Analysis I Summary

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1 Real numbers, euclidean spaces

Archimedes' principle. If $x \in \mathbb{R}$ with $x > 0$ and $y \in \mathbb{R}$, then $\exists n \in \mathbb{N}$ $(y \leq n \cdot x)$

Thm. (i) $|x| \geq 0 \quad \forall x \in \mathbb{R}$

(ii) $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$

(iii) $|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$

(iv) $|x + y| \geq ||x| - |y|| \quad \forall x, y \in \mathbb{R}$

Young's inequality. $\forall \epsilon > 0, \forall x, y \in \mathbb{R}$:

$$2|xy| \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$$

2 Sequences

2.1 Convergence

$(a_n)_{n\geq1}$ converges to $L = \lim_{n\to\infty} a_n$
$\iff \forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n > N \, (|a_n - L| < \epsilon)$

Def (Convergence). $(a_n)_{n\geq1}$ converges
$\iff \exists L \in \mathbb{R} \, \forall \epsilon > 0 \, (\{n \in \mathbb{N} \mid |a_n - L| \geq \epsilon\})$ is finite.

Hint. Let $(a_n)_{n\geq1}$, $(b_n)_{n\geq1}$ converge with limit a and b:

1. $(a_n + b_n)_{n\geq1}$ converges with limit $a + b$
2. $(a_n \cdot b_n)_{n\geq1}$ converges with limit $a \cdot b$.
3. $(\frac{a_n}{b_n})_{n\geq1}$ converges with limit $\frac{a}{b}$
4. $\exists K \geq 1 \, \forall n \geq K : a_n \leq b_n \implies a \leq b$

2.1.1 Tips & Tricks

- a_n convergent $\implies$ a_n bounded
- a_n convergent $\iff$ a_n bounded and $\liminf a_n = \limsup a_n$

Monotone Convergence. $(a_n)_{n\geq1}$ monotone increasing and upper bounded $\implies \lim a_n = \sup\{a_n \mid n \geq 1\}$
$(a_n)_{n\geq1}$ monotone decreasing and lower bounded $\implies \lim a_n = \inf\{a_n \mid n \geq 1\}$

Lemma (Bernoulli Inequation).

$$(1 + x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}, x > -1$$

Def.

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} (\inf\{a_k \mid k \geq n\})$$
$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} (\sup\{a_k \mid k \geq n\})$$

Cauchy Criteria. a_n converges iff $\forall \epsilon > 0 \, \exists N \geq 1$ s.t. $|a_n - a_m| < \epsilon \, \forall n, m \geq N$ (cauchy sequence).

(i) Each Cauchy sequence is bounded

(ii) $(a_n)_{n\geq1}$ conv. $\implies$ $(a_n)_{n\geq1}$ cauchy

(iii) $(a_n)_{n\geq1}$ cauchy $\implies$ $(a_n)_{n\geq1}$ conv.

Bolzano-Weierstrass. Each bounded sequence contains a convergent sub sequence.

Sandwich. If $\lim a_n = \alpha$, $\lim b_n = \alpha$, $k \in \mathbb{N}$ and $a_n \leq c_n \leq b_n \, \forall n \geq k$, then $\lim c_n = \alpha$

Cauchy-Cantor. Let $I_1 \supseteq I_2 \supseteq \cdots I_n \cdots$ be a sequence of proper intervals with $\mathcal{L} < +\infty$, then $\bigcap_{n\geq1} I_n \neq 0$. And if $\lim_{n\to\infty} \mathcal{L}(I_n) = 0$ then $|\bigcap_{n\geq1} I_n| = 1$

Cor. *Let (a_n) be bounded, then for each subsequence (b_n): $\liminf a_n \leq \lim b_n \leq \limsup a_n$.*

Each subsequence (b_n) of a convergent (a_n) converges and $\lim b_n = \lim a_n$.

3 Series

Def. "$\sum_{k=1}^{\infty} a_k$" converges, if the sequence $(S_n)_{n\geq1}$ of partial sums converges and $\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$

Thm. $\sum_{k=1}^{\infty} a_k$ and $\sum_{j=1}^{\infty} b_j$ convergent:

- $\sum_{k=1}^{\infty}(a_k + b_k) = (\sum_{k=1}^{\infty} a_k) + (\sum_{k=1}^{\infty} b_k)$
- $\sum_{k=1}^{\infty} \alpha \cdot a_k = \alpha \sum_{k=1}^{\infty} a_k$

Couchy Criteria. $\sum_{k=1}^{\infty} a_k$ conv. $\iff \forall \epsilon > 0 \, \exists N \geq 1 :$ $|\sum_{k=n}^{m} a_k| = |S_m - S_n| < \epsilon \, \forall m \geq n \geq N$

Zero Sequence Criteria. $\sum_{k=1}^{\infty} a_k$ conv. $\implies \lim a_k = 0$

Comparison Theorem. Let $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ series s.t. $0 \leq a_k \leq b_k \quad \forall k \geq 1$:

$$\sum_{k=1}^{\infty} b_k \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges} \implies \sum_{k=1}^{\infty} b_k \text{ diverges}$$

Thm. Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0 \quad \forall k \in \mathbb{N}^*$

$$\sum_{k=1}^{\infty} \text{ converges} \iff (S_n)_{n\geq1} \text{ upper bounded}$$

Def (Asolute Convergence). $\sum_{k=1}^{\infty} a_k$ absolute converges if $\sum_{k=1}^{\infty} |a_k|$ converges.

