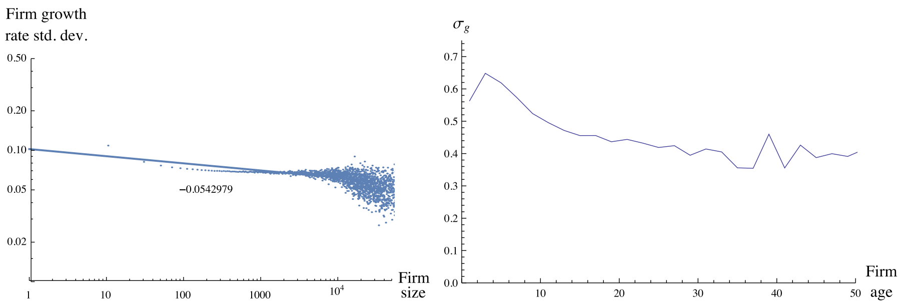

FIGURE 28

Dependence of the standard deviation of g on firm size (left) and firm age (right).

explanations for $0 \leq \kappa \leq 1/2$ have been proposed (Buldyrev et al., 1997; Amaral et al., 1998; Sutton, 2002; Wyart and Bouchaud, 2002; Fu et al., 2005; Riccaboni et al., 2008), all involving firms having internal structure. Note that no internal structure exists for the firms in our model, since they are simply collections of agents, yet dependence of the standard deviation of g on size is present nonetheless. In our model the weak correlation that develops between firm fluctuations has to do with the flow of workers between firms.

Over any epoch of time some firms grow and others decline. Expanding firms may shed some workers while shrinking firms may do some hiring. Fig. 29 shows that growing firms in our model experiences some separations while declining firms continue to hire, even when separations are the norm. These results are quite similar to U.S. data (Davis et al., 2006). The 'hiring' line from the model is quite comparable to the empirical result, but the 'separations' line is somewhat different for declining firms—there are too few separations in the model. Having explored firms cross-sectionally we now turn to the properties of the population of agents in the steady-state configuration of the model.

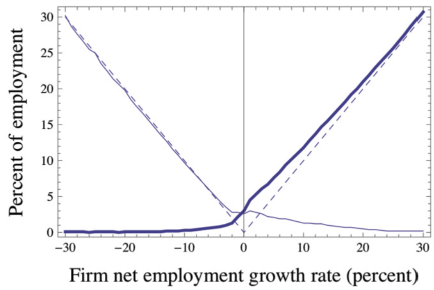

FIGURE 29

Labor transitions as a function of firm growth rate, model output.

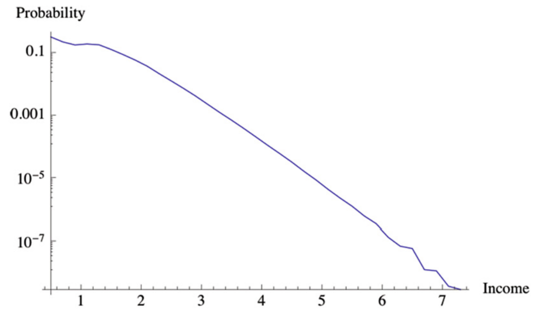

FIGURE 30

Wage distribution (arbitrary units).

3.5 THE STEADY-STATE POPULATION OF AGENTS: WAGES EARNED, JOB TENURE, AND EMPLOYMENT AS A FUNCTION OF FIRM SIZE AND AGE

Worker behavior in firms is characterized here across the agent population. While each agent's situation adjusts uniquely and idiosyncratically, at the population level there emerge robust statistical features.

Wage Distribution

While income and wealth are famously heavy-tailed (Pareto, 1971; Wolff, 1994), *wages* are less so. A recent empirical examination of U.S. adjusted gross incomes argues that an exponential distribution fits the data below about $125K, while a power law better fits the upper tail (Yakovenko and Rosser, 2009). Fig. 30 gives the income distribution from the model. Since incomes are nearly linear in this semi-log coordinate system, they are approximately exponentially-distributed.

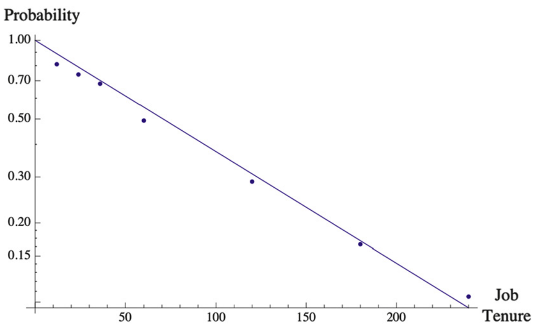

FIGURE 31

Job tenure (months) is exponentially-distributed in the U.S. (dots, binned) and in the model (line). Source: BLS and author calculations.

Job Tenure Distribution

Job tenure in the U.S. has a median near 4 years and a mean of about 8.5 years (BLS Job Tenure, 2010). The complementary-cumulative distribution for 2010 is Fig. 31 (points) with the straight line being the model output. As with income, these data are well-approximated by an exponential distribution. The base case of the model is calibrated to make these distributions nearly coincide. That is, the number of agent activations per period is specified in order to make the line go through the points, thus defining the meaning of one unit of time in the model, here a month. The many other dimensions of the model having to do with time—e.g., firm growth rates, ages—derive from this basic calibration.

