

# Analysis I Summary

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## 1 Real numbers, euclidean spaces

**Archimedes' principle.** If  $x \in \mathbb{R}$  with  $x > 0$  and  $y \in \mathbb{R}$ , then  $\exists n \in \mathbb{N}$  ( $y \leq n \cdot x$ )

**Thm.** (i)  $|x| \leq 0 \quad \forall x \in \mathbb{R}$   
(ii)  $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$   
(iii)  $|x + y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$   
(iv)  $|x + y| \geq ||x| - |y|| \quad \forall x, y \in \mathbb{R}$

**Young's inequality.**  $\forall \epsilon > 0, \forall x, y \in \mathbb{R}$ :

$$2|xy| \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$$

## 2 Sequences

### 2.1 Convergence

$(a_n)_{n \geq 1}$  converges to  $L = \lim_{n \rightarrow \infty} a_n \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N (|a_n - L| < \epsilon)$

**Def (Convergence).**  $(a_n)_{n \geq 1}$  convergent  $\iff \exists L \in \mathbb{R} \forall \epsilon > 0 (\{n \in \mathbb{N} \mid |a_n - L| \geq \epsilon\})$  is finite.

#### 2.1.1 Tips & Tricks

- $a_n$  convergent  $\implies a_n$  bounded
- $a_n$  convergent  $\iff a_n$  bounded and  $\liminf a_n = \limsup a_n$

**Monotone Convergence.**  $(a_n)_{n \geq 1}$  monotone increasing and upper bounded  $\implies \lim a_n = \sup\{a_n \mid n \geq 1\}$   
 $(a_n)_{n \geq 1}$  monotone decreasing and lower bounded  $\implies \lim a_n = \inf\{a_n \mid n \geq 1\}$

**Lemma** (Bernoulli Inequation).

$$(1 + x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}, x > -1$$

**Def.**

$$\liminf_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} (\inf\{a_k \mid k \geq n\})$$

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} (\sup\{a_k \mid k \geq n\})$$

**Cauchy Criteria.**  $a_n$  converges iff  $\forall \epsilon > 0 \exists N \geq 1$  s.t.  $|a_n - a_m| < \epsilon \quad \forall n, m \geq N$ .

- (i) Each Cauchy sequence is bounded
- (ii)  $(a_n)_{n \geq 1}$  conv.  $\implies (a_n)_{n \geq 1}$  cauchy
- (iii)  $(a_n)_{n \geq 1}$  cauchy  $\implies (a_n)_{n \geq 1}$  conv.

**Bolzano-Weierstrass.** Each bounded sequence contains a convergent sub sequence.

**Sandwich.** If  $\lim a_n = \alpha, \lim b_n = \alpha, k \in \mathbb{N}$  and  $a_n \leq c_n \leq b_n \quad \forall n \geq k$ , then  $\lim c_n = \alpha$

**Cauchy-Cantor.** Let  $I_1 \supseteq I_2 \subseteq \dots \supseteq I_n \dots$  be a sequence of proper intervals with  $\mathcal{L} < +\infty$ , then  $\bigcap_{n \geq 1} I_n \neq \emptyset$ . And if  $\lim_{n \rightarrow \infty} \mathcal{L}(I_n) = 0$  then  $|\bigcap_{n \geq 1} I_n| = 1$

**Cor.** Let  $(a_n)$  be bounded, then for each subsequence  $(b_n)$ :  $\liminf a_n \leq \lim b_n \leq \limsup a_n$ .

For each subsequence  $(b_n)$  of a convergent  $(a_n)$ :  $(b_n)$  converges and  $\lim b_n = \lim a_n$ .

## 3 Series

**Def.** " $\sum_{k=1}^{\infty} a_k$ " converges, if the sequence  $(S_n)_{n \geq 1}$  of partial sums converges and  $\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} S_n$

### 3.1 Important Series

- Geometric:  $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$  if  $|q| < 1$
- Harmonic:  $\sum_{n=1}^{\infty} \frac{1}{k}$  diverges
- Telescope:  $\sum_{n=0}^{\infty} \frac{1}{k(k+1)} = 1$
- $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n \rightarrow \infty} (1 + \frac{z}{n})^n = e^z$
- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  converges  $s > 1$
- $p(z) = \sum_{k=0}^{\infty} c_k z^k$  converges absolute for  $|z| < \rho$

$$\bullet \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$$

**Thm.**  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{j=1}^{\infty} b_j$  convergent:

- $\sum_{k=1}^{\infty} (a_k + b_k) = (\sum_{k=1}^{\infty} a_k) + (\sum_{k=1}^{\infty} b_k)$
- $\sum_{k=1}^{\infty} \alpha \cdot a_k = \alpha \sum_{k=1}^{\infty} a_k$

**Cauchy Criteria.**  $\sum_{k=1}^{\infty} a_k$  converges  $\iff \forall \epsilon > 0 \exists N \geq 1 (|\sum_{k=n}^m a_k| < \epsilon) \quad \forall m \geq n \geq N$

**Zero Sequence Criteria.**  $\sum_{k=1}^{\infty} a_k$  conv.  $\implies \lim a_k = 0$

**Comparison Theorem.** Let  $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$  series s.t.  $0 \leq a_k \leq b_k \quad \forall k \geq 1$ :

$$\sum_{k=1}^{\infty} b_k \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges} \implies \sum_{k=1}^{\infty} b_k \text{ diverges}$$

