

Analysis I Summary

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1 Real numbers, euclidean spaces

Archimedes' principle. If $x \in \mathbb{R}$ with $x > 0$ and $y \in \mathbb{R}$, then $\exists n \in \mathbb{N}$ ($y \leq n \cdot x$)

Thm. (i) $|x| \geq 0 \quad \forall x \in \mathbb{R}$
(ii) $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$
(iii) $|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$
(iv) $|x + y| \geq ||x| - |y|| \quad \forall x, y \in \mathbb{R}$

Young's inequality. $\forall \epsilon > 0, \forall x, y \in \mathbb{R}$:

$$2|xy| \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$$

2 Sequences

2.1 Convergence

$(a_n)_{n \geq 1}$ converges to $L = \lim_{n \rightarrow \infty} a_n$
 $\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N (|a_n - L| < \epsilon)$

Def (Convergence). $(a_n)_{n \geq 1}$ converges
 $\iff \exists L \in \mathbb{R} \forall \epsilon > 0 (\{n \in \mathbb{N} \mid |a_n - L| \geq \epsilon\})$ is finite.

Hint. Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$ converge with limit a and b :

1. $(a_n + b_n)_{n \geq 1}$ converges with limit $a + b$
2. $(a_n \cdot b_n)_{n \geq 1}$ converges with limit $a \cdot b$.
3. $(\frac{a_n}{b_n})_{n \geq 1}$ converges with limit $\frac{a}{b}$
4. $\exists K \geq 1 \forall n \geq K : a_n \leq b_n \implies a \leq b$

2.1.1 Tips & Tricks

- a_n convergent $\implies a_n$ bounded
- a_n convergent $\iff a_n$ bounded and $\liminf a_n = \limsup a_n$

Monotone Convergence. $(a_n)_{n \geq 1}$ monotone increasing and upper bounded $\implies \lim a_n = \sup\{a_n \mid n \geq 1\}$
 $(a_n)_{n \geq 1}$ monotone decreasing and lower bounded $\implies \lim a_n = \inf\{a_n \mid n \geq 1\}$

Lemma (Bernoulli Inequation).

$$(1 + x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}, x > -1$$

Def (Limit inferior / Limit superior).

$$\liminf_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} (\inf\{a_k \mid k \geq n\})$$

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} (\sup\{a_k \mid k \geq n\})$$

Cauchy Criteria. a_n converges iff $\forall \epsilon > 0 \exists N \geq 1$ s.t. $|a_n - a_m| < \epsilon \forall n, m \geq N$ (cauchy sequence).

- (i) Each Cauchy sequence is bounded
- (ii) $(a_n)_{n \geq 1}$ conv. $\implies (a_n)_{n \geq 1}$ cauchy
- (iii) $(a_n)_{n \geq 1}$ cauchy $\implies (a_n)_{n \geq 1}$ conv.

Bolzano-Weierstrass. Each bounded sequence contains a convergent sub sequence.

Sandwich. If $\lim a_n = \alpha, \lim b_n = \alpha, k \in \mathbb{N}$ and $a_n \leq c_n \leq b_n \forall n \geq k$, then $\lim c_n = \alpha$

Cauchy-Cantor. Let $I_1 \supseteq I_2 \supseteq \dots I_n \dots$ be a sequence of proper intervals with $\mathcal{L} < +\infty$, then $\bigcap_{n \geq 1} I_n \neq \emptyset$. And if $\lim_{n \rightarrow \infty} \mathcal{L}(I_n) = 0$ then $|\bigcap_{n \geq 1} I_n| = 1$

Cor. Let (a_n) be bounded, then for each subsequence (b_n) : $\liminf a_n \leq \lim b_n \leq \limsup a_n$.

Each subsequence (b_n) of a convergent (a_n) converges and $\lim b_n = \lim a_n$.

3 Series

Def. " $\sum_{k=1}^{\infty} a_k$ " converges, if the sequence $(S_n)_{n \geq 1}$ of partial sums converges and $\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$

Thm. $\sum_{k=1}^{\infty} a_k$ and $\sum_{j=1}^{\infty} b_j$ convergent:

- $\sum_{k=1}^{\infty} (a_k + b_k) = (\sum_{k=1}^{\infty} a_k) + (\sum_{k=1}^{\infty} b_k)$
- $\sum_{k=1}^{\infty} \alpha \cdot a_k = \alpha \sum_{k=1}^{\infty} a_k$

Cauchy Criteria. $\sum_{k=1}^{\infty} a_k$ conv. $\iff \forall \epsilon > 0 \exists N \geq 1 : |\sum_{k=n}^m a_k| = |S_m - S_n| < \epsilon \forall m \geq n \geq N$

Zero Sequence Criteria. $\sum_{k=1}^{\infty} a_k$ conv. $\implies \lim a_k = 0$

Comparison Theorem. Let $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$ series s.t. $0 \leq a_k \leq |a_k| \leq b_k \quad \forall k \geq 1$:

$$\sum_{k=1}^{\infty} b_k \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges absolute}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges} \implies \sum_{k=1}^{\infty} b_k \text{ diverges}$$

Thm. Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0 \quad \forall k \in \mathbb{N}^*$

$$\sum_{k=1}^{\infty} \text{ converges} \iff (S_n)_{n \geq 1} \text{ upper bounded}$$

Def (Absolute Convergence). $\sum_{k=1}^{\infty} a_k$ absolute converges if $\sum_{k=1}^{\infty} |a_k|$ converges.

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \not\Rightarrow \sum_{k=1}^{\infty} |a_k| \text{ converges}$$

(Dirichlet) If a series converges absolute, then each permutation of the series converges with the same limit.

(Riemann) If a series only converges, then there exists a permutation such that:

$$\sum_{k=1}^{\infty} a_{\phi(k)} = x \quad \forall x \in \mathbb{R} \cup \{\infty\}$$

Thm.

$$\left| \sum_{k=1}^{\infty} a_k \right| \leq \sum_{k=1}^{\infty} |a_k|$$

Leibniz. (a_n) mon. dec. s.t. $a_n \geq 0 \forall n \geq 1 \wedge \lim a_n = 0$:

$$S := \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges.}$$

Furthermore: $a_1 - a_2 \leq S \leq a_1$

Ratio Test. Let $(a_n)_{n \geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$:

$$\limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$$

$$\liminf_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Lemma. $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$
- $L > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$
- $L = 1 \implies \text{no information}$

Root Test. Let $(a_n)_{n \geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$:

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$$

$$\liminf_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} |a_n| \text{ diverge.}$$

Lemma. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$
- $L > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$
- $L = 1 \implies \text{no information}$

Def (Cauchy Product). $\sum_{i=0}^{\infty} a_i, \sum_{j=0}^{\infty} b_j$:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n a_{n-j} b_j \right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots$$

Thm. $\sum_{i=0}^{\infty} a_i, \sum_{j=0}^{\infty} b_j \text{ conv. abs.} \implies \text{Cauchy prod. conv. abs.:}$

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n a_{n-j} b_j \right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_i b_j$$

Hint (Strategy: Convergence of Series).