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \not\Rightarrow \sum_{k=1}^{\infty} |a_k| \text{ converges}$$

(**Dirichlet**) If a series converges absolute, then each permutation of the series converges with the same limit.
(**Riemann**) If a series only converges, then there exists a permutation such that:

$$\sum_{k=1}^{\infty} a_{\phi(k)} = x \quad \forall x \in \mathbb{R} \cup \{\infty\}$$

Thm.

$$\left|\sum_{k=1}^{\infty} a_k\right| \leq \sum_{k=1}^{\infty} |a_k|$$

Leibniz. $(a_n)_{n\geq 1}$ monotone decreasing s.t. $a_n \geq 0\, \forall n \geq 1$ and $\lim a_n = 0$:

$$S := \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges.}$$

Furthermore: $a_1 - a_2 \leq S \leq a_1$

Ratio Test. Let $(a_n)_{n\geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$:
$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolute.
$\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges

Lemma. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n$ *converges absolute.*
- $L > 1 \implies \sum_{n=1}^{\infty} a_n$ *diverges.*
- $L = 1 \implies$ *no information*

Root Test. Let $(a_n)_{n\geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$:
$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolute.
$\liminf_{n\to\infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} |a_n|$ diverge.

Lemma. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n$ *converges absolute.*
- $L > 1 \implies \sum_{n=1}^{\infty} a_n$ *diverges.*
- $L = 1 \implies$ *no information*

Def (Cauchy Product)**.** $\sum_{i=0}^{\infty} a_i$, $\sum_{j=0}^{\infty} b_j$:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{n-j} b_j\right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots$$

Thm. $\sum\limits_{i=0}^{\infty} a_i$, $\sum\limits_{j=0}^{\infty} b_j$ conv. abs. $\Rightarrow$ Couchy prod. conv. abs.:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_{n-j} b_j\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_i b_j$$

Hint (Strategy: Convergence of Series)**.**

1. Check for known types (Telescope, Geometric, etc.)
2. $\lim a_n \neq 0 \implies$ divergence
3. Ratio Test
4. Root Test
5. Search convergent majors: $0 \leq a_n \leq b_n$
6. If divergent minors $\implies$ divergence
7. Be creative

4 Functions

$\mathbb{R}^D = \{f : D \to \mathbb{R} \mid f \text{ is function}\}$, $(\mathbb{R}^D; +, \cdot)$ is V.R.

4.1 Continuity

Def (Continuity)**.** A function f is continuous in x_0 if:

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in D\ (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon)$$

$$\iff$$

$$\forall (a_n)_{n\geq 1} \text{ with } \lim a_n = x_0 \text{ holds } \lim f(a_n) = f(\lim a_n)$$

Def. A function $f : D \to \mathbb{R}$ is continuous if it is continuous in all $x_0 \in D$

Hint. To prove continuity try to filter $|x - x_0|$ out of $|f(x) - f(x_0)|$ and choose δ, such that the rest term disappears. Be aware that δ is part of ϵ and normally $|x_0|$ as well. But not x!

Cor. *$f, g : D \to \mathbb{R}$ continuous in $x_0 \in D$. Then:*

- *$fg, \lambda f, f \pm g$ continuous in x_0*
- *$\frac{f}{g} : D \setminus \{x \in D \mid g(x) = 0\} \to \mathbb{R}$ continuous in x_0 (if $g(x_0) \neq 0$)*
- *$|f|, \max(f, g), \min(f, g)$ continuous in x_0*
- *$P(x) = a_n x^n + \cdots + a_0$ continuous on $\mathbb{R}$*
- *$\frac{P(x)}{Q(x)}$ continuous on $\mathbb{R} \setminus \{x_1, \cdots, x_m\}$ if $x_1, \cdots, x_m$ are roots of $Q(x)$*

Thm. Let $f : D_1 \to D2 \subset \mathbb{R}, g : D_2 \to \mathbb{R}$ be continuous $\implies g \circ f : D_1 \to \mathbb{R}$ continuous.

Bolzano (Intermediate value theorem). Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ and $a, b \in I$. For each c between $f(a)$ and $f(b)$ there is a $z \in [a, b]$ with $f(z) = c$

Min-Max. Let $f : I = [a, b] \to \mathbb{R}$ be continuous.

$$\exists u, v \in I\ \forall x \in I\ (f(u) \leq f(x) \leq f(v))$$

In particular $f([a, b]) \subset [f(u), f(v)]$ is bounded.

Cor. *$I = [a, b]$, $f : I \to \mathbb{R}$ continuous, then $Im(f) = f(I)$ is a compact interval $J = [\min f, \max f] = [f(u), f(v)]$*

Inverse Mapping. Let $f : I \to \mathbb{R}$ be continuous and strict monotone increasing. Then $J := f(I) \subseteq \mathbb{R}$ is an interval and $f^{-1} : J \to I$ is continuous and strict monotone.

4.2 Exponential function

$\exp : \mathbb{R} \to]0, \infty[$ is continuous, strictly monotone increasing, surjective.

- $\exp(x) \geq 1 + x \quad \forall x \in \mathbb{R}$
- For $x > 0, a \in \mathbb{R} : x^a := \exp(a \ln x)$
- $x^0 = 1 \quad \forall x > 0$

Def. The inverse mapping of $\exp(x)$ is called the natural logarithm:

$$\ln :]0, \infty[\to \mathbb{R}, \quad x \mapsto \ln x$$

It is strictly monotone increasing, continuous and bijective.

