

# Analysis I Summary

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## 1 Real numbers, euclidean spaces

**Archimedes' principle.** If  $x \in \mathbb{R}$  with  $x > 0$  and  $y \in \mathbb{R}$ , then  $\exists n \in \mathbb{N}$  ( $y \leq n \cdot x$ )

**Thm.** (i)  $|x| \geq 0 \quad \forall x \in \mathbb{R}$   
(ii)  $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$   
(iii)  $|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$   
(iv)  $|x + y| \geq ||x| - |y|| \quad \forall x, y \in \mathbb{R}$

**Young's inequality.**  $\forall \epsilon > 0, \forall x, y \in \mathbb{R}$ :

$$2|xy| \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$$

## 2 Sequences

### 2.1 Convergence

$(a_n)_{n \geq 1}$  converges to  $L = \lim_{n \rightarrow \infty} a_n$   
 $\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N (|a_n - L| < \epsilon)$

**Def** (Convergence).  $(a_n)_{n \geq 1}$  converges  
 $\iff \exists L \in \mathbb{R} \forall \epsilon > 0 (\{n \in \mathbb{N} \mid |a_n - L| \geq \epsilon\})$  is finite.

**Hint.** Let  $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$  converge with limit  $a$  and  $b$ :

1.  $(a_n + b_n)_{n \geq 1}$  converges with limit  $a + b$
2.  $(a_n \cdot b_n)_{n \geq 1}$  converges with limit  $a \cdot b$ .
3.  $(\frac{a_n}{b_n})_{n \geq 1}$  converges with limit  $\frac{a}{b}$
4.  $\exists K \geq 1 \forall n \geq K : a_n \leq b_n \implies a \leq b$

#### 2.1.1 Tips & Tricks

- $a_n$  convergent  $\implies a_n$  bounded
- $a_n$  convergent  $\iff a_n$  bounded and  $\liminf a_n = \limsup a_n$

**Monotone Convergence.**  $(a_n)_{n \geq 1}$  monotone increasing and upper bounded  $\implies \lim a_n = \sup\{a_n \mid n \geq 1\}$   
 $(a_n)_{n \geq 1}$  monotone decreasing and lower bounded  $\implies \lim a_n = \inf\{a_n \mid n \geq 1\}$

**Lemma** (Bernoulli Inequation).

$$(1 + x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}, x > -1$$

**Def** (Limit inferior / Limit superior).

$$\liminf_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} (\inf\{a_k \mid k \geq n\})$$

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} (\sup\{a_k \mid k \geq n\})$$

**Cauchy Criteria.**  $a_n$  converges iff  $\forall \epsilon > 0 \exists N \geq 1$  s.t.  $|a_n - a_m| < \epsilon \forall n, m \geq N$  (cauchy sequence).

- (i) Each Cauchy sequence is bounded
- (ii)  $(a_n)_{n \geq 1}$  conv.  $\implies (a_n)_{n \geq 1}$  cauchy
- (iii)  $(a_n)_{n \geq 1}$  cauchy  $\implies (a_n)_{n \geq 1}$  conv.

**Bolzano-Weierstrass.** Each bounded sequence contains a convergent sub sequence.

**Sandwich.** If  $\lim a_n = \alpha, \lim b_n = \alpha, k \in \mathbb{N}$  and  $a_n \leq c_n \leq b_n \forall n \geq k$ , then  $\lim c_n = \alpha$

**Cauchy-Cantor.** Let  $I_1 \supseteq I_2 \supseteq \dots I_n \dots$  be a sequence of proper intervals with  $\mathcal{L} < +\infty$ , then  $\bigcap_{n \geq 1} I_n \neq \emptyset$ . And if  $\lim_{n \rightarrow \infty} \mathcal{L}(I_n) = 0$  then  $|\bigcap_{n \geq 1} I_n| = 1$

**Cor.** Let  $(a_n)$  be bounded, then for each subsequence  $(b_n)$ :  $\liminf a_n \leq \lim b_n \leq \limsup a_n$ .

Each subsequence  $(b_n)$  of a convergent  $(a_n)$  converges and  $\lim b_n = \lim a_n$ .

## 3 Series

**Def.** " $\sum_{k=1}^{\infty} a_k$ " converges, if the sequence  $(S_n)_{n \geq 1}$  of partial sums converges and  $\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$

**Thm.**  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{j=1}^{\infty} b_j$  convergent:

- $\sum_{k=1}^{\infty} (a_k + b_k) = (\sum_{k=1}^{\infty} a_k) + (\sum_{k=1}^{\infty} b_k)$
- $\sum_{k=1}^{\infty} \alpha \cdot a_k = \alpha \sum_{k=1}^{\infty} a_k$

**Cauchy Criteria.**  $\sum_{k=1}^{\infty} a_k$  conv.  $\iff \forall \epsilon > 0 \exists N \geq 1 : |\sum_{k=n}^m a_k| = |S_m - S_n| < \epsilon \forall m \geq n \geq N$

**Zero Sequence Criteria.**  $\sum_{k=1}^{\infty} a_k$  conv.  $\implies \lim a_k = 0$

**Comparison Theorem.** Let  $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$  series s.t.  $0 \leq a_k \leq b_k \quad \forall k \geq 1$ :

$$\sum_{k=1}^{\infty} b_k \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges} \implies \sum_{k=1}^{\infty} b_k \text{ diverges}$$

**Thm.** Let  $\sum_{k=1}^{\infty} a_k$  be a series with  $a_k \geq 0 \quad \forall k \in \mathbb{N}^*$

$$\sum_{k=1}^{\infty} \text{converges} \iff (S_n)_{n \geq 1} \text{ upper bounded}$$

**Def** (Absolute Convergence).  $\sum_{k=1}^{\infty} a_k$  absolute converges if  $\sum_{k=1}^{\infty} |a_k|$  converges.

