Analysis I Summary

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Real numbers, euclidean spaces

Archimedes' principle. If $x \in \mathbb{R}$ with x > 0 and $y \in \mathbb{R}$, then $\exists n \in \mathbb{N} \ (y < n \cdot x)$

(i) $|x| < 0 \quad \forall x \in \mathbb{R}$ Thm.

- (ii) $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$
- (iii) $|x+y| < |x| + |y| \quad \forall x, y \in \mathbb{R}$
- (iv) $|x+y| \ge ||x| |y|| \quad \forall x, y \in \mathbb{R}$

Young's inequality. $\forall \epsilon > 0, \forall x, y \in \mathbb{R}$:

$$2|xy| \le \epsilon x^2 + \frac{1}{\epsilon}y^2$$

Sequences

Convergence

 $(a_n)_{n\geq 1}$ converges to $L=\lim_{n\to\infty}a_n\iff \forall \epsilon>0\ \exists N\in$ $\mathbb{N} \ \forall n > N \ (|a_n - L| < \epsilon)$

Def (Convergence). $(a_n)_{n\geq 1}$ converges $\iff \exists L\in\mathbb{R}\ \forall \epsilon > 1$ $0 (\{n \in \mathbb{N} \mid |a_n - L| \ge \epsilon\})$ is finite.

Hint. Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ converge with limit a and b:

- 1. $(a_n + b_n)_{n \ge 1}$ converges with limit a + b
- 2. $(a_n \cdot b_n)_{n \ge 1}$ converges with limit $a \cdot b$.
- 3. $(\frac{a_n}{b})_{n>1}$ converges with limit $\frac{a}{b}$
- 4. $\exists K \geq 1 \ \forall n \geq K : a_n \leq b_n \implies a \leq b$

2.1.1 Tips & Tricks

- a_n convergent $\implies a_n$ bounded
- a_n convergent \iff a_n bounded and $\liminf a_n =$ $\limsup a_n$

Monotone Convergence. $(a_n)_{n\geq 1}$ monotone increasing and upper bounded $\implies \lim a_n = \sup\{a_n \mid n \ge 1\}$ $(a_n)_{n\geq 1}$ monotone decreasing and lower bounded \Longrightarrow $\lim a_n = \inf\{a_n \mid n \ge 1\}$

Lemma (Bernoulli Inequation).

$$(1+x)^n \ge 1 + nx \quad \forall n \in \mathbb{N}, x > -1$$

Def.

$$\liminf_{n \to \infty} a_n := \lim_{n \to \infty} (\inf\{a_k \mid k \ge n\})$$

$$\limsup_{n \to \infty} a_n := \lim_{n \to \infty} (\sup\{a_k \mid k \ge n\})$$

Cauchy Criteria. a_n converges iff $\forall \epsilon > 0 \ \exists N \geq 1 \ \text{s.t.}$ $|a_n - a_m| < \epsilon \ \forall n, m \ge N$ (cauchy sequence).

- (i) Each Cauchy sequence is bounded
- (ii) $(a_n)_{n\geq 1}$ conv. $\Longrightarrow (a_n)_{n\geq 1}$ cauchy
- (iii) $(a_n)_{n>1}$ cauchy $\implies (a_n)_{n>1}$ conv.

Bolzano-Weierstrass. Each bounded sequence contains a convergent sub sequence.

Sandwich. If $\lim a_n = \alpha$, $\lim b_n = \alpha$, $k \in \mathbb{N}$ and $a_n \leq \infty$ $c_n \leq b_n \ \forall n \geq k$, then $\lim c_n = \alpha$

Cauchy-Cantor. Let $I_1 \supseteq I_2 \subseteq \cdots I_n \cdots$ be a sequence of proper intervals with $\mathcal{L} < +\infty$, then $\bigcap_{n>1} I_n \neq 0$. And if $\lim_{n\to\infty} \mathcal{L}(I_n) = 0$ then $|\bigcap_{n>1} I_n| = 1$

Cor. Let (a_n) be bounded, then for each subsequence (b_n) : $\lim \inf a_n \leq \lim b_n \leq \lim \sup a_n$.

Each subsequence (b_n) of a convergent (a_n) converges and $\lim b_n = \lim a_n$.

Series

Def. " $\sum_{k=1}^{\infty} a_k$ " converges, if the sequence $(S_n)_{n\geq 1}$ of partial sums converges and $\sum_{k=1}^{\infty} a_k := \lim_{n\to\infty} S_n$

- **Thm.** $\sum_{k=1}^{\infty} a_k$ and $\sum_{j=1}^{\infty} b_j$ convergent: $\sum_{k=1}^{\infty} (a_k + b_k) = (\sum_{k=1}^{\infty} a_k) + (\sum_{k=1}^{\infty} b_k)$
 - $\sum_{k=1}^{\infty} \alpha \cdot a_k = \alpha \sum_{k=1}^{\infty} a_k$

Couchy Criteria. $\sum_{k=1}^{\infty} a_k$ conv. $\iff \forall \epsilon > 0 \ \exists N \geq 1$: $\left|\sum_{k=n}^{m} a_k\right| < \epsilon \ \forall m \ge n \ge N$

Zero Sequence Criteria. $\sum_{k=1}^{\infty} a_k \text{ conv.} \implies \lim a_k = 0$

Comparison Theorem. Let $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ series s.t. $0 < a_k < b_k \quad \forall k > 1$:

$$\sum_{k=1}^{\infty} b_k \text{ converges } \Longrightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges } \Longrightarrow \sum_{k=1}^{\infty} b_k \text{ diverges}$$

Thm. A series converges if the sequence of partial sums $(s_n)_{n\geq 1}$ converges. $s_n:=\sum_{k=1}^n a_k$

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \sum_{k=1}^n a_k =: \sum_{k=1}^\infty a_k$$