Employment as a Function of Firm Size and Age

Calling $f(s)$ the firm size distribution function (*PDF or PMF*), the worker-weighted firm size distribution, $w(s) = sf(s)$. Whereas each firm is one data 'point' in $f(s)$, each worker is a 'point' in $w(s)$, so large firms 'count' more, in proportion to their size. Because $f(s)$ is a power law, so is $w(s)$, which can usefully be thought of as the fraction of total employment as a function of firm size. It turns out that the worker-weighted firm size distribution has some interesting properties. For example, the so-called *Florence median* (Pryor, 2001) is the firm size for which half of the labor force works in larger firms and half in smaller—it is the number you would get if you asked all 120 million private sector employees how big their firm was and then averaged the result. This quantity is about 500 for the U.S., roughly invariant over time—half of the American workforce is employed in firms having more than 500 employees and half in firms having 500 or fewer. It turns out that since the firm size distribution produced by the model is very close to the empirical data (Fig. 17, left), the weighted firm size distribution is also quite similar.

A related notion is the dependence of employment on firm *age*. In Fig. 32 the fraction of total employment as a function of firm age is shown. About half of Amer-

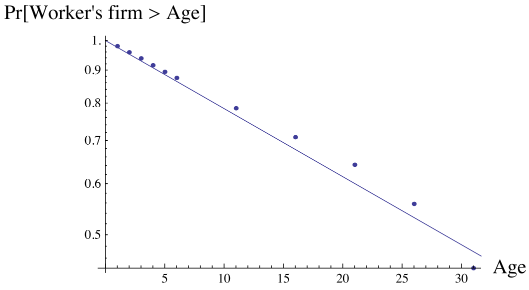

FIGURE 32

Counter cumulative distribution of employment by firm age in years in the U.S. (line) and in the model (dots). Source: BLS (BDM), available online.

ican workers are in firms younger than 25 years of age, half in older. The U.S. data are shown as a counter-cumulative distribution while the model output is shown as points. Again there is good agreement between the model and the data. Employment changes by age (Haltiwanger et al., 2008) have also been studied with our model.

3.6 STEADY-STATE JOB-TO-JOB FLOWS: THE LABOR FLOW NETWORK

In the model, as in the real world, workers regularly move between jobs, as shown in Fig. 16. Here the *structure* of such flows is studied, using a graph theoretic representation of inter-firm labor flows. Let each firm be a node (vertex) in such a graph, and an edge (link) exists between two firms if a worker has migrated between them. Elsewhere this has been called the *labor flow network* (Guerrero and Axtell, 2013). In Fig. 33 four properties of this network for the base case of the model are shown. The upper left panel gives the degree distribution, while the upper right is the distribution of edge weights. The lower plots are the clustering coefficient (left) and the assortativity (average neighbor degree), each as a function of the degree. These closely reproduce data from Finland and Mexico (Guerrero and Axtell, 2013), shown as insets.[27] Three of these plots are in log–log coordinates. The heavy-tailed character of the relationships reflects the underlying Pareto distribution of firm sizes.

3.7 STEADY-STATE AGENT WELFARE

Each time an agent is activated it seeks higher utility, which is bounded from below by the singleton utility. Therefore, it must be the case that all agents prefer the non-equilibrium state to one in which each is working alone—the state of all firms being size one is Pareto-dominated by the dynamical configurations above.

[27] Comparable data for the U.S. are not available at this time.

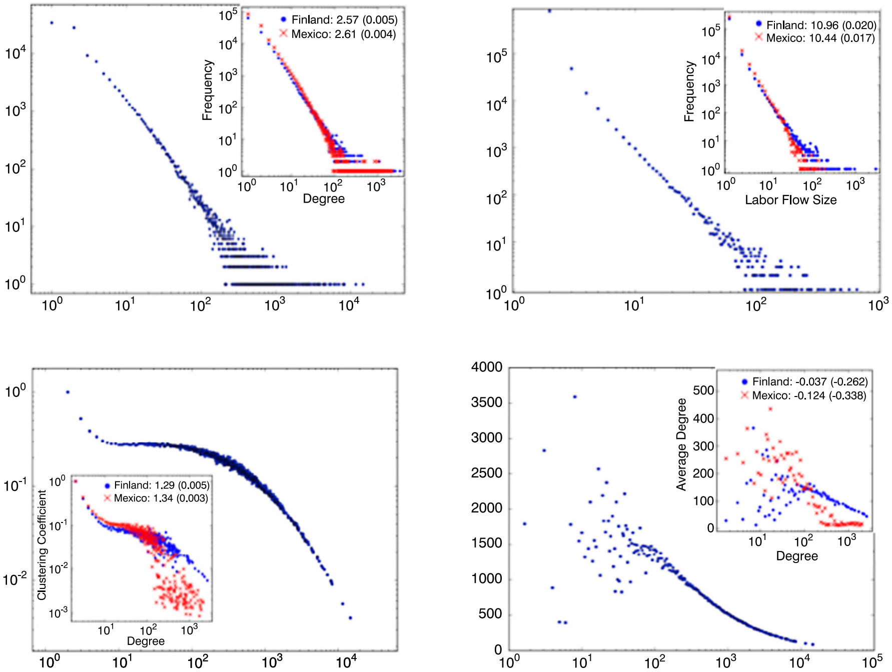

FIGURE 33

Properties of the labor flow network: degree distribution (upper left), edge weight distribution (upper right), clustering as a function of degree (lower left), and average neighbor degree (assortativity) vs. degree (lower right).

To analyze welfare of agents, consider homogeneous groups of maximum stable size, having utility levels shown in Fig. 6, replotted in Fig. 34. Overlaid on these smooth curves is the cross-section of utilities in realized groups. The main result here is that most agents prefer the non-equilibrium world to the equilibrium outcome with homogeneous groups.

4 MODEL VARIATIONS: SENSITIVITY AND ROBUSTNESS

In this section the base model of Table 2 is varied in certain ways and the effects described. The main lesson is that some aspects of the model can be modified while preserving the empirical character of the results, relaxation of certain core model specifications, individually, is sufficient to break its connection to the data. Another way to say this is that the combination of the models' main components—increasing returns, agent heterogeneity, imperfect compensation, limited information—is a minimal set of specifications that is sufficient to produce a close connection to the empirical data.