4.3 Converge of function sequences

$$\mathbb{N} \to \mathbb{R}^D = \{f : D \to \mathbb{R}\}, \quad n \mapsto f(n)$$

Def (pointwise convergence)**.** $(f_n)_{n\geq 0}$ converges pointwise to a function $f : D \to \mathbb{R}$, if $\forall x \in D : \lim_{n\to\infty} f_n(x) = f(x)$

$$\iff$$

$$\forall x \in D\ \forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N\ (|f_n(x) - f(x)| < \epsilon)$$

Def (uniform convergence (Weierstrass))**.** $f_n : D \to \mathbb{R}$ converges uniformly in D to $f : D \to \mathbb{R}$ if:

$$\forall \epsilon > 0\ \exists N \geq 1 \text{ s.t. } \forall n \geq N\ \forall x \in D\ (|f_n(x) - f(x)| < \epsilon)$$

$$\iff$$

$$\lim_{n\to\infty} \sup_{x\in D} |f_n(x) - f(x)| = 0$$

The function sequence (f_n) is uniformly convergent if for all $x \in D$ the limit $\lim_{n\to\infty} f_n(x) = f(x)$ exists and the sequence (f_n) uniformly converges to f. Furthermore if $\forall \epsilon > 0\ \exists N \geq 1\ \forall n, m \geq N\ \forall x \in D : |f_n(x) - f_m(x)| < \epsilon$.

The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly (in D), if the function sequence $S_n(x) := \sum_{k=0}^{n} f_k(x)$ converges uniformly.

Thm. let $D \subseteq \mathbb{R}$ and $f_n : D \to \mathbb{R}$ a function sequence containing (in D) continuous functions which converge (in D) uniformly against a function $f : D \to \mathbb{R}$, then f (in D) is continuous.

Hint (not uniform convergent)**.** $(f_n)_{n\geq 0}$ converges not uniformly if: $\forall \epsilon > 0\ \forall N \in \mathbb{N}\ \exists x \in D(|f_n(x) - f(x)| \geq \epsilon)$

Hint. Check the function and try to construct x (dependent on N in general), such that $|f_n(x) - f(x)|$ is always greater than a specific ϵ and afterwards choose the ϵ.

Def (Power Functions)**.** $\sum_{k=0}^{\infty} c_k x^k$ has positive convergence radius if $\limsup\limits_{k\to\infty} \sqrt[k]{|c_k|}$ exists.

$$\rho = \begin{cases} +\infty & \text{, if } \limsup\limits_{k\to\infty} \sqrt[k]{|c_k|} = 0 \\ \frac{1}{\limsup\limits_{k\to\infty} \sqrt[k]{|c_k|}} & \text{, if } \limsup\limits_{k\to\infty} \sqrt[k]{|c_k|} > 0 \end{cases}$$

Thm. Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $\rho > 0$ and let $f(x) = \sum_{k=0}^{\infty} c_k x^k, |x| < \rho$ Then: $\forall 0 \leq r < \rho$ converges $\sum_{k=0}^{\infty} c_k x^k$ uniformly on $[-r, r]$, furthermore $f :]-\rho, \rho[\to \mathbb{R}$ is continuous.

4.4 Trigonometric Functions

sin and cos are continuous functions $\mathbb{R} \to \mathbb{R}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Thm.

1. $\exp(iz) = \cos(z) + i\sin(z) \quad \forall z \in \mathbb{C}$
2. $\cos z = \cos(-z)$ und $\sin(-z) = -\sin(z) \quad \forall z \in \mathbb{C}$
3. $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$
4. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
   $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
5. $\cos(z)^2 + \sin(z)^2 = 1 \quad \forall z \in \mathbb{C}$

Cor.

$$\sin(2z) = 2\sin(z)\cos(z)$$
$$\cos(2z) = \cos(z)^2 - \sin(z)^2$$
$$\sin(x) - \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Def (π)**.** $\pi := \inf\{t > 0 \mid \sin t = 0\}$

(i) $\sin\pi = 0, \pi \in]2, 4[$

(ii) $\forall x \in]0, \pi[: \sin x > 0$

(iii) $e^{\frac{i\pi}{2}} = i$

Cor. $x \geq \sin x \geq x - \frac{x^3}{3!} \quad \forall 0 \leq x \leq \sqrt{6}$

Cor. *1.* $e^{i\pi} = -1, \qquad e^{2i\pi} = 1$

2. $\sin(x + \frac{\pi}{2}) = \cos(x), \quad \cos(1 + \frac{\pi}{2}) = -\sin(x) \quad \forall x \in \mathbb{R}$

3. $\sin(x + \pi) = -\sin(x), \quad \sin(x + 2\pi) = \sin(x) \quad \forall x \in \mathbb{R}$

4. $\cos(x + \pi) = -\cos(x), \quad \cos(x + 2\pi) = \cos(x) \quad \forall x \in \mathbb{R}$

5. *Roots of sinus* $= \{\pi \cdot k \mid k \in \mathbb{Z}\}$
$\sin(x) > 0\ \forall x \in]2k\pi, (2k+1)\pi[, \quad k \in Z$
$\sin(x) < 0\ \forall x \in](2k+1)\pi, (2k+2)\pi[, \quad k \in \mathbb{Z}$

6. *Roots of cosine* $= \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$
$\cos(x) > 0\ \forall x \in]-\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + (2k+1)\pi[, \quad k \in \mathbb{Z}$
$\cos(x) < 0\ \forall x \in]-\frac{\pi}{2} + (2k+1)\pi, -\frac{\pi}{2} + (2k+2)\pi[, \quad k \in \mathbb{Z}$

4.5 Limit of Functions

Def (accumulation point)**.** $x_0 \in \mathbb{R}$ is an accumulation point of D if $\forall \delta > 0$: $(]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) \cap D \neq \varnothing$

Def (Limit of Function)**.** if $f : D \to \mathbb{R}, x_0 \in \mathbb{R}$ an accumulation point of D, then $A \in \mathbb{R}$ is the limit of $f(x)$ for $x \to x_0$, written as $\lim_{x\to x_0} f(x) = A$. If $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in D \cap (]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) : \ |f(x) - A| < \epsilon$

Important Rules. Let $f : D \to \mathbb{R}$ and x_0 is an accumulation point of D.