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \not\Rightarrow \sum_{k=1}^{\infty} |a_k| \text{ converges}$$

**(Dirichlet)** If a series converges absolute, then each permutation of the series converges with the same limit.

**(Riemann)** If a series only converges, then there exists a permutation such that:

$$\sum_{k=1}^{\infty} a_{\phi(k)} = x \quad \forall x \in \mathbb{R} \cup \{\infty\}$$

**Thm.**

$$\left| \sum_{k=1}^{\infty} a_k \right| \leq \sum_{k=1}^{\infty} |a_k|$$

**Leibniz.**  $(a_n)_{n \geq 1}$  monotone decreasing s.t.  $a_n \geq 0 \forall n \geq 1$  and  $\lim a_n = 0$ :

$$S := \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges.}$$

Furthermore:  $a_1 - a_2 \leq S \leq a_1$

**Ratio Test.** Let  $(a_n)_{n \geq 1}$  with  $a_n \neq 0 \quad \forall n \geq 1$ :

$$\limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$$

$$\liminf_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$$

**Lemma.**  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$ :

- $L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$
- $L > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$
- $L = 1 \implies \text{no information}$

**Root Test.** Let  $(a_n)_{n \geq 1}$  with  $a_n \neq 0 \quad \forall n \geq 1$ :

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$$

$$\liminf_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} |a_n| \text{ diverge.}$$

**Lemma.**  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ :

- $L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$
- $L > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$
- $L = 1 \implies \text{no information}$

**Def (Cauchy Product).**  $\sum_{i=0}^{\infty} a_i, \sum_{j=0}^{\infty} b_j$ :

$$\sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} a_{n-j} b_j \right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots$$

**Thm.**  $\sum_{i=0}^{\infty} a_i, \sum_{j=0}^{\infty} b_j \text{ conv. abs.} \implies \text{Cauchy prod. conv. abs.:}$

$$\sum_{n=0}^{\infty} \left( \sum_{j=0}^n a_{n-j} b_j \right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_i b_j$$

**Hint (Strategy: Convergence of Series).**

1. Check for known types (Telescope, Geometric, etc.)
2.  $\lim a_n \neq 0 \implies \text{divergence}$

3. Ratio Test

4. Root Test

5. Search convergent majors:  $0 \leq a_n \leq b_n$

6. If divergent minors  $\implies \text{divergence}$

7. Be creative

## 4 Functions

$\mathbb{R}^D = \{f : D \rightarrow \mathbb{R} \mid f \text{ is function}\}, (\mathbb{R}^D; +, \cdot) \text{ is V.R.}$

### 4.1 Continuity

**Def (Continuity).** A function  $f$  is continuous in  $x_0$  if:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon)$$

$$\iff$$

$$\forall (a_n)_{n \geq 1} \text{ with } \lim a_n = x_0 \text{ holds } \lim f(a_n) = f(\lim a_n)$$

**Def.** A function  $f : D \rightarrow \mathbb{R}$  is continuous if it is continuous in all  $x_0 \in D$

**Hint.** To prove continuity try to filter  $|x - x_0|$  out of  $|f(x) - f(x_0)|$  and choose  $\delta$ , such that the rest term disappears. Be aware that  $\delta$  is part of  $\epsilon$  and normally  $|x_0|$  as well. But not  $x$ !

**Cor.**  $f, g : D \rightarrow \mathbb{R}$  continuous in  $x_0 \in D$ . Then:

- $f, g, \lambda f, f \pm g$  continuous in  $x_0$
- $\frac{f}{g} : D \setminus \{x \in D \mid g(x) = 0\} \rightarrow \mathbb{R}$  continuous in  $x_0$  (if  $g(x_0) \neq 0$ )
- $|f|, \max(f, g), \min(f, g)$  continuous in  $x_0$
- $P(x) = a_n x^n + \dots + a_0$  continuous on  $\mathbb{R}$
- $\frac{P(x)}{Q(x)}$  continuous on  $\mathbb{R} \setminus \{x_1, \dots, x_m\}$  if  $x_1, \dots, x_m$  are roots of  $Q(x)$

**Thm.** Let  $f : D_1 \rightarrow D_2 \subset \mathbb{R}, g : D_2 \rightarrow \mathbb{R}$  be continuous  $\implies g \circ f : D_1 \rightarrow \mathbb{R}$  continuous.

**Bolzano (Intermediate value theorem).** Let  $I \subseteq \mathbb{R}, f : I \rightarrow \mathbb{R}$  and  $a, b \in I$ . For each  $c$  between  $f(a)$  and  $f(b)$  there is a  $z \in [a, b]$  with  $f(z) = c$

**Min-Max.** Let  $f : I = [a, b] \rightarrow \mathbb{R}$  be continuous.

$$\exists u, v \in I \forall x \in I (f(u) \leq f(x) \leq f(v))$$

In particular  $f([a, b]) \subset [f(u), f(v)]$  is bounded.

**Cor.**  $I = [a, b], f : I \rightarrow \mathbb{R}$  continuous, then  $\text{Im}(f) = f(I)$  is a compact interval  $J = [\min f, \max f] = [f(u), f(v)]$

**Inverse Mapping.** Let  $f : I \rightarrow \mathbb{R}$  be continuous and strict monotone increasing. Then  $J := f(I) \subseteq \mathbb{R}$  is an interval and  $f^{-1} : J \rightarrow I$  is continuous and strict monotone.

### 4.2 Exponential function

$\exp : \mathbb{R} \rightarrow ]0, \infty[$  is continuous, strictly monotone increasing, surjective.

- $\exp(x) \geq 1 + x \quad \forall x \in \mathbb{R}$
- For  $x > 0, a \in \mathbb{R} : x^a := \exp(a \ln x)$
- $x^0 = 1 \quad \forall x > 0$

**Def.** The inverse mapping of  $\exp(x)$  is called the natural logarithm:

$$\ln : ]0, \infty[ \rightarrow \mathbb{R}, \quad x \mapsto \ln x$$

It is strictly monotone increasing, continuous and bijective.

### 4.3 Converge of function sequences

$$\mathbb{N} \rightarrow \mathbb{R}^D = \{f : D \rightarrow \mathbb{R}\}, \quad n \mapsto f(n)$$

**Def (pointwise convergence).**  $(f_n)_{n \geq 0}$  converges pointwise to a function  $f : D \rightarrow \mathbb{R}$ , if  $\forall x \in D : \lim_{n \rightarrow \infty} f_n(x) = f(x)$

$$\iff$$

$$\forall x \in D \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N (|f_n(x) - f(x)| < \epsilon)$$

**Def (uniform convergence (Weierstrass)).**  $f_n : D \rightarrow \mathbb{R}$  converges uniformly in  $D$  to  $f : D \rightarrow \mathbb{R}$  if:

$$\forall \epsilon > 0 \exists N \geq 1 \text{ s.t. } \forall n \geq N \forall x \in D (|f_n(x) - f(x)| < \epsilon)$$

$$\iff$$

$$\lim_{n \rightarrow \infty} \sup_{x \in D} |f_n(x) - f(x)| = 0$$

The function sequence  $(f_n)$  is uniformly convergent if for all  $x \in D$  the limit  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  exists and the sequence  $(f_n)$  uniformly converges to  $f$ . Furthermore if  $\forall \epsilon > 0 \exists N \geq 1 \forall n, m \geq N \forall x \in D : |f_n(x) - f_m(x)| < \epsilon$ .

