

Analysis I Summary

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1 Real numbers, euclidean spaces

Archimedes' principle. If $x \in \mathbb{R}$ with $x > 0$ and $y \in \mathbb{R}$, then $\exists n \in \mathbb{N}$ ($y \leq n \cdot x$)

Thm. (i) $|x| \leq 0 \quad \forall x \in \mathbb{R}$
(ii) $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$
(iii) $|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$
(iv) $|x + y| \geq ||x| - |y|| \quad \forall x, y \in \mathbb{R}$

Young's inequality. $\forall \epsilon > 0, \forall x, y \in \mathbb{R}$:

$$2|xy| \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$$

2 Sequences

2.1 Convergence

$(a_n)_{n \geq 1}$ converges to $L = \lim_{n \rightarrow \infty} a_n \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N (|a_n - L| < \epsilon)$

Def (Convergence). $(a_n)_{n \geq 1}$ convergent $\iff \exists L \in \mathbb{R} \forall \epsilon > 0 (\{n \in \mathbb{N} \mid |a_n - L| \geq \epsilon\})$ is finite.

2.1.1 Tips & Tricks

- a_n convergent $\implies a_n$ bounded
- a_n convergent $\iff a_n$ bounded and $\liminf a_n = \limsup a_n$

Monotone Convergence. $(a_n)_{n \geq 1}$ monotone increasing and upper bounded $\implies \lim a_n = \sup\{a_n \mid n \geq 1\}$

$(a_n)_{n \geq 1}$ monotone decreasing and lower bounded $\implies \lim a_n = \inf\{a_n \mid n \geq 1\}$

Lemma (Bernoulli Inequation).

$$(1 + x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}, x > -1$$

Def.

$$\liminf_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} (\inf\{a_k \mid k \geq n\})$$

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} (\sup\{a_k \mid k \geq n\})$$

Cauchy Criteria. a_n converges iff $\forall \epsilon > 0 \exists N \geq 1$ s.t. $|a_n - a_m| < \epsilon \quad \forall n, m \geq N$.

- (i) Each Cauchy sequence is bounded
- (ii) $(a_n)_{n \geq 1}$ conv. $\implies (a_n)_{n \geq 1}$ cauchy
- (iii) $(a_n)_{n \geq 1}$ cauchy $\implies (a_n)_{n \geq 1}$ conv.

Bolzano-Weierstrass. Each bounded sequence contains a convergent sub sequence.

Sandwich. If $\lim a_n = \alpha$, $\lim b_n = \alpha$, $k \in \mathbb{N}$ and $a_n \leq c_n \leq b_n \quad \forall n \geq k$, then $\lim c_n = \alpha$

Cauchy-Cantor. Let $I_1 \supseteq I_2 \subseteq \dots I_n \dots$ be a sequence of proper intervals with $\mathcal{L} < +\infty$, then $\bigcap_{n \geq 1} I_n \neq \emptyset$. And if $\lim_{n \rightarrow \infty} \mathcal{L}(I_n) = 0$ then $|\bigcap_{n \geq 1} I_n| = 1$

Cor. Let (a_n) be bounded, then for each subsequence (b_n) : $\liminf a_n \leq \lim b_n \leq \limsup a_n$.

For each subsequence (b_n) of a convergent (a_n) : (b_n) converges and $\lim b_n = \lim a_n$.

3 Series

Def. " $\sum_{k=1}^{\infty} a_k$ " converges, if the sequence $(S_n)_{n \geq 1}$ of partial sums converges and $\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} S_n$

3.1 Important Series

- Geometric: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $|q| < 1$
- Harmonic: $\sum_{n=1}^{\infty} \frac{1}{k}$ diverges
- Telescope: $\sum_{n=0}^{\infty} \frac{1}{k(k+1)} = 1$
- $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n \rightarrow \infty} (1 + \frac{z}{n})^n = e^z$

- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges $s > 1$
- $p(z) = \sum_{k=0}^{\infty} c_k z^k$ converges absolute for $|z| < \rho$
- $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$

Thm. $\sum_{k=1}^{\infty} a_k$ and $\sum_{j=1}^{\infty} b_j$ convergent:

- $\sum_{k=1}^{\infty} (a_k + b_k) = (\sum_{k=1}^{\infty} a_k) + (\sum_{k=1}^{\infty} b_k)$
- $\sum_{k=1}^{\infty} \alpha \cdot a_k = \alpha \sum_{k=1}^{\infty} a_k$

Cauchy Criteria. $\sum_{k=1}^{\infty} a_k$ converges $\iff \forall \epsilon > 0 \exists N \geq 1$ ($|\sum_{k=n}^m a_k| < \epsilon$) $\forall m \geq n \geq N$