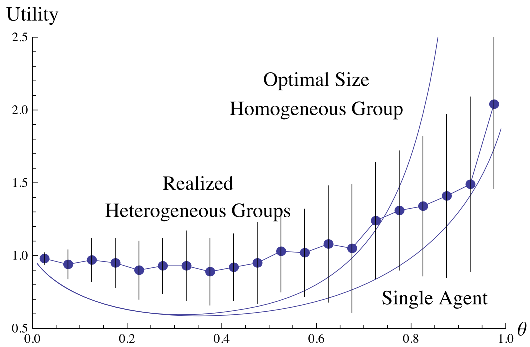

FIGURE 34

Utility in single agent firms, optimal homogeneous firms, and realized firms (averages and standard deviations), by θ.

First we investigate the importance of purposive behavior. Since certain stochastic growth processes are known to yield power law distributions, perhaps the model described above is simply a complicated way to generate random behavior. That is, although the agents are behaving purposively, this may be just noise at the macro level. If agent behavior were simply random, would this yield realistic firms too? We have investigated this in two ways. First, consider that agents randomly select whether to stay in their current firm, leave for another firm, or start-up a new firm, while still picking an optimal effort where they end up. It turns out that this specification yields only small firms, under size 10. Second, if agents select the best firm to work in but then choose an effort level at random, again nothing like skew size distributions arise. These results suggest that any systematic departure from (locally) purposive behavior is unrealistic.

One specification found to have no effect on the model in the long run is the initial condition. Starting the agents in groups seems to modify only the duration of the initial transient.

Next how does the number of agents matter? While the base case of the model has been realized for 120 million agents, Fig. 35 gives the dependence of the largest firm realized as the population size is varied. The maximum firm size rises sublinearly with the size of the population for the parameters of Table 2. This means that a less than full-scale model would produce somewhat different statistics than the base model of Table 2, e.g., a 1000 agent model can yield a 100 person largest firm, too large proportionally.

Next, consider alternative agent activation schemes. While it is well-known that *synchronous* activation can produce anomalous output (Huberman and Glance, 1993), *asynchronous* activation can also lead to subtle effects based on whether agents are activated randomly or uniformly (Axtell et al., 1996). Moving from uniform to ran-

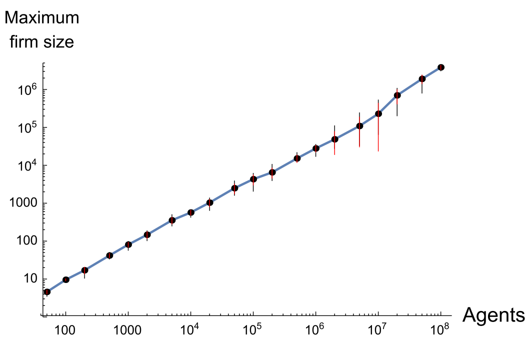

FIGURE 35

Largest firm size realized as a function of the number of agents.

dom activation produces *quantitative* changes in output but only small *qualitative* changes.

How does the specification of production matter? Of the three parameters that specify the production function, a, b, and β, as increasing returns are made stronger, larger firms are realized and average firm size increases. For $\beta > 2$, very large firms arise; these are 'too big' empirically.[28]

Are the results presented above robust to different kinds of agent heterogeneity? With preferences distributed uniformly on (0, 1) in the base case a certain number of extreme agents exist: those with $\theta \approx 0$ are leisure lovers and those with $\theta \approx 1$ love income. Other distributions (e.g., beta, triangular) were investigated and found to change the results *quantitatively* but not *qualitatively*. Removing agents with extreme preferences from the population may result in too few large firms forming, but this can be repaired by increasing β. If agent preferences are *too homogeneous* the model output is qualitatively different from the empirical data. Finally, CES preferences do not alter the general character of the results. Overall, the results are robust to alternative specifications of heterogeneous preferences, as long as there is sufficient heterogeneity.

Social networks play an important role in the model. In the base case each agent has 2 to 6 friends. This number is a measure of the size of an agent's search or information space, since the agent queries these other agents when active to assess the feasibility of joining their firms. The main qualitative impact of increasing the number of friends is to slow model execution. However, when agents query *firms* for jobs something different happens. Asking an agent about a job may lead to working at a big firm. But asking a firm at random usually leads to small firms and empirically-irrelevant model output because most firms are small.

[28] If β is sufficiently large the model can occasionally 'run away' to a single firm employing all the agents!

How does compensation matter to the results? Pay proportional to effort[29]:

$$U_i^p\left(e_i; \theta_i, E_{\sim i}\right) = \left(\frac{e_i}{E} O\left(E\right)\right)^{\theta_i} \left(\omega_i - e_i\right)^{1-\theta_i}$$

leads to a breakdown in the basic model results, possibly with one giant firm forming. The reason for this is that there are great advantages from the increasing returns to being in a large firm and if everyone is compensated in proportion to their effort level no one can do better away from the one large firm. Thus, while there is a certain 'perfection' in the microeconomics of this pay scheme, it completely destroys all connections of the model to empirical data. Consider instead a mixture of compensation schemes, with workers paid partially in proportion to how hard they work and partially based on total output. Calling the U of Eq. (1) U_i^e, a convex combination of utility functions is

$$\begin{aligned} U_i(e_i) &= f U_i^e(e_i) + (1-f) U_i^p(e_i) \\ &= \left(\frac{f}{n^{\theta_i}} + \frac{(1-f)e_i^{\theta_i}}{(E_{\sim i} + e_i)^{\theta_i}}\right) \left[O(e_i, E_{\sim i})\right]^{\theta_i} (\omega_i - e_i)^{1-\theta_i}. \end{aligned}$$

Parameter f moves compensation between 'equal' and 'proportional'. This expression can be maximized analytically for $\beta = 2$, but produces a messy result. Experiments varying f show the *qualitative* character of the model is insensitive to the value of f except in the limit of f approaching 0. Additional sensitivity tests and model extensions are described in the appendix, including variants in which one agent in each firm acts as a residual claimant and hires and fires workers, relaxing the free entry and exit character of the base model.