Def (Asolute Convergence). $\sum_{k=1}^{\infty} a_k$ absolute converges if $\sum_{k=1}^{\infty} |a_k|$ converges.

$$\sum_{k=1}^{\infty} |a_k| \text{ converges } \Longrightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \Rightarrow \sum_{k=1}^{\infty} |a_k| \text{ converges}$$

(**Dirichlet**) If a series converges absolute, then each permutation of the series converges with the same limit. (Riemann) If a series only converges, then there exists a permutation such that:

$$\sum_{k=1}^{\infty} a_{\phi(k)} = x \quad \forall x \in \mathbb{R} \cup \{\infty\}$$

Thm.

$$\left| \sum_{k=1}^{\infty} a_k \right| \le \sum_{k=1}^{\infty} |a_k|$$

Leibniz. $(a_n)_{n\geq 1}$ monotone decreasing s.t. $a_n\geq 1 \quad \forall n\geq 1$ and $\lim a_n=0$:

$$S := \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges.}$$

Furthermore: $a_1 - a_2 \le S \le a_1$

Ratio Test. Let $(a_n)_{n\geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$: $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolute. $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges

Lemma. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolute.
- $L > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.
- $L = 1 \implies no information$

Root Test. Let $(a_n)_{n\geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$: $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolute. $\liminf_{n\to\infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.

Lemma. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolute.
- $L > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges.
- $L = 1 \implies no information$

Def (Cauchy Product). $\sum_{i=0}^{\infty} a_i$, $\sum_{j=0}^{\infty} b_j$:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{n-j} b_j \right) = a_0 b_0 + \left(a_0 b_1 + a_1 b_0 \right) + \cdots$$

Thm. If $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$ converge absolute:

$$\sum_{n=0}^{\infty} (\sum_{j=0}^{\infty} a_{n-j} b_j) = (\sum_{i=0}^{\infty} a_i) (\sum_{j=0}^{\infty} b_j) = \sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j$$

3.1 Strategy: Convergence of Series

- 1. Check for known types (Telescope, Geometric, etc.)
- 2. $\lim a_n \neq 0 \implies \text{divergence}$

- 3. Ratio Test
- 4. Root Test
- 5. Search convergent majors: $0 \le a_n \le b_n$
- 6. If divergent minors \implies divergence
- 7. Be creative

4 Functions

 $\mathbb{R}^D = \{ f : D \to \mathbb{R} \mid f \text{ is function} \}$

4.1 Continuity

Def (Continuity). A function f is continuous in x_0 if:

 $\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \in D \; (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon)$

 \Leftrightarrow

 $\forall (a_n)_{n\geq 1} \text{ with } \lim a_n = x_0 \text{ holds } \lim f(a_n) = f(\lim a_n)$

Def. A function $f: D \to \mathbb{R}$ is continuous if it is continuous in all $x_0 \in D$

Hint. To prove continuity try to filter $|x-x_0|$ out of $|f(x)-f(x_0)|$ and choose δ , such that the rest term disappears. Be aware that δ is part of ϵ and normally $|x_0|$ as well. But no x!

Cor. $f, g: D \to \mathbb{R}$ continuous in $x_0 \in D$. Then:

- $fg, \lambda f, f \pm g$ continuous in x_0
- $\frac{f}{g}: D \setminus \{x \in D \mid g(x) = 0\} \to \mathbb{R} \text{ continuous in } x_0 \text{ (if } g(x_0) \neq 0)$
- $|f|, \max(f, g), \min(f, g)$ continuous in x_0
- $P(x) = a_n x^n + \dots + a_0$ continuous on \mathbb{R}
- $\frac{P(x)}{Q(x)}$ continuous on $\mathbb{R}\setminus\{x_1,\dots,x_m\}$ if x_1,\dots,x_m are roots of Q(x)

Thm. Let $f: D_1 \to D2 \subset \mathbb{R}, g: D_2 \to \mathbb{R}$ be continuous $\Rightarrow g \circ f: D_1 \to \mathbb{R}$ continuous.

Bolzano (Intermediate value theorem). Let $I \subseteq \mathbb{R}$, $f: I \to \mathbb{R}$ and $a, b \in I$. For each c between f(a) and f(b) there is a $z \in [a, b]$ with f(z) = c

Min-Max. Let $f: I = [a, b] \to \mathbb{R}$ be continuous.

$$\exists u, v \in I \ \forall x \in I \ (f(u) \le f(x) \le f(v))$$

In particular $f([a,b]) \subset [f(u),f(v)]$ is bounded.

Cor. $I = [a,b], f: I \to \mathbb{R}$ continuous, then $\Im(f) = f(I)$ is a compact interval $J = [\min f, \max f] = [f(u), f(v)]$

Inverse Mapping. Let $f: I \to \mathbb{R}$ be continuous and strict monotone increasing. Then $J:=f(I)\subseteq \mathbb{R}$ is an interval and $f^{-1}: J \to I$ is continuous and strict monotone.

4.2 Exponential function

 $\exp: \mathbb{R} \to]0, \infty[$ is continuous, strictly monotone increasing, surjective.

- $\exp(x) \ge 1 + x \quad \forall x \in \mathbb{R}$
- For $x > 0, a \in \mathbb{R} : x^a := \exp(a \ln x)$
- $x^0 = 1 \quad \forall x > 0$

Def. The inverse mapping of $\exp(x)$ is called the natural logarithm:

$$\ln :]0, \infty[\to \mathbb{R}, \quad x \mapsto \ln x$$

It is strictly monotone increasing, continuous and bijective.