Zero Sequence Criteria. $\sum_{k=1}^{\infty} a_k$ conv. $\implies \lim a_k = 0$

Comparison Theorem. Let $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$ series s.t. $0 \leq a_k \leq b_k \quad \forall k \geq 1$:

$$\sum_{k=1}^{\infty} b_k \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges} \implies \sum_{k=1}^{\infty} b_k \text{ diverges}$$

Thm. A series converges if the sequence of partial sums $(s_n)_{n \geq 1}$ converges. $s_n := \sum_{k=1}^n a_k$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k =: \sum_{k=1}^{\infty} a_k$$

Def (Absolute Convergence). $\sum_{k=1}^{\infty} a_k$ absolute converges if $\sum_{k=1}^{\infty} |a_k|$ converges.

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \not\Rightarrow \sum_{k=1}^{\infty} |a_k| \text{ converges}$$

(Dirichlet) If a series converges absolute, then each permutation of the series converges with the same limit. **(Riemann)** If a series only converges, then there exists a permutation such that:
 $\sum_{k=1}^{\infty} a_{\phi(k)} = x \quad \forall x \in \mathbb{R} \cup \{\infty\}$

Thm.

$$\left| \sum_{k=1}^{\infty} a_k \right| \leq \sum_{k=1}^{\infty} |a_k|$$

Leibniz. $(a_n)_{n \geq 1}$ monotone decreasing s.t. $a_n \geq 1 \quad \forall n \geq 1$ and $\lim a_n = 0$:

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges}$$

Thm (Ratio Test). Let $(a_n)_{n \geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$:

$$\limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$$

$$\liminf_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Lemma. $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$
- $L > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$
- $L = 1 \implies \text{no information}$

Root Test. Let $(a_n)_{n \geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$:

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$$

$$\liminf_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Lemma. $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$
- $L > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$
- $L = 1 \implies \text{no information}$

Def (Cauchy Product). $\sum_{i=0}^{\infty} a_i, \sum_{j=0}^{\infty} b_j$:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{n-j} b_j \right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots$$

Thm. If $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$ converge absolute:

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{n-j} b_j \right) = \left(\sum_{i=0}^{\infty} a_i \right) \left(\sum_{j=0}^{\infty} b_j \right) = \sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j$$

3.2 Strategy: Convergence of Series

1. Check for known types (Telescope, Geometric, etc.)
2. $\lim a_n \neq 0 \implies \text{divergence}$
3. Ratio Test
4. Root Test
5. Search convergent majors: $0 \leq a_n \leq b_n$
6. If divergent minors $\implies \text{divergence}$
7. Be creative

4 Functions

$$\mathbb{R}^D = \{f : D \rightarrow \mathbb{R} \mid f \text{ is function}\}$$

4.1 Continuity

Def (Continuity). A function f is continuous in x_0 if:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon)$$

$$\iff$$

$$\forall (a_n)_{n \geq 1} \text{ with } \lim a_n = x_0 \text{ holds } \lim f(a_n) = f(\lim a_n)$$

Def. A function $f : D \rightarrow \mathbb{R}$ is continuous if it is continuous in all $x_0 \in D$

Cor. $f, g : D \rightarrow \mathbb{R}$ continuous in $x_0 \in D$. Then:

- $f, g, \lambda f, f \pm g$ continuous in x_0
- $\frac{f}{g} : D \setminus \{x \in D \mid g(x) = 0\} \rightarrow \mathbb{R}$ continuous in x_0 (if $g(x_0) \neq 0$)
- $|f|, \max(f, g), \min(f, g)$ continuous in x_0
- $P(x) = a_n x^n + \dots + a_0$ continuous on \mathbb{R}
- $\frac{P(x)}{Q(x)}$ continuous on $\mathbb{R} \setminus \{x_1, \dots, x_m\}$ if x_1, \dots, x_m are roots of $Q(x)$

Thm. Let $f : D_1 \rightarrow D_2 \subset \mathbb{R}, g : D_2 \rightarrow \mathbb{R}$ be continuous $\implies g \circ f : D_1 \rightarrow \mathbb{R}$ continuous.

Bolzano (Intermediate value theorem). Let $I \subseteq \mathbb{R}, f : I \rightarrow \mathbb{R}$ and $a, b \in I$. For each c between $f(a)$ and $f(b)$ there is a $z \in [a, b]$ with $f(z) = c$

Min-Max. Let $f : I = [a, b] \rightarrow \mathbb{R}$ be continuous.

$$\exists u, v \in I \forall x \in I (f(u) \leq f(x) \leq f(v))$$

In particular $f([a, b]) \subset [f(u), f(v)]$ is bounded.