5 SUMMARY AND CONCLUSIONS

A model in which individual agents form firms has been analyzed mathematically, realized computationally, and tested empirically. Stable equilibrium configurations of firms *do not exist* in this model. Rather, agents constantly adapt to their economic circumstances, changing jobs when it is in their self-interest to do so. Firms are born and enter the economy, they live and age, and then they exit. No firm lives forever. This multi-level model, consisting of a large number of simple agents in an environment of increasing returns, is sufficient to generate macro-statistics on firm sizes, ages, growth rates, job tenure, wages, networks, etc., that closely resemble some three dozen data, summarized in Table 3.

[29] Encinosa et al. (1997) studied compensation systems empirically for team production environments in medical practices. They find that "group norms" are important in determining pay practices. Garen (1998) empirically links pay systems to monitoring costs. More recent work is Shaw and Lazear (2008).

Table 3 Empirical data to which the model output is compared; similar data similarly colored

	Datum or data compared	Source	In text
1	Size of the U.S. workforce: 120 million	Census	Table 2
2	Number of firms with employees: ~6 million	Census	Fig. 13
3	Number of new firms monthly: ~100 thousand	Fairlie (2012)	Fig. 13
4	Number of exiting firms monthly: ~100 thousand	Fairlie (2012)	Fig. 13
5	Variance higher for exiting firms than new firms	various	Fig. 13
6	Average firm size: 20 employees/firm	Census	Fig. 14
7	Maximum firm size: ~1 million employees	Fortune	Fig. 14
8	Number of job-to-job changes monthly: ~3+ million	Fallick and Fleischman (2004)	Fig. 16
9	Number of jobs created monthly: ~2 million	Fallick/Fed spreadsheet	Fig. 16
10	Number of jobs destroyed monthly: ~2 million	Fallick/Fed spreadsheet	Fig. 16
11	Variance higher for jobs destroyed than jobs created	Davis et al. (1996)	Fig. 16
12	Firm size distribution (employees): ~Zipf	Axtell (2001)	Fig. 17 (left)
13	Firm size distribution (output): ~Zipf	Axtell (2001)	Fig. 17 (right)
14	Aggregate returns to scale: constant	Basu and Fernald (1997)	Fig. 18
15	Productivity distribution: Pareto tail	Souma et al. (2009)	Fig. 19
16	Firm age distribution: Weibull; mean ~14 years	Bureau of Labor Statistics	Fig. 20
17	Firm survival probability: increasing with age	Bureau of Labor Statistics	Fig. 21
18	Joint dist. of firms, size and age: linear in age, log size	Haltiwanger et al. (2011)	Fig. 23

(continued on next page)

Table 3 (*continued*)

	Datum or data compared	Source	In text
19	Firm growth rates depend on firm size	various, see text	Figs. 24, 25
20	Log firm growth rates (g) are Subbotin-distributed	Stanley et al. (1996)	Figs. 24, 25
21	Mode(g) = 0.0, many firms do not grow	various, see text	Figs. 24, 25
22	More variance for firm decline than firm growth	Davis et al. (1996)	Fig. 24, 25
23	Mean of $\overline{g}$ near 0.0, + for small firms, − for large	Birch (1981), others	Fig. 26 (left)
24	Variance of g declines with firm size	Stanley et al. (1996)	Fig. 28 (left)
25	Mean of g declines with firm age	Haltiwanger et al. (2011)	Fig. 26 (right)
26	Variance of g declines with firm age	Evans (1987a, 1987b)	Fig. 28 (right)
27	Mean of g as function of size, age: young firms grow	Haltiwanger et al. (2011)	Fig. 27
28	Simultaneous hiring and separation	Davis et al. (2006)	Fig. 29
29	Wage distribution: exponential	Yakovenko and Rosser (2009)	Fig. 30
30	Job tenure dist.: exponential with mean 90 months	Bureau of Labor Statistics	Fig. 31
31	Employment vs. age: exp. with mean 25 years	Bureau of Labor Statistics	Fig. 32
32	Florence (firm size weighted) median: 500 employees	Census	Around Fig. 32
33	Degree distribution of the labor flow network (LFN)	Guerrero and Axtell (2013)	Fig. 33
34	Edge weight distribution of the LFN	Guerrero and Axtell (2013)	Fig. 33
35	Clustering coefficient vs. firm size in the LFN	Guerrero and Axtell (2013)	Fig. 33
36	Assortativity (degree of neighbors) vs. firm size, LFN	Guerrero and Axtell (2013)	Fig. 33

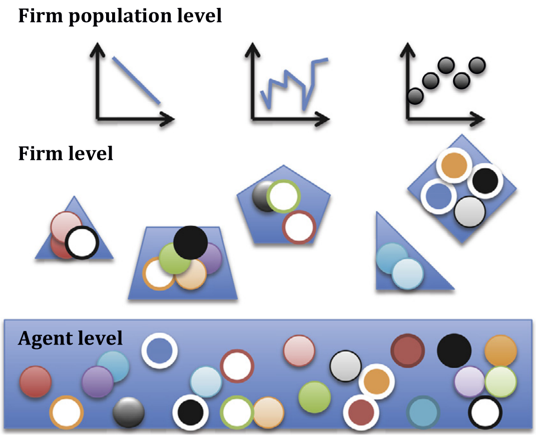

FIGURE 36

Multi-level schematic of firm formation from agents.