1. Check for known types. Telescope, Geometric, etc.
2. $\lim |a_n| \neq 0 \implies \text{divergence}$

3. Ratio Test

4. Root Test

5. Search convergent majors: $0 \leq a_n \leq b_n$

6. If divergent minors \implies divergence

7. Be creative

4 Functions

$\mathbb{R}^D = \{f : D \rightarrow \mathbb{R} \mid f \text{ is function}\}, (\mathbb{R}^D; +, \cdot) \text{ is V.R.}$

4.1 Continuity

Def (Continuity). A function f is continuous in x_0 if:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon)$$

$$\iff$$

$$\forall (a_n) \text{ with } \lim_{n \rightarrow \infty} a_n = x_0 : \lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n) = f(x_0)$$

Def. A function $f : D \rightarrow \mathbb{R}$ is continuous if it is continuous in all $x_0 \in D$

Hint. To prove continuity try to filter $|x - x_0|$ out of $|f(x) - f(x_0)|$ and choose δ , such that the rest term disappears. Be aware that δ is part of ϵ and normally $|x_0|$ as well. But not x !

Cor. $f, g : D \rightarrow \mathbb{R}$ continuous in $x_0 \in D$. Then:

- $fg, \lambda f, f \pm g$ continuous in x_0
- $\frac{f}{g} : D \setminus \{x \in D \mid g(x) = 0\} \rightarrow \mathbb{R}$ continuous in x_0 ($g(x_0) \neq 0$)
- $|f|, \max(f, g), \min(f, g)$ continuous in x_0
- $P(x) = a_n x^n + \dots + a_0$ continuous on \mathbb{R}
- $\frac{P(x)}{Q(x)}$ continuous on $\mathbb{R} \setminus \{x_1, \dots, x_m\}$ if x_1, \dots, x_m are roots of $Q(x)$

Thm. Let $f : D_1 \rightarrow D_2 \subset \mathbb{R}, g : D_2 \rightarrow \mathbb{R}$ be continuous $\implies g \circ f : D_1 \rightarrow \mathbb{R}$ continuous.

Bolzano (Intermediate value theorem). Let $I \subseteq \mathbb{R}, f : I \rightarrow \mathbb{R}$ continuous and $a, b \in I$. For each c between $f(a)$ and $f(b)$ there is a $z \in [a, b]$ with $f(z) = c$

Min-Max. Let $f : I = [a, b] \rightarrow \mathbb{R}$ be continuous.

$$\exists u, v \in I \forall x \in I (f(u) \leq f(x) \leq f(v))$$

In particular $f([a, b]) \subset [f(u), f(v)]$ is bounded.

Cor. $I = [a, b], f : I \rightarrow \mathbb{R}$ continuous, then $\text{Im}(f) = f(I)$ is a compact interval $J = [\min f, \max f] = [f(u), f(v)]$

Inverse Mapping. Let $f : I \rightarrow \mathbb{R}$ be continuous and strict monotone increasing. Then $J := f(I) \subseteq \mathbb{R}$ is an interval and $f^{-1} : J \rightarrow I$ is continuous and strict monotone.

4.2 Exponential function

$\exp : \mathbb{R} \rightarrow]0, \infty[$ is continuous, strictly monotone increasing, surjective.

- $\exp(x) \geq 1 + x \quad \forall x \in \mathbb{R}$
- For $x > 0, a \in \mathbb{R} : x^a := \exp(a \ln x)$
- $x^0 = 1 \quad \forall x > 0$

Def. The inverse mapping of $\exp(x)$ is called the natural logarithm:

$$\ln :]0, \infty[\rightarrow \mathbb{R}, \quad x \mapsto \ln x$$

It is strictly monotone increasing, continuous and bijective.

4.3 Converge of function sequences

$$\mathbb{N} \rightarrow \mathbb{R}^D = \{f : D \rightarrow \mathbb{R}\}, \quad n \mapsto f(n)$$

Def (pointwise convergence). $(f_n)_{n \geq 0}$ converges pointwise to a function $f : D \rightarrow \mathbb{R}$, if $\forall x \in D : \lim_{n \rightarrow \infty} f_n(x) = f(x)$

$$\iff$$

$$\forall x \in D \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N (|f_n(x) - f(x)| < \epsilon)$$

Def (uniform convergence (Weierstrass)). $f_n : D \rightarrow \mathbb{R}$ converges uniformly in D to $f : D \rightarrow \mathbb{R}$ if:

$$\forall \epsilon > 0 \exists N \geq 1 \text{ s.t. } \forall n \geq N \forall x \in D (|f_n(x) - f(x)| < \epsilon)$$

$$\iff$$

$$\lim_{n \rightarrow \infty} \sup_{x \in D} |f_n(x) - f(x)| = 0$$

The function sequence (f_n) is uniformly convergent if for all $x \in D$ the limit $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ exists and the sequence (f_n) uniformly converges to f . Furthermore if $\forall \epsilon > 0 \exists N \geq 1 \forall n, m \geq N \forall x \in D : |f_n(x) - f_m(x)| < \epsilon$.

The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly (in D), if the function sequence $S_n(x) := \sum_{k=0}^n f_k(x)$ converges uniformly.

Thm. let $D \subseteq \mathbb{R}$ and $f_n : D \rightarrow \mathbb{R}$ a function sequence containing (in D) continuous functions which converge (in D) uniformly against a function $f : D \rightarrow \mathbb{R}$, then f (in D) is continuous.

Hint (not uniform convergent). $(f_n)_{n \geq 0}$ converges not uniformly if: $\forall \epsilon > 0 \forall N \in \mathbb{N} \exists x \in D (|f_n(x) - f(x)| \geq \epsilon)$

Hint. Check the function and try to construct x (dependent on N in general), such that $|f_n(x) - f(x)|$ is always greater than a specific ϵ and afterwards choose the ϵ .

Def (Power Functions). $\sum_{k=0}^{\infty} c_k x^k$ has positive convergence radius if $\limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}$ exists.

$$\rho = \begin{cases} +\infty & , \text{ if } \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} = 0 \\ \frac{1}{\limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}} & , \text{ if } \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} > 0 \end{cases}$$

Thm. Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $\rho > 0$ and let $f(x) = \sum_{k=0}^{\infty} c_k x^k, |x| < \rho$ Then: $\forall 0 \leq r < \rho$ converges $\sum_{k=0}^{\infty} c_k x^k$ uniformly on $[-r, r]$, furthermore $f :]-\rho, \rho[\rightarrow \mathbb{R}$ is continuous.

4.4 Trigonometric Functions

\sin and \cos are continuous functions $\mathbb{R} \rightarrow \mathbb{R}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Thm. 1. $\exp(iz) = \cos(z) + i \sin(z) \quad \forall z \in \mathbb{C}$
2. $\cos z = \cos(-z)$ und $\sin(-z) = -\sin(z) \quad \forall z \in \mathbb{C}$

3. $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$
4. $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
 $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
5. $\cos(z)^2 + \sin(z)^2 = 1 \quad \forall z \in \mathbb{C}$

Cor.

$$\sin(2z) = 2\sin(z)\cos(z)$$

$$\cos(2z) = \cos(z)^2 - \sin(z)^2$$

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Def (π). $\pi := \inf\{t > 0 \mid \sin t = 0\}$

- (i) $\sin \pi = 0, \pi \in]2, 4[$
- (ii) $\forall x \in]0, \pi[: \sin x > 0$
- (iii) $e^{\frac{i\pi}{2}} = i$

Cor. $x \geq \sin x \geq x - \frac{x^3}{3!} \quad \forall 0 \leq x \leq \sqrt{6}$