1. $\lim\limits_{x\to x_0} f(x) = A \iff \forall (a_n)_{n\geq 1}$ in $D \setminus \{x_0\}$ with $\lim\limits_{n\to\infty} a_n = x_0 \implies \lim\limits_{n\to\infty} f(a_n) = A$.
2. Let $x_0 \in D$. Then f is continuous in x_0 $\iff \lim\limits_{x\to x_0} f(x) = f(x_0)$
3. $f, g : D \to \mathbb{R}$ and $\exists \lim\limits_{x\to x_0} f(x), \exists \lim\limits_{x\to x_0} g(x) \implies$
$$\lim_{x\to x_0} (f + g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$$
$$\lim_{x\to x_0} (f \cdot g)(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$$
4. $f, g : D \to \mathbb{R}$ and $f \leq g$, then if both limit exists
$$\lim_{x\to x_0} f(x) \leq \lim_{x\to x_0} g(x)$$
5. If $g_1 \leq f \leq g_2$ and $\lim\limits_{x\to x_0} g_1(x) = \lim\limits_{x\to x_0} g_2(x)$ then $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g_1(x)$

Hint. Sometimes it can be really helpful to convert known functions to their power series to calculate a limit. E.g.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \ldots}{x} = \lim_{x\to 0} 1 - \frac{x^2}{3!} + \ldots = 1$$

Hint ($e^{\log}$)**.** Transform ugly function with this trick.

$$\lim_{x\to x_0} f(x)^{g(x)} = \lim_{x\to x_0} e^{g(x)\log(f(x))} = e^{\lim\limits_{x\to x_0} g(x)\log(f(x))}$$

5 Differentiable Functions

Def (Differentiable)**.** f is in x_0 differentiable, if the limit $\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$ exists. f is differentiable if $\forall x_0 \in D$ f is differentiable.

Weierstrass. $f : D \to \mathbb{R}$, $x_0 \in D$ accumulation point. Equivalent statements:

1. f is in x_0 differentiable
2. It exists $c \in R$ ($c = f'(x_0)$) and $r : D \to \mathbb{R}$ s.t.:
   2.1 $f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0)$
   2.2 $r(x_0) = 0$ and r continuous in x_0.

Cor. *f diff. in x_0 $\implies$ f continuous in x_0*

Thm. f diff. in $x_0 \iff \exists \phi : D \to \mathbb{R}$ continuous in x_0 s.t. $\forall x \in D : f(x) = f(x_0) + \phi(x)(x - x_0)$. Then $\phi(x_0) = f'(x_0)$.

Derivative rules.
Linearity: $(\alpha \cdot f(x) + g(x))' = \alpha \cdot f'(x) + g'(x)$
Product rule: $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$
Chain rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Cor. *f bijective and in x_0 differentiable s.t. $f'(x_0) \neq 0$. f^{-1} is continuous in $y_0 = f(x_0)$. Then f^{-1} is differentiable in y_0 and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$.*

5.1 Derivative Implications

1. x_0 is local minimum if $f'(x_0) = 0 \wedge f''(x_0) > 0$ or the sign of f' changes from $-$ to $+$.
2. x_0 is local maximum if $f'(x_0) = 0 \wedge f''(x_0) < 0$ or the sign of f' changes from $+$ to $-$.
3. x_0 is local extremum if $f'(x_0) = 0 \wedge f''(x_0) \neq 0$
4. x_0 is a saddle point if $f'(x_0) = 0$ and $f''(x_0) = 0$
5. x_0 is a inflection point if $f''(x_0) = 0$
6. $f'(x_0) = f^{(2)}(x_0) = \ldots = f^{(n)}(x_0) = 0$
   6.1 n odd and $f^{(n+1)}(x_0) > 0 \implies x_0$ strict local minimum
   6.2 n odd and $f^{(n+1)}(x_0) < 0 \implies x_0$ strict local maximum

5.2 Derivative Theorems

Rolle. Let $f : [a, b] \to \mathbb{R}$ continuous and in $]a, b[$ differentiable. If $f(a) = f(b)$, then there exists $\xi \in]a, b[$ with $f'(\xi) = 0$.

Mean Value / Lagrange. Let $f : [a, b] \to \mathbb{R}$ continuous and in $]a, b[$ differentiable, then there exists $\xi \in]a, b[$ with $f(b) - f(a) = f'(\xi)(b - a)$.
There exists points ξ with $f'(\xi)$ equal to the gradient of the secant between a to b.

Cor. *Let $f, g : [a, b] \to \mathbb{R}$ cont. and diff. in $]a, b[$.*

1. *$\forall \xi \in]a, b[: f'(\xi) = 0 \implies f$ is constant*
2. *$\forall \xi \in]a, b[: f'(\xi) = g'(x) \implies \exists c \in \mathbb{R} \forall x \in [a, b] : f(x) = g(x) + c$*
3. *$\forall \xi \in]a, b[: f'(\xi) \geq 0 \Rightarrow f$ in $[a, b]$ mon. inc. $\Rightarrow$ conv.*
4. *$\forall \xi \in]a, b[: f'(\xi) > 0 \implies f$ in $[a, b]$ str. mon. inc.*
5. *$\forall \xi \in]a, b[: f'(\xi) \leq 0 \Rightarrow f$ in $[a, b]$ mon. dec. $\Rightarrow$ conc.*
6. *$\forall \xi \in]a, b[: f'(\xi) < 0 \implies f$ in $[a, b]$ str. mon. dec.*
7. *$\exists M \geq 0 \; \forall \xi \in]a, b[: |f'(\xi)| \leq M \implies \forall x_1, x_2 \in [a, b] : |f(x_1) - f(x_2)| \leq M|x_1 - x_2|$*