4.3 Converge of function sequences

$$\mathbb{N} \to \mathbb{R}^D = \{ f : D \to \mathbb{R} \}, \quad n \mapsto f(n)$$

Def (pointwise convergence). $(f_n)_{n\geq 0}$ converges pointwise to a function $f: D \to \mathbb{R}$, if $\forall x \in D: \lim_{n\to\infty} f_n(x) = f(x)$

 \iff

$$\forall x \in D \ \forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n \ge N \ (|f_n(x) - f(x)| < \epsilon)$$

Def (uniform convergence (Weierstrass)). $f_n: D \to \mathbb{R}$ converges uniformly in D to $f: D \to \mathbb{R}$ if:

$$\forall \epsilon > 0 \ \exists N \ge 1 \ \text{s.t.} \ \forall n \ge N \ \forall x \in D \ (|f_n(x) - f(x)| < \epsilon)$$

 $\lim_{n \to \infty} \sup_{x \in D} |f_n(x) - f(x)| = 0$

The function sequence (f_n) converges uniformly if for all $x \in D$ the limit $\lim_{n\to\infty} f_n(x) = f(x)$ exists and the sequence (f_n) uniformly converges to f.

The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly (in D), if the function sequence $S_n(x) := \sum_{k=0}^n f_k(x)$ converges uniformly.

Thm. let $D \subseteq \mathbb{R}$ and $f_n : D \to \mathbb{R}$ a function sequence containing (in D) continuous functions which converge (in D) uniformly against a function $f : D \to \mathbb{R}$, then f (in D) is continuous.

Hint (not uniform convergent). $(f_n)_{n\geq 0}$ converges not uniformly if:

$$\forall \epsilon > 0 \ \forall N \in \mathbb{N} \ \exists x \in D(|f_n(x) - f(x)| \ge \epsilon)$$

Hint. Check the function and try to construct x (dependent on N in general), such that $|f_n(x) - f(x)|$ is always greater than a specific ϵ and afterwards choose the ϵ .

Def (Power Functions). $\sum_{k=0}^{\infty} c_k x^k$ has positive convergence radius if \limsup

$$\rho = \begin{cases} +\infty & \text{, if } \limsup_{k \to \infty} \sqrt[k]{|c_k|} = 0\\ \frac{1}{\limsup_{k \to \infty} \sqrt[k]{|c_k|}} & \text{, if } \limsup_{k \to \infty} \sqrt[k]{|c_k|} > 0 \end{cases}$$

Thm. Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $\rho > 0$ and let $f(x) = \sum_{k=0}^{\infty} c_k x^k, |x| < \rho$ Then: $\forall 0 \leq r < \rho$ converges $\sum_{k=0}^{\infty} c_k x^k$ uniformly on [-r, r], furthermore $f:]-\rho, \rho[\to \mathbb{R}$ is continuous.

4.4 Trigonometric Functions

sin and cos are continuous functions $\mathbb{R} \to \mathbb{R}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Thm. 1. $\exp(iz) = \cos(z) + i\sin(z) \quad \forall z \in \mathbb{C}$

2.
$$\cos z = \cos(-z)$$
 und $\sin(-z) = -\sin(z) \quad \forall z \in \mathbb{C}$

3.
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

4.
$$\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$
$$\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

5.
$$\cos(z)^2 + \sin(z)^2 = 1 \quad \forall z \in \mathbb{C}$$

Cor.

$$\sin(2z) = 2\sin(z)\cos(z)$$
$$\cos(2z) = \cos(z)^2 - \sin(z)^2$$
$$\sin(x) - \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Def (π) . $\pi := \inf\{t > 0 \mid \sin t = 0\}$

- (i) $\sin \pi = 0, \pi \in]2, 4[$
- (ii) $\forall x \in]0, \pi[: \sin x > 0]$
- (iii) $e^{\frac{i\pi}{2}} = i$

Cor. $x \ge \sin x \ge x - \frac{x^3}{3!} \quad \forall 0 \le x \le \sqrt{6}$

Cor. 1.
$$e^{i\pi} = -1$$
, $e^{2i\pi} = 1$

2.
$$\sin(x + \frac{\pi}{2}) = \cos(x)$$
, $\cos(1 + \frac{\pi}{2}) = -\sin(x) \quad \forall x \in \mathbb{R}$

3.
$$\sin(x+\pi) = -\sin(x)$$
, $\sin(x+2\pi) = \sin(x)$ $\forall x \in \mathbb{R}$

4.
$$\cos(x+\pi) = -\cos(x)$$
, $\cos(x+2\pi) = \cos(x)$ $\forall x \in \mathbb{R}$

5. Roots of sinus =
$$\{\pi \cdot k \mid k \in \mathbb{Z}\}\$$

 $\sin(x) > 0 \ \forall x \in]2k\pi, (2k+1)\pi[, k \in \mathbb{Z}]$
 $\sin(x) < 0 \ \forall x \in](2k+1)\pi, (2k+2)\pi[, k \in \mathbb{Z}]$

6. Roots of $cosine = \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}\$ $cos(x) > 0 \ \forall x \in]-\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + (2k+1)\pi[, \quad k \in \mathbb{Z}$ $cos(x) < 0 \ \forall x \in]-\frac{\pi}{2} + (2k+1)\pi, -\frac{\pi}{2} + (2k+2)\pi[, \ k \in \mathbb{Z}$

4.5 Limit of Functions

Def (accumulation point). $x_0 \in \mathbb{R}$ is an accumulation point of D if $\forall \delta > 0$: $(|x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) \cap D \neq \emptyset$

Def (Limit of Function). if $f: D \to \mathbb{R}, x_0 \in \mathbb{R}$ an accumulation point of D, then $A \in \mathbb{R}$ is the limit of f(x) for $x \to x_0$, written as $\lim_{x \to x_0} f(x) = A$. If $\forall \epsilon > 0 \ \exists \delta > 0 \ \text{s.t.}$ $\forall x \in D \cap (|x_0 - \delta, x_0 + \delta| \setminus \{x_0\}) : |f(x) - A| < \epsilon$

Important Rules. Let $f: D \to \mathbb{R}$ and x_0 is an accumulation point of D.