Cor. 1. $e^{i\pi} = -1, \quad e^{2i\pi} = 1$

2. $\sin(x + \frac{\pi}{2}) = \cos(x), \quad \cos(x + \frac{\pi}{2}) = -\sin(x) \quad \forall x \in \mathbb{R}$
3. $\sin(x + \pi) = -\sin(x), \quad \sin(x + 2\pi) = \sin(x) \quad \forall x \in \mathbb{R}$
4. $\cos(x + \pi) = -\cos(x), \quad \cos(x + 2\pi) = \cos(x) \quad \forall x \in \mathbb{R}$
5. *Roots of sinus* $= \{\pi \cdot k \mid k \in \mathbb{Z}\}$
 $\sin(x) > 0 \quad \forall x \in]2k\pi, (2k+1)\pi[, \quad k \in \mathbb{Z}$
 $\sin(x) < 0 \quad \forall x \in [(2k+1)\pi, (2k+2)\pi[, \quad k \in \mathbb{Z}$
6. *Roots of cosine* $= \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$
 $\cos(x) > 0 \quad \forall x \in]-\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + (2k+1)\pi[, \quad k \in \mathbb{Z}$
 $\cos(x) < 0 \quad \forall x \in [-\frac{\pi}{2} + (2k+1)\pi, -\frac{\pi}{2} + (2k+2)\pi[, \quad k \in \mathbb{Z}$

4.5 Limit of Functions

Def (accumulation point). $x_0 \in \mathbb{R}$ is an accumulation point of D if $\forall \delta > 0: (]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) \cap D \neq \emptyset$

Def (Limit of Function). if $f : D \rightarrow \mathbb{R}, x_0 \in \mathbb{R}$ an accumulation point of D , then $A \in \mathbb{R}$ is the limit of $f(x)$ for $x \rightarrow x_0$, written as $\lim_{x \rightarrow x_0} f(x) = A$. If $\forall \epsilon > 0 \exists \delta > 0$ s.t. $\forall x \in D \cap (]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) : |f(x) - A| < \epsilon$

Important Rules. Let $f : D \rightarrow \mathbb{R}$ and x_0 is an accumulation point of D .

1. $\lim_{x \rightarrow x_0} f(x) = A \iff \forall (a_n)_{n \geq 1} \text{ in } D \setminus \{x_0\} \text{ with}$

$$\lim_{n \rightarrow \infty} a_n = x_0 \implies \lim_{n \rightarrow \infty} f(a_n) = A.$$

2. Let $x_0 \in D$. Then f is continuous in x_0
 $\iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$

3. $f, g : D \rightarrow \mathbb{R}$ and $\exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \implies$

$$\lim_{x \rightarrow x_0} (f+g)(x) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$$

$$\lim_{x \rightarrow x_0} (f \cdot g)(x) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x)$$

4. $f, g : D \rightarrow \mathbb{R}$ and $f \leq g$, then if both limit exists

$$\lim_{x \rightarrow x_0} f(x) \leq \lim_{x \rightarrow x_0} g(x)$$

5. If $g_1 \leq f \leq g_2$ and $\lim_{x \rightarrow x_0} g_1(x) = \lim_{x \rightarrow x_0} g_2(x)$ then
 $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g_1(x)$

Hint. Sometimes it can be really helpful to convert known functions to their power series to calculate a limit. E.g.

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \dots}{x} = \lim_{x \rightarrow 0} 1 - \frac{x^2}{3!} + \dots = 1$$

Hint (e^{\log}). Transform ugly function with this trick.

$$\lim_{x \rightarrow x_0} f(x)^{g(x)} = \lim_{x \rightarrow x_0} e^{g(x) \log(f(x))} = e^{\lim_{x \rightarrow x_0} g(x) \log(f(x))}$$

5 Differentiable Functions

Def (Differentiable). f is in x_0 differentiable, if the limit $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$ exists. f is differentiable if $\forall x_0 \in D$ f is differentiable.

Weierstrass. $f : D \rightarrow \mathbb{R}, x_0 \in D$ accumulation point. Equivalent statements:

1. f is in x_0 differentiable
2. It exists $c \in \mathbb{R}$ ($c = f'(x_0)$) and $r : D \rightarrow \mathbb{R}$ s.t.:

$$2.1 \quad f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0)$$

$$2.2 \quad r(x_0) = 0 \text{ and } r \text{ continuous in } x_0.$$

Cor. f diff. in $x_0 \implies f$ continuous in x_0

Thm. f diff. in $x_0 \iff \exists \phi : D \rightarrow \mathbb{R}$ continuous in x_0 s.t. $\forall x \in D : f(x) = f(x_0) + \phi(x)(x - x_0)$. Then $\phi(x_0) = f'(x_0)$.

Derivative rules.

Linearity: $(\alpha \cdot f(x) + g(x))' = \alpha \cdot f'(x) + g'(x)$
 Product rule: $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
 Quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$
 Chain rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Cor. f bijective and in x_0 differentiable s.t. $f'(x_0) \neq 0$. f^{-1} is continuous in $y_0 = f(x_0)$. Then f^{-1} is differentiable in y_0 and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$.

5.1 Derivative Implications

- x_0 is local minimum if $f'(x_0) = 0 \wedge f''(x_0) > 0$ or the sign of f' changes from $-$ to $+$.
- x_0 is local maximum if $f'(x_0) = 0 \wedge f''(x_0) < 0$ or the sign of f' changes from $+$ to $-$.
- x_0 is local extremum if $f'(x_0) = 0 \wedge f''(x_0) \neq 0$
- x_0 is a saddle point if $f'(x_0) = 0$ and $f''(x_0) = 0$
- x_0 is a inflection point if $f''(x_0) = 0 \wedge f^{(3)}(x_0) \neq 0$
- $f'(x_0) = f^{(2)}(x_0) = \dots = f^{(n)}(x_0) = 0$
 - n odd and $f^{(n+1)}(x_0) > 0 \implies x_0$ strict local minimum
 - n odd and $f^{(n+1)}(x_0) < 0 \implies x_0$ strict local maximum

5.2 Derivative Theorems

Rolle. Let $f : [a, b] \rightarrow \mathbb{R}$ continuous and in $]a, b[$ differentiable. If $f(a) = f(b)$, then there exists $\xi \in]a, b[$ with $f'(\xi) = 0$.

Mean Value / Lagrange. Let $f : [a, b] \rightarrow \mathbb{R}$ continuous and in $]a, b[$ differentiable, then there exists $\xi \in]a, b[$ with $f(b) - f(a) = f'(\xi)(b - a)$.

There exists points ξ with $f'(\xi)$ equal to the gradient of the secant between a to b .

Cor. Let $f, g : [a, b] \rightarrow \mathbb{R}$ cont. and diff. in $]a, b[$.

- $\forall \xi \in]a, b[: f'(\xi) = 0 \implies f$ is constant

- $\forall \xi \in]a, b[: f'(\xi) = g'(x) \implies \exists c \in \mathbb{R} \forall x \in [a, b] : f(x) = g(x) + c$
- $\forall \xi \in]a, b[: f'(\xi) \geq 0 \implies f$ in $[a, b]$ mon. inc.
- $\forall \xi \in]a, b[: f'(\xi) > 0 \implies f$ in $[a, b]$ str. mon. inc.
- $\forall \xi \in]a, b[: f'(\xi) \leq 0 \implies f$ in $[a, b]$ mon. dec.
- $\forall \xi \in]a, b[: f'(\xi) < 0 \implies f$ in $[a, b]$ str. mon. dec.
- $\exists M \geq 0 \forall \xi \in]a, b[: |f'(\xi)| \leq M \implies \forall x_1, x_2 \in [a, b] : |f(x_1) - f(x_2)| \leq M|x_1 - x_2|$

Cauchy. $f, g : [a, b] \rightarrow \mathbb{R}$ continuous and in $]a, b[$ diff. Then there exists $\xi \in]a, b[$ with $\frac{g'(b) - g(a)}{g(b) - g(a)} = \frac{f'(b) - f(a)}{f(b) - f(a)}$.