The series  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly (in  $D$ ), if the function sequence  $S_n(x) := \sum_{k=0}^n f_k(x)$  converges uniformly.

**Thm.** let  $D \subseteq \mathbb{R}$  and  $f_n : D \rightarrow \mathbb{R}$  a function sequence containing (in  $D$ ) continuous functions which converge (in  $D$ ) uniformly against a function  $f : D \rightarrow \mathbb{R}$ , then  $f$  (in  $D$ ) is continuous.

**Hint** (not uniform convergent).  $(f_n)_{n \geq 0}$  converges not uniformly if:  $\forall \epsilon > 0 \forall N \in \mathbb{N} \exists x \in D (|f_n(x) - f(x)| \geq \epsilon)$

**Hint.** Check the function and try to construct  $x$  (dependent on  $N$  in general), such that  $|f_n(x) - f(x)|$  is always greater than a specific  $\epsilon$  and afterwards choose the  $\epsilon$ .

**Def** (Power Functions).  $\sum_{k=0}^{\infty} c_k x^k$  has positive convergence radius if  $\limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}$  exists.

$$\rho = \begin{cases} +\infty & , \text{ if } \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} = 0 \\ \frac{1}{\limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}} & , \text{ if } \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} > 0 \end{cases}$$

**Thm.** Let  $\sum_{k=0}^{\infty} c_k x^k$  be a power series with positive convergence radius  $\rho > 0$  and let  $f(x) = \sum_{k=0}^{\infty} c_k x^k, |x| < \rho$  Then:  $\forall 0 \leq r < \rho$  converges  $\sum_{k=0}^{\infty} c_k x^k$  uniformly on  $[-r, r]$ , furthermore  $f : ]-\rho, \rho[ \rightarrow \mathbb{R}$  is continuous.

## 4.4 Trigonometric Functions

$\sin$  and  $\cos$  are continuous functions  $\mathbb{R} \rightarrow \mathbb{R}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

**Thm.** 1.  $\exp(iz) = \cos(z) + i \sin(z) \quad \forall z \in \mathbb{C}$   
2.  $\cos z = \cos(-z)$  und  $\sin(-z) = -\sin(z) \quad \forall z \in \mathbb{C}$

3.  $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$
4.  $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$   
 $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
5.  $\cos(z)^2 + \sin(z)^2 = 1 \quad \forall z \in \mathbb{C}$

**Cor.**

$$\sin(2z) = 2\sin(z)\cos(z)$$

$$\cos(2z) = \cos(z)^2 - \sin(z)^2$$

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

**Def** ( $\pi$ ).  $\pi := \inf\{t > 0 \mid \sin t = 0\}$

- (i)  $\sin \pi = 0, \pi \in ]2, 4[$
- (ii)  $\forall x \in ]0, \pi[: \sin x > 0$
- (iii)  $e^{\frac{i\pi}{2}} = i$

**Cor.**  $x \geq \sin x \geq x - \frac{x^3}{3!} \quad \forall 0 \leq x \leq \sqrt{6}$

**Cor.** 1.  $e^{i\pi} = -1, \quad e^{2i\pi} = 1$

2.  $\sin(x + \frac{\pi}{2}) = \cos(x), \quad \cos(1 + \frac{\pi}{2}) = -\sin(x) \quad \forall x \in \mathbb{R}$
3.  $\sin(x + \pi) = -\sin(x), \quad \sin(x + 2\pi) = \sin(x) \quad \forall x \in \mathbb{R}$
4.  $\cos(x + \pi) = -\cos(x), \quad \cos(x + 2\pi) = \cos(x) \quad \forall x \in \mathbb{R}$
5. *Roots of sinus*  $= \{\pi \cdot k \mid k \in \mathbb{Z}\}$   
 $\sin(x) > 0 \quad \forall x \in ]2k\pi, (2k+1)\pi[, \quad k \in \mathbb{Z}$   
 $\sin(x) < 0 \quad \forall x \in [(2k+1)\pi, (2k+2)\pi[, \quad k \in \mathbb{Z}$
6. *Roots of cosine*  $= \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$   
 $\cos(x) > 0 \quad \forall x \in ]-\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + (2k+1)\pi[, \quad k \in \mathbb{Z}$   
 $\cos(x) < 0 \quad \forall x \in [-\frac{\pi}{2} + (2k+1)\pi, -\frac{\pi}{2} + (2k+2)\pi[, \quad k \in \mathbb{Z}$

## 4.5 Limit of Functions

**Def** (accumulation point).  $x_0 \in \mathbb{R}$  is an accumulation point of  $D$  if  $\forall \delta > 0: (]x_0 - \delta, x_0 + \delta[ \setminus \{x_0\}) \cap D \neq \emptyset$

**Def** (Limit of Function). if  $f : D \rightarrow \mathbb{R}, x_0 \in \mathbb{R}$  an accumulation point of  $D$ , then  $A \in \mathbb{R}$  is the limit of  $f(x)$  for  $x \rightarrow x_0$ , written as  $\lim_{x \rightarrow x_0} f(x) = A$ . If  $\forall \epsilon > 0 \exists \delta > 0$  s.t.  $\forall x \in D \cap (]x_0 - \delta, x_0 + \delta[ \setminus \{x_0\}) : |f(x) - A| < \epsilon$

**Important Rules.** Let  $f : D \rightarrow \mathbb{R}$  and  $x_0$  is an accumulation point of  $D$ .

1.  $\lim_{x \rightarrow x_0} f(x) = A \iff \forall (a_n)_{n \geq 1} \text{ in } D \setminus \{x_0\} \text{ with}$

$$\lim_{n \rightarrow \infty} a_n = x_0 \implies \lim_{n \rightarrow \infty} f(a_n) = A.$$

2. Let  $x_0 \in D$ . Then  $f$  is continuous in  $x_0$   
 $\iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$

3.  $f, g : D \rightarrow \mathbb{R}$  and  $\exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \implies$

$$\lim_{x \rightarrow x_0} (f+g)(x) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$$

$$\lim_{x \rightarrow x_0} (f \cdot g)(x) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x)$$