Cor. $I = [a, b], f : I \rightarrow \mathbb{R}$ continuous, then $\Im(f) = f(I)$ is a compact interval $J = [\min f, \max f] = [f(u), f(v)]$

Inverse Mapping. Let $f : I \rightarrow \mathbb{R}$ be continuous and strict monotone increasing. Then $J := f(I) \subseteq \mathbb{R}$ is an interval and $f^{-1} : J \rightarrow I$ is continuous and strict monotone.

4.2 Exponential function

$\exp : \mathbb{R} \rightarrow]0, \infty[$ is continuous, strictly monotone increasing, surjective.

- $\exp(x) \geq 1 + x \quad \forall x \in \mathbb{R}$
- For $x > 0, a \in \mathbb{R} : x^a := \exp(a \ln x)$
- $x^0 = 1 \quad \forall x > 0$

Def. The inverse mapping of $\exp(x)$ is called the natural logarithm:

$$\ln :]0, \infty[\rightarrow \mathbb{R}, \quad x \mapsto \ln x$$

It is strictly monotone increasing, continuous and bijective.

4.3 Converge of function sequences

$$\mathbb{N} \rightarrow \mathbb{R}^D = \{f : D \rightarrow \mathbb{R}\}, \quad n \mapsto f(n)$$

Def (pointwise convergence). $(f_n)_{n \geq 0}$ converges pointwise to a function $f : D \rightarrow \mathbb{R}$, if $\forall x \in D (\lim_{n \rightarrow \infty} f_n(x) = f(x))$

$$\iff$$

$$\forall x \in D \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N (|f_n(x) - f(x)| < \epsilon)$$

Def (uniform convergence (Weierstrass)). $f_n : D \rightarrow \mathbb{R}$ converges uniformly in D to $f : D \rightarrow \mathbb{R}$ if:

$$\forall \epsilon > 0 \exists N \geq 1 \text{ s.t. } \forall n \geq N \forall x \in D (|f_n(x) - f(x)| < \epsilon)$$

$$\iff$$

$$\lim_{n \rightarrow \infty} \sup_{x \in D} |f_n(x) - f(x)| = 0$$

The function sequence (f_n) converges uniformly if for all $x \in D$ the limit $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ exists and the sequence (f_n) uniformly converges to f .

The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly (in D), if the function sequence $S_n(x) := \sum_{k=0}^n f_k(x)$ converges uniformly.

Thm. let $D \subseteq \mathbb{R}$ and $f_n : D \rightarrow \mathbb{R}$ a function sequence containing (in D) continuous functions which converge (in D) uniformly against a function $f : D \rightarrow \mathbb{R}$, then f (in D) is continuous.

Hint (not uniform convergent). $(f_n)_{n \geq 0}$ converges not uniformly if:

$$\forall \epsilon > 0 \forall N \in \mathbb{N} \exists x \in D (|f_n(x) - f(x)| \geq \epsilon)$$

Hint. Check the function and try to construct x (dependent on N in general), such that $|f_n(x) - f(x)|$ is always greater than a specific ϵ and afterwards choose the ϵ .

Def (Power Functions). $\sum_{k=0}^{\infty} c_k x^k$ has positive convergence radius if $\limsup_{k \rightarrow \infty}$

$$\rho = \begin{cases} +\infty & , \text{ if } \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} = 0 \\ \frac{1}{\limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}} & , \text{ if } \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} > 0 \end{cases}$$

Thm. Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $\rho > 0$ and let $f(x) = \sum_{k=0}^{\infty} c_k x^k$, $|x| < \rho$. Then: $\forall 0 \leq r < \rho$ converges $\sum_{k=0}^{\infty} c_k x^k$ uniformly on $[-r, r]$, furthermore $f :]-\rho, \rho[\rightarrow \mathbb{R}$ is continuous.

4.4 Trigonometric Functions

\sin and \cos are continuous functions $\mathbb{R} \rightarrow \mathbb{R}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Thm. 1. $\exp(iz) = \cos(z) + i \sin(z) \quad \forall z \in \mathbb{C}$

2. $\cos z = \cos(-z)$ und $\sin(-z) = -\sin(z) \quad \forall z \in \mathbb{C}$

$$3. \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$4. \sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w) \\ \cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

$$5. \cos(z)^2 + \sin(z)^2 = 1 \quad \forall z \in \mathbb{C}$$

Cor.

$$\sin(2z) = 2\sin(z)\cos(z)$$

$$\cos(2z) = \cos(z)^2 - \sin(z)^2$$

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Def (π) . $\pi := \inf\{t > 0 \mid \sin t = 0\}$

$$(i) \sin \pi = 0, \pi \in]2, 4[$$

$$(ii) \forall x \in]0, \pi[: \sin x > 0$$

$$(iii) e^{\frac{i\pi}{2}} = i$$

Cor. $x \geq \sin x \geq x - \frac{x^3}{3!} \quad \forall 0 \leq x \leq \sqrt{6}$