If $\forall x \in]a, b[: g'(x) \neq 0$ it implies that $g(a) \neq g(b)$ and $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$

l'Hôpital. $f, g :]a, b[\rightarrow \mathbb{R}$ diff. with $\forall x \in]a, b[: g'(x) \neq 0$. If $\lim_{x \rightarrow b^-} f(x) = 0, \lim_{x \rightarrow b^-} g(x) = 0$ and $\lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)} =: \lambda$ exists, then $\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)}$.

Hint. Only use l'Hospital if either $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

- Def.**
- (strictly) convex: $(x \leq y) : f(\lambda x + (1 - \lambda)y) (<) \leq \lambda f(x) + (1 - \lambda)f(y)$
 - (strictly) concave: $(x \leq y) : f(\lambda x + (1 - \lambda)y) (>) \geq \lambda f(x) + (1 - \lambda)f(y)$

Lemma. $f : I \rightarrow \mathbb{R}$. f is convex $\iff \forall x_0 < x < x_1 \in I : \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(x_1) - f(x)}{x_1 - x}$. Strictly convex if $<$.

Lemma. $f :]a, b[\rightarrow \mathbb{R}$ diff.

- f' (strictly) mon. inc. dec. $\Rightarrow f$ (strictly) conv. conc.
- $f'' \geq (>) 0 \implies f$ (strictly) conv. ($< / \leq$ for conc.)

5.3 Higher Derivatives

- For $n \geq 2$ is f **n -times differentiable in D** if $f^{(n-1)}$ in D is differentiable. Then $f^{(n)} := (f^{(n-1)})'$ and is the n -th derivative of f
- f is **n -times continuous differentiable in D** if f is n -times differentiable and if $f^{(n)}$ is continuous in D
- f is in D **smooth** if $\forall n \geq 1, f$ is n -times differentiable.

Smooth Functions. $\exp, \sin, \cos, \sinh, \cosh, \tanh, \ln, \arcsin, \arccos, \operatorname{arccot}, \arctan$ and all polynomials. \tan is smooth on $\mathbb{R} \setminus \{\pi/2 + k\pi\}$ and \cot on $\mathbb{R} \setminus \{k\pi\}$

Thm. $f, g : D \rightarrow \mathbb{R}$ are n -times diff. in D .

- $(f + g)^{(n)} = f^{(n)} + g^{(n)}$
- $(f \cdot g)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$
- $(g \circ f)^{(n)}(x) = \sum_{k=1}^n A_{n,k}(x) (g^{(k)} \circ f)(x)$ with $A_{n,k}$ as polynomial in the functions $f', f^{(2)}, \dots, f^{(n+1-k)}$

5.4 Power Series and Taylor approximation

Thm. Let $f_n :]a, b[$ be a function sequence with f_n one time in $]a, b[$ continuous diff. $\forall n \geq 1$. Assume that $(f_n)_{n \geq 1}, (f'_n)_{n \geq 1}$ uniformly convergent in $]a, b[$ with $\lim_{n \rightarrow \infty} f_n =: f$ and $\lim_{n \rightarrow \infty} f'_n =: p$, then f is continuously diff. and $f' = p$.

Thm. Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series with convergent radius $\rho > 0$. Then $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ is differentiable on $]x_0 - \rho, x_0 + \rho[$ and $\forall x \in]x_0 - \rho, x_0 + \rho[: f'(x) = \sum_{k=0}^{\infty} k c_k (x - x_0)^{k-1}$

Cor. $f^{(j)} = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!} (x - x_0)^{k-j}$. Furthermore $c_j = \frac{f^{(j)}(x_0)}{j!}$. Power series can be differentiated part by part in their converge area.

Def (Taylor Polynomial). The n -th Taylor-polynomial of cont. $n + 1$ times diff. in $[c, d]$ f is defined as $T_n(f, x, a)$ with center $a \in]c, d[$ and error $R_n(f, x, a)$. $\forall x \in [a, b] \exists \xi \in]x, a[\cup]a, x[:$

$$T_n(f, x, a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} \cdot (x - a)^k$$

$$R_n(f, x, a) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$$

$$f(x) = T_n(f, x, a) + R_n(f, x, a)$$

Hint. The error can be approximated as

$$|R_n(f, x, a)| \leq \sup_{a < c < x} \left| \frac{f^{(n+1)}(c)(x - a)^{n+1}}{(n+1)!} \right|$$

Def (Taylor Series). $T_{\infty}(f, x, x_0) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$

6 Riemann Integral

$a < b$, $I = [a, b]$, $\mathcal{P}(I) = \{P \mid P \subsetneq I \wedge \{a, b\} \in P \wedge |P| \in \mathbb{N}\}$

Def (Partition). • Partition: $P \in \mathcal{P}(I)$

- $\delta_i := x_i - x_{i-1}$ length of $I_i := [x_{i-1}, x_i]$, $i \geq 1$
- Mesh of partition: $\delta(P) := \max_{1 \leq i \leq n} (x_i - x_{i-1})$
- $\xi := \{\xi_1, \dots, \xi_n\}$, $\xi_i \in I_i$
- P' refines P if $P \subseteq P'$

Def (Riemann Sums). $S(f, P, \xi) := \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$

- Lower sum: $\underline{S}(f, P) := \sum_{i=1}^n (\inf_{x \in I_i} f(x)) (x_i - x_{i-1})$
- Upper sum: $\overline{S}(f, P) := \sum_{i=1}^n (\sup_{x \in I_i} f(x)) (x_i - x_{i-1})$

It holds: $-M(b-a) \leq \underline{S}(f, P) \leq \overline{S}(f, P) \leq M(b-a)$

Lemma. $P \subseteq P'$: $\underline{S}(f, P) \leq \underline{S}(f, P') \leq \overline{S}(f, P') \leq \overline{S}(f, P)$

Lemma. $\forall P_1, P_2 \in \mathcal{P}(I)$: $\underline{S}(f, P_1) \leq \overline{S}(f, P_2)$

Def (Lower Riemann Integral). $\underline{S}(f) := \sup_{P \in \mathcal{P}(I)} \underline{S}(f, P)$

Def (Upper Riemann Integral). $\overline{S}(f) := \inf_{P \in \mathcal{P}(I)} \overline{S}(f, P)$

6.1 Integrability criteria

Def (Integrable). Bounded $f : [a, b] \rightarrow \mathbb{R}$ is integrable if $\underline{S}(f) = \overline{S}(f)$ and the shared value is $\int_a^b f(x) dx$.

Riemann Criteria. Bounded $f : I \rightarrow \mathbb{R}$ is integrable. Let $\mathcal{P}_\delta(I) := \{P \in \mathcal{P}(I) \mid \delta(P) < \delta\}$.