**Thm.** A series converges if the sequence of partial sums  $(s_n)_{n \geq 1}$  converges.  $s_n := \sum_{k=1}^n a_k$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k =: \sum_{k=1}^{\infty} a_k$$

**Def (Absolute Convergence).**  $\sum_{k=1}^{\infty} a_k$  absolute converges if  $\sum_{k=1}^{\infty} |a_k|$  converges.

$$\sum_{k=1}^{\infty} |a_k| \text{ converges} \implies \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges} \not\Rightarrow \sum_{k=1}^{\infty} |a_k| \text{ converges}$$

**(Dirichlet)** If a series converges absolute, then each permutation of the series converges with the same limit. **(Riemann)** If a series only converges, then there exists a permutation such that:

$$\sum_{k=1}^{\infty} a_{\phi(k)} = x \quad \forall x \in \mathbb{R} \cup \{\infty\}$$

**Thm.**

$$|\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k|$$

**Leibniz.**  $(a_n)_{n \geq 1}$  monotone decreasing s.t.  $a_n \geq 1 \quad \forall n \geq 1$  and  $\lim a_n = 0$ :

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges}$$

**Thm (Ratio Test).** Let  $(a_n)_{n \geq 1}$  with  $a_n \neq 0 \quad \forall n \geq 1$ :  
 $\limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$   
 $\liminf_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$

**Lemma.**  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$ :

- $L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$
- $L > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$
- $L = 1 \implies \text{no information}$

**Root Test.** Let  $(a_n)_{n \geq 1}$  with  $a_n \neq 0 \quad \forall n \geq 1$ :  
 $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$   
 $\liminf_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$

**Lemma.**  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ :

- $L < 1 \implies \sum_{n=1}^{\infty} a_n \text{ converges absolute.}$
- $L > 1 \implies \sum_{n=1}^{\infty} a_n \text{ diverges.}$
- $L = 1 \implies \text{no information}$

**Def (Cauchy Product).**  $\sum_{i=0}^{\infty} a_i, \sum_{j=0}^{\infty} b_j$ :

$$\sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} a_{n-j} b_j \right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots$$

**Thm.** If  $\sum_{i=0}^{\infty} a_i$  and  $\sum_{j=0}^{\infty} b_j$  converge absolute:

$$\sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} a_{n-j} b_j \right) = \left( \sum_{i=0}^{\infty} a_i \right) \left( \sum_{j=0}^{\infty} b_j \right) = \sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j$$

### 3.2 Strategy: Convergence of Series

1. Check for known types (Telescope, Geometric, etc.)
2.  $\lim a_n \neq 0 \implies \text{divergence}$
3. Ratio Test
4. Root Test

5. Search convergent majors:  $0 \leq a_n \leq b_n$
6. If divergent minors  $\implies \text{divergence}$
7. Be creative

## 4 Functions

$\mathbb{R}^D = \{f : D \rightarrow \mathbb{R} \mid f \text{ is function}\}$

### 4.1 Continuity

**Def (Continuity).** A function  $f$  is continuous in  $x_0$  if:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in D (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon)$$

$\iff$

$\forall (a_n)_{n \geq 1}$  with  $\lim a_n = x_0$  holds  $\lim f(a_n) = f(\lim a_n)$

**Def.** A function  $f : D \rightarrow \mathbb{R}$  is continuous if it is continuous in all  $x_0 \in D$

**Cor.**  $f, g : D \rightarrow \mathbb{R}$  continuous in  $x_0 \in D$ . Then:

- $f, g, \lambda f, f \pm g$  continuous in  $x_0$
- $\frac{f}{g} : D \setminus \{x \in D \mid g(x) = 0\} \rightarrow \mathbb{R}$  continuous in  $x_0$  (if  $g(x_0) \neq 0$ )
- $|f|, \max(f, g), \min(f, g)$  continuous in  $x_0$
- $P(x) = a_n x^n + \dots + a_0$  continuous on  $\mathbb{R}$
- $\frac{P(x)}{Q(x)}$  continuous on  $\mathbb{R} \setminus \{x_1, \dots, x_m\}$  if  $x_1, \dots, x_m$  are roots of  $Q(x)$

**Thm.** Let  $f : D_1 \rightarrow D_2 \subset \mathbb{R}, g : D_2 \rightarrow \mathbb{R}$  be continuous  $\implies g \circ f : D_1 \rightarrow \mathbb{R}$  continuous.

**Bolzano (Intermediate value theorem).** Let  $I \subseteq \mathbb{R}, f : I \rightarrow \mathbb{R}$  and  $a, b \in I$ . For each  $c$  between  $f(a)$  and  $f(b)$  there is a  $z \in [a, b]$  with  $f(z) = c$

**Min-Max.** Let  $f : I = [a, b] \rightarrow \mathbb{R}$  be continuous.

$$\exists u, v \in I \forall x \in I (f(u) \leq f(x) \leq f(v))$$

In particular  $f([a, b]) \subset [f(u), f(v)]$  is bounded.