4.  $f, g : D \rightarrow \mathbb{R}$  and  $f \leq g$ , then if both limit exists

$$\lim_{x \rightarrow x_0} f(x) \leq \lim_{x \rightarrow x_0} g(x)$$

5. If  $g_1 \leq f \leq g_2$  and  $\lim_{x \rightarrow x_0} g_1(x) = \lim_{x \rightarrow x_0} g_2(x)$  then  
 $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g_1(x)$

**Hint.** Sometimes it can be really helpful to convert known functions to their power series to calculate a limit. E.g.

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \dots}{x} = \lim_{x \rightarrow 0} 1 - \frac{x^2}{3!} + \dots = 1$$

**Hint** ( $e^{\log}$ ). Transform ugly function with this trick.

$$\lim_{x \rightarrow x_0} f(x)^{g(x)} = \lim_{x \rightarrow x_0} e^{g(x) \log(f(x))} = e^{\lim_{x \rightarrow x_0} g(x) \log(f(x))}$$

## 5 Differentiable Functions

**Def** (Differentiable).  $f$  is in  $x_0$  differentiable, if the limit  $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$  exists.  $f$  is differentiable if  $\forall x_0 \in D$   $f$  is differentiable.

**Weierstrass.**  $f : D \rightarrow \mathbb{R}, x_0 \in D$  accumulation point. Equivalent statements:

1.  $f$  is in  $x_0$  differentiable
2. It exists  $c \in \mathbb{R}$  ( $c = f'(x_0)$ ) and  $r : D \rightarrow \mathbb{R}$  s.t.:

$$2.1 \quad f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0)$$

$$2.2 \quad r(x_0) = 0 \text{ and } r \text{ continuous in } x_0.$$

**Cor.**  $f$  diff. in  $x_0 \implies f$  continuous in  $x_0$

**Thm.**  $f$  diff. in  $x_0 \iff \exists \phi : D \rightarrow \mathbb{R}$  continuous in  $x_0$  s.t.  $\forall x \in D : f(x) = f(x_0) + \phi(x)(x - x_0)$ . Then  $\phi(x_0) = f'(x_0)$ .

### Derivative rules.

Linearity:  $(\alpha \cdot f(x) + g(x))' = \alpha \cdot f'(x) + g'(x)$   
 Product rule:  $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$   
 Quotient rule:  $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$   
 Chain rule:  $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

**Cor.**  $f$  bijective and in  $x_0$  differentiable s.t.  $f'(x_0) \neq 0$ .  $f^{-1}$  is continuous in  $y_0 = f(x_0)$ . Then  $f^{-1}$  is differentiable in  $y_0$  and  $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$ .

## 5.1 Derivative Implications

- $x_0$  is local minimum if  $f'(x_0) = 0 \wedge f''(x_0) > 0$  or the sign of  $f'$  changes from  $-$  to  $+$ .
- $x_0$  is local maximum if  $f'(x_0) = 0 \wedge f''(x_0) < 0$  or the sign of  $f'$  changes from  $+$  to  $-$ .
- $x_0$  is local extremum if  $f'(x_0) = 0 \wedge f''(x_0) \neq 0$
- $x_0$  is a saddle point if  $f'(x_0) = 0$  and  $f''(x_0) = 0$
- $x_0$  is a inflection point if  $f''(x_0) = 0$
- $f'(x_0) = f^{(2)}(x_0) = \dots = f^{(n)}(x_0) = 0$ 
  - $n$  odd and  $f^{(n+1)}(x_0) > 0 \implies x_0$  strict local minimum
  - $n$  odd and  $f^{(n+1)}(x_0) < 0 \implies x_0$  strict local maximum

## 5.2 Derivative Theorems

**Rolle.** Let  $f : [a, b] \rightarrow \mathbb{R}$  continuous and in  $]a, b[$  differentiable. If  $f(a) = f(b)$ , then there exists  $\xi \in ]a, b[$  with  $f'(\xi) = 0$ .

**Mean Value / Lagrange.** Let  $f : [a, b] \rightarrow \mathbb{R}$  continuous and in  $]a, b[$  differentiable, then there exists  $\xi \in ]a, b[$  with  $f(b) - f(a) = f'(\xi)(b - a)$ .

There exists points  $\xi$  with  $f'(\xi)$  equal to the gradient of the secant between  $a$  to  $b$ .

**Cor.** Let  $f, g : [a, b] \rightarrow \mathbb{R}$  cont. and diff. in  $]a, b[$ .

- $\forall \xi \in ]a, b[ : f'(\xi) = 0 \implies f$  is constant

- $\forall \xi \in ]a, b[ : f'(\xi) = g'(x) \implies \exists c \in \mathbb{R} \forall x \in [a, b] : f(x) = g(x) + c$
- $\forall \xi \in ]a, b[ : f'(\xi) \geq 0 \implies f$  in  $[a, b]$  mon. inc.
- $\forall \xi \in ]a, b[ : f'(\xi) > 0 \implies f$  in  $[a, b]$  str. mon. inc.
- $\forall \xi \in ]a, b[ : f'(\xi) \leq 0 \implies f$  in  $[a, b]$  mon. dec.
- $\forall \xi \in ]a, b[ : f'(\xi) < 0 \implies f$  in  $[a, b]$  str. mon. dec.
- $\exists M \geq 0 \forall \xi \in ]a, b[ : |f'(\xi)| \leq M \implies \forall x_1, x_2 \in [a, b] : |f(x_1) - f(x_2)| \leq M|x_1 - x_2|$

**Cauchy.**  $f, g : [a, b] \rightarrow \mathbb{R}$  continuous and in  $]a, b[$  diff. Then there exists  $\xi \in ]a, b[$  with  $g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a))$ .

If  $\forall x \in ]a, b[ : g'(x) \neq 0$  it implies that  $g(a) \neq g(b)$  and  $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$

**l'Hôpital.**  $f, g : ]a, b[ \rightarrow \mathbb{R}$  diff. with  $\forall x \in ]a, b[ : g'(x) \neq 0$ . If  $\lim_{x \rightarrow b^-} f(x) = 0, \lim_{x \rightarrow b^-} g(x) = 0$  and

$\lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)} =: \lambda$  exists, then  $\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)}$ .

**Hint.** Only use l'Hospital if either  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ !

- Def.**
- (strictly) convex:  $(x \leq y) : f(\lambda x + (1 - \lambda)y) (<) \leq \lambda f(x) + (1 - \lambda)f(y)$
  - (strictly) concave:  $(x \leq y) : f(\lambda x + (1 - \lambda)y) (>) \geq \lambda f(x) + (1 - \lambda)f(y)$

**Lemma.**  $f : I \rightarrow \mathbb{R}$ .  $f$  is convex  $\iff \forall x_0 < x < x_1 \in I : \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(x_1) - f(x)}{x_1 - x}$ . Strictly convex if  $<$ .