Overall, firms are vehicles through which agents realize greater utility than they would by working alone. The general character of these results is robust to many model variations. However, it is possible to sever connections to empirical data with agents who are too homogeneous, too random, or too rational.

5.1 EMERGENCE OF FIRMS, OUT OF MICROECONOMIC EQUILIBRIUM

The main result of this research is to connect an explicit microeconomic model of team formation to emerging micro-data on the population of U.S. business firms. Agent behavior is specified at the micro-level with firms emerging at a meso-level, and the population of firms studied at the aggregate level (Fig. 36). This micro–meso–macro picture has been created with agent computing, realized at full-scale with the U.S. private sector.[30] However, despite the vast scale of the model, its specification is actually very *minimal*, so spare as to seem rather unrealistic[31]—no product markets are modeled, no prices computed, no consumption represented, no industries appear, and agent behavior is relatively simple. Furthermore, there is no technological change and thus no economic growth—all the dynamics are produced simply through rearrangements of firm personnel to achieve local improvements in the *social* technology of production (Beinhocker, 2006). How is it that such a stripped-down model could ever resemble empirical data?

This model works because its *dynamics* capture elements of the real world more closely than conventional models involving static equilibria, either with or without external shocks. This is so despite the agents being unequipped to figure out optimal

[30] It is folk wisdom that agent models are 'macroscopes,' illuminating macro patterns from the micro rules.
[31] In this it is reminiscent of Gode and Sunder and their zero-intelligence traders (Gode and Sunder, 1993).

multi-period strategies. In defense of such *simple* agents, the environments in which they find themselves are *too complex* for them to compute rational behaviors—each agent's team contains difficult-to-forecast contingencies concerning co-worker effort levels, the tenure of current colleagues, the arrival of new personnel, and fluctuating outside opportunities.[32] A major finding of this research is that we are able to *neglect* strategic behavior at the firm level—price and quantity-setting, for instance—yet explain many empirical properties of firms. Strategic decisions certainly matter for the fortunes of individual firms, but seem to not be needed to explain the gross properties of the *population* of firms.

More generally, the belief that social (aggregate) equilibria require agent-level equilibria is problematical (Foley, 1994; Axtell, 2015), a classical fallacy of division. The goal of social science is to explain *social* regularities realized at some level 'above' the agent behavioral level. While agent-level equilibria are commonly treated as *necessary*, such equilibria are, in fact, only *sufficient*—macroscopic regularities that have the character of statistical steady-states (e.g., stationary distributions) may result when there do not exist stable agent-level equilibria, as we have seen above. The assumption of similarity across levels, whether explicitly made or implicitly followed as a social norm, can be fallacious. Important regularities and patterns may arise at the macro-level without the agent level being in Nash or Walrasian equilibrium. Furthermore, when stable equilibria exist but require huge amounts of time to be realized, one may be better off looking for regularities in long-lived transients. This is particularly relevant to coalition formation games in large populations, where the number of coalitions is given by the unimaginably vast Bell numbers, meaning that anything like optimal coalitions could never be realized during agent lifetimes. Perpetual flux in the composition of groups leads naturally to the conclusion that microeconomic equilibria have little explanatory power.

5.2 FROM THEORIES OF THE FIRM TO A THEORY OF FIRMS

Unfortunately, most extant theories of the firm are steeped in this kind of micro-to-macro homogeneity. They begin innocuously enough, with firms conceived of as being composed of a few actors. They then go on to derive firm performance in response to rivals, strategic uncertainty, information processing constraints, and so on. But these derivations interpret the overall performance of multi-agent groups and organizations in terms of a few agents in equilibrium,[33] and have little connection to the empirical regularities documented above.[34]

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[32] Anderlini and Felli (1994) assert the impossibility of complete contracts due to the complexity of nature. Anderlini (1998) describes the kinds of forecasting errors that are intrinsic in such environments.

[33] Least guilty of this charge is the evolutionary paradigm.

[34] For example, the industrial organization textbooks of both Shy (1995) and Cabral (2000) fail to make any mention whatsoever of firm size, age, or growth rate distributions, nor do they note either the number of firms or the average firm size, either in the U.S. or in other countries!

There are two senses in which the model described above is a theory of firms. First, from a purely descriptive point of view, the model reproduces the gross features of U.S. firms, while extant theories of the firm do not.[35] Nor are most theories sufficiently explicit to be operationalized, mathematically or computationally, their focus on equilibrium leaving behavior away from equilibrium unspecified.[36] In the language of Simon (1976), these theories are substantively rational, not procedurally so. Or, if micro-mechanisms are given, the model is only notionally related to data (e.g., Hopenhayn, 1992; Kremer, 1993; Rajan and Zingales, 2001), or else the model generates the wrong patterns (e.g., Cooley and Quadrini, 2001 get *exponential* firm sizes; Klette and Kortum, 2004 get *logarithmic* sizes and incorrect dependence of firm growth rate variance on size).

The second sense in which my model is a theory of firms is that agent models are *explanations* of the phenomena they reproduce.[37] In the philosophy of science an explanation is defined with respect to a theory,[38] which has to be general enough to provide explanations of whole classes of phenomena, while not being so vague that it can rationalize all phenomena. Each parameterization of an agent-based model is an instance of a more general agent 'theory'. Executing an instance yields patterns that can be compared to data, thus making it falsifiable.[39]

My 'explanation' for firms is simple: purposive agents in increasing returns environments form quasi-stable coalitions. The ability of agents to move between such transient teams 'arbitrages' away superlinear returns. In effect, firms compete for high effort individuals. Successful firms in this environment are ones that can attract and keep productive workers. This model, suitably parameterized, can be compared directly to emerging micro-data on firms. Today we do not have a *mathematical derivation* of the aggregate (firm population) properties of our model from the micro (agent behavioral) specifications, so for now we must content ourselves with the *computational discovery* that such firms result from purposive agents in economic environments having increasing returns.