**Cor.**  $I = [a, b], f : I \rightarrow \mathbb{R}$  continuous, then  $\Im(f) = f(I)$  is a compact interval  $J = [\min f, \max f] = [f(u), f(v)]$

**Inverse Mapping.** Let  $f : I \rightarrow \mathbb{R}$  be continuous and strict monotone increasing. Then  $J := f(I) \subseteq \mathbb{R}$  is an interval and  $f^{-1} : J \rightarrow I$  is continuous and strict monotone.

### 4.2 Exponential function

$\exp : \mathbb{R} \rightarrow ]0, \infty[$  is continuous, strictly monotone increasing, surjective.

- $\exp(x) \geq 1 + x \quad \forall x \in \mathbb{R}$
- For  $x > 0, a \in \mathbb{R} : x^a := \exp(a \ln x)$
- $x^0 = 1 \quad \forall x > 0$

**Def.** The inverse mapping of  $\exp(x)$  is called the natural logarithm:

$$\ln : ]0, \infty[ \rightarrow \mathbb{R}, \quad x \mapsto \ln x$$

It is strictly monotone increasing, continuous and bijective.

### 4.3 Converge of function sequences

$$\mathbb{N} \rightarrow \mathbb{R}^D = \{f : D \rightarrow \mathbb{R}\}, \quad n \mapsto f(n)$$

**Def (pointwise convergence).**  $(f_n)_{n \geq 0}$  converges pointwise to a function  $f : D \rightarrow \mathbb{R}$ , if  $\forall x \in D (\lim_{n \rightarrow \infty} f_n(x) = f(x))$

$\iff$

$$\forall x \in D \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N (|f_n(x) - f(x)| < \epsilon)$$

**Def (uniform convergence (Weierstrass)).**  $f_n : D \rightarrow \mathbb{R}$  converges uniformly in  $D$  to  $f : D \rightarrow \mathbb{R}$  if:

$$\forall \epsilon > 0 \exists N \geq 1 \text{ s.t. } \forall n \geq N \forall x \in D (|f_n(x) - f(x)| < \epsilon)$$

$\iff$

$$\lim_{n \rightarrow \infty} \sup_{x \in D} |f_n(x) - f(x)| = 0$$

The function sequence  $(f_n)$  converges uniformly if for all  $x \in D$  the limit  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  exists and the sequence  $(f_n)$  uniformly converges to  $f$ .

The series  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly (in  $D$ ), if the function sequence  $S_n(x) := \sum_{k=0}^n f_k(x)$  converges uniformly.

**Thm.** let  $D \subseteq \mathbb{R}$  and  $f_n : D \rightarrow \mathbb{R}$  a function sequence containing (in  $D$ ) continuous functions which converge (in  $D$ ) uniformly against a function  $f : D \rightarrow \mathbb{R}$ , then  $f$  (in  $D$ ) is continuous.

**Hint** (not uniform convergent).  $(f_n)_{n \geq 0}$  converges not uniformly if:

$$\forall \epsilon > 0 \forall N \in \mathbb{N} \exists x \in D (|f_n(x) - f(x)| \geq \epsilon)$$

**Hint.** Check the function and try to construct  $x$  (dependent on  $N$  in general), such that  $|f_n(x) - f(x)|$  is always greater than a specific  $\epsilon$  and afterwards choose the  $\epsilon$ .

**Def** (Power Functions).  $\sum_{k=0}^{\infty} c_k x^k$  has positive convergence radius if  $\limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} > 0$ .

$$\rho = \begin{cases} +\infty & , \text{ if } \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} = 0 \\ \frac{1}{\limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|}} & , \text{ if } \limsup_{k \rightarrow \infty} \sqrt[k]{|c_k|} > 0 \end{cases}$$

**Thm.** Let  $\sum_{k=0}^{\infty} c_k x^k$  be a power series with positive convergence radius  $\rho > 0$  and let  $f(x) = \sum_{k=0}^{\infty} c_k x^k, |x| < \rho$ . Then:  $\forall 0 \leq r < \rho$  converges  $\sum_{k=0}^{\infty} c_k x^k$  uniformly on  $[-r, r]$ , furthermore  $f : ]-\rho, \rho[ \rightarrow \mathbb{R}$  is continuous.

## 4.4 Trigonometric Functions

$\sin$  and  $\cos$  are continuous functions  $\mathbb{R} \rightarrow \mathbb{R}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

- Thm.**
- $\exp(iz) = \cos(z) + i \sin(z) \quad \forall z \in \mathbb{C}$
  - $\cos z = \cos(-z)$  und  $\sin(-z) = -\sin(z) \quad \forall z \in \mathbb{C}$
  - $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$
  - $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$   
 $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
  - $\cos(z)^2 + \sin(z)^2 = 1 \quad \forall z \in \mathbb{C}$