Cor. 1. $e^{i\pi} = -1, \quad e^{2i\pi} = 1$

$$2. \sin(x + \frac{\pi}{2}) = \cos(x), \quad \cos(1 + \frac{\pi}{2}) = -\sin(x) \quad \forall x \in \mathbb{R}$$

$$3. \sin(x + \pi) = -\sin(x), \quad \sin(x + 2\pi) = \sin(x) \quad \forall x \in \mathbb{R}$$

$$4. \cos(x + \pi) = -\cos(x), \quad \cos(x + 2\pi) = \cos(x) \quad \forall x \in \mathbb{R}$$

$$5. \text{Roots of sinus} = \{\pi \cdot k \mid k \in \mathbb{Z}\}$$

$$\sin(x) > 0 \quad \forall x \in]2k\pi, (2k+1)\pi[, \quad k \in \mathbb{Z}$$

$$\sin(x) < 0 \quad \forall x \in [(2k+1)\pi, (2k+2)\pi[, \quad k \in \mathbb{Z}$$

$$6. \text{Roots of cosine} = \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$$

$$\cos(x) > 0 \quad \forall x \in]-\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + (2k+1)\pi[, \quad k \in \mathbb{Z}$$

$$\cos(x) < 0 \quad \forall x \in [-\frac{\pi}{2} + (2k+1)\pi, -\frac{\pi}{2} + (2k+2)\pi[, \quad k \in \mathbb{Z}$$

4.5 Limit of Functions

Def (accumulation point). $x_0 \in \mathbb{R}$ is an accumulation point of D if $\forall \delta > 0: (]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) \cap D \neq \emptyset$

Def (Limit of Function). if $f : D \rightarrow \mathbb{R}, x_0 \in \mathbb{R}$ an accumulation point of D , then $A \in \mathbb{R}$ is the limit of $f(x)$ for $x \rightarrow x_0$, written as $\lim_{x \rightarrow x_0} f(x) = A$. If $\forall \epsilon > 0 \exists \delta > 0$ s.t. $\forall x \in D \cap (]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) : |f(x) - A| < \epsilon$

Important Rules. Let $f : D \rightarrow \mathbb{R}$ and x_0 is an accumulation point of D .

$$1. \lim_{x \rightarrow x_0} f(x) = A \iff \forall (a_n)_{n \geq 1} \text{ in } D \setminus \{x_0\} \text{ with}$$

$$\lim_{n \rightarrow \infty} a_n = x_0 \implies \lim_{n \rightarrow \infty} f(a_n) = A.$$

2. Let $x_0 \in D$. Then f is continuous in x_0

$$\iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

3. $f, g : D \rightarrow \mathbb{R}$ and $\exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \implies$

$$\lim_{x \rightarrow x_0} (f + g)(x) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$$

$$\lim_{x \rightarrow x_0} (f \cdot g)(x) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x)$$

4. $f, g : D \rightarrow \mathbb{R}$ and $f \leq g$, then if both limit exists

$$\lim_{x \rightarrow x_0} f(x) \leq \lim_{x \rightarrow x_0} g(x)$$

5. If $g_1 \leq f \leq g_2$ and $\lim_{x \rightarrow x_0} g_1(x) = \lim_{x \rightarrow x_0} g_2(x)$ then $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g_1(x)$

5 Differentiable Functions

Def (Differentiable). f is in x_0 differentiable, if the limit $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$ exists. f is differentiable if $\forall x_0 \in D$ f is differentiable.

Weierstrass. $f : D \rightarrow \mathbb{R}, x_0 \in D$ accumulation point. Equivalent statements:

1. f is in x_0 differentiable

2. It exists $c \in \mathbb{R}$ ($c = f'(x_0)$) and $r : D \rightarrow \mathbb{R}$ s.t.:

$$2.1 \quad f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0)$$

$$2.2 \quad r(x_0) = 0 \text{ and } r \text{ continuous in } x_0.$$

Cor. f diff. in $x_0 \iff f$ continuous in x_0

Derivative rules.

Linearity: $(\alpha \cdot f(x) + g(x))' = \alpha \cdot f'(x) + g'(x)$

Product rule: $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$

Chain rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Cor. f bijective and in x_0 differentiable s.t. $f'(x_0) \neq 0$. f^{-1} is continuous in $y_0 = f(x_0)$. Then f^{-1} is differentiable in y_0 and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$.