**Lemma.**  $f : ]a, b[ \rightarrow \mathbb{R}$  diff.

- $f'$  (strictly) mon. inc. dec.  $\Rightarrow f$  (strictly) conv. conc.
- $f'' \geq (>) 0 \implies f$  (strictly) conv. ( $< / \leq$  for conc.)

## 5.3 Higher Derivatives

- For  $n \geq 2$  is  $f$   **$n$ -times differentiable in  $D$**  if  $f^{(n-1)}$  in  $D$  is differentiable. Then  $f^{(n)} := (f^{(n-1)})'$  and is the  $n$ -th derivative of  $f$
- $f$  is  **$n$ -times continuous differentiable in  $D$**  if  $f$  is  $n$ -times differentiable and if  $f^{(n)}$  is continuous in  $D$
- $f$  is in  $D$  **smooth** if  $\forall n \geq 1, f$  is  $n$ -times differentiable.

**Smooth Functions.**  $\exp, \sin, \cos, \sinh, \cosh, \tanh, \ln, \arcsin, \arccos, \operatorname{arccot}, \arctan$  and all polynomials.  $\tan$  is smooth on  $\mathbb{R} \setminus \{\pi/2 + k\pi\}$  and  $\cot$  on  $\mathbb{R} \setminus \{k\pi\}$

**Thm.**  $f, g : D \rightarrow \mathbb{R}$  are  $n$ -times diff. in  $D$ .

- $(f + g)^{(n)} = f^{(n)} + g^{(n)}$
- $(f \cdot g)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$
- $(g \circ f)^{(n)}(x) = \sum_{k=1}^n A_{n,k}(x) (g^{(k)} \circ f)(x)$  with  $A_{n,k}$  as polynomial in the functions  $f', f^{(2)}, \dots, f^{(n+1-k)}$

## 5.4 Power Series and Taylor approximation

**Thm.** Let  $f_n : ]a, b[$  be a function sequence with  $f_n$  one time in  $]a, b[$  continuous diff.  $\forall n \geq 1$ . Assume that  $(f_n)_{n \geq 1}, (f'_n)_{n \geq 1}$  uniformly convergent in  $]a, b[$  with  $\lim_{n \rightarrow \infty} f_n =: f$  and  $\lim_{n \rightarrow \infty} f'_n =: p$ , then  $f$  is continuously diff. and  $f' = p$ .

**Thm.** Let  $\sum_{k=0}^{\infty} c_k$  be a power series with convergent radius  $\rho > 0$ . Then  $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$  is differentiable on  $]x_0 - \rho, x_0 + \rho[$  and  $\forall x \in ]x_0 - \rho, x_0 + \rho[ : f'(x) = \sum_{k=0}^{\infty} k c_k (x - x_0)^{k-1}$

**Cor.**  $f^{(j)} = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!} (x - x_0)^{k-j}$ . Furthermore

$c_j = \frac{f^{(j)}(x_0)}{j!}$ . Power series can be differentiated part by part in their converge area.

**Def (Taylor Polynomial).** The  $n$ -th Taylor-polynomial of cont.  $n + 1$  times diff. in  $[c, d]$   $f$  is defined as  $T_n(f, x, a)$  with center  $a \in ]c, d[$  and error  $R_n(f, x, a)$ .  $\forall x \in [a, b] \exists \xi \in ]x, a[ \cup ]a, x[ :$

$$T_n(f, x, a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} \cdot (x - a)^k$$

$$R_n(f, x, a) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$$

$$f(x) = T_n(f, x, a) + R_n(f, x, a)$$

**Hint.** The error can be approximated as

$$|R_n(f, x, a)| \leq \sup_{a < c < x} \left| \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} \right|$$

**Def (Taylor Series).**  $T_{\infty}(f, x, x_0) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$

## 6 Riemann Integral

$a < b$ ,  $I = [a, b]$ ,  $\mathcal{P}(I) = \{P \mid P \subsetneq I \wedge \{a, b\} \in P \wedge |P| \in \mathbb{N}\}$

**Def** (Partition). • Partition:  $P \in \mathcal{P}(I)$

- $\delta_i := x_i - x_{i-1}$  length of  $I_i := [x_{i-1}, x_i]$ ,  $i \geq 1$
- Mesh of partition:  $\delta(P) := \max_{1 \leq i \leq n} (x_i - x_{i-1})$
- $\xi := \{\xi_1, \dots, \xi_n\}$ ,  $\xi_i \in I_i$
- $P'$  refines  $P$  if  $P \subseteq P'$

**Def** (Riemann Sums).  $S(f, P, \xi) := \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$

- Lower sum:  $\underline{S}(f, P) := \sum_{i=1}^n (\inf_{x \in I_i} f(x)) (x_i - x_{i-1})$
- Upper sum:  $\overline{S}(f, P) := \sum_{i=1}^n (\sup_{x \in I_i} f(x)) (x_i - x_{i-1})$

It holds:  $-M(b-a) \leq \underline{S}(f, P) \leq \overline{S}(f, P) \leq M(b-a)$

**Lemma.**  $P \subseteq P'$ :  $\underline{S}(f, P) \leq \underline{S}(f, P') \leq \overline{S}(f, P') \leq \overline{S}(f, P)$

**Lemma.**  $\forall P_1, P_2 \in \mathcal{P}(I)$ :  $\underline{S}(f, P_1) \leq \overline{S}(f, P_2)$

**Def** (Lower Riemann Integral).  $\underline{S}(f) := \sup_{P \in \mathcal{P}(I)} \underline{S}(f, P)$

**Def** (Upper Riemann Integral).  $\overline{S}(f) := \inf_{P \in \mathcal{P}(I)} \overline{S}(f, P)$

### 6.1 Integrability criteria

**Def** (Integrable). Bounded  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if  $\underline{S}(f) = \overline{S}(f)$  and the shared value is  $\int_a^b f(x) dx$ .

**Riemann Criteria.** Bounded  $f : I \rightarrow \mathbb{R}$  is integrable. Let  $\mathcal{P}_\delta(I) := \{P \in \mathcal{P}(I) \mid \delta(P) < \delta\}$ .