**Cor.**

$$\begin{aligned} \sin(2z) &= 2 \sin(z) \cos(z) \\ \cos(2z) &= \cos(z)^2 - \sin(z)^2 \end{aligned}$$

$$\sin(x) - \sin(y) = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

**Def** ( $\pi$ ).  $\pi := \inf\{t > 0 \mid \sin t = 0\}$

- (i)  $\sin \pi = 0, \pi \in ]2, 4[$

- (ii)  $\forall x \in ]0, \pi[: \sin x > 0$

- (iii)  $e^{\frac{i\pi}{2}} = i$

**Cor.**  $x \geq \sin x \geq x - \frac{x^3}{3!} \quad \forall 0 \leq x \leq \sqrt{6}$

**Cor.** 1.  $e^{i\pi} = -1, \quad e^{2i\pi} = 1$

$$2. \sin(x + \frac{\pi}{2}) = \cos(x), \quad \cos(1 + \frac{\pi}{2}) = -\sin(x) \quad \forall x \in \mathbb{R}$$

$$3. \sin(x + \pi) = -\sin(x), \quad \sin(x + 2\pi) = \sin(x) \quad \forall x \in \mathbb{R}$$

$$4. \cos(x + \pi) = -\cos(x), \quad \cos(x + 2\pi) = \cos(x) \quad \forall x \in \mathbb{R}$$

$$5. \text{Roots of sinus} = \{\pi \cdot k \mid k \in \mathbb{Z}\}$$

$$\sin(x) > 0 \quad \forall x \in ]2k\pi, (2k+1)\pi[, \quad k \in \mathbb{Z}$$

$$\sin(x) < 0 \quad \forall x \in [(2k+1)\pi, (2k+2)\pi[, \quad k \in \mathbb{Z}$$

$$6. \text{Roots of cosine} = \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$$

$$\cos(x) > 0 \quad \forall x \in ]-\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + (2k+1)\pi[, \quad k \in \mathbb{Z}$$

$$\cos(x) < 0 \quad \forall x \in ]-\frac{\pi}{2} + (2k+1)\pi, -\frac{\pi}{2} + (2k+2)\pi[, \quad k \in \mathbb{Z}$$

## 4.5 Limit of Functions

**Def** (accumulation point).  $x_0 \in \mathbb{R}$  is an accumulation point of  $D$  if  $\forall \delta > 0: (]x_0 - \delta, x_0 + \delta[ \setminus \{x_0\}) \cap D \neq \emptyset$

**Def** (Limit of Function). if  $f : D \rightarrow \mathbb{R}, x_0 \in \mathbb{R}$  an accumulation point of  $D$ , then  $A \in \mathbb{R}$  is the limit of  $f(x)$  for  $x \rightarrow x_0$ , written as  $\lim_{x \rightarrow x_0} f(x) = A$ . If  $\forall \epsilon > 0 \exists \delta > 0$  s.t.  $\forall x \in D \cap (]x_0 - \delta, x_0 + \delta[ \setminus \{x_0\}) : |f(x) - A| < \epsilon$

**Important Rules.** Let  $f : D \rightarrow \mathbb{R}$  and  $x_0$  is an accumulation point of  $D$ .

- $\lim_{x \rightarrow x_0} f(x) = A \iff \forall (a_n)_{n \geq 1} \text{ in } D \setminus \{x_0\} \text{ with } \lim_{n \rightarrow \infty} a_n = x_0 \implies \lim_{n \rightarrow \infty} f(a_n) = A.$
- Let  $x_0 \in D$ . Then  $f$  is continuous in  $x_0$   
 $\iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$
- $f, g : D \rightarrow \mathbb{R}$  and  $\exists \lim_{x \rightarrow x_0} f(x), \exists \lim_{x \rightarrow x_0} g(x) \implies$

$$\lim_{x \rightarrow x_0} (f + g)(x) = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$$

$$\lim_{x \rightarrow x_0} (f \cdot g)(x) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x)$$

4.  $f, g : D \rightarrow \mathbb{R}$  and  $f \leq g$ , then if both limit exists

$$\lim_{x \rightarrow x_0} f(x) \leq \lim_{x \rightarrow x_0} g(x)$$

5. If  $g_1 \leq f \leq g_2$  and  $\lim_{x \rightarrow x_0} g_1(x) = \lim_{x \rightarrow x_0} g_2(x)$  then  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g_1(x)$

## 5 Differentiable Functions

**Def** (Differentiable).  $f$  is in  $x_0$  differentiable, if the limit  $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$  exists.  $f$  is differentiable if  $\forall x_0 \in D$   $f$  is differentiable.

**Weierstrass.**  $f : D \rightarrow \mathbb{R}, x_0 \in D$  accumulation point. Equivalent statements:

- $f$  is in  $x_0$  differentiable
- It exists  $c \in \mathbb{R}$  ( $c = f'(x_0)$ ) and  $r : D \rightarrow \mathbb{R}$  s.t.:

$$2.1 \quad f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0)$$

$$2.2 \quad r(x_0) = 0 \text{ and } r \text{ continuous in } x_0.$$

**Cor.**  $f$  diff. in  $x_0 \iff f$  continuous in  $x_0$

**Derivative rules.**

Linearity:  $(\alpha \cdot f(x) + g(x))' = \alpha \cdot f'(x) + g'(x)$

Product rule:  $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient rule:  $\left(\frac{f}{g}\right)'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$

Chain rule:  $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

**Cor.**  $f$  bijective and in  $x_0$  differentiable s.t.  $f'(x_0) \neq 0$ .  $f^{-1}$  is continuous in  $y_0 = f(x_0)$ . Then  $f^{-1}$  is differentiable in  $y_0$  and  $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$ .