- $\Leftrightarrow \forall \epsilon > 0 \exists P \in \mathcal{P}(I) : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon$
- $\Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 \forall P \in \mathcal{P}_\delta(I) : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon$
- $\Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 \forall P \in \mathcal{P}$ with $\delta(P) < \delta$:

$$\left| A - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon$$

Hint. Bounded $f : [a, b] \rightarrow \mathbb{R}$ is int. if $\lim_{\delta(P) \rightarrow 0} S(f, P, \xi)$ exists for all P with $\delta(P) \rightarrow 0$. It follows that $\lim_{\delta(P) \rightarrow 0} S(f, P, \xi) = \int_a^b f(x) dx$

6.2 Integrable Functions

1. f (bounded) cont. in $[a, b] \implies f$ int. over $[a, b]$
2. f monotone in $[a, b] \implies f$ int. over $[a, b]$
3. If f, g bounded and int., then integrable as well:

$$f + g, \lambda \cdot f, f \cdot g, |f|, \min(f, g), \max(f, g), \frac{f}{g}$$

4. All polynomials are integrable, even $\frac{P(x)}{Q(x)}$ if $Q(x)$ has no root in $[a, b]$

Hint. Let $V := \{f : I \rightarrow \mathbb{R} \mid f \text{ is a mapping}\}$. $(V, +, \cdot)$ is a vector space. Then it implies that $W := \{f : I \rightarrow \mathbb{R} \mid f \text{ is integrable}\}$ is a subspace of V .

6.3 Integration Inequalities and Theorems

Def (Uniform continuous).

$\forall \epsilon > 0 \exists \delta > 0 \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$

Thm. $f : [a, b] \rightarrow \mathbb{R}$ cont. $\implies f$ is uni. cont. in $[a, b]$.

Thm. f uni. cont. $\implies f$ cont.

Thm. $f, g : [a, b] \rightarrow \mathbb{R}$ bounded and integrable and $\forall x \in [a, b] : f(x) \leq g(x) \implies \int_a^b f(x) dx \leq \int_a^b g(x) dx$

Cauchy-Schwarz.

$$\left| \int_a^b f(x)g(x) dx \right| \leq \sqrt{\int_a^b f^2(x) dx} \sqrt{\int_a^b g^2(x) dx}$$

Hint. $\langle f, g \rangle := \int_a^b f(x)g(x) dx$ is a scalar product. $\|f\|^2 = \langle f, f \rangle = \int_a^b f^2(x) dx$.

Mean Value Theorem. $f : [a, b] \rightarrow \mathbb{R}$ continuous $\implies \exists \xi \in [a, b] : \int_a^b f(x) dx = f(\xi)(b - a)$

Cauchy. $f, g : [a, b] \rightarrow \mathbb{R}$ with f continuous and g bounded and integrable with $g(x) \geq 0, \forall x \in [a, b]$

$$\implies \exists \xi \in [a, b] : \int_a^b f(x)g(x) dx = f(\xi) \int_a^b g(x) dx$$

6.4 Integration Properties

Additive Property.

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Linearity.

$$\int_a^b (\alpha f_1 + \beta f_2) dx = \alpha \int_a^b f_1(x) dx + \beta \int_a^b f_2(x) dx$$

Preservation of Order.

$$\forall x \in [a, b] : f(x) \leq g(x) \implies \int_a^b f(x) dx \leq \int_a^b g(x) dx$$

Triangle Inequality.

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

6.5 Primitive Functions

Def (Primitive Function). $F : [a, b] \rightarrow \mathbb{R}$ is a primitive function of f if F is cont. diff. and $F' = f$.

Hint. f is integrable \nRightarrow exists a primitive function for f .

HID. Let $a < b$, $f : [a, b] \rightarrow \mathbb{R}$ continuous. The function

$$F(x) := \int_a^x f(t) dt \quad a \leq x \leq b$$

is cont. diff. in $[a, b]$ and $F'(x) = f(x) \forall x \in [a, b]$.

Fundamental theorem of calculus. $f : [a, b] \rightarrow \mathbb{R}$ continuous. Then there exists a unique (except a constant term) primitive function F of f , such that

$$\int_a^b f(x) dx = F(b) - F(a).$$

6.6 Integration Methods

Partial Inegration.

$$\begin{aligned}\int_a^b f(x)g'(x) dx &= f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x) dx \\ \int_a^b f(x)g'(x) dx &= (f \cdot g)|_a^b - \int_a^b f'(x)g(x) dx \\ \int_a^b f(x)g'(x) dx &= f(x)g(x) - \int_a^b f'(x)g(x) dx\end{aligned}$$

- Choose g' : exp \rightarrow trig \rightarrow poly \rightarrow inverse trig. \rightarrow logs
- Choose f : logs \rightarrow inverse trig. \rightarrow poly \rightarrow trig \rightarrow exp
- Sometimes it is necessary to multiply by 1. E.g.: $\int \ln x dx = \int \ln x \cdot 1 dx \implies f(x) = \ln x, g'(x) = 1.$
- Sometimes it is necessary to do it multiple times

Substitution. Let $a < b, \phi : [a, b] \rightarrow \mathbb{R}$, cont. diff, $I \subseteq \mathbb{R}$ with $\phi([a, b]) \subseteq I$ and $f : I \rightarrow \mathbb{R}$ a cont. function. Then it follows:

$$\int_{\phi(a)}^{\phi(b)} f(x) dx = \int_a^b f(\phi(t))\phi'(t)dt = (F \circ \phi)(b) - (F \circ \phi)(a)$$

since $F' = f$ then $f(\phi(t))\phi'(t) = (F \circ \phi)'(t).$

Partial Fraction Decomposition. Let $P(x), Q(x)$ be two polynomials. $\int \frac{P(x)}{Q(x)}$ can be calculated as follows:

1. If $\deg(P) \geq \deg(Q) \implies$ poly. div. $\frac{P(x)}{Q(x)} = a(x) + \frac{r(x)}{Q(x)}$
2. Calculate all roots of $Q(x)$
3. Create a partial fraction per root
 - Simple real root: $x_1 \rightarrow \frac{A}{x-x_1}$
 - n -fold real root: $x_1 \rightarrow \frac{A_1}{x-x_1} + \dots + \frac{A_r}{(x-x_1)^r}$
 - Simple i -root: $x^2 + px + q \rightarrow \frac{Ax+B}{x^2+px+q}$
 - n -fold i -root: $x^2 + px + q \rightarrow \frac{A_1x+B_1}{x^2+px+q} + \dots + \frac{A_rx+B_r}{(x^2+px+q)^r}$
4. Calculate parameters A_1, \dots, A_n . (Insert the root as s , transform and solve)

Hint (Odd functions). $\int_{-\lambda}^{\lambda} f(x) dx = 0.$

Cor. $\int_{a+c}^{b+c} f(x) dx = \int_a^b f(t+c)dt$

Cor. $\int_a^b f(ct)dt = \frac{1}{c} \int_{ac}^{bc} f(x) dx$

6.7 Integration of convergent series

Thm. Let $f_n : [a, b] \rightarrow \mathbb{R}$ be a sequence of bounded, integrable functions which converge uniformly against a function $f : [a, b] \rightarrow \mathbb{R}$. Then f is bounded and integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = \int_a^b f(x) dx$$

$$\sum_{n=0}^{\infty} \int_a^b f_n(x) dx = \int_a^b \left(\sum_{n=0}^{\infty} f_n(x) \right) dx$$

Thm. Let $f(x) := \sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $p > 0$. Then $\forall 0 \leq r < p$ f is integrable on $[-r, r]$ and $\forall x \in]-p, p[$: $\int_0^x f(t)dt = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$

6.8 Improper Integral

$$f : [a, \infty) \rightarrow \mathbb{R}, f : [-\infty, a] \rightarrow \mathbb{R}, f : (-\infty, \infty) \rightarrow \mathbb{R}$$

Def. Let $f : [a, \infty[\rightarrow \mathbb{R}$ be bounded and integrable on $[a, b]$ for all $b > a$. If $\lim_{b \rightarrow \infty} \int_a^b f(x) dx$ exists, the limit is defined as $\int_a^{\infty} f(x) dx$ and one can say that f is integrable on $[a, +\infty[$. If the limit does not exist, one can say that $\int_a^{\infty} f(x) dx$ diverges.