5.1 Derivative Implications

1. x_0 is local minimum if $f'(x_0) = 0 \wedge f''(x_0) > 0$ or the sign of f' changes from $-$ to $+$.
2. x_0 is local maximum if $f'(x_0) = 0 \wedge f''(x_0) < 0$ or the sign of f' changes from $+$ to $-$.
3. x_0 is local extremum if $f'(x_0) = 0 \wedge f''(x_0) \neq 0$
4. x_0 is a saddle point if $f'(x_0) = 0$ and $f''(x_0) = 0$
5. x_0 is a inflection point if $f''(x_0) = 0$
6. $f'(x_0) = f^{(2)}(x_0) = \dots = f^{(n)}(x_0) = 0$

- 6.1 n odd and $f^{(n+1)}(x_0) > 0 \implies x_0$ strict local minimum
- 6.2 n odd and $f^{(n+1)}(x_0) < 0 \implies x_0$ strict local maximum

5.2 Derivative Theorems

Rolle. Let $f : [a, b] \rightarrow \mathbb{R}$ continuous and in $]a, b[$ differentiable. If $f(a) = f(b)$, then there exists $\xi \in]a, b[$ with $f'(\xi) = 0$.

Mean Value / Lagrange. Let $f : [a, b] \rightarrow \mathbb{R}$ continuous and in $]a, b[$ differentiable, then there exists $\xi \in]a, b[$ with $f(b) - f(a) = f'(\xi)(b - a)$.

There exists points ξ with $f'(\xi)$ equal to the gradient of the secant between a to b .

Cor. Let $f, g : [a, b] \rightarrow \mathbb{R}$ continuous and diff. in $]a, b[$.

1. $\forall \xi \in]a, b[: f'(\xi) = 0 \implies f$ is constant
2. $\forall \xi \in]a, b[: f'(\xi) = g'(x) \implies \exists c \in \mathbb{R} \forall x \in [a, b] : f(x) = g(x) + c$
3. $\forall \xi \in]a, b[: f'(\xi) \geq 0 \implies f$ in $[a, b]$ mon. inc.
4. $\forall \xi \in]a, b[: f'(\xi) \geq 0 \implies f$ in $[a, b]$ str. mon. inc.
5. $\forall \xi \in]a, b[: f'(\xi) \leq 0 \implies f$ in $[a, b]$ mon. dec.
6. $\forall \xi \in]a, b[: f'(\xi) < 0 \implies f$ in $[a, b]$ str. mon. dec.
7. $\exists M \geq 0 \forall \xi \in]a, b[: |f'(\xi)| \leq M \implies \forall x_1, x_2 \in [a, b] : |f(x_1) - f(x_2)| \leq M|x_1 - x_2|$

Cauchy. $f, g, : [a, b] \rightarrow \mathbb{R}$ continuous and in $]a, b[$ diff. Then there exists $\xi \in]a, b[$ with $g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a))$.

If $\forall x \in]a, b[: g'(x) \neq 0$ it implies that $g(a) \neq g(b)$ and $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$

l'Hospital. $f, g :]a, b[\rightarrow \mathbb{R}$ diff. with $\forall x \in]a, b[: g'(x) \neq 0$. If $\lim_{x \rightarrow b^-} f(x) = 0, \lim_{x \rightarrow b^-} g(x) = 0$ and $\lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)} =: \lambda$ exists, then $\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)}$.

Hint. Only use l'Hospital if either $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

- Def.**
1. convex: $(x \leq y): f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$
 2. strict convex: $(x \leq y): f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$
 3. concave: $(x \leq y): f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$
 4. strict concave: $(x \leq y): f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$

Lemma. $f : I \rightarrow \mathbb{R}$. f is convex $\iff \forall x_0 < x < x_1 \in I : \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(x_1) - f(x)}{x_1 - x}$. Strictly convex if $<$.

Thm. $f :]a, b[\rightarrow \mathbb{R}$ in $]a, b[$ diff. Function is (strictly) convex if f' is (strictly) monotonically increasing.

5.3 Higher Derivatives

1. For $n \geq 2$ is f n -times differentiable in D if $f^{(n-1)}$ in D is differentiable. Then $f^{(n)} := (f^{(n-1)})'$ and is the n -th derivative of f
2. f is n -times **continuous differentiable** if f is n -times differentiable and if $f^{(n)}$ is continuous in D
3. f is in D smooth if $\forall n \geq 1, f$ is n -times differentiable.

Smooth Functions. $\exp, \sin, \cos, \sinh, \cosh, \tanh, \ln, \arcsin, \arccos, \operatorname{arccot}, \arctan$ and all polynomials. \tan is smooth on $\mathbb{R} \setminus \{\pi/2 + k\pi\}$ and \cot on $\mathbb{R} \setminus \{k\pi\}$

Thm. $f, g : D \rightarrow \mathbb{R}$ are n -times diff. in D .