Cauchy. $f, g : [a, b] \to \mathbb{R}$ continuous and in $]a, b[$ diff. Then there exists $\xi \in]a, b[$ with
$g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a))$.
If $\forall x \in]a, b[: g'(x) \neq 0$ it implies that $g(a) \neq g(b)$ and $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$

l'Hôspital. $f, g :]a, b[\to \mathbb{R}$ diff. with $\forall x \in]a, b[: g'(x) \neq 0$.
If $\lim\limits_{x \to b^-} f(x) = 0$, $\lim\limits_{x \to b^-} g(x) = 0$ and
$\lim\limits_{x \to b^-} \frac{f'(x)}{g'(x)} =: \lambda$ exists, then $\lim\limits_{x \to b^-} \frac{f(x)}{g(x)} = \lim\limits_{x \to b^-} \frac{f'(x)}{g'(x)}$.

Hint. Only use l'Hospital if either $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

Def.
1. convex: $(x \leq y)$: $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$
2. strict convex: $(x \leq y)$: $f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$
3. concave: $(x \leq y)$: $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$
4. strict concave: $(x \leq y)$: $f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$

Lemma. *$f : I \to \mathbb{R}$. f is convex $\iff \forall x_0 < x < x_1 \in I : \frac{f(x)-f(x_0)}{x-x_0} \leq \frac{f(x_1)-f(x)}{x_1-x}$. Strictly convex if $<$.*

5.3 Higher Derivatives

1. For $n \geq 2$ is f n-times differentiable in D if $f^{(n-1)}$ in D is differentiable. Then $f^{(n)} := (f^{(n-1)})'$ and is the n-th derivative of f
2. f is n-times **continuous differentiable** if f is n-times differentiable and if $f^{(n)}$ is continuous in D
3. f is in D smooth if $\forall n \geq 1$, f is n-times differentiable.

Smooth Functions. exp, sin, cos, sinh, cosh, tanh, ln, arcsin, arccos, arccot, arctan and all polynomials. tan is smooth on $\mathbb{R} \setminus \{\pi/2 + k\pi\}$ and cot on $\mathbb{R} \setminus \{k\pi\}$

Thm. $f, g : D \to \mathbb{R}$ are n-times diff. in D.

1. $(f + g)^{(n)} = f^{(n)} + g^{(n)}$
2. $(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$
3. $(g \circ f)^{(n)}(x) = \sum_{k=1}^{n} A_{n,k}(x)(g^{(k)} \circ f)(x)$ with $A_{n,k}$ as polynom of in the functions $f', f^{(2)}, \ldots, f^{(n+1-k)}$

5.4 Power Series and Taylor approximation

Thm. Let $f_n :]a, b[$ be a function sequence with f_n one time in $]a, b[$ continuous diff. $\forall n \geq 1$. Assume that $(f_n)_{n \geq 1}$, $(f_n')_{n \geq 1}$ uniformly convergent in $]a, b[$ with $\lim_{n \to \infty} f_n =: f$ and $\lim_{n \to \infty} f_n' =: p$, then f is continuously diff. and $f' = p$.

Thm. Let $\sum_{k=0}^{\infty}$ be a power series with convergent radius $\rho > 0$. Then $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ is differentiable on $]x_0 - \rho, x_0 + \rho[$ and $\forall x \in]x_0 - \rho, x_0 + \rho[$:
$f'(x) = \sum_{k=0}^{\infty} k c_k (x - x_0)^{k-1}$

Cor. *$f^{(j)} = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!}(x - x_0)^{k-j}$. Furthermore $c_j = \frac{f^{(j)}(x_0)}{j!}$. Power series can be differentiated part by part in their converge area.*

Def (Taylor Polynomial)**.** The n-th Taylor-polynomial of cont. $n + 1$ times diff. in $[c, d]$ f is defined as $T_n(f, x, a)$ with center $a \in]c, d[$ and error $R_n(f, x, a)$.
$\forall x \in [a, b] \; \exists \xi \in]x, a[\cup]a, x[$:

$$T_n(f, x, a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (x - a)^k$$
$$R_n(f, x, a) = \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1}$$
$$f(x) = T_n(f, x, a) + R_n(f, x, a)$$

Hint. The error can be approximated as
$$|R_n(f, x, a)| \leq \sup_{a<c<x} \left| \frac{f^{(n+1)}(x)(x-a)^{n+1}}{(n + 1)!} \right|$$

Def (Taylor Series)**.** $T_\infty(f, x, x_0) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$

6 Riemann Integral

$a < b$, $I = [a, b]$, $\mathcal{P}(I) = \{P \mid P \subsetneq I \wedge \{a, b\} \in P \wedge |P| \in \mathbb{N}\}$

Def (Partition).
- Partition: $P \in \mathcal{P}(I)$
- $\delta_i := x_i - x_{i-1}$ length of $I_i := [x_{i-1}, x_i]$, $i \geq 1$
- Mesh of partition: $\delta(P) := \max_{1 \leq i \leq n}(x_i, x_{i-1})$
- $\xi := \{\xi_1, \ldots, \xi_n\}$, $\xi_i \in I_i$
- P' refines P if $P \subset P'$

Def (Riemann Sums). $S(f, P, \xi) := \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$
- Lower sum: $\underline{S}(f, P) := \sum_{i=1}^n (\inf_{x \in I_i} f(x))(x_i - x_{i-1})$
- Upper sum: $\overline{S}(f, P) := \sum_{i=1}^n (\sup_{x \in I_i} f(x))(x_i - x_{i-1})$