Comparison Theorem. Let $f : [a, \infty[\rightarrow \mathbb{R}$ be bounded and integrable on $[a, b] \forall b \in \mathbb{R}, b > a$.

1. If $\forall x \geq a : |f(x)| \leq g(x)$ and $g(x)$ is integrable on $[a, \infty[\implies f$ is integrable on $[a, \infty[$.
2. If $0 \leq g(x) \leq f(x)$ and $\int_a^{\infty} g(x) dx$ diverges $\implies \int_a^{\infty} f(x) dx$ diverges.

Hint. Sometimes an integral can be split into a normal integral and an improper integral:

$$\int_0^{\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx$$

McLaurin. Let $f : [1, \infty[\rightarrow [0, \infty[$ be mon. dec.

$$\sum_{n=1}^{\infty} f(n) \text{ converges } \iff \int_1^{\infty} f(x) dx \text{ converges.}$$

The following holds:

$$0 \leq \sum_{k=1}^{\infty} f(k) - \int_1^{\infty} f(x) dx \leq f(1)$$

Def. Let f be a function which is bounded and integrable on all intervals $[a + \epsilon, b] \forall \epsilon > 0$. $f :]a, b] \rightarrow \mathbb{R}$ is integrable if $\lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b f(x) dx$ exists. In this case the limit is defined as $\int_a^b f(x) dx$. (The comparison theorem can be used for such integrals as well.)

6.9 Indefinite Integrals

Let $f : I \rightarrow \mathbb{R}$ be defined on the interval $I \subseteq \mathbb{R}$. If f is continuous there exists a primitive function F .

$$\int f(x) dx = F(x) + C$$

The indefinite integral is the inverse of the derivative.

Hint. $\int_{-\infty}^{\infty} f(x) dx = \lim_{b \rightarrow -\infty} \int_b^a f(x) dx + \lim_{c \rightarrow \infty} \int_a^c f(x) dx.$

$$\int_{-\infty}^{\infty} f(x) dx \text{ conv. } \iff \int_{-\infty}^a f(x) dx \text{ conv. } \wedge \int_a^{\infty} f(x) dx \text{ conv.}$$

In general: Let $f :]a, b[\rightarrow \mathbb{R}$ such that it is integrable on each compact interval $[\tilde{a}, \tilde{b}]$. Then

$$\int_a^b f(x) dx := \lim_{\tilde{a} \searrow a} \lim_{\tilde{b} \nearrow b} \int_{\tilde{a}}^{\tilde{b}} f(x) dx$$

6.10 Euler Gamma Function

Def. For $s > 0$:

$$\Gamma(s) := \int_0^{\infty} e^{-x} x^{s-1} dx.$$

The gamma function interpolates the function $n \mapsto (n-1)!$. It converges for all $s > 0$.

Useful Listings

Limits

$$\begin{array}{ll} \lim_{x \rightarrow \infty} \frac{1}{x} = 0 & \lim_{x \rightarrow \infty} 1 + \frac{1}{x} = 1 \\ \lim_{x \rightarrow \infty} e^x = \infty & \lim_{x \rightarrow -\infty} e^x = 0 \\ \lim_{x \rightarrow \infty} e^{-x} = 0 & \lim_{x \rightarrow -\infty} e^{-x} = \infty \\ \lim_{x \rightarrow \infty} \frac{e^x}{x^m} = \infty & \lim_{x \rightarrow -\infty} x e^x = 0 \\ \lim_{x \rightarrow \infty} \ln(x) = \infty & \lim_{x \rightarrow 0} \ln(x) = -\infty \\ \lim_{x \rightarrow \infty} (1+x)^{\frac{1}{x}} = 1 & \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \\ \lim_{x \rightarrow \infty} (1 \pm \frac{1}{x})^\lambda = 1 & \lim_{x \rightarrow \infty} (1 + \frac{\lambda}{x})^x = e^\lambda \\ \lim_{x \rightarrow \infty} x^\lambda q^x = 0, \forall 0 \leq q < 1 & \lim_{x \rightarrow \infty} \sqrt[x]{x} = 1 \\ \lim_{x \rightarrow \pm \infty} (1 + \frac{1}{x})^x = e & \lim_{x \rightarrow \infty} (1 - \frac{1}{x})^x = \frac{1}{e} \\ \lim_{x \rightarrow \pm \infty} (1 + \frac{\lambda}{x})^{\alpha x} = e^{\lambda \alpha} & \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \\ \lim_{x \rightarrow 0} \frac{1}{\cos(x)} = 1 & \lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0 \\ \lim_{x \rightarrow 0} \frac{\log(1-x)}{x} = -1 & \lim_{x \rightarrow \infty} \frac{\sin(x)}{x} = 0 \\ \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2} & \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \\ \lim_{x \rightarrow 0} \frac{x}{\arctan(x)} = 1 & \lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2} \\ \lim_{x \rightarrow \infty} (\frac{x}{x+\lambda})^x = e^{-\lambda} & \lim_{h \rightarrow 0} (1 + \frac{h}{x})^{\frac{x}{h}} = e \\ \lim_{x \rightarrow 0} \frac{\lambda^x - 1}{x} = \ln(\lambda), \lambda > 0 & \lim_{x \rightarrow 0} \frac{e^{\lambda x} - 1}{x} = \lambda \\ \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1 & \lim_{x \rightarrow 1} \frac{\ln(x)}{x-1} = 1 \\ \lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = 0 & \lim_{x \rightarrow \infty} \frac{\log(x)}{x^\lambda} = 0 \\ \lim_{x \rightarrow \infty} \frac{\lambda x}{\lambda^x} = 0 & \lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) = +\infty \\ \lim_{x \rightarrow \frac{\pi}{2}^+} \tan(x) = -\infty & \lim_{x \rightarrow 0} x \log x = 0 \\ \lim_{x \rightarrow 0^+} x \ln x = 0 & \text{Stirling Formula} \\ & \lim_{x \rightarrow \infty} \frac{x!}{(\frac{x}{e})^x \sqrt{2\pi x}} = 1 \end{array}$$

Series

- Geometric: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $|q| < 1$
- Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
- Telescope: $\sum_{n=0}^{\infty} \frac{1}{n(n+1)} = 1$
- $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n \rightarrow \infty} (1 + \frac{z}{n})^n = e^z$
- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges $s > 1$ ($\frac{1}{1-\frac{1}{2^s-1}}$)
- $p(z) = \sum_{k=0}^{\infty} c_k z^k$ conv. abs. $|z| < \rho = \frac{1}{\limsup |c_k|^{1/k}}$

$$\begin{array}{ll} \sum_{i=1}^n i = \frac{n(n+1)}{2} & \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4} & \sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \end{array}$$

Taylor Series

$$\begin{array}{l} e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5) \\ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \mathcal{O}(x^7) \\ \sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7) \\ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \mathcal{O}(x^6) \\ \cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6) \\ \tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7) \\ \tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \mathcal{O}(x^7) \\ \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5) \\ \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \mathcal{O}(x^4) \end{array}$$

Parity of Functions

Even: $f(-x) = f(x) \quad \forall x \in D$ $|x|, \cos x, x^2$
Odd: $f(-x) = -f(x) \quad \forall x \in D$ x, \sin, \tan, x^3

Hint. Chaining odd functions results in an odd function.