### 5.1 Derivative Implications

- $x_0$  is local minimum if  $f'(x_0) = 0 \wedge f''(x_0) > 0$  or the sign of  $f'$  changes from  $-$  to  $+$ .
- $x_0$  is local maximum if  $f'(x_0) = 0 \wedge f''(x_0) < 0$  or the sign of  $f'$  changes from  $+$  to  $-$ .
- $x_0$  is local extremum if  $f'(x_0) = 0 \wedge f''(x_0) \neq 0$
- $x_0$  is a saddle point if  $f'(x_0) = 0$  and  $f''(x_0) = 0$
- $x_0$  is a inflection point if  $f''(x_0) = 0$
- $f'(x_0) = f^{(2)}(x_0) = \dots = f^{(n)}(x_0) = 0$

$$6.1 \quad n \text{ odd and } f^{(n+1)}(x_0) > 0 \implies x_0 \text{ strict local minimum}$$

$$6.2 \quad n \text{ odd and } f^{(n+1)}(x_0) < 0 \implies x_0 \text{ strict local maximum}$$

## 5.2 Derivative Theorems

**Rolle.** Let  $f : [a, b] \rightarrow \mathbb{R}$  continuous and in  $]a, b[$  differentiable. If  $f(a) = f(b)$ , then there exists  $\xi \in ]a, b[$  with  $f'(\xi) = 0$ .

**Mean Value / Lagrange.** Let  $f : [a, b] \rightarrow \mathbb{R}$  continuous and in  $]a, b[$  differentiable, then there exists  $\xi \in ]a, b[$  with  $f(b) - f(a) = f'(\xi)(b - a)$ .

There exists points  $\xi$  with  $f'(\xi)$  equal to the gradient of the secant between  $a$  to  $b$ .

**Cor.** Let  $f, g : [a, b] \rightarrow \mathbb{R}$  continuous and diff. in  $]a, b[$ .

1.  $\forall \xi \in ]a, b[: f'(\xi) = 0 \implies f$  is constant
2.  $\forall \xi \in ]a, b[: f'(\xi) = g'(x) \implies \exists c \in \mathbb{R} \forall x \in [a, b] : f(x) = g(x) + c$
3.  $\forall \xi \in ]a, b[: f'(\xi) \geq 0 \implies f$  in  $[a, b]$  mon. inc.
4.  $\forall \xi \in ]a, b[: f'(\xi) \geq 0 \implies f$  in  $[a, b]$  str. mon. inc.
5.  $\forall \xi \in ]a, b[: f'(\xi) \leq 0 \implies f$  in  $[a, b]$  mon. dec.
6.  $\forall \xi \in ]a, b[: f'(\xi) < 0 \implies f$  in  $[a, b]$  str. mon. dec.
7.  $\exists M \geq 0 \forall \xi \in ]a, b[: |f'(\xi)| \leq M \implies \forall x_1, x_2 \in [a, b] : |f(x_1) - f(x_2)| \leq M|x_1 - x_2|$

**Cauchy.**  $f, g : [a, b] \rightarrow \mathbb{R}$  continuous and in  $]a, b[$  diff. Then there exists  $\xi \in ]a, b[$  with  $g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a))$ .

If  $\forall x \in ]a, b[: g'(x) \neq 0$  it implies that  $g(a) \neq g(b)$  and  $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$

**l'Hospital.**  $f, g : ]a, b[ \rightarrow \mathbb{R}$  diff. with  $\forall x \in ]a, b[: g'(x) \neq 0$ . If  $\lim_{x \rightarrow b^-} f(x) = 0, \lim_{x \rightarrow b^-} g(x) = 0$  and  $\lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)} =: \lambda$  exists, then  $\lim_{x \rightarrow b^-} \frac{f(x)}{g(x)} = \lim_{x \rightarrow b^-} \frac{f'(x)}{g'(x)}$ .

**Hint.** Only use l'Hospital if either  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ !

**Def.** 1. convex:  $(x \leq y): f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$   
 2. strict convex:  $(x \leq y): f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$   
 3. concave:  $(x \leq y): f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$

4. strict concave:  $(x \leq y): f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$

**Lemma.**  $f : I \rightarrow \mathbb{R}$ .  $f$  is convex  $\iff \forall x_0 < x < x_1 \in I : \frac{f(x)-f(x_0)}{x-x_0} \leq \frac{f(x_1)-f(x)}{x_1-x}$ . Strictly convex if  $<$ .

**Thm.**  $f : ]a, b[ \rightarrow \mathbb{R}$  in  $]a, b[$  diff. Function is (strictly) convex if  $f'$  is (strictly) monotonically increasing.

## 5.3 Higher Derivatives

1. For  $n \geq 2$  is  $f$   $n$ -times differentiable in  $D$  if  $f^{(n-1)}$  in  $D$  is differentiable. Then  $f^{(n)} := (f^{(n-1)})'$  and is the  $n$ -th derivative of  $f$
2.  $f$  is  $n$ -times **continuous differentiable** if  $f$  is  $n$ -times differentiable and if  $f^{(n)}$  is continuous in  $D$
3.  $f$  is in  $D$  smooth if  $\forall n \geq 1, f$  is  $n$ -times differentiable.