1.
$$\lim_{x\to x_0} f(x) = A \iff \forall (a_n)_{n\geq 1} \text{ in } D\setminus \{x_0\} \text{ with }$$

$$\lim_{n\to\infty} a_n = x_0 \implies \lim_{n\to\infty} f(a_n) = A.$$

- 2. Let $x_0 \in D$. Then f is continuous in $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0)$
- 3. $f, g: D \to \mathbb{R}$ and $\exists \lim_{x \to x_0} f(x), \exists \lim_{x \to x_0} g(x) \Longrightarrow$

$$\lim_{x \to x_0} (f+g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$
$$\lim_{x \to x_0} (f \cdot g)(x) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

4. $f, g: D \to \mathbb{R}$ and $f \leq g$, then if both limit exists

$$\lim_{x \to x_0} f(x) \le \lim_{x \to x_0} g(x)$$

5. If $g_1 \leq f \leq g_2$ and $\lim_{x \to x_0} g_1(x) = \lim_{x \to x_0} g_2(x)$ then $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g_1(x)$

Hint. Sometimes it can be really helpful to convert known functions to their power series to calculate a limit. E.g.

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \dots}{x} = \lim_{x \to 0} 1 - \frac{x^2}{3!} + \dots = 1$$

Hint (e^{\log}) . Transform ugly function with this trick.

$$\lim_{x \to x_0} f(x)^{g(x)} = \lim_{x \to x_0} e^{g(x)\log(f(x))} = e^{\lim_{x \to x_0} g(x)\log(f(x))}$$

5 Differentiable Functions

Def (Differentiable). f is in x_0 differentiable, if the limit $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$ exists. f is differentiable if $\forall x_0 \in D$ f is differentiable.

Weierstrass. $f: D \to \mathbb{R}, x_0 \in D$ accumulation point. Equivalent statements:

- 1. f is in x_0 differentiable
- 2. It exists $c \in R$ $(c = f'(x_0))$ and $r : D \to \mathbb{R}$ s.t.:

2.1
$$f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0)$$

2.2 $r(x_0) = 0$ and r continuous in x_0 .

Cor. f diff. in $x_0 \iff f$ continuous in x_0

Derivative rules.

Linearity: $(\alpha \cdot f(x) + g(x))' = \alpha \cdot f'(x) + g'(x)$ Product rule: $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ Quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$

Chain rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Cor. f bijective and in x_0 differentiable s.t. $f'(x_0) \neq 0$. f^{-1} is continuous in $y_0 = f(x_0)$. Then f^{-1} is differentiable in y_0 and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$.

5.1 Derivative Implications

- 1. x_0 is local minimum if $f'(x_0) = 0 \land f''(x_0) > 0$ or the sign of f' changes from - to +.
- 2. x_0 is local maximum if $f'(x_0) = 0 \land f''(x_0) < 0$ or the sign of f' changes from + to -.
- 3. x_0 is local extremum if $f'(x_0) = 0 \land f''(x_0) \neq 0$
- 4. x_0 is a saddle point if $f'(x_0) = 0$ and $f''(x_0) = 0$
- 5. x_0 is a inflection point if $f''(x_0) = 0$
- 6. $f'(x_0) = f^{(2)}(x_0) = \dots = f^{(n)}(x_0) = 0$
 - 6.1 n odd and $f^{(n+1)}(x_0) > 0 \implies x_0$ strict local minimum
 - 6.2 n odd and $f^{(n+1)}(x_0) < 0 \implies x_0$ strict local maximum

Derivative Theorems

Rolle. Let $f:[a,b]\to\mathbb{R}$ continuous and in [a,b] differentiable. If f(a) = f(b), then there exists $\xi \in]a,b[$ with $f'(\xi) = 0.$

Mean Value / Lagrange. Let $f:[a,b] \to \mathbb{R}$ continuous and in [a, b] differentiable, then there exists $\xi \in [a, b]$ with $f(b) - f(a) = f'(\xi)(b - a).$

There exists points ξ with $f'(\xi)$ equal to the gradient of the secant between a to b.

Cor. Let $f, g : [a, b] \to \mathbb{R}$ continuous and diff. in [a, b].

- 1. $\forall \xi \in]a, b[: f'(\xi) = 0 \implies f \text{ is constant}$
- 2. $\forall \xi \in]a,b[: f'(\xi) = g'(x) \implies \exists c \in \mathbb{R} \forall x \in [a,b]:$ f(x) = g(x) + c

- 3. $\forall \xi \in]a, b[: f'(\xi) > 0 \implies f \text{ in } [a, b] \text{ mon. inc.}$
- 4. $\forall \xi \in]a,b[:f'(\xi) \geq 0 \implies f \text{ in } [a,b] \text{ str. mon. inc.}$
- 5. $\forall \xi \in]a,b[:f'(\xi) < 0 \implies f \text{ in } [a,b] \text{ mon. dec.}$
- 6. $\forall \xi \in]a,b[:f'(\xi)<0 \implies f \text{ in } [a,b] \text{ str. mon. dec.}$
- 7. $\exists M > 0 \ \forall \xi \in]a, b[: |f'(\xi)| < M \implies$ $\forall x_1, x_2 \in [a, b] : |f(x_1) - f(x_2)| < M|x_1 - x_2|$

Cauchy. $f, g, : [a, b] \to \mathbb{R}$ continuous and in [a, b] diff. Then there exists $\xi \in]a, b[$ with $q'(\xi)(f(b) - f(a)) = f'(\xi)(q(b) - f(a))$ q(a)).

If $\forall x \in]a,b[:g'(x) \neq 0$ it implies that $g(a) \neq g(b)$ and

l'Hôspital. $f, g:]a, b[\to \mathbb{R} \text{ diff. with } \forall x \in]a, b[: g'(x) \neq 0.$ If $\lim_{x\to b^-} f(x) = 0$, $\lim_{x\to b^-} g(x) = 0$ and $\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = 0$ λ exists, then $\lim_{x\to b^-} \frac{f(x)}{g(x)} = \lim_{x\to b^-} \frac{f'(x)}{g'(x)}$.