Tips and Tricks from PVW script

(Inductive sequences) $a_1 := C, a_{n+1} := f(a_n) (\forall n > 1)$

1. Show monotonicity (either by induction over \mathbb{N} or show $b_n = a_{n+1} - a_n$ (strictly) pos/neg $\forall n > 1$)
2. Show that the sequence is bounded (by induction)
3. Monotone convergence theorem $\implies \exists \lim_{n \rightarrow \infty} a_n$
4. $\lim_{n \rightarrow \infty} a_n = a \implies \forall \text{subseq. } l(n) : \lim_{n \rightarrow \infty} a_{l(n)} = a$
Choose $l(n) = n+1 \implies \lim_{n \rightarrow \infty} a_{n+1} = a$
And thus $a = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n)$

Hint. Do no forget to write: f as composition of two continuous functions is continuous when using MVT.

(Change of variables) Let f and g be functions with f continuous in y_0 and g continuous in x_0 with $y_0 = \lim_{x \rightarrow x_0} g(x)$:

$$\lim_{x \rightarrow x_0} f(g(x)) = \lim_{y \rightarrow y_0} f(y)$$

(Powerrule) Let $f, g : D \rightarrow \mathbb{R}$ be cont. in x_0 with $\lim_{x \rightarrow x_0} f(x) = f(x_0) > 0$ and $\lim_{x \rightarrow x_0} g(x) = g(x_0)$ (both exist), then:

$$\lim_{x \rightarrow x_0} f(x)^{g(x)} = f(x_0)^{g(x_0)}$$

(Taylor Polynomial $\sin(4)$ with precision d)

1. Choose a fitting center and denote the new function
 - $f(x) = \sin(x)$ with $x_0 = 4, a = \pi$ **or**
 - $f(x) = \sin(\pi + x)$ with $x_0 = 4 - \pi, a = 0$
2. Calculate the $(N+1)$ -th derivative of $f(x)$
3. Calculate the error $|R_n(f, x_0, a)|$
4. Solve the inequality $|R_n(f, x_0, a)| < d$ to N .
5. Solve the taylor polynomial for the given N .

Hint (Convergence of integrals). An integral does not converge if it is not bounded in the given interval.

$$\lim_{x \rightarrow 2} \frac{1}{(x-2)^2} = \infty \implies \int_0^\infty \frac{1}{(x-2)^2} dx \text{ diverges}$$

Trigonometry

Periodicity

$$\sin(x) = \sin(x + 2\pi) \quad \cos(x) = \cos(x + 2\pi)$$

$$\tan(x) = \tan(x + \pi) \quad \cot(x) = \cot(x + \pi)$$

Parity

$$\sin(-x) = -\sin(x) \quad \cos(-x) = \cos(x)$$

$$\tan(-x) = -\tan(x) \quad \cot(-x) = -\cot(x)$$

Complement

$$\sin(\pi - x) = \sin(x) \quad \cos(\pi - x) = -\cos(x)$$

$$\tan(\pi - x) = -\tan(x) \quad \cot(\pi - x) = -\cot(x)$$

Multiple-angles formulae

$$\sin(2x) = 2 \sin x \cos x \quad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x} \quad \cot(2x) = \frac{\cot x - \tan x}{2}$$

$$\sin(3x) = 3 \sin x - 4 \sin^3 x \quad \cos(3x) = 4 \cos^3 x - 3 \cos x$$

Addition Theorems

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$$

Multiplication

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$$

$$\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$$

$$\sin x \cos y = \frac{1}{2}(\sin(x - y) + \sin(x + y))$$

Powers

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \sin^3 x = \frac{3 \sin x - \sin(3x)}{4}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \cos^3 x = \frac{3 \cos x + \cos(3x)}{4}$$

$$\tan^2 x = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

Sum of functions

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = 2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

Miscellaneous

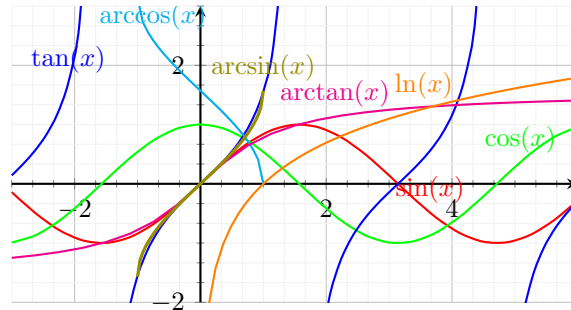
$$\sin^2 x + \cos^2 x = 1 \quad \cosh^2 x - \sinh^2 x = 1$$

$$\sin x^{(n)} = \sin \left(x + \frac{n\pi}{2} \right) \quad \cos x^{(n)} = \cos \left(x + \frac{n\pi}{2} \right)$$

Angles

deg	0	30	45	60	90	120	135	150	180	270	360
rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	-	0

Important Functions



Useful bound for sin

$$\forall x \in \mathbb{R}_0^+ : \sin(x) \leq x$$

Proof. Let $g(x) = x - \sin(x)$ with $g'(x) = 1 - \cos(x) \geq 0$ \square

Derivatives and Integrals ([src: dcamenisch](#))

$F(x)$	$f(x)$
c	0
x^a	$a \cdot x^{a-1}$
$\frac{1}{a+1} x^{a+1}$	x^a
$\frac{1}{a \cdot (n+1)} (ax+b)^{n+1}$	$(ax+b)^n$
$\frac{x^{a+1}}{a+1}$	$x^a, a \neq -1$
$\frac{1}{x}$	$-\frac{1}{x^2}$
\sqrt{x}	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n} x^{\frac{1}{n}-1}$
$\frac{2}{3} x^{\frac{3}{2}}$	\sqrt{x}
$\frac{n}{n+1} x^{\frac{1}{n}+1}$	$\sqrt[n]{x}$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x \ln(a)} = \log_a(e^{\frac{1}{x}})$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x) = \frac{\cos(x)}{\sin(x)}$	$-\frac{1}{\sin^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\sinh(x) = \frac{e^x + e^{-x}}{2}$	$\cosh(x)$
$\cosh(x) = \frac{e^x - e^{-x}}{2}$	$\sinh(x)$
$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\frac{1}{f(x)}$	$-\frac{f'(x)}{(f(x))^2}$
a^{cx}	$a^{cx} \cdot c \ln(a)$
x^x	$x^x \cdot (1 + \ln(x)), x > 0$
$(x^x)^x$	$(x^x)^x (x + 2x \ln(x)), x > 0$
x^{x^x}	$x^{x^x} (x^{x-1} + \ln(x) \cdot x^x (1 + \ln(x)))$