It holds: $-M(b-a) \leq \underline{S}(f, P) \leq \overline{S}(f, P) \leq M(b-a)$

Lemma. $P \subset P'$: $\underline{S}(f, P) \leq \underline{S}(f, P') \leq \overline{S}(f, P') \leq \overline{S}(f, P)$

Lemma. $\forall P_1, P_2 \in \mathcal{P}(I) : \underline{S}(f, P_1) \leq \overline{S}(f, P_2)$

Def (Lower Riemann Integral). $\underline{S}(f) := \sup_{P \in \mathcal{P}(I)} \underline{S}(f, P)$

Def (Upper Riemann Integral). $\overline{S}(f) := \inf_{P \in \mathcal{P}(I)} \overline{S}(f, P)$

6.1 Integrability criteria

Def (Integrable). Bounded $f : [a, b] \to \mathbb{R}$ is integrable if $\underline{S}(f) = \overline{S}(f)$ and the shared value is $\int_a^b f(x)\, dx$.

Riemann Criteria. Bounded $f : I \to \mathbb{R}$ is integrable. Let $\mathcal{P}_\delta(I) := \{P \in \mathcal{P}(I) \mid \delta(P) < \delta\}$.

$$\Leftrightarrow \quad \forall \epsilon > 0\ \exists P \in \mathcal{P}(I) : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon$$

$$\Leftrightarrow \quad \forall \epsilon > 0\ \exists \delta > 0\ \forall P \in \mathcal{P}_\delta(I) : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon$$

$$\Leftrightarrow \quad \forall \epsilon > 0\ \exists \delta > 0\ \forall P \in \mathcal{P} \text{ with } \delta(P) < \delta :$$

$$\left| A - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon$$

Hint. Bounded $f : [a, b] \to \mathbb{R}$ is int. if $\lim_{\delta(P) \to 0} S(f, P, \xi)$ exists for all P with $\delta(P) \to 0$. It follows that $\lim_{\delta(P) \to 0} S(f, P, \xi) = \int_a^b f(x)\, dx$

6.2 Integrable Functions

1. f (bounded) cont. in $[a, b] \implies f$ int. over $[a, b]$
2. f monotone in $[a, b] \implies f$ int. over $[a, b]$
3. If f, g bounded and int., then integrable as well:
$$f + g, \lambda \cdot f, f \cdot g, |f|, \min(f, g), \max(f, g), \frac{f}{g}$$
4. All polynomials are integrable, even $\frac{P(x)}{Q(x)}$ if $Q(x)$ has no root in $[a, b]$

Hint. Let $V := \{f : I \to \mathbb{R} \mid f \text{ is a mapping}\}$. $(V, +, \cdot)$ is a vector space. Then it implies that $W := \{f : I \to \mathbb{R} \mid f \text{ is integrable}\}$ is a subspace of V.

6.3 Integration Inequalities and Theorems

Def (Uniform continuous).

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Thm. $f : [a, b] \to \mathbb{R}$ cont. $\implies f$ is uni. cont. in $[a, b]$.

Thm. f uni. cont. $\implies f$ cont.

Thm. $f, g : [a, b] \to \mathbb{R}$ bounded and integrable and $\forall x \in [a, b] : f(x) \leq g(x) \implies \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$

Cauchy-Schwarz.

$$\left| \int_a^b f(x) g(x)\, dx \right| \leq \sqrt{\int_a^b f^2(x)\, dx} \sqrt{\int_a^b g^2(x)\, dx}$$

Hint. $< f, g > := \int_a^b f(x) g(x)\, dx$ is a scalar product. $||f||^2 = < f, f > = \int_a^b f^2(x)\, dx$.

Mean Value Theorem. $f : [a, b] \to \mathbb{R}$ continuous $\implies \exists \xi \in [a, b] : \int_a^b f(x)\, dx = f(\xi)(b - a)$

Cauchy. $f, g : [a, b] \to \mathbb{R}$ with f continuous and g bounded and integrable with $g(x) \geq 0$, $\forall x \in [a, b]$

$$\implies \exists \xi \in [a, b] : \int_a^b f(x) g(x)\, dx = f(\xi) \int_a^b g(x)\, dx$$

6.4 Integration Properties

Additive Property.

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Linearity.

$$\int_a^b (\alpha f_1 + \beta f_2)\, dx = \alpha \int_a^b f_1(x)\, dx + \beta \int_a^b f_2(x)\, dx$$

Preservation of Order.

$$\forall x \in [a, b] : f(x) \leq g(x) \implies \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

Triangle Inequality.

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

6.5 Primitive Functions

Def (Primitive Function). $F : [a, b] \to \mathbb{R}$ is a primitive function of f if F is cont. diff. and $F' = f$.

Hint. f is integrable $\not\Rightarrow$ exists a primitive function for f.

HID. Let $a < b$, $f : [a, b] \to \mathbb{R}$ continuous. The function

$$F(x) := \int_a^x f(t) dt \quad a \leq x \leq b$$

is cont. diff. in $[a, b]$ and $F'(x) = f(x)\ \forall x \in [a, b]$.

Fundamental theorem of calculus. $f : [a, b] \to \mathbb{R}$ continuous. Then there exists a unique (except a constant term) primitive function F of f, such that

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

6.6 Integration Methods

Partial Inegration.