Hint. Only use l'Hospital if either $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

- **Def.** 1. convex: (x < y): $f(\lambda x + (1 \lambda)y) < \lambda f(x) + (1 \lambda)y$ $(1-\lambda)f(y)$
 - 2. strict convex: $(x \le y)$: $f(\lambda x + (1 \lambda)y) < \lambda f(x) + (1 \lambda)y$ $(1-\lambda)f(y)$
 - 3. concave: (x < y): $f(\lambda x + (1 \lambda)y) > \lambda f(x) + (1 \lambda)y$ $\lambda)f(y)$
 - 4. strict concave: $(x \le y)$: $f(\lambda x + (1 \lambda)y) > \lambda f(x) +$ $(1-\lambda)f(y)$

Lemma. $f: I \to \mathbb{R}$. f is convex $\iff \forall x_0 < x < x_1 \in I$: $\frac{f(x)-f(x_0)}{x-x_0} \leq \frac{f(x_1)-f(x)}{x_1-x}$. Strictly convex if <.

Thm. $f:]a, b[\to \mathbb{R} \text{ in }]a, b[\text{ diff. Function is (strictly) convex}]$ if f' is (strictly) monotonically increasing.

5.3 Higher Derivatives

- 1. For n > 2 is f n-times differentiable in D if $f^{(n-1)}$ in D is differentiable. Then $f^{(n)} := (f^{(n-1)})'$ and is the n-th derivative of f
- 2. f is n-times continuous differentiable if f is ntimes differentiable and if $f^{(n)}$ is continuous in D
- 3. f is in D smooth if $\forall n \geq 1$, f is n-times differentiable.

Smooth Functions. exp. sin, cos, sinh, cosh, tanh, ln, arcsin, arccos, arccot, arctan and all polynomials. tan is smooth on $\mathbb{R} \setminus \{\pi/2 + k\pi\}$ and cot on $\mathbb{R} \setminus \{k\pi\}$

Thm. $f, g: D \to \mathbb{R}$ are *n*-times diff. in D.

- 1. $(f+q)^{(n)} = f^{(n)} + q^{(n)}$
- 2. $(f \cdot g)^{(n)} = \sum_{k=0}^{n} {n \choose k} f^{(k)} g^{(n-k)}$
- 3. $(g \circ f)^{(n)}(x) = \sum_{k=1}^{n} A_{n,k}(x)(g^{(k)} \circ f)(x)$ with $A_{n,k}$ as polynom of in the functions $f', f^{(2)}, \dots, f^{(n+1-k)}$

5.4 Power Series and Taylor approximation

Thm. Let $f_n:]a,b[$ be a function sequence with f_n one time in a, b continuous diff. $\forall n \geq 1$. Assume that $(f_n)_{n\geq 1}$ $(f'_n)_{n\geq 1}$ uniformly convergent in]a,b[with $\lim_{n\to\infty}f_n=:f$ and $\lim_{n\to\infty} f'_n =: p$, then f is continuously diff. and f' = p.

Thm. Let $\sum_{k=0}^{\infty}$ be a power series with convergent radius p > 0. Then $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ is differentiable on $]x_0 - p, x_0 + p[$ and $\forall x \in]x_0 - p, x_0 + p[: f'(x) = \sum_{k=0}^{\infty} kc_k(x - y)]$

Cor. $f(j) = \sum_{k=1}^{\infty} c_k \frac{k!}{(k-j)!} (x-x_0)^{k-j}$. Furthermore $c_j =$ $\frac{f(j)(x_0)}{i!}$. Power series can be differentiated part by part in their converge area.

Def (Taylor Polynomial). The *n*-th Taylor-polynomial of cont. n+1 times diff. in [c,d] f is defined as $T_n(f,x,a)$ with center $a \in]c,d[$ and error $R_n(f,x,a)$. $\forall x \in [a,b] \exists \xi \in$ $]x,a[\ \cup\]a,x[\ :$

$$T_n(f, x, a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} \cdot (x - a)^k$$
$$R_n(f, x, a) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$$
$$f(x) = T_n(f, x, a) + R_n(f, x, a)$$

Hint. The error can be approximated as $|R_n(f,x,a)| <$ $\sup_{a < c < x} \left| \frac{f^{(n+1)(x)(x-a)^{n+1}}}{f^{(n+1)(x)(x-a)^{n+1}}} \right|$

Def (Taylor Series). $T_{\infty}(f, x, x_0) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$

Riemann Integral

 $a < b, I = [a, b], \mathcal{P}(I) = \{P \mid P \subsetneq I \land \{a, b\} \in P \land |P| \in \mathbb{N}\}$

Def (Partition). • Partition: $P \in \mathcal{P}(I)$

- $\delta_i := x_i x_{i-1}$ length of $I_i := [x_{i-1}, x_i], i > 1$
- Mesh of partition: $\delta(P) := \max_{1 \le i \le n} (x_i, x_{i-1})$
- $\xi := \{\xi_1, \dots, \xi_n\}, \, \xi_i \in I_i$
- P' refines P if $P \subset P'$

Def (Riemann Sums). $S(f, P, \xi) := \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$ • Lower sum: $\underline{S}(f, P) := \sum_{i=1}^{n} (\inf_{x \in I_i} f(x))(x_i - x_{i-1})$

- Upper sum: $\overline{S}(f,P) := \sum_{i=1}^{n} (\sup_{x \in I_i} f(x))(x_i x_{i-1})$ It holds: $-M(b-a) < S(f,P) < \overline{S}(f,P) < M(b-a)$

Lemma. $P \subset P' : S(f, P) \leq S(f, P') \leq \overline{S}(f, P') \leq \overline{S}(f, P)$

Lemma. $\forall P_1, P_2 \in \mathcal{P}(I) : S(f, P_1) < \overline{S}(f, P_2)$

Def (Lower Riemann Integral). $S(f) := \sup_{P \in \mathcal{P}(I)} S(f, P)$

Def (Upper Riemann Integral). $\overline{S}(f) := \inf_{P \in \mathcal{P}(I)} \overline{S}(f, P)$

6.1 Integrability criteria

Def (Integrable). Bounded $f:[a,b]\to\mathbb{R}$ is integrable if $\underline{S}(f) = \overline{S}(f)$ and the shared value is $\int_a^b f(x) dx$.