This model is a first step toward a more realistic, dynamical theory of the firm, one with explicit micro-foundations. Clearly this approach produces empirically-rich results. We have produced these results computationally. Today computation is used by economists in many ways, to numerically *solve* equations (e.g., Judd, 1998), to *execute* mathematical programs (Scarf, 1973; Scarf and Shoven, 1984; Scarf, 1990), to *run* regressions (e.g., Sala-i-Martin, 1997), to *simulate* stochastic processes (e.g., Bratley et al., 1987), or to *perform* micro-simulations (e.g., Bergmann,

[35] A variety of models aim for one of these targets, often the firm size distribution (e.g., Lucas, 1978; Kwasnicki, 1998) and only a handful attempt to get more (Luttmer, 2007, 2011; Arkolakis, 2013).

[36] I began this work with the expectation of drawing heavily on extant theory. While I did not expect to be able to turn Coase's elegant prose into software line-for-line, I did expect to find significant guidance on the micro-mechanisms of firm formation. These hopes were soon dashed.

[37] According to Simon (Ijiri and Simon, 1977, p. 118): "To 'explain' an empirical regularity is to discover a set of simple mechanisms that would produce the former in any system governed by the latter."

[38] This is the so-called deductive-nomological (D-N) view of explanation; see Hempel (1966).

[39] In models that are intrinsically stochastic, multiple realizations must be made to find robust regularities.

1990)—all complementary to conventional theorizing. Agent computing enriches these approaches. Like microsimulation, it facilitates heterogeneity, so representative agents (Kirman, 1992) are not needed. Unlike microsimulation, it features direct (local) interactions, so networks (Kirman, 1997; Vega-Redondo, 2007) are natural to consider. Agents possess limited information and are of necessity boundedly rational, since full rationality is computationally intractable (Papadimitriou and Yannakakis, 1994). This encourages experimentally-grounded behavioral specifications. Aggregation happens, as in the real world, by summing over agents and firms. Macro-relationships *emerge* and are not limited *a priori* to what the 'armchair economist' (Simon, 1986) can first imagine and then solve for analytically. There is no need to postulate the attainment of equilibrium since one merely interrogates a model's output for patterns, which may or may not include stable equilibria. Indeed, agent computing is a natural technique for studying economic processes that are far from (agent-level) equilibrium (Arthur, 2006, 2015).

5.3 ECONOMICS OF COMPUTATION AND COMPUTATIONAL ECONOMICS

We have entered the age of *computational synthesis*. Across the sciences, driven by massive reduction in the cost of computing, researchers have begun to reproduce fundamental structures and phenomena in their fields using large-scale computation. In chemistry, complex molecules have their structure and properties investigated digitally before they are manufactured in the lab (Lewars, 2011). In biology, whole cell simulation, involving thousands of genes and millions of molecules, has recently been demonstrated (Karr et al., 2012). In fluid mechanics, turbulence has resisted analytical solution despite the governing equations being known since the 19th century. Today turbulent flows are studied computationally using methods that permit transient internal structures (e.g., eddies, vortices) to arise spontaneously (Hoffman and Johnson, 2007). In climate science whole Earth models couple atmospheric and ocean circulation dynamics to study global warming at ever-finer spatio-temporal resolution (Lau and Ploshay, 2013). In planetary science the way the moon formed after a large Earth impact event has been simulated in great detail (Canup, 2012; Cuk and Stewart, 2012). In neuroscience high frequency modeling of billions of neurons is now possible, leading to the drive for whole brain models (Markram, 2006, 2012).

Surely economics cannot be far behind. Across the social sciences people are utilizing 'big data' in a variety of ways (Lazer et al., 2009; Watts, 2013; Alvarez, 2016). The time has come for a computational research program focused on creating economies in software at full scale with real economies. Perhaps such a new endeavor needs a name—*synthetic economics* might work. More than a generation ago an empirically-rich computational model of a specific firm was created and described by Cyert and March (1963) in their book entitled *A Behavioral Theory of the Firm*. I hope the present work can begin for the *population* of U.S. firms what Cyert and March accomplished for an *individual* organization. At this point we have merely scratched the surface of the rich intersection between large-scale agent computing and economics. Let the computing begin!

APPENDIX A GENERALIZED PREFERENCE SPECIFICATIONS

The functional forms of Section 2 can be relaxed without altering the main conclusions. Consider each agent having preferences for income, I, and leisure, Λ, with more of each being preferred to less. Agent i's income is monotone non-decreasing in its effort level e_i as well as that of the other agents in the group, $E_{\sim i}$. Its leisure is a non-decreasing function of $\omega_i - e_i$. The agent's utility is thus $U_i(e_i; E_i) = U_i(I(e_i; E_{\sim i}), \Lambda(\omega_i - e_i))$, with $\partial U_i/\partial I > 0$, $\partial U_i/\partial \Lambda > 0$, and $\partial I(e_i; E_{\sim i})/\partial e_i > 0$, $\partial \Lambda(e_i)/\partial e_i < 0$. Furthermore, assuming $U_i(I = 0, \cdot) = U_i(\cdot, \Lambda = 0) = 0$, U is single-peaked. Each agent selects the effort that maximizes its utility. The first-order condition is straightforward. From the inverse function theorem there exists a solution to this equation of the form $e_i^* = \max[0, \zeta(E_{\sim i})]$. From the implicit function theorem both ζ and e_i^* are continuous, non-increasing functions of $E_{\sim i}$.

Team effort equilibrium corresponds to each agent contributing its e_i^*, and that the other agents are doing so as well, i.e., substituting $E_{\sim i}^*$ for $E_{\sim i}$. Since each e_i^* is a continuous function of $E_{\sim i}$ so is the vector of optimal efforts, $e^* \in [0, \omega]^n$, a compact, convex set. By the Leray–Schauder–Tychonoff theorem an effort fixed point exists. Such a solution constitutes a Nash equilibrium, which is Pareto-dominated by effort vectors having larger amounts of effort for all agents.