**Smooth Functions.** exp, sin, cos, sinh, cosh, tanh, ln, arcsin, arccos, arccot, arctan and all polynomials. tan is smooth on  $\mathbb{R} \setminus \{\pi/2 + k\pi\}$  and cot on  $\mathbb{R} \setminus \{k\pi\}$

**Thm.**  $f, g : D \rightarrow \mathbb{R}$  are  $n$ -times diff. in  $D$ .

1.  $(f + g)^{(n)} = f^{(n)} + g^{(n)}$
2.  $(f \cdot g)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}$
3.  $(g \circ f)^{(n)}(x) = \sum_{k=1}^n A_{n,k}(x) (g^{(k)} \circ f)(x)$  with  $A_{n,k}$  as polynomial of in the functions  $f', f^{(2)}, \dots, f^{(n+1-k)}$

## 5.4 Power Series and Taylor approximation

**Thm.** Let  $f_n : ]a, b[$  be a function sequence with  $f_n$  one time in  $]a, b[$  continuous diff.  $\forall n \geq 1$ . Assume that  $(f_n)_{n \geq 1}, (f'_n)_{n \geq 1}$  uniformly convergent in  $]a, b[$  with  $\lim_{n \rightarrow \infty} f_n =: f$  and  $\lim_{n \rightarrow \infty} f'_n =: p$ , then  $f$  is continuously diff. and  $f' = p$ .

**Thm.** Let  $\sum_{k=0}^{\infty} c_k (x - x_0)^k$  be a power series with convergent radius  $p > 0$ . Then  $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$  is differentiable on  $]x_0 - p, x_0 + p[$  and  $\forall x \in ]x_0 - p, x_0 + p[: f'(x) = \sum_{k=0}^{\infty} k c_k (x - x_0)^{k-1}$

**Cor.**  $f^{(j)} = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!} (x - x_0)^{k-j}$ . Furthermore  $c_j = \frac{f^{(j)}(x_0)}{j!}$ . Power series can be differentiated part by part in their converge area.

**Def (Taylor Polynomial).** The  $n$ -th Taylor-polynomial of cont.  $n + 1$  times diff. in  $[c, d]$   $f$  is defined as  $T_n(f, x, a)$  with center  $a \in ]c, d[$  and error  $R_n(f, x, a)$ .  $\forall x \in [a, b] \exists \xi \in ]x, a[ \cup ]a, x[ :$

$$T_n(f, x, a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} \cdot (x - a)^k$$

$$R_n(f, x, a) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - a)^{n+1}$$

$$f(x) = T_n(f, x, a) + R_n(f, x, a)$$

**Hint.** The error can be approximated as  $|R_n(f, x, a)| \leq \sup_{a < c < x} \left| \frac{f^{(n+1)}(x)(x-a)^{n+1}}{(n+1)!} \right|$

**Def (Taylor Series).**  $T_{\infty}(f, x, x_0) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$

## 6 Riemann Integral

$a < b, I = [a, b], \mathcal{P}(I) = \{P \mid P \subsetneq I \wedge \{a, b\} \in P \wedge |P| \in \mathbb{N}\}$

**Def (Partition).** • Partition:  $P \in \mathcal{P}(I)$

- $\delta_i := x_i - x_{i-1}$  length of  $I_i := [x_{i-1}, x_i], i \geq 1$
- Mesh of partition:  $\delta(P) := \max_{1 \leq i \leq n} (x_i - x_{i-1})$
- $\xi := \{\xi_1, \dots, \xi_n\}, \xi_i \in I_i$
- $P'$  refines  $P$  if  $P \subset P'$

**Def (Riemann Sums).**  $S(f, P, \xi) := \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$   
 • Lower sum:  $\underline{S}(f, P) := \sum_{i=1}^n (\inf_{x \in I_i} f(x)) (x_i - x_{i-1})$

• Upper sum:  $\overline{S}(f, P) := \sum_{i=1}^n (\sup_{x \in I_i} f(x)) (x_i - x_{i-1})$   
 It holds:  $-M(b - a) \leq \underline{S}(f, P) \leq \overline{S}(f, P) \leq M(b - a)$

**Lemma.**  $P \subset P' : \underline{S}(f, P) \leq \underline{S}(f, P') \leq \overline{S}(f, P') \leq \overline{S}(f, P)$

**Lemma.**  $\forall P_1, P_2 \in \mathcal{P}(I) : \underline{S}(f, P_1) \leq \overline{S}(f, P_2)$

**Def (Lower Riemann Integral).**  $\underline{S}(f) := \sup_{P \in \mathcal{P}(I)} \underline{S}(f, P)$

**Def (Upper Riemann Integral).**  $\overline{S}(f) := \inf_{P \in \mathcal{P}(I)} \overline{S}(f, P)$

## 6.1 Integrability criteria

**Def** (Integrable). Bounded  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if  $\underline{S}(f) = \overline{S}(f)$  and the shared value is  $\int_a^b f(x)dx$ .