$F(x)$	$f(x)$
$\frac{1}{a} \ln(ax+b)$	$\frac{1}{ax+b}$
$\frac{ax}{c} - \frac{ad-bc}{c^2} \ln(cx+d)$	$\frac{ax+b}{cx+d}$
$\frac{1}{2a} \ln\left(\left \frac{x-a}{x+a}\right \right)$	$\frac{1}{x^2-a^2}$
$\frac{x}{2} \sqrt{a^2+x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2+x^2})$	$\sqrt{a^2+x^2}$
$\frac{x}{2} \sqrt{a^2-x^2} - \frac{a^2}{2} \arcsin\left(\frac{x}{ a }\right)$	$\sqrt{a^2-x^2}$
$\frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \ln(x + \sqrt{x^2-a^2})$	$\sqrt{x^2-a^2}$
$\ln(x + \sqrt{x^2 \pm a^2})$	$\frac{1}{\sqrt{x^2 \pm a^2}}$
$\arcsin\left(\frac{x}{ a }\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\frac{1}{a} \arctan\left(\frac{x}{a}\right)$	$\frac{1}{x^2+a^2}$
$-\frac{1}{a} \cos(ax+b)$	$\sin(ax+b)$
$\frac{1}{a} \sin(ax+b)$	$\cos(ax+b)$
$-\ln(\cos(x))$	$\tan(x)$
$\ln(\sin(x))$	$\cot(x)$
$\ln\left(\left \tan\left(\frac{x}{2}\right)\right \right)$	$\frac{1}{\sin(x)}$
$\ln\left(\left \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right \right)$	$\frac{1}{\cos(x)}$
$\frac{1}{2}(x - \sin(x) \cos(x))$	$\sin^2(x)$
$\frac{1}{2}(x + \sin(x) \cos(x))$	$\cos^2(x)$
$\tan(x) - x$	$\tan^2(x)$
$-\cot(x) - x$	$\cot^2(x)$
$x \arcsin(x) + \sqrt{1-x^2}$	$\arcsin(x)$
$x \arccos(x) - \sqrt{1-x^2}$	$\arccos(x)$
$x \arctan(x) - \frac{1}{2} \ln(1+x^2)$	$\arctan(x)$
$\ln(\cosh(x))$	$\tanh(x)$
$\ln(f(x))$	$\frac{f'(x)}{f(x)}$
$x(\ln(x) - 1)$	$\ln(x)$
$\frac{1}{n+1} (\ln x)^{n+1} \quad n \neq -1$	$\frac{1}{x} (\ln x)^n$
$\frac{1}{2n} (\ln x^n)^2 \quad n \neq 0$	$\frac{1}{x} \ln x^n$
$\ln(\ln(x)) \quad x > 0, x \neq 1$	$\frac{1}{x \ln(x)}$

$F(x)$	$f(x)$
$\frac{1}{b \ln(a)} a^{bx}$	a^{bx}
$\frac{cx-1}{c^2} \cdot e^{cx}$	$x \cdot e^{cx}$
$\frac{x^{n+1}}{n+1} \left(\ln(x) - \frac{1}{n+1} \right) \quad n \neq -1$	$x^n \ln(x)$
$\frac{e^{cx} (c \sin(ax+b) - a \cos(ax+b))}{a^2+c^2}$	$e^{cx} \sin(ax+b)$
$\frac{e^{cx} (c \cos(ax+b) + a \sin(ax+b))}{a^2+c^2}$	$e^{cx} \cos(ax+b)$
$\frac{\sin^2(x)}{2}$	$\sin(x) \cos(x)$

Multiple Choice Questions

This section contains only the correct answers.

Sequences and Series

$\sum_{k=1}^{\infty}$ converges absolute and $\sum_{k=1}^{\infty}$ converges:

◇ $\sum_{k=1}^{\infty} |a_k|^2$ converges **always** absolute.

◇ $\sum_{k=1}^{\infty} a_k b_k$ converges **always** absolute.

Let f, g be monotonically increasing functions

✗ $f \cdot g$ monotonically increasing

✗ $\frac{f}{g}$ or $\frac{g}{f}$ monotonically increasing

Functions

$f: X \rightarrow Y, g: Y \rightarrow Z$ and $g \circ f: X \rightarrow Z$ bijective:

◇ f is injective, g is surjective

Let $a, b \in \mathbb{R}$ with $a < b$ and $f:]a, b[\rightarrow \mathbb{R}$:

◇ $|f|$ cont. ✗ f cont. (**CE** $f: -1 \leq x \leq 0, 1 > x > 0$)

◇ If f^2 and f^3 diff. in $]a, b[$ and $f(x) \neq 0 \forall x \implies f$ diff.
since $\frac{h}{g}$ is diff. if both $h = f^3$ and $g = f^2$ are diff.

Let $f: [a, b] \rightarrow \mathbb{R}$ be a function and f_n a function sequence with converges uniformly to f :

◇ If f_n for all $n \geq 1$ in $x_0 \in [a, b]$ bounded, then for all partitions P der limit of lower sums exists
 $\lim_{n \rightarrow \infty} \underline{S}(f_n, P) = \underline{S}(f, P)$

◇ If f_n for all $n \geq 1$ in x_0 is continuous, then f is uniformly continuous

◇ If f_n for all $n \geq 1$ in x_0 convex, then f is convex.

◇ If f_n for all $n \geq 1$ is diff. in x_0 ✗ f is diff. in x_0

General Exercises

Sequences and Series

Investigate if $\sum_{n=1}^{\infty} \log(\frac{n}{n+1})$ converges:

◇ $\sum_{n=1}^{\infty} \log(\frac{n}{n+1}) = (\log(1) - \log(2)) + (\log(2) - \log(3)) + \dots$
This is a telescopic series and thus alle terms cancel out except for $\log(1) - \log(n+1)$ and thus for $n \rightarrow \infty$ it the series must diverge.

Let $\sum_{k=1}^{\infty} a_k$ be abs. conv. and $\sum_{k=1}^{\infty} b_k$ conv. does $\sum_{k=1}^{\infty} b_k \sin(a_k)$ converge?

◇ Proof $\lim_{k \rightarrow \infty} b_k = 0$ by using c_k where $c_1 = 0$ and $c_k = b_{k-1}$ for $k \geq 2$.

◇ $\lim_{n \rightarrow \infty} \sum_{k=1}^n b_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n c_k \implies$
 $\lim_{n \rightarrow \infty} b_k = \lim_{n \rightarrow \infty} (\sum_{k=1}^n b_k - \sum_{k=1}^n c_k) = 0$

◇ Since $|\sin(x)| \leq |x|$, then by comparison theorem $\sum_{k=1}^{\infty} \sin(a_k)$ converges absolutely.

◇ Since b_k is bounded by constant C : $|b_n \sin(a_n)| \leq C|a_n|$.

Functions

Show that $f(x) = \begin{cases} x^x, & x > 0 \\ 1, & x = 0 \\ (-x)^{-x} & x < 0 \end{cases}$ is continuous.

◇ Because x^x is cont. for $x > 0$ and $(-x)^{-x}$ is cont. for $x < 0$ it suffices to prove $\lim_{x \rightarrow 0} f(x) = 1$.

◇ $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = 0 \Rightarrow x^x \xrightarrow{\lim} 1$

◇ Since the function is even, we have $\lim_{x \rightarrow 0^-} x^x = 1$.

Show that $f(x) = \tan(x^3), x \in ((-\frac{\pi}{2})^{\frac{1}{3}}, (\frac{\pi}{2})^{\frac{1}{3}})$ is monotone and calculate its inverse:

◇ $f'(x) = \tan'(x^3)3x^2 = \frac{3x^2}{\cos^2(x^3)} > 0$ since $\cos(y)$ is strictly positive for $-\frac{\pi}{2} < y < \frac{\pi}{2}$. $x = 0$ does not break the monotonicity. $f^{-1}(x) = \arctan(x)^{\frac{1}{3}}$.