- $\Leftrightarrow \forall \epsilon > 0 \exists P \in \mathcal{P}(I) : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon$
- $\Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 \forall P \in \mathcal{P}_\delta(I) : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon$
- $\Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 \forall P \in \mathcal{P}$  with  $\delta(P) < \delta$ :

$$\left| A - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon$$

**Hint.** Bounded  $f : [a, b] \rightarrow \mathbb{R}$  is int. if  $\lim_{\delta(P) \rightarrow 0} S(f, P, \xi)$  exists for all  $P$  with  $\delta(P) \rightarrow 0$ . It follows that  $\lim_{\delta(P) \rightarrow 0} S(f, P, \xi) = \int_a^b f(x) dx$

### 6.2 Integrable Functions

1.  $f$  (bounded) cont. in  $[a, b] \implies f$  int. over  $[a, b]$
2.  $f$  monotone in  $[a, b] \implies f$  int. over  $[a, b]$
3. If  $f, g$  bounded and int., then integrable as well:

$$f + g, \lambda \cdot f, f \cdot g, |f|, \min(f, g), \max(f, g), \frac{f}{g}$$

4. All polynomials are integrable, even  $\frac{P(x)}{Q(x)}$  if  $Q(x)$  has no root in  $[a, b]$

**Hint.** Let  $V := \{f : I \rightarrow \mathbb{R} \mid f \text{ is a mapping}\}$ .  $(V, +, \cdot)$  is a vector space. Then it implies that  $W := \{f : I \rightarrow \mathbb{R} \mid f \text{ is integrable}\}$  is a subspace of  $V$ .

### 6.3 Integration Inequalities and Theorems

**Def** (Uniform continuous).

$\forall \epsilon > 0 \exists \delta > 0 \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$

**Thm.**  $f : [a, b] \rightarrow \mathbb{R}$  cont.  $\implies f$  is uni. cont. in  $[a, b]$ .

**Thm.**  $f$  uni. cont.  $\implies f$  cont.

**Thm.**  $f, g : [a, b] \rightarrow \mathbb{R}$  bounded and integrable and  $\forall x \in [a, b] : f(x) \leq g(x) \implies \int_a^b f(x) dx \leq \int_a^b g(x) dx$

**Cauchy-Schwarz.**

$$\left| \int_a^b f(x)g(x) dx \right| \leq \sqrt{\int_a^b f^2(x) dx} \sqrt{\int_a^b g^2(x) dx}$$

**Hint.**  $\langle f, g \rangle := \int_a^b f(x)g(x) dx$  is a scalar product.  $\|f\|^2 = \langle f, f \rangle = \int_a^b f^2(x) dx$ .

**Mean Value Theorem.**  $f : [a, b] \rightarrow \mathbb{R}$  continuous  $\implies \exists \xi \in [a, b] : \int_a^b f(x) dx = f(\xi)(b - a)$

**Cauchy.**  $f, g : [a, b] \rightarrow \mathbb{R}$  with  $f$  continuous and  $g$  bounded and integrable with  $g(x) \geq 0, \forall x \in [a, b]$

$$\implies \exists \xi \in [a, b] : \int_a^b f(x)g(x) dx = f(\xi) \int_a^b g(x) dx$$

### 6.4 Integration Properties

**Additive Property.**

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

**Linearity.**

$$\int_a^b (\alpha f_1 + \beta f_2) dx = \alpha \int_a^b f_1(x) dx + \beta \int_a^b f_2(x) dx$$

**Preservation of Order.**

$$\forall x \in [a, b] : f(x) \leq g(x) \implies \int_a^b f(x) dx \leq \int_a^b g(x) dx$$

**Triangle Inequality.**

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

### 6.5 Primitive Functions

**Def** (Primitive Function).  $F : [a, b] \rightarrow \mathbb{R}$  is a primitive function of  $f$  if  $F$  is cont. diff. and  $F' = f$ .

**Hint.**  $f$  is integrable  $\nRightarrow$  exists a primitive function for  $f$ .

**HID.** Let  $a < b$ ,  $f : [a, b] \rightarrow \mathbb{R}$  continuous. The function

$$F(x) := \int_a^x f(t) dt \quad a \leq x \leq b$$

is cont. diff. in  $[a, b]$  and  $F'(x) = f(x) \forall x \in [a, b]$ .

**Fundamental theorem of calculus.**  $f : [a, b] \rightarrow \mathbb{R}$  continuous. Then there exists a unique (except a constant term) primitive function  $F$  of  $f$ , such that

$$\int_a^b f(x) dx = F(b) - F(a).$$



## 6.6 Integration Methods

### Partial Inegration.

$$\begin{aligned}\int_a^b f(x)g'(x) dx &= f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x) dx \\ \int_a^b f(x)g'(x) dx &= (f \cdot g)|_a^b - \int_a^b f'(x)g(x) dx \\ \int_a^b f(x)g'(x) dx &= f(x)g(x) - \int_a^b f'(x)g(x) dx\end{aligned}$$

- Choose  $g'$ : exp  $\rightarrow$  trig  $\rightarrow$  poly  $\rightarrow$  inverse trig.  $\rightarrow$  logs
- Choose  $f$ : logs  $\rightarrow$  inverse trig.  $\rightarrow$  poly  $\rightarrow$  trig  $\rightarrow$  exp
- Sometimes it is necessary to multiply by 1. E.g.:  $\int \ln x dx = \int \ln x \cdot 1 dx \implies f(x) = \ln x, g'(x) = 1$ .
- Sometimes it is necessary to do it multiple times

**Substitution.** Let  $a < b, \phi : [a, b] \rightarrow \mathbb{R}$ , cont. diff,  $I \subseteq \mathbb{R}$  with  $\phi([a, b]) \subseteq I$  and  $f : I \rightarrow \mathbb{R}$  a cont. function. Then it follows:

$$\int_{\phi(a)}^{\phi(b)} f(x) dx = \int_a^b f(\phi(t))\phi'(t)dt = (F \circ \phi)(b) - (F \circ \phi)(a)$$

since  $F' = f$  then  $f(\phi(t))\phi'(t) = (F \circ \phi)'(t)$ .