**Riemann Criteria.** Bounded  $f : I \rightarrow \mathbb{R}$  is integrable

$$\begin{aligned} \Leftrightarrow \quad & \forall \epsilon > 0 \exists P \in \mathcal{P}(I) : \overline{S}(f, P) - \underline{S}(f, P) < \epsilon \\ \Leftrightarrow \quad & \forall \epsilon > 0 \exists \delta > 0 \forall P \in \mathcal{P}_\delta(I) := \{P \in \mathcal{P}(I) \mid \delta(P) < \delta\} : \\ & \overline{S}(f, P) - \underline{S}(f, P) < \epsilon \\ \Leftrightarrow \quad & \forall \epsilon > 0 \exists \delta > 0 \forall P \in \mathcal{P} \text{ with } \delta(P) < \delta : \\ & \left| A - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon \end{aligned}$$

**Hint.** Bounded  $f : [a, b] \rightarrow \mathbb{R}$  is int. if  $\lim_{\delta(P) \rightarrow 0} S(f, P, \xi)$  exists for all  $P$  with  $\delta(P) \rightarrow 0$ . It follows that  $\lim_{\delta(P) \rightarrow 0} S(f, P, \xi) = \int_a^b f(x)dx$

## 6.2 Integrable Functions

1.  $f$  (bounded) cont. in  $[a, b] \implies f$  int. over  $[a, b]$
2.  $f$  monotone in  $[a, b] \implies f$  int. over  $[a, b]$
3. If  $f, g$  bounded and int., then

$$f + g, \lambda \cdot f, f \cdot g, |f|, \max(f, g), \frac{f}{g}$$

are integrable.

4. All polynomials are integrable, even  $\frac{P(x)}{Q(x)}$  if  $Q(x)$  has no root in  $[a, b]$

**Hint.** Let  $V := \{f : I \rightarrow \mathbb{R} \mid f \text{ is a mapping}\}$ .  $(V, +, \cdot)$  is a vector space. Then it implies that  $W := \{f : I \rightarrow \mathbb{R} \mid f \text{ is integrable}\}$  is a subspace of  $V$ .

## 6.3 Integration Inequalities and Theorems

**Def** (Uniform continuous).

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, y \in D : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

**Thm.**  $f : [a, b] \rightarrow \mathbb{R}$  cont.  $\implies f$  is uni. cont. in  $[a, b]$ .

**Thm.**  $f, g : [a, b] \rightarrow \mathbb{R}$  bounded and integrable and  $\forall x \in [a, b] : f(x) \leq g(x) \implies$

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

$$\text{Cor. } \left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

**Cauchy-Schwarz.**

$$\left| \int_a^b f(x)g(x)dx \right| \leq \sqrt{\int_a^b f^2(x)dx} \sqrt{\int_a^b g^2(x)dx}$$

**Hint.**  $\langle f, g \rangle := \int_a^b f(x)g(x)dx$  is a scalar product.  
 $\|f\|^2 = \langle f, f \rangle = \int_a^b f^2(x)dx$ .

**Mean Value Theorem.**  $f : [a, b] \rightarrow \mathbb{R}$  continuous  $\implies \exists \xi \in [a, b] : \int_a^b f(x)dx = f(\xi)(b - a)$

**Cauchy.**  $f, g : [a, b] \rightarrow \mathbb{R}$  with  $f$  cont. and  $g$  bounded and integrable with  $g(x) \geq 0, \forall x \in [a, b] \implies \exists c \in [a, b] : \int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$ .

## 6.4 Integration Properties

**Additive Property.**

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

**Linearity.**

$$\int_a^b (\alpha f_1 + \beta f_2)dx = \alpha \int_a^b f_1(x)dx + \beta \int_a^b f_2(x)dx$$

## 6.5 Primitive Functions

**Def** (Primitive Function).  $F : [a, b] \rightarrow \mathbb{R}$  is a primitive function of  $f$  if  $F$  is cont. diff. and  $F' = f$ .

**Hint.**  $f$  is integrale  $\nRightarrow$  exists a primitive function for  $f$ .

**HID.** Let  $a < b, f : [a, b] \rightarrow \mathbb{R}$  continuous. The function

$$F(x) := \int_a^x f(t)dt \quad a \leq x \leq b$$

is cont. diff. in  $[a, b]$  and  $F'(x) = f(x) \forall x \in [a, b]$ .

**Fundamental theorem of calculus.**  $f : [a, b] \rightarrow \mathbb{R}$  continuous. Dann there exists a unique (except a constant term) primitive function  $F$  of  $f$ , such that

$$\int_a^b f(x)dx = F(b) - F(a).$$

## 6.6 Integration Methods

**Partial Inegration.**

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

$$\int_a^b f(x)g'(x)dx = (f \cdot g)|_a^b - \int_a^b f'(x)g(x)dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

- Choose  $g'$ : exp  $\rightarrow$  trig  $\rightarrow$  poly  $\rightarrow$  inverse trig.  $\rightarrow$  logs
- Choose  $f$ : logs  $\rightarrow$  inverse trig.  $\rightarrow$  poly  $\rightarrow$  trig  $\rightarrow$  exp
- Sometimes it is necessary to multiply by 1. E.g.:  $\int \ln x dx = \int \ln x \cdot 1 dx \implies f(x) = \ln x, g'(x) = 1$ .
- Sometimes it is necessary to to it multiple times