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

$$\int_a^b f(x)g'(x)\,dx = (f\cdot g)|_a^b - \int_a^b f'(x)g(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

- Choose g': exp $\to$ trig $\to$ poly $\to$ inverse trig. $\to$ logs
- Choose f: logs $\to$ inverse trig. $\to$ poly $\to$ trig $\to$ exp
- Sometimes it is necessary to multiply by 1. E.g.: $\int \ln x \;\; dx = \int \ln x \cdot 1 \;\; dx \implies f(x) = \ln x, \;\; g'(x) = 1$.
- Sometimes it is necessary to do it multiple times

Substitution. Let $a < b$, $\phi : [a,b] \to \mathbb{R}$, cont. diff, $I \subseteq \mathbb{R}$ with $\phi([a,b]) \subseteq I$ and $f : I \to \mathbb{R}$ a cont. function. Then it follows:

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\phi'(t)dt = (F\circ\phi)(b) - (F\circ\phi)(a)$$

since $F' = f$ then $f(\phi(t))\phi'(t) = (F\circ\phi)'(t)$.

Partial Fraction Decomposition. Let $P(x)$, $Q(x)$ be two polynomials. $\int \frac{P(x)}{Q(x)}$ can be calculated as follows:

1. If $\deg(P) \geq Q(P)$ use polynomial division to get
$$\frac{P(x)}{Q(x)} = S(x) + \frac{\hat{P}(x)}{Q(x)}$$
2. Calculate all roots of $Q(x)$
3. Create a partial fraction per root
    - Simple real root: $x_1 \to \frac{A}{x-x_1}$
    - n-fold real root: $x_1 \to \frac{A_1}{x-x_1} + \ldots + \frac{A_r}{(x-x_1)^r}$
4. Calculate parameters $A_1, \ldots, A_n$. (Insert the root as s, transform and solve)

Hint (Odd functions). $\int_{-\lambda}^{\lambda} f(x)\,dx = 0$.

Cor. $\int_{a+c}^{b+c} f(x)\,dx = \int_a^b f(t+c)dt$

Cor. $\int_a^b f(ct)dt = \frac{1}{c}\int_{ac}^{bc} f(x)\,dx$

6.7 Integration of convergent series

Thm. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of bounded, integrable functions which converge uniformly against a function $f : [a,b] \to \mathbb{R}$. Then f is bounded and integrable and

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

$$\sum_{n=0}^{\infty}\int_a^b f_n(x)\,dx = \int_a^b \left(\sum_{n=0}^{\infty} f_n(x)\,dx\right)$$

Thm. Let $f(x) := \sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $p > 0$. Then $\forall 0 \leq r < p$ f is integrable on $[-r,r]$ and $\forall x \in]-p,p[$: $\int\limits_0^x f(t)dt = \sum\limits_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$

6.8 Improper Integral

$$f : [a,\infty) \to \mathbb{R}, f : [-\infty,a] \to \mathbb{R}, f : (-\infty,\infty) \to \mathbb{R}$$

Def. Let $f : [a,\infty[\to \mathbb{R}$ be bounded and integrable on $[a,b]$ for all $b > a$. If $\lim_{b\to\infty}\int_a^b f(x)\,dx$ exists, the limit is defined as $\int_a^\infty f(x)\,dx$ and one can say that f is integrable on $[a,+\infty[$. If the limit does not exists, one can say that $\int_a^\infty f(x)\,dx$ diverges.

Comparison Theorem. Let $f : [a,\infty[\to \mathbb{R}$ be bounded and integrable on $[a,b]$ $\forall b \in \mathbb{R}, b > a$.

1. If $\forall x \geq a : |f(x)| \leq g(x)$ and $g(x)$ is integrable on $[a,\infty[\implies f$ is integrable on $[a,\infty[$.
2. If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)\,dx$ diverges $\implies$ $\int_a^\infty f(x)\,dx$ diverges.

Hint. Sometimes an integral can be split into a normal integral and an improper integral:

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

McLaurin. Let $f : [1,\infty[\to [0,\infty[$ be mon. dec.

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^\infty f(x)\,dx \text{ converges.}$$

The following holds:

$$0 \leq \sum_{k=1}^{\infty} f(k) - \int_1^\infty f(x)\,dx \leq f(1)$$

Def. Let f be a function which is bounded and integrable on all intervals $[a+\epsilon, b]$ $\forall \epsilon > 0$. $f :]a,b] \to \mathbb{R}$ is integrable if $\lim\limits_{\epsilon\to 0}\int_{a+\epsilon}^b f(x)\,dx$ exists. In this case the limit is defined as $\int_a^b f(x)\,dx$. (The comparison theorem can be used for such integrals as well.)

6.9 Indefinite Integrals

Let $f : I \to \mathbb{R}$ be defined on the interval $I \subseteq \mathbb{R}$. If f is continuous there exists a primitive function F.

$$\int f(x)\,dx = F(x) + C$$

The indefinite integral is the inverse of the derivative.

Hint. $\int_{-\infty}^\infty f(x)\,dx = \lim\limits_{b\to-\infty}\int_b^a f(x)\,dx + \lim\limits_{c\to\infty}\int_a^c f(x)\,dx$.

$$\int_{-\infty}^\infty \text{conv.} \iff \int_{-\infty}^a f(x)\,dx \text{ conv.} \wedge \int_a^\infty f(x)\,dx \text{ conv.}$$

In general: Let $f :]a,b[\to \mathbb{R}$ such that it is integrable on each compact interval $[\tilde{a},\tilde{b}]$. Then

$$\int_a^b f(x)\,dx := \lim_{\tilde{a}\searrow a}\lim_{\tilde{b}\nearrow b}\int_{\tilde{a}}^{\tilde{b}} f(x)\,dx$$

6.10 Euler Gamma Function

Def. For $s > 0$:

$$\Gamma(s) := \int_0^\infty e^{-x}x^{s-1}\,dx.$$

The gamma function interpolates the function $n \mapsto (n-1)!$. It converges for all $s > 0$.