For any effort adjustment function $e_i(t+1) = h_i(E_{\sim i}(t))$, such that

$$\frac{dh_i\left(E_{\sim i}\right)}{dE_{\sim i}} = \frac{\partial h_i\left(E_{\sim i}\right)}{\partial e_j} \leq 0,$$

for all $j \neq i$, there may exist an upper bound on firm size. Under these circumstances the Jacobian matrix retains the structure described in Section 2.2, where each row contains $N - 1$ identical entries and a 0 on the diagonal. The bounds on the dominant eigenvalue derived in Section 2.2 guarantee that there exists an upper bound on the stable group size, as long as the previous inequality is strict, thus establishing the onset of instability above some critical size.

APPENDIX B GENERALIZED COMPENSATION AND NASH STABILITY

It was asserted in Section 4 that proportional or piecemeal compensation breaks our basic results. What it does is dramatically reduce the incentive problems of team production. To see this we redo Fig. 1 for this compensation function, as shown in

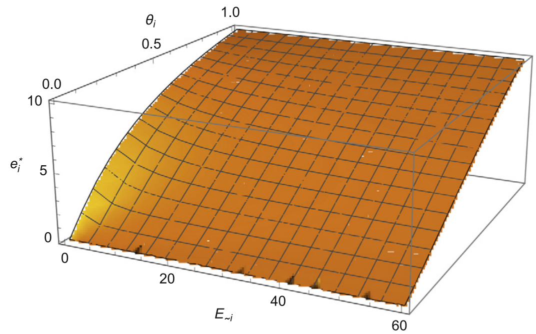

FIGURE 37

Dependence of e_i^* on $E_{\sim i}$ and θ_i for $a = 1$, $b = 1$, $\omega_i = 10$.

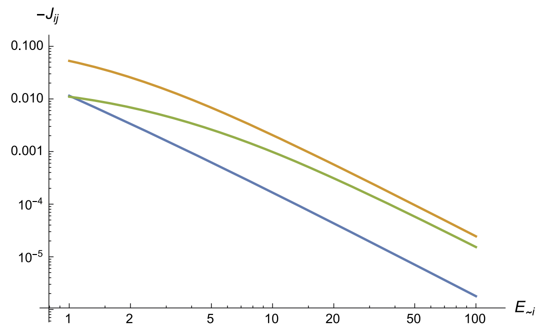

FIGURE 38

Dependence of the elements of the Jacobian matrix on $E_{\sim i}$ for $a = 1$, $b = 1$, and $\omega_i = 1$, for three values of θ_i (0.1, 0.5, and 0.9).

Fig. 37. Note that there is no longer a region of zero effort. We next compute the Jacobian matrix and evaluate its elements as the size of the group increases. This is shown in Fig. 38. The values decline sufficiently rapidly (note the log–log coordinates) that no instability will be induced by the dynamical effort level adjustments of the agents to one another, no matter how large the group.

For mixtures of compensation we recover the general properties of equal compensation. The way that effort, e_i^*, depends on $E_{\sim i}$ and θ_i for $f = 1/2$ is shown in Fig. 39. Note the region of zero effort for agents with low preference for income. For this mixture of compensation policies the eigenvalues of the Jacobian matrix can be computed numerically for various values of n (Fig. 40). While these values still decline as an approximate power law, they do so sufficiently slowly that it becomes possible to produce eigenvalues outside the unit circle, particularly for large n, since the matrix entries begin plateauing then.

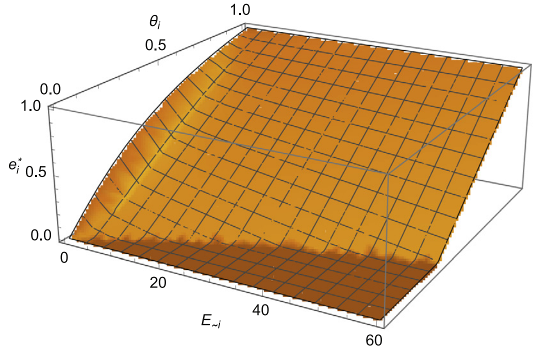

FIGURE 39

Dependence of e_i^* on $E_{\sim i}$ and θ_i for $a = 1$, $b = 1$, $\omega_i = 10$, and $f = 1/2$.

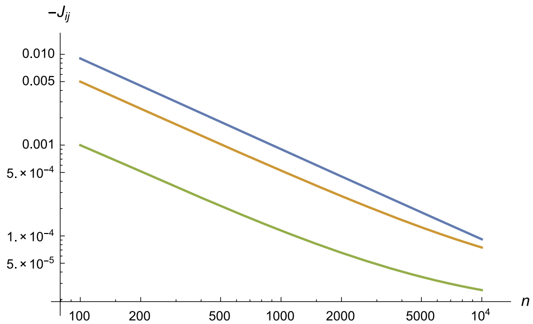

FIGURE 40

Dependence of the elements of the Jacobian matrix on n for $a = 1$, $b = 1$, $\omega_i = 1$, $f = 1/2$, and $E_{\sim i} = 100$, for three values of θ_i (0.1 (blue), 0.5 (orange), and 0.9 (green)).

APPENDIX C SENSITIVITY TO ‘STICKY’ EFFORT ADJUSTMENT

In the base model agents adjust their effort levels to anywhere within the feasible range $[0, \omega]$. A different behavioral model involves agents making only small changes from their current effort level each time they are activated. Think of this as a kind of prevailing *work ethic* within the group or *individual habit* that constrains the agents to keep doing what they have been, with small changes.