1. $(f + g)^{(n)} = f^{(n)} + g^{(n)}$

$$2. (f \cdot g)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$$

$$3. (g \circ f)^{(n)}(x) = \sum_{k=1}^n A_{n,k}(x) (g^{(k)} \circ f)(x) \text{ with } A_{n,k} \text{ as polynomial of in the functions } f', f^{(2)}, \dots, f^{(n+1-k)}$$

5.4 Power Series and Taylor approximation

Thm. Let $f_n :]a, b[$ be a function sequence with f_n one time in $]a, b[$ continuous diff. $\forall n \geq 1$. Assume that $(f_n)_{n \geq 1}, (f'_n)_{n \geq 1}$ uniformly convergent in $]a, b[$ with $\lim_{n \rightarrow \infty} f_n =: f$ and $\lim_{n \rightarrow \infty} f'_n =: p$, then f is continuously diff. and $f' = p$.

Thm. Let $\sum_{k=0}^{\infty} c_k (x - x_0)^k$ be a power series with convergent radius $p > 0$. Then $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ is differentiable on $]x_0 - p, x_0 + p[$ and $\forall x \in]x_0 - p, x_0 + p[: f'(x) = \sum_{k=0}^{\infty} k c_k (x - x_0)^{k-1}$

Cor. $f^{(j)} = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!} (x - x_0)^{k-j}$. Furthermore $c_j = \frac{f^{(j)}(x_0)}{j!}$. Power series can be differentiated part by part in their converge area.

Def (Taylor Polynomial). The n -th Taylor-polynomial of cont. $n + 1$ times diff. in $[c, d]$ f is defined as $T_n(f, x, a)$ with center $a \in]c, d[$ and error $R_n(f, x, a)$. $\forall x \in [a, b] \exists \xi \in]x, a[\cup]a, x[:$

$$T_n(f, x, a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} \cdot (x - a)^k$$

$$R_n(f, x, a) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$$

$$f(x) = T_n(f, x, a) + R_n(f, x, a)$$

Hint. The error can be approximated as $|R_n(f, x, a)| \leq \sup_{a < c < x} \left| \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!} \right|$

Def (Taylor Series). $T_{\infty}(f, x, x_0) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$

6 Riemann Integral

$a < b, I = [a, b], \mathcal{P}(I) = \{P \mid P \subsetneq I \wedge \{a, b\} \in P \wedge |P| \in \mathbb{N}\}$

Def (Partition). • Partition: $P \in \mathcal{P}(I)$

- $\delta_i := x_i - x_{i-1}$ length of $I_i := [x_{i-1}, x_i], i \geq 1$
- Mesh of partition: $\delta(P) := \max_{1 \leq i \leq n} (x_i, x_{i-1})$

- $\xi := \{\xi_1, \dots, \xi_n\}$, $\xi_i \in I_i$
- P' refines P if $P \subset P'$

Def (Riemann Sums). $S(f, P, \xi) := \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$

- Lower sum: $\underline{S}(f, P) := \sum_{i=1}^n (\inf_{x \in I_i} f(x))(x_i - x_{i-1})$
- Upper sum: $\overline{S}(f, P) := \sum_{i=1}^n (\sup_{x \in I_i} f(x))(x_i - x_{i-1})$

It holds: $-M(b-a) \leq \underline{S}(f, P) \leq \overline{S}(f, P) \leq M(b-a)$

Lemma. $P \subset P' : \underline{S}(f, P) \leq \underline{S}(f, P') \leq \overline{S}(f, P') \leq \overline{S}(f, P)$

Lemma. $\forall P_1, P_2 \in \mathcal{P}(I) : \underline{S}(f, P_1) \leq \overline{S}(f, P_2)$

Def (Lower Riemann Integral). $\underline{S}(f) := \sup_{P \in \mathcal{P}(I)} \underline{S}(f, P)$

Def (Upper Riemann Integral). $\overline{S}(f) := \inf_{P \in \mathcal{P}(I)} \overline{S}(f, P)$

6.1 Integrability criteria

Def (Integrable). Bounded $f : [a, b] \rightarrow \mathbb{R}$ is integrable if $\underline{S}(f) = \overline{S}(f)$ and the shared value is $\int_a^b f(x)dx$.