Useful Listings

Limits

$\lim_{x\to\infty} \frac{1}{x} = 0$	$\lim_{x\to\infty} 1 + \frac{1}{x} = 1$
$\lim_{x\to\infty} e^x = \infty$	$\lim_{x\to-\infty} e^x = 0$
$\lim_{x\to\infty} e^{-x} = 0$	$\lim_{x\to-\infty} e^{-x} = \infty$
$\lim_{x\to\infty} \frac{e^x}{x^m} = \infty$	$\lim_{x\to-\infty} xe^x = 0$
$\lim_{x\to\infty} \ln(x) = \infty$	$\lim_{x\to 0} \ln(x) = -\infty$
$\lim_{x\to\infty} (1+x)^{\frac{1}{x}} = 1$	$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$
$\lim_{x\to\infty} (1 \pm \frac{1}{x})^\lambda = 1$	$\lim_{x\to\infty} (1+\frac{\lambda}{x})^x = e^\lambda$
$\lim_{x\to\infty} x^\lambda q^x = 0, \forall 0 \leq q < 1$	$\lim_{x\to\infty} \sqrt[x]{x} = 1$
$\lim_{x\to\pm\infty} (1+\frac{1}{x})^x = e$	$\lim_{x\to\infty} (1-\frac{1}{x})^x = \frac{1}{e}$
$\lim_{x\to\pm\infty} (1+\frac{\lambda}{x})^{\alpha x} = e^{\lambda\alpha}$	$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
$\lim_{x\to 0} \frac{1}{\cos(x)} = 1$	$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0$
$\lim_{x\to 0} \frac{\log(1)-x}{x} = -1$	$\lim_{x\to 0} x \log x = 0$
$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$	$\lim_{x\to 0} \frac{e^x-1}{x} = 1$
$\lim_{x\to 0} \frac{x}{\arctan(x)} = 1$	$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$
$\lim_{x\to\infty} (\frac{x}{x+\lambda})^x = e^{-\lambda}$	$\lim_{x\to 0} \frac{e^x-1}{x} = 1$
$\lim_{x\to 0} \frac{\lambda^x-1}{x} = \ln(\lambda), \lambda > 0$	$\lim_{x\to 0} \frac{e^{\lambda x}-1}{x} = \lambda$
$\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$	$\lim_{x\to 1} \frac{\ln(x)}{x-1} = 1$
$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$	$\lim_{x\to\infty} \frac{\log(x)}{x^\lambda} = 0$
$\lim_{x\to\infty} \frac{\lambda x}{\lambda^x} = 0$	$\lim_{x\to\frac{\pi}{2}^-} \tan(x) = +\infty$
$\lim_{x\to\frac{\pi}{2}^+} \tan(x) = -\infty$	$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$
$\lim_{x\to 0^+} x \ln x = 0$	Strirling Formula
	$\lim_{x\to\infty} \frac{x!}{(\frac{x}{e})^x\sqrt{2\pi x}} = 1$

Series

- Geometric: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $|q| < 1$
- Harmonic: $\sum_{n=1}^{\infty} \frac{1}{k}$ diverges
- Telescope: $\sum_{n=0}^{\infty} \frac{1}{k(k+1)} = 1$
- $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n\to\infty}(1+\frac{z}{n})^n = e^z$
- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges $s > 1$ $(\frac{1}{1-\frac{1}{2^{s-1}}})$
- $\mathrm{p}(z) = \sum_{k=0}^{\infty} c_k z^k$ conv. abs. $|z| < \rho = \frac{1}{\limsup |c_k|^{1/k}}$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$	$\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Taylor Polynomials

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \mathcal{O}(x^7)$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \mathcal{O}(x^6)$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6)$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7)$$

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \mathcal{O}(x^7)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \mathcal{O}(x^4)$$

Parity of Functions

Even: $f(-x) = f(x) \quad \forall x \in D$ $\qquad |x|,\ \cos x,\ x^2$

Odd: $f(-x) = -f(x) \quad \forall x \in D$ $\qquad x, \sin, \tan, x^3$

Hint. Chaining odd functions results in an odd function.

Common Derivatives and Integrals

$\mathbf{F(x)}$	$\mathbf{f(x)}$
c	0
x^a	$a \cdot x^{a-1}$
$\frac{1}{a+1}x^{a+1}$	x^a
$\frac{1}{a\cdot(n+1)}(ax+b)^{n+1}$	$(ax+b)^n$
$\frac{x^{a+1}}{a+1}$	$x^a,\ a \neq -1$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n}x^{\frac{1}{n}-1}$
$\frac{2}{3}x^{\frac{3}{2}}$	$\sqrt{x}$
$\frac{n}{n+1}x^{\frac{1}{n}+1}$	$\sqrt[n]{x}$
e^x	e^x
$\ln(\|x\|)$	$\frac{1}{x}$
$\log_a(\|x\|)$	$\frac{1}{x\ln(a)} = \log_a(e^{\frac{1}{x}})$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x) = \frac{\cos(x)}{\sin(x)}$	$\frac{1}{-\sin^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\sinh(x) = \frac{e^x+e^{-x}}{2}$	$\cosh(x)$
$\cosh(x) = \frac{e^x-e^{-x}}{2}$	$\sinh(x)$
$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\frac{1}{f(x)}$	$\frac{-f'(x)}{(f(x))^2}$
a^{cx}	$a^{ck} \cdot c\ln(a)$
x^x	$x^x \cdot (1 + \ln(x)),\ x > 0$
$(x^x)^x$	$(x^x)^x(x + 2x\ln(x)),\ x > 0$
x^{x^x}	$x^{x^x}(x^{x-1} + \ln(x) \cdot x^x(1 + \ln(x)))$