Experiments have been conducted for each agent searching over a range of 0.10 around its current effort level: an agent working with effort e_i picks its new effort from the range $[e_L, e_H]$, where $e_L = \max(0, e_i - 0.05)$ and $e_H = \min(e_i + 0.05, 1)$. This slows down the dynamics somewhat, yielding larger firms. This is because as large firms tend toward non-cooperation, this kind of sticky effort adjustment dampens the downhill spiral to free riding. I have also experimented with agents who ‘grope’ for welfare gains by randomly perturbing current effort levels, yielding similar results.

APPENDIX D EXTENSION: STABILIZING EFFECT OF AGENT LOYALTY

In the basic model an agent moves immediately to a new firm when its subjective evaluation is that it will be better off by doing so. Behaviorally, this seems implausible for certain kinds of workers, especially those who feel some *loyalty* to their firm. The formulation of agent loyalty used here involves agents not changing jobs right away, as soon as they figure out that they can do better elsewhere. Rather, they let χ better job opportunities arrive before separating from their current firm. Think of an agent's χ as a kind of counter. It starts off with some value and each time the agent determines there are higher payoffs elsewhere but does not leave its firm the value of χ declines by 1. When $\chi = 0$ the next preferable position it can find it takes and χ is reset. The base case of the model corresponds to no loyalty, that is, $\chi = 0$.

I have experimented with homogeneous and heterogeneous χs, in the range from [0, 10]. Even a modest amount of loyalty reduces worker turnover and firm volatility, especially in large firms, and increases job tenure, firm age, and firm lifetime, holding other parameters constant. Increasing loyalty makes large firms bigger while reducing labor flows. In order to maintain the close connection of the model output to empirical data in the presence of agent loyalty it would be necessary to recalibrate the model, something reserved for future work.

APPENDIX E EXTENSION: HIRING

One aspect of the base model is very unrealistic: that agents can join whatever firms they want, as if there is no barrier to getting hired by any firm. The model can be made more realistic by instituting local hiring policies.

Let us say that one agent in each firm does all hiring, perhaps the agent who founded the firm or the one with the most seniority. We will call this agent the 'boss'. A simple hiring policy has the boss compare current productivity to what would be generated by the addition of a new worker, *assuming that no agents adjust their effort levels*. The boss computes the minimum effort, $\phi E/n$, for a new hire to raise productivity as a function of a, b, β, E, and n, where ϕ is a fraction:

$$\frac{aE + bE^{\beta}}{n} < \frac{a\left(E + \phi\frac{E}{n}\right) + b\left(E + \phi\frac{E}{n}\right)^{\beta}}{n+1} = \frac{aE\left(1 + \frac{\phi}{n}\right) + bE^{\beta}\left(1 + \frac{\phi}{n}\right)^{\beta}}{n+1}. \quad \text{(A.8)}$$

For $\beta = 2$ this can be solved explicitly for the minimum ϕ necessary

$$\phi_* = \frac{-n(a + 2bE) + \sqrt{n^2(a + 2bE)^2 + 4bEn(a + bE)}}{2bE}.$$

Table 4 Dependence of the minimum fraction of average effort on firm size, n, and increasing returns parameter, β

n	β			
	1.0	**1.5**	**2.0**	**2.5**
1	1.0	0.59	0.41	0.32
2	1.0	0.62	0.45	0.35
5	1.0	0.65	0.48	0.38
10	1.0	0.66	0.49	0.39
100	1.0	0.67	0.50	0.40

For all values of ϕ_* exceeding this level the prospective worker is hired. For the case of $a = 0$, (A.8) can be solved for any value of β:

$$\phi_* = n\left(\frac{n+1}{n}\right)^{1/\beta} - n;$$

this is independent of b and E. The dependence of ϕ_* on β and n is show in Table 4.

As n increases for a given β, ϕ_* increases. In the limit of large n, ϕ_* equals $1/\beta$. So with sufficient increasing returns the boss will hire just about any agent who wants a job! These results can be generalized to hiring multiple workers.

Adding this functionality to the computational model changes the behavior of individual firms and the life trajectories of individual agents but does not substantially alter the overall macrostatistics of the artificial economy.

APPENDIX F EXTENSION: EFFORT MONITORING AND WORKER TERMINATION

In the base model, shirking goes completely undetected and unpunished. Effort level monitoring is important in real firms, and a large literature has grown up studying it; see Olson (1965), the models of mutual monitoring of Varian (1990), Bowles and Gintis (1998), and Dong and Dow (1993b), the effect of free exit (Dong and Dow, 1993a), and endowment effects (Legros and Newman, 1996); Ostrom (1990) describes mutual monitoring in institutions of self-governance.

It is possible to *perfectly* monitor workers and fire the shirkers, but this breaks the model by pushing it toward static equilibrium. All real firms suffer from imperfect monitoring. Indeed, many real-world compensation systems can be interpreted as ways to manage incentive problems by substituting reward for supervision, from efficiency wages to profit-sharing (Bowles and Gintis, 1996). Indeed, if incentive problems in team production were perfectly handled by monitoring there would be no need for corporate law (Blair and Stout, 1999).

To introduce involuntary separations, say the residual claimant knows the effort of each agent and can thus determine if the firm would be better off if the least hard working one were let go. Analogous to hiring we have:

$$\frac{aE + bE^{\beta}}{n} < \frac{a\left(E - \phi\frac{E}{n}\right) + b\left(E - \phi\frac{E}{n}\right)^{\beta}}{n-1} = \frac{aE\left(1 - \frac{\phi}{n}\right) + bE^{\beta}\left(1 - \frac{\phi}{n}\right)^{\beta}}{n-1}.$$

Introducing this logic into the code there results unemployment: agents are terminated and do not immediately find another firm to join. Experiments with terminations and unemployment have been undertaken and many new issues are raised, so we leave full investigation of this for future work.

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