**Partial Fraction Decomposition.** Let  $P(x), Q(x)$  be two polynomials.  $\int \frac{P(x)}{Q(x)}$  can be calculated as follows:

1. If  $\deg(P) \geq \deg(Q)$  use polynomial division to get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{\hat{P}(x)}{Q(x)}$$

2. Calculate all roots of  $Q(x)$
3. Create a partial fraction per root
  - Simple real root:  $x_1 \rightarrow \frac{A}{x-x_1}$
  - $n$ -fold real root:  $x_1 \rightarrow \frac{A_1}{x-x_1} + \dots + \frac{A_r}{(x-x_1)^r}$
4. Calculate parameters  $A_1, \dots, A_n$ . (Insert the root as  $s$ , transform and solve)

**Hint** (Odd functions).  $\int_{-\lambda}^{\lambda} f(x) dx = 0$ .

**Cor.**  $\int_{a+c}^{b+c} f(x) dx = \int_a^b f(t+c)dt$

**Cor.**  $\int_a^b f(ct)dt = \frac{1}{c} \int_{ac}^{bc} f(x) dx$

### 6.7 Integration of convergent series

**Thm.** Let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a sequence of bounded, integrable functions which converge uniformly against a function  $f : [a, b] \rightarrow \mathbb{R}$ . Then  $f$  is bounded and integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = \int_a^b f(x) dx$$

$$\sum_{n=0}^{\infty} \int_a^b f_n(x) dx = \int_a^b \left( \sum_{n=0}^{\infty} f_n(x) \right) dx$$

**Thm.** Let  $f(x) := \sum_{k=0}^{\infty} c_k x^k$  be a power series with positive convergence radius  $p > 0$ . Then  $\forall 0 \leq r < p$   $f$  is integrable on  $[-r, r]$  and  $\forall x \in ]-p, p[$ :  $\int_0^x f(t)dt = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$

### 6.8 Improper Integral

$$f : [a, \infty) \rightarrow \mathbb{R}, f : [-\infty, a] \rightarrow \mathbb{R}, f : (-\infty, \infty) \rightarrow \mathbb{R}$$

**Def.** Let  $f : [a, \infty[ \rightarrow \mathbb{R}$  be bounded and integrable on  $[a, b]$  for all  $b > a$ . If  $\lim_{b \rightarrow \infty} \int_a^b f(x) dx$  exists, the limit is defined as  $\int_a^{\infty} f(x) dx$  and one can say that  $f$  is integrable on  $[a, +\infty[$ . If the limit does not exist, one can say that  $\int_a^{\infty} f(x) dx$  diverges.

**Comparison Theorem.** Let  $f : [a, \infty[ \rightarrow \mathbb{R}$  be bounded and integrable on  $[a, b] \forall b \in \mathbb{R}, b > a$ .

1. If  $\forall x \geq a : |f(x)| \leq g(x)$  and  $g(x)$  is integrable on  $[a, \infty[ \implies f$  is integrable on  $[a, \infty[$ .
2. If  $0 \leq g(x) \leq f(x)$  and  $\int_a^{\infty} g(x) dx$  diverges  $\implies \int_a^{\infty} f(x) dx$  diverges.

**Hint.** Sometimes an integral can be split into a normal integral and an improper integral:

$$\int_0^{\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx$$

**McLaurin.** Let  $f : ]1, \infty[ \rightarrow ]0, \infty[$  be mon. dec.

$$\sum_{n=1}^{\infty} f(n) \text{ converges } \iff \int_1^{\infty} f(x) dx \text{ converges.}$$

The following holds:

$$0 \leq \sum_{k=1}^{\infty} f(k) - \int_1^{\infty} f(x) dx \leq f(1)$$

**Def.** Let  $f$  be a function which is bounded and integrable on all intervals  $[a + \epsilon, b] \forall \epsilon > 0$ .  $f : ]a, b] \rightarrow \mathbb{R}$  is integrable if  $\lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b f(x) dx$  exists. In this case the limit is defined as  $\int_a^b f(x) dx$ . (The comparison theorem can be used for such integrals as well.)

### 6.9 Indefinite Integrals

Let  $f : I \rightarrow \mathbb{R}$  be defined on the interval  $I \subseteq \mathbb{R}$ . If  $f$  is continuous there exists a primitive function  $F$ .

$$\int f(x) dx = F(x) + C$$

The indefinite integral is the inverse of the derivative.

**Hint.**  $\int_{-\infty}^{\infty} f(x) dx = \lim_{b \rightarrow -\infty} \int_b^a f(x) dx + \lim_{c \rightarrow \infty} \int_a^c f(x) dx$ .

$$\int_{-\infty}^{\infty} \text{conv.} \iff \int_{-\infty}^a f(x) dx \text{ conv.} \wedge \int_a^{\infty} f(x) dx \text{ conv.}$$

In general: Let  $f : ]a, b[ \rightarrow \mathbb{R}$  such that it is integrable on each compact interval  $[\tilde{a}, \tilde{b}]$ . Then

$$\int_a^b f(x) dx := \lim_{\tilde{a} \searrow a} \lim_{\tilde{b} \nearrow b} \int_{\tilde{a}}^{\tilde{b}} f(x) dx$$

### 6.10 Euler Gamma Function

**Def.** For  $s > 0$ :

$$\Gamma(s) := \int_0^{\infty} e^{-x} x^{s-1} dx.$$

The gamma function interpolates the function  $n \mapsto (n-1)!$ . It converges for all  $s > 0$ .