Riemann Criteria. Bounded $f: I \to \mathbb{R}$ is integrable. Let $\mathcal{P}_{\delta}(I) := \{ P \in \mathcal{P}(I) \mid \delta(P) < \delta \}.$

- $\Leftrightarrow \forall \epsilon > 0 \; \exists P \in \mathcal{P}(I) : \overline{S}(f, P) S(f, P) < \epsilon$
- $\forall \epsilon > 0 \; \exists \delta > 0 \; \forall P \in \mathcal{P}_{\delta}(I) : \overline{S}(f, P) S(f, P) < \epsilon$
- $\Leftrightarrow \forall \epsilon > 0 \ \exists \delta > 0 \ \forall P \in \mathcal{P} \text{ with } \delta(P) < \delta$:

$$\left| A - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon$$

Hint. Bounded $f:[a,b]\to\mathbb{R}$ is int. if $\lim_{\delta(P)\to 0}S(f,\overline{P},\xi)$ exists for all P with $\delta(P) \to 0$. It follows that $\lim_{\delta(P)\to 0} S(f,P,\xi) = \int_a^b f(x) dx$

Integrable Functions

- 1. f (bounded) cont. in $[a, b] \implies f$ int. over [a, b]
- 2. f monotone in $[a,b] \implies f$ int. over [a,b]
- 3. If f, g bounded and int., then

$$f + g, \lambda \cdot f, f \cdot g, |f|, \max(f, g), \frac{f}{g}$$

are integrable.

4. All polynomials are integrable, even $\frac{P(x)}{Q(x)}$ if Q(x) has no root in [a, b]

Hint. Let $V := \{f : I \to \mathbb{R} \mid f \text{ is a mapping}\}.$ $(V, +, \cdot)$ is a vector space. Then it implies that $W := \{f : I \to I\}$ $\mathbb{R} \mid f$ is integrable} is a subspace of V.

Integration Inequalities and Theorems

Def (Uniform continuous).

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Thm. $f:[a,b]\to\mathbb{R}$ cont. \Longrightarrow f is uni. cont. in [a,b].

Thm. $f,g:[a,b]\to\mathbb{R}$ bounded and integrable and $\forall x\in$ $[a,b]: f(x) \leq g(x) \implies$

$$\int_{a}^{b} f(x) \, dx \le \int_{a}^{b} g(x) \, dx$$

Cauchy-Schwarz.

$$\left| \int_a^b f(x)g(x) \, dx \right| \le \sqrt{\int_a^b f^2(x) \, dx} \sqrt{\int_a^b g^2(x) \, dx}$$

Hint. $\langle f, g \rangle := \int_a^b f(x)g(x) dx$ is a scalar product. $||f||^2 = \langle f, f \rangle = \int_a^b f^2(x) dx.$

Mean Value Theorem. $f:[a,b]\to\mathbb{R}$ continuous \Longrightarrow $\exists \xi \in [a,b]: \int_a^b f(x) dx = f(\xi)(b-a)$

Cauchy. $f, g : [a, b] \to \mathbb{R}$ with f continuous and g bounded

and integrable with $q(x) > 0, \forall x \in [a, b]$

$$\implies \exists \xi \in [a,b]: \int_a^b f(x)g(x)\,dx = f(\xi) \int_a^b g(x)\,dx$$

6.4 Integration Properties

Additive Property.

$$\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$$

Linearity.

$$\int_{a}^{b} (\alpha f_1 + \beta f_2) \, dx = \alpha \int_{a}^{b} f_1(x) \, dx + \beta \int_{a}^{b} f_2(x) \, dx$$

Preservation of Order.

$$\forall x \in [a,b]: f(x) \leq g(x) \implies \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$$

Triangle Inequality.

$$\left| \int_{a}^{b} f(x) \, dx \right| \le \int_{a}^{b} |f(x)| \, dx$$

6.5 Primitive Functions

Def (Primitive Function). $F:[a,b] \to \mathbb{R}$ is a primitive function of f if F is cont. diff. and F' = f.

Hint. f is integrable $\not\Rightarrow$ exists a primitive function for f.

HID. Let $a < b, f : [a, b] \to \mathbb{R}$ continuous. The function

$$F(x) := \int_{a}^{x} f(t)dt \quad a \le x \le b$$

is cont. diff. in [a, b] and $F'(x) = f(x) \ \forall x \in [a, b]$.