Riemann Criteria. Bounded $f : I \rightarrow \mathbb{R}$ is integrable

- $\Leftrightarrow \forall \epsilon > 0 \exists P \in \mathcal{P}(I) : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon$
- $\Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 \forall P \in \mathcal{P}_\delta(I) := \{P \in \mathcal{P}(I) \mid \delta(P) < \delta\} : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon$
- $\Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 \forall P \in \mathcal{P} \text{ with } \delta(P) < \delta : \left| A - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon$

Hint. Bounded $f : [a, b] \rightarrow \mathbb{R}$ is int. if $\lim_{\delta(P) \rightarrow 0} S(f, P, \xi)$ exists for all P with $\delta(P) \rightarrow 0$. It follows that $\lim_{\delta(P) \rightarrow 0} S(f, P, \xi) = \int_a^b f(x)dx$

6.2 Integrable Functions

1. f (bounded) cont. in $[a, b] \implies f$ int. over $[a, b]$
2. f monotone in $[a, b] \implies f$ int. over $[a, b]$
3. If f, g bounded and int., then

$$f + g, \lambda \cdot f, f \cdot g, |f|, \max(f, g), \frac{f}{g}$$

are integrable.

4. All polynomials are integrable, even $\frac{P(x)}{Q(x)}$ if $Q(x)$ has no root in $[a, b]$

Hint. Let $V := \{f : I \rightarrow \mathbb{R} \mid f \text{ is a mapping}\}$. $(V, +, \cdot)$ is a vector space. Then it implies that $W := \{f : I \rightarrow \mathbb{R} \mid f \text{ is integrable}\}$ is a subspace of V .

6.3 Integration Inequalities and Theorems

Def (Uniform continuous).

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Thm. $f : [a, b] \rightarrow \mathbb{R}$ cont. $\implies f$ is uni. cont. in $[a, b]$.

Thm. $f, g : [a, b] \rightarrow \mathbb{R}$ bounded and integrable and $\forall x \in [a, b] : f(x) \leq g(x) \implies$

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Cor. $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

Cauchy-Schwarz.

$$\left| \int_a^b f(x)g(x)dx \right| \leq \sqrt{\int_a^b f^2(x)dx} \sqrt{\int_a^b g^2(x)dx}$$

Hint. $\langle f, g \rangle := \int_a^b f(x)g(x)dx$ is a scalar product. $\|f\|^2 = \langle f, f \rangle = \int_a^b f^2(x)dx$.

Mean Value Theorem. $f : [a, b] \rightarrow \mathbb{R}$ continuous $\implies \exists \xi \in [a, b] : \int_a^b f(x)dx = f(\xi)(b-a)$

Cauchy. $f, g : [a, b] \rightarrow \mathbb{R}$ with f cont. and g bounded and integrable with $g(x) \geq 0, \forall x \in [a, b] \implies \exists c \in [a, b] : \int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$.

6.4 Integration Properties

Additive Property.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Linearity.

$$\int_a^b (\alpha f_1 + \beta f_2)dx = \alpha \int_a^b f_1(x)dx + \beta \int_a^b f_2(x)dx$$

6.5 Primitive Functions

Def (Primitive Function). $F : [a, b] \rightarrow \mathbb{R}$ is a primitive function of f if F is cont. diff. and $F' = f$.

Hint. f is integrable \nRightarrow exists a primitive function for f .

HID. Let $a < b, f : [a, b] \rightarrow \mathbb{R}$ continuous. The function

$$F(x) := \int_a^x f(t)dt \quad a \leq x \leq b$$

is cont. diff. in $[a, b]$ and $F'(x) = f(x) \forall x \in [a, b]$.

Fundamental theorem of calculus. $f : [a, b] \rightarrow \mathbb{R}$ continuous. Dann there exists a unique (except a constant term) primitive function F of f , such that

$$\int_a^b f(x)dx = F(b) - F(a).$$

6.6 Integration Methods

Partial Inegration.

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

$$\int_a^b f(x)g'(x)dx = (f \cdot g)|_a^b - \int_a^b f'(x)g(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

- Choose g' : exp \rightarrow trig \rightarrow poly \rightarrow inverse trig. \rightarrow logs
- Choose f : logs \rightarrow inverse trig. \rightarrow poly \rightarrow trig \rightarrow exp
- Sometimes it is necessary to multiply by 1. E.g.: $\int \ln x dx = \int \ln x \cdot 1 dx \implies f(x) = \ln x, g'(x) = 1$.
- Sometimes it is necessary to do it multiple times

Substitution. Let $a < b, \phi : [a, b] \rightarrow \mathbb{R}$, cont. diff, $I \subseteq \mathbb{R}$ with $\phi([a, b]) \subseteq I$ and $f : I \rightarrow \mathbb{R}$ a cont. function. Then it

follows:

$$\int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt = (F \circ \phi)(b) - (F \circ \phi)(a)$$

since $F' = f$ then $f(\phi(t))\phi'(t) = (F \circ \phi)'(t)$.
t.b.d

Partial Fraction Decomposition. Let $P(x)$, $Q(x)$ be two polynomials. $\int \frac{P(x)}{Q(x)}$ can be calculated as follows:

1. If $\deg(P) \geq \deg(Q)$ use polynomial division to get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{\hat{P}(x)}{Q(x)}$$

2. Calculate all roots of $Q(x)$
3. Create a partial fraction per root
 - Simple real root: $x_1 \rightarrow \frac{A}{x-x_1}$
 - n -fold real root: $x_1 \rightarrow \frac{A_1}{x-x_1} + \dots + \frac{A_r}{(x-x_1)^r}$
4. Calculate parameters A_1, \dots, A_n . (Insert the root as s , transform and solve)

Cor. $\int_{a+c}^{b+c} f(x)dx = \int_a^b f(t+c)dt$

Cor. $\int_a^b f(ct)dt = \frac{1}{c} \int_{ac}^{bc} f(x)dx$

6.7 Integration of convergent series

Thm. Let $f_n : [a, b] \rightarrow \mathbb{R}$ be a sequence of bounded, integrable functions which converges uniformly against a function $f : [a, b] \rightarrow \mathbb{R}$. Then f is bounded and integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x)dx = \int_a^b f(x)dx$$

Thm. Let $f(x) := \sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $p > 0$. Then $\forall 0 \leq r < p$ f is integrable on $[-r, r]$ and $\forall x \in]-p, p[$:

$$\int_0^x f(t)dt = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$$

6.8 Improper Integral

$$f : [a, \infty) \rightarrow \mathbb{R}, f : [-\infty, a] \rightarrow \mathbb{R}, f : (-\infty, \infty) \rightarrow \mathbb{R}$$

Def. Let $f : [a, \infty[\rightarrow \mathbb{R}$ be bounded and integrable on $[a, b]$ for all $b > a$. If $\lim_{b \rightarrow \infty} \int_a^b f(x)dx$ exists, the limit is defined as $\int_a^{\infty} f(x)dx$ and one can say that f is integrable on $[a, +\infty[$. If the limit does not exist, one can say that $\int_a^{\infty} f(x)dx$ diverges.

Comparison Theorem. Let $f : [a, \infty[\rightarrow \mathbb{R}$ be bounded and integrable on $[a, b] \forall b \in \mathbb{R}, b > a$.

1. If $\forall x \geq a : |f(x)| \leq g(x)$ and $g(x)$ is integrable on $[a, \infty[\Rightarrow f$ is integrable on $[a, \infty[$.
2. If $0 \leq g(x) \leq f(x)$ and $\int_a^{\infty} g(x)dx$ diverges $\Rightarrow \int_a^{\infty} f(x)dx$ diverges.

Hint. Sometimes an integral can be split into a normal integral and an improper integral:

$$\int_0^{\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx$$

McLaurin. Let $f : [1, \infty[\rightarrow [0, \infty[$ be monotonically decreasing.

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^{\infty} f(x)dx \text{ converges.}$$

The following holds:

$$0 \leq \sum_{k=1}^{\infty} f(k) - \int_1^{\infty} f(x)dx \leq f(1)$$

Def. Let f be a function which is bounded and integrable on all intervals $[a+\epsilon, b] \forall \epsilon > 0$. $f :]a, b] \rightarrow \mathbb{R}$ is integrable if $\lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b f(x)dx$ exists. In this case the limit is defined as $\int_a^b f(x)dx$.

The comparison theorem can be used for such integrals as well.

6.9 Indefinite Integrals

Let $f : I \rightarrow \mathbb{R}$ be defined on the interval $I \subseteq \mathbb{R}$. If f is continuous there exists a primitive function F .

$$\int f(x)dx = F(x) + C$$

The indefinite integral is the inverse of the derivative.

Hint. $\int_{-\infty}^{\infty} f(x)dx = \lim_{b \rightarrow -\infty} \int_b^a f(x)dx + \lim_{c \rightarrow \infty} \int_a^c f(x)dx.$

$$\int_{-\infty}^{\infty} \text{conv.} \iff \int_{-\infty}^a f(x)dx \text{ conv.} \wedge \int_a^{\infty} f(x)dx \text{ conv.}$$

In general: Let $f :]a, b[\rightarrow \mathbb{R}$ such that it is integrable on each compact interval $[\tilde{a}, \tilde{b}]$. Then

$$\int_a^b f(x)dx := \lim_{\tilde{a} \searrow a} \lim_{\tilde{b} \nearrow b} \int_{\tilde{a}}^{\tilde{b}} f(x)dx$$

6.10 Euler Gamma Function

Def. For $s > 0$:

$$\Gamma(s) := \int_0^{\infty} e^{-x} x^{s-1} dx.$$

The gamma function interpolates the function $n \mapsto (n-1)!$. It converges for all $s > 0$.