# Useful Listings

## Limits

$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$	$\lim_{x \rightarrow \infty} 1 + \frac{1}{x} = 1$
$\lim_{x \rightarrow \infty} e^x = \infty$	$\lim_{x \rightarrow -\infty} e^x = 0$
$\lim_{x \rightarrow \infty} e^{-x} = 0$	$\lim_{x \rightarrow -\infty} e^{-x} = \infty$
$\lim_{x \rightarrow \infty} \frac{e^x}{x^m} = \infty$	$\lim_{x \rightarrow -\infty} x e^x = 0$
$\lim_{x \rightarrow \infty} \ln(x) = \infty$	$\lim_{x \rightarrow 0} \ln(x) = -\infty$
$\lim_{x \rightarrow \infty} (1+x)^{\frac{1}{x}} = 1$	$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$
$\lim_{x \rightarrow \infty} (1 \pm \frac{1}{x})^\lambda = 1$	$\lim_{x \rightarrow \infty} (1 + \frac{\lambda}{x})^x = e^\lambda$
$\lim_{x \rightarrow \infty} x^\lambda q^x = 0, \forall 0 \leq q < 1$	$\lim_{x \rightarrow \infty} \sqrt[x]{x} = 1$
$\lim_{x \rightarrow \pm \infty} (1 + \frac{1}{x})^x = e$	$\lim_{x \rightarrow \infty} (1 - \frac{1}{x})^x = \frac{1}{e}$
$\lim_{x \rightarrow \pm \infty} (1 + \frac{\lambda}{x})^{\alpha x} = e^{\lambda \alpha}$	$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$
$\lim_{x \rightarrow 0} \frac{1}{\cos(x)} = 1$	$\lim_{x \rightarrow 0} \frac{\cos(x)-1}{x} = 0$
$\lim_{x \rightarrow 0} \frac{\log(1-x)}{x} = -1$	$\lim_{x \rightarrow 0} x \log x = 0$
$\lim_{x \rightarrow 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$	$\lim_{x \rightarrow 0} \frac{e^x-1}{x} = 1$
$\lim_{x \rightarrow 0} \frac{x}{\arctan(x)} = 1$	$\lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}$
$\lim_{x \rightarrow \infty} (\frac{x}{x+\lambda})^x = e^{-\lambda}$	$\lim_{x \rightarrow 0} \frac{e^x-1}{x} = 1$
$\lim_{x \rightarrow 0} \frac{\lambda^x-1}{x} = \ln(\lambda), \lambda > 0$	$\lim_{x \rightarrow 0} \frac{e^{\lambda x}-1}{x} = \lambda$
$\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1$	$\lim_{x \rightarrow 1} \frac{\ln(x)}{x-1} = 1$
$\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = 0$	$\lim_{x \rightarrow \infty} \frac{\log(x)}{x^\lambda} = 0$
$\lim_{x \rightarrow \infty} \frac{\lambda x}{\lambda^x} = 0$	$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) = +\infty$
$\lim_{x \rightarrow \frac{\pi}{2}^+} \tan(x) = -\infty$	$\lim_{x \rightarrow \infty} \frac{\sin(x)}{x} = 0$
$\lim_{x \rightarrow 0^+} x \ln x = 0$	Stirling Formula
	$\lim_{x \rightarrow \infty} \frac{x!}{(\frac{x}{e})^x \sqrt{2\pi x}} = 1$

## Series

- Geometric:  $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$  if  $|q| < 1$
- Harmonic:  $\sum_{n=1}^{\infty} \frac{1}{k}$  diverges
- Telescope:  $\sum_{n=0}^{\infty} \frac{1}{k(k+1)} = 1$
- $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n \rightarrow \infty} (1 + \frac{z}{n})^n = e^z$
- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  converges  $s > 1$  ( $\frac{1}{1-\frac{1}{2^{s-1}}}$ )
- $p(z) = \sum_{k=0}^{\infty} c_k z^k$  conv. abs.  $|z| < \rho = \frac{1}{\limsup |c_k|^{1/k}}$

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$\sum_{i=1}^n i = \frac{n(n+1)}{2}$	$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$	$\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

## Taylor Polynomials

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5)$
$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \mathcal{O}(x^7)$
$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \mathcal{O}(x^6)$
$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6)$
$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7)$
$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \mathcal{O}(x^7)$
$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5)$
$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \mathcal{O}(x^4)$

## Parity of Functions

<b>Even:</b> $f(-x) = f(x) \quad \forall x \in D$	$ x , \cos x, x^2$
<b>Odd:</b> $f(-x) = -f(x) \quad \forall x \in D$	$x, \sin, \tan, x^3$

**Hint.** Chaining odd functions results in an odd function.

## Common Derivatives and Integrals

<b>F(x)</b>	<b>f(x)</b>
$c$	0
$x^a$	$a \cdot x^{a-1}$
$\frac{1}{a+1} x^{a+1}$	$x^a$
$\frac{1}{a \cdot (n+1)} (ax+b)^{n+1}$	$(ax+b)^n$
$\frac{x^{a+1}}{a+1}$	$x^a, a \neq -1$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n} x^{\frac{1}{n}-1}$
$\frac{2}{3} x^{\frac{3}{2}}$	$\sqrt{x}$
$\frac{n}{n+1} x^{\frac{1}{n}+1}$	$\sqrt[n]{x}$
$e^x$	$e^x$
$\ln( x )$	$\frac{1}{x}$
$\log_a( x )$	$\frac{1}{x \ln(a)} = \log_a(e^{\frac{1}{x}})$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x) = \frac{\cos(x)}{\sin(x)}$	$\frac{1}{-\sin^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\sinh(x) = \frac{e^x + e^{-x}}{2}$	$\cosh(x)$
$\cosh(x) = \frac{e^x - e^{-x}}{2}$	$\sinh(x)$
$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\frac{1}{f(x)}$	$\frac{-f'(x)}{(f(x))^2}$
$a^{cx}$	$a^{cx} \cdot c \ln(a)$
$x^x$	$x^x \cdot (1 + \ln(x)), x > 0$
$(x^x)^x$	$(x^x)^x (x + 2x \ln(x)), x > 0$
$x^{x^x}$	$x^{x^x} (x^{x-1} + \ln(x) \cdot x^x (1 + \ln(x)))$

Trigonometry

Periodicity

$$\sin(x) = \sin(x + 2\pi)$$
$$\tan(x) = \tan(x + \pi)$$

$$\tan(x) = \cos(x + 2\pi)$$
$$\cot(x) = \cot(x + \pi)$$

Parity

$$\sin(-x) = -\sin(x)$$
$$\tan(-x) = -\tan(x)$$

$$\cos(-x) = \cos(x)$$
$$\cot(-x) = -\cot(x)$$

Complement

$$\sin(\pi - x) = \sin(x)$$
$$\tan(\pi - x) = -\tan(x)$$

$$\cos(\pi - x) = -\cos(x)$$
$$\cot(\pi - x) = -\cot(x)$$

Multiple-angles formulae

$$\sin(2x) = 2 \sin x \cos x$$
$$\sin(3x) = 3 \sin x - 4 \sin^3 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$\cos(3x) = 3 \cos^3 x - 3 \cos x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\cot(2x) = \frac{\cot x - \tan x}{2}$$

Addition Theorems

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin x$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$$

Multiplication

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$$
$$\sin x \cos y = \frac{1}{2}(\sin(x - y) + \sin(x + y))$$

$$\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$$

Powers

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$
$$\tan^2 x = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$\sin^3 x = \frac{3 \sin x - \sin(3x)}{4}$$
$$\cos^3 x = \frac{3 \cos x + \cos(4x)}{4}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

Miscellaneous

$$\sin^2 x + \cos^2 x = 1$$
$$\sin x^{(n)} = \sin \left( x + \frac{n\pi}{2} \right)$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$\cos x^{(n)} = \cos \left( x + \frac{n\pi}{2} \right)$$

Angles

deg	0°	30°	45°	60°	90°	180°	270°	360°
rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\pi$	$\frac{3}{2}\pi$	$2\pi$
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0	1
$\tan(x)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	NaN	0	NaN	45