Fundamental theorem of calculus. $f:[a,b] \to \mathbb{R}$ continuous. Dann there exists a unique (except a constant term) primitive function F of f, such that

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

6.6 Integration Methods

Partial Inegration.

$$\int_{a}^{b} f(x)g'(x) \, dx = f(b)g(b) - f(a)g(a) - \int_{a}^{b} f'(x)g(x) \, dx$$

$$\int_{a}^{b} f(x)g'(x) \, dx = (f \cdot g)|_{a}^{b} - \int_{a}^{b} f'(x)g(x) \, dx$$

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx$$

- Choose $g': \exp \to \operatorname{trig} \to \operatorname{poly} \to \operatorname{inverse} \operatorname{trig}. \to \operatorname{logs}$
- Choose $f: \log s \to \text{inverse trig.} \to \text{poly} \to \text{trig} \to \exp$
- Sometimes it is necessary to multiply by 1. E.g.: $\int \ln x \ dx = \int \ln x \cdot 1 \ dx \implies f(x) = \ln x, \ g'(x) = 1.$
- Sometimes it is necessary to to it multiple times

Substitution. Let $a < b, \phi : [a, b] \to \mathbb{R}$, cont. diff, $I \subseteq \mathbb{R}$ with $\phi([a, b]) \subseteq I$ and $f : I \to \mathbb{R}$ a cont. function. Then it follows:

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\phi'(t)dt = (F\circ\phi)(b) - (F\circ\phi)(a)$$

since F' = f then $f(\phi(t))\phi'(t) = (F \circ \phi)'(t)$.

Partial Fraction Decomposition. Let P(x), Q(x) be two polynomials. $\int \frac{P(x)}{Q(x)}$ can be calculated as follows:

1. If $deg(P) \ge Q(P)$ use polynomial division to get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{\hat{P}(x)}{Q(x)}$$

- 2. Calculate all roots of Q(x)
- 3. Create a partial fraction per root
 - Simple real root: $x_1 \to \frac{A}{x-x_1}$
 - *n*-fold real root: $x_1 \to \frac{A_1}{x-x_1} + \ldots + \frac{A_r}{(x-x_1)^r}$
- 4. Calculate parameters A_1, \ldots, A_n . (Insert the root as s, transform and solve)

Hint (Odd functions). $\int_{-\lambda}^{\lambda} f(x) dx = 0$.

Cor. $\int_{a+c}^{b+c} f(x) dx = \int_a^b f(t+c) dt$

Cor. $\int_a^b f(ct)dt = \frac{1}{c} \int_{ac}^{bc} f(x) dx$

6.7 Integration of convergent series

Thm. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of bounded, integrable functions which converges uniformly against a function $f : [a, b] \to \mathbb{R}$. Then f is bounded and integrable and

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b \lim_{n \to \infty} f_n(x) \, dx = \int_a^b f(x) \, dx$$

Thm. Let $f(x) := \sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius p > 0. Then $\forall 0 \leq r < p$ f is integrable on [-r, r] and $\forall x \in]-p, p[$:

$$\int_{0}^{x} f(t)dt = \sum_{k=0}^{\infty} c_{k} \frac{x^{k+1}}{k+1}$$

6.8 Improper Integral

$$f:[a,\infty)\to\mathbb{R}, f:[-\infty,a]\to\mathbb{R}, f:(-\infty,\infty)\to\mathbb{R}$$

Def. Let $f:[a,\infty[\to\mathbb{R}]$ be bounded and integrable on [a,b] for all b>a. If $\lim_{b\to\infty}\int_a^b f(x)\,dx$ exists, the limit is defined as $\int_a^\infty f(x)\,dx$ and one can say that f is integrable on $[a,+\infty[$. If the limit does not exists, one can say that $\int_a^\infty f(x)\,dx$ diverges.

Comparison Theorem. Let $f:[a,\infty[\to \mathbb{R}]]$ be bounded and integrable on [a,b] $\forall b\in\mathbb{R},b>a$.

- 1. If $\forall x \geq a : |f(x)| \leq g(x)$ and g(x) is integrable on $[a, \infty[$ $\Longrightarrow f$ is integrable on $[a, \infty[$.
- 2. If $0 \le g(x) \le f(x)$ and $\int_a^\infty g(x) dx$ diverges $\Longrightarrow \int_a^\infty g(x) dx$ diverges.

Hint. Sometimes an integral can be split into a normal integral and an improper integral:

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx$$

McLaurin. Let $f:[1,\infty[\to[0,\infty[$ be monotonically de-

creasing.

$$\sum_{n=1}^{\infty} f(n) \text{ converges } \iff \int_{1}^{\infty} f(x) dx \text{ converges.}$$

The following holds:

$$0 \le \sum_{k=1}^{\infty} f(k) - \int_{1}^{\infty} f(x) \, dx \le f(1)$$

Def. Let f be a function which is bounded and integrable on all intervals $[a+\epsilon,b] \ \forall \epsilon > 0$. $f:]a,b] \to \mathbb{R}$ is integrable if $\lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x) \, dx$ exists. In this case the limit is defined as $\int_a^b f(x) \, dx$.

The comparison theorem can be used for such integrals as well.

6.9 Indefinite Integrals

Let $f: I \to \mathbb{R}$ be defined on the interval $I \subseteq \mathbb{R}$. If f is continuous there exists a primitive function F.

$$\int f(x) \, dx = F(x) + C$$

The indefinite integral is the inverse of the derivative.

Hint.
$$\int_{-\infty}^{\infty} f(x) dx = \lim_{b \to -\infty} \int_{b}^{a} f(x) dx + \lim_{c \to \infty} \int_{a}^{c} f(x) dx.$$

$$\int_{-infty}^{\infty} \text{ conv. } \iff \int_{-\infty}^{a} f(x) \, dx \, \text{conv.} \land \int_{a}^{\infty} f(x) \, dx \, \text{conv.}$$

In general: Let $f:]a, b[\to \mathbb{R}$ such that it is integrable on each compact interval $[\tilde{a}, \tilde{b}]$. Then

$$\int_{a}^{b} f(x) dx := \lim_{\tilde{a} \searrow a} \lim_{\tilde{b} \nearrow b} \int_{\tilde{a}}^{\tilde{b}} f(x) dx$$

6.10 Euler Gamma Function

Def. For s > 0:

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} \, dx.$$

The gamma function interpolates the function $n \mapsto (n-1)!$. It converges for all s > 0.

Useful Listings

Limits

Limits	
$\lim_{x \to \infty} \frac{1}{x} = 0$	$\lim_{x \to \infty} 1 + \frac{1}{x} = 1$
$\lim_{x \to \infty} e^x = \infty$	$\lim_{x \to -\infty} e^x = 0$
$\lim_{x \to \infty} e^{-x} = 0$	$\lim_{x \to -\infty} e^{-x} = \infty$
$\lim_{x \to \infty} \frac{e^x}{x^m} = \infty$	$\lim_{x \to -\infty} x e^x = 0$
$\lim_{x \to \infty} \ln(x) = \infty$	$\lim_{x \to 0} \ln(x) = -\infty$
$\lim_{x \to \infty} (1+x)^{\frac{1}{x}} = 1$	$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$
$\lim_{x \to \infty} (1 \pm \frac{1}{x})^{\lambda} = 1$	$\lim_{x \to \infty} (1 + \frac{\lambda}{x})^x = e^{\lambda}$
$\lim_{x \to \infty} x^{\lambda} q^x = 0, \forall 0 \le q < 1$	$\lim_{x \to \infty} \sqrt[x]{x} = 1$
$\lim_{x \to \pm \infty} (1 + \frac{1}{x})^x = e$	$\lim_{x \to \infty} (1 - \frac{1}{x})^x = \frac{1}{e}$
$\lim_{x \to \pm \infty} (1 + \frac{\lambda}{x})^{\alpha x} = e^{\lambda \alpha}$	$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
$\lim_{x \to 0} \frac{1}{\cos(x)} = 1$	$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$
$\lim_{x \to 0} \frac{\log(1) - x}{x} = -1$	$\lim_{x \to 0} x \log x = 0$
$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$	$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
$\lim_{x \to 0} \frac{x}{\arctan(x)} = 1$	$\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}$
$\lim_{x \to \infty} (\frac{x}{x+\lambda})^x = e^{-\lambda}$	$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
$\lim_{x \to 0} \frac{\lambda^x - 1}{x} = \ln(\lambda), \lambda > 0$	$\lim_{x \to 0} \frac{e^{\lambda x} - 1}{x} = \lambda$
$\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$	$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$
$\lim_{x \to \infty} \frac{\ln(x)}{x} = 0$	$\lim_{x \to \infty} \frac{\log(x)}{x^{\lambda}} = 0$
$\lim_{x \to \infty} \frac{\lambda x}{\lambda^x} = 0$	$\lim_{x \to \frac{\pi}{2}^{-}} \tan(x) = +\infty$

 $\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty \qquad \qquad \lim_{x \to \infty} \frac{\sin(x)}{x} = 0$

 $\lim x \ln x = 0$

Series

- Geometric: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if |q| < 1
- Harmonic: $\sum_{n=1}^{\infty} \frac{1}{k}$ diverges
- Telescope: $\sum_{n=0}^{\infty} \frac{1}{k(k+1)} = 1$
- $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n \to \infty} (1 + \frac{z}{n})^n = e^z$
- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges s > 1
- $p(z) = \sum_{k=0}^{\infty} c_k z^k$ converges absolute for $|z| < \rho$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$	$\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Taylor Polynomials

$$e^{x} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \frac{x^{4}}{4!} + \mathcal{O}(x^{5})$$

$$\sin(x) = x - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} - \mathcal{O}(x^{7})$$

$$\sinh(x) = x + \frac{x^{3}}{3!} + \frac{x^{5}}{5!} + \mathcal{O}(x^{7})$$

$$\cos(x) = 1 - \frac{x^{2}}{2!} + \frac{x^{4}}{4!} - \mathcal{O}(x^{6})$$

$$\cosh(x) = 1 + \frac{x^{2}}{2!} + \frac{x^{4}}{4!} + \mathcal{O}(x^{6})$$

$$\tan(x) = x + \frac{x^{3}}{3} + \frac{2x^{5}}{15} + \mathcal{O}(x^{7})$$

$$\tanh(x) = x - \frac{x^{3}}{3} + \frac{2x^{5}}{15} - \mathcal{O}(x^{7})$$

$$\log(1 + x) = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} - \frac{x^{4}}{d} + \mathcal{O}(x^{5})$$

$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^{2}}{8} + \frac{x^{3}}{16} - \mathcal{O}(x^{4})$$

Parity of Functions

Even: $f(-x) = f(x) \quad \forall x \in D$ $|x|, \cos x, x^2$ **Odd:** $f(-x) = -f(x) \quad \forall x \in D$ x, \sin, \tan, x^3

Hint. Chaining odd functions results in an odd function.

Common Derivatives and Integrals

$\mathbf{F}(\mathbf{x})$	$\mathbf{f}(\mathbf{x})$
c	0
x^a	$a \cdot x^{a-1}$
$\frac{1}{a+1}x^{a+1}$	x^a
$\frac{1}{a \cdot (n+1)} (ax+b)^{n+1}$	$(ax+b)^n$
$\frac{x^{a+1}}{a+1}$	$x^a, a \neq -1$
\sqrt{x}	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n}x^{\frac{1}{n}-1}$
$\frac{2}{3}x^{\frac{3}{2}}$	\sqrt{x}
$\frac{n}{n+1}x^{\frac{1}{n}+1}$	$\sqrt[n]{x}$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x\ln(a)} = \log_a(e)^{\frac{1}{x}}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
tan(x)	$\frac{1}{\cos^2(x)=1+\tan^2(x)}$
$\cot(x)$	$\frac{1}{-\sin^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
tanh(x)	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\frac{1}{f(x)}$	$\frac{-f'(x)}{(f(x))^2}$
a^{cx}	$a^{ck} \cdot c \ln(a)$
x^x	$x^x \cdot (1 + \ln(x)), \ x > 0$
$(x^x)^x$	$(x^x)^x(x+2x\ln(x)), x>0$
x^{x^x}	$x^{x^{x}}(x^{x-1} + \ln(x) \cdot x^{x}(1 + \ln(x)))$