**Substitution.** Let  $a < b, \phi : [a, b] \rightarrow \mathbb{R}$ , cont. diff,  $I \subseteq \mathbb{R}$  with  $\phi([a, b]) \subseteq I$  and  $f : I \rightarrow \mathbb{R}$  a cont. function. Then it follows:

$$\int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt = (F \circ \phi)(b) - (F \circ \phi)(a)$$

since  $F' = f$  then  $f(\phi(t))\phi'(t) = (F \circ \phi)'(t)$ .

t.b.d

**Partial Fraction Decomposition.** Let  $P(x), Q(x)$  be two polynomials.  $\int \frac{P(x)}{Q(x)}$  can be calculated as follows:

1. If  $\deg(P) \geq \deg(Q)$  use polynomial division to to get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{\hat{P}(x)}{Q(x)}$$

2. Calculate all roots of  $Q(x)$
3. Create a partial fraction per root

- Simple real root:  $x_1 \rightarrow \frac{A}{x - x_1}$
- $n$ -fold real root:  $x_1 \rightarrow \frac{A_1}{x - x_1} + \dots + \frac{A_r}{(x - x_1)^r}$

4. Calculate parameters  $A_1, \dots, A_n$ . (Insert the root as  $s$ , transform and solve)

$$\text{Cor. } \int_{a+c}^{b+c} f(x)dx = \int_a^b f(t+c)dt$$

$$\text{Cor. } \int_a^b f(ct)dt = \frac{1}{c} \int_{ac}^{bc} f(x)dx$$

## 6.7 Integration of convergent series

**Thm.** Let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a sequence of bounded, integrable functions which converges uniformly against a function  $f : [a, b] \rightarrow \mathbb{R}$ . Then  $f$  is bounded and integrable and

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = \int_a^b f(x) dx$$

**Thm.** Let  $f(x) := \sum_{k=0}^{\infty} c_k x^k$  be a power series with positive convergence radius  $p > 0$ . Then  $\forall 0 \leq r < p$   $f$  is integrable on  $[-r, r]$  and  $\forall x \in ]-p, p[$ :

$$\int_0^x f(t) dt = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$$

## 6.8 Improper Integral

$$f : [a, \infty) \rightarrow \mathbb{R}, f : [-\infty, a] \rightarrow \mathbb{R}, f : (-\infty, \infty) \rightarrow \mathbb{R}$$

**Def.** Let  $f : [a, \infty[ \rightarrow \mathbb{R}$  be bounded and integrable on  $[a, b]$  for all  $b > a$ . If  $\lim_{b \rightarrow \infty} \int_a^b f(x) dx$  exists, the limit is defined as  $\int_a^{\infty} f(x) dx$  and one can say that  $f$  is integrable on  $[a, +\infty[$ . If the limit does not exist, one can say that  $\int_a^{\infty} f(x) dx$  diverges.

**Comparison Theorem.** Let  $f : [a, \infty[ \rightarrow \mathbb{R}$  be bounded and integrable on  $[a, b] \forall b \in \mathbb{R}, b > a$ .

1. If  $\forall x \geq a : |f(x)| \leq g(x)$  and  $g(x)$  is integrable on  $[a, \infty[ \implies f$  is integrable on  $[a, \infty[$ .
2. If  $0 \leq g(x) \leq f(x)$  and  $\int_a^{\infty} g(x) dx$  diverges  $\implies \int_a^{\infty} f(x) dx$  diverges.

**Hint.** Sometimes an integral can be split into a normal integral and an improper integral:

$$\int_0^{\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx$$

**McLaurin.** Let  $f : [1, \infty[ \rightarrow [0, \infty[$  be monotonically decreasing.

$$\sum_{n=1}^{\infty} f(n) \text{ converges } \iff \int_1^{\infty} f(x) dx \text{ converges.}$$

The following holds:

$$0 \leq \sum_{k=1}^{\infty} f(k) - \int_1^{\infty} f(x) dx \leq f(1)$$

**Def.** Let  $f$  be a function which is bounded and integrable on all intervals  $[a + \epsilon, b] \forall \epsilon > 0$ .  $f : ]a, b] \rightarrow \mathbb{R}$  is integrable if  $\lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b f(x) dx$  exists. In this case the limit is defined as  $\int_a^b f(x) dx$ .

The comparison theorem can be used for such integrals as well.

## 6.9 Indefinite Integrals

Let  $f : I \rightarrow \mathbb{R}$  be defined on the interval  $I \subseteq \mathbb{R}$ . If  $f$  is continuous there exists a primitive function  $F$ .

$$\int f(x) dx = F(x) + C$$

The indefinite integral is the inverse of the derivative.

**Hint.**  $\int_{-\infty}^{\infty} f(x) dx = \lim_{b \rightarrow -\infty} \int_b^a f(x) dx + \lim_{c \rightarrow \infty} \int_a^c f(x) dx$ .

$$\int_{-\infty}^{\infty} \text{conv.} \iff \int_{-\infty}^a f(x) dx \text{ conv.} \wedge \int_a^{\infty} f(x) dx \text{ conv.}$$

In general: Let  $f : ]a, b[ \rightarrow \mathbb{R}$  such that it is integrable on each compact interval  $[\tilde{a}, \tilde{b}]$ . Then

$$\int_a^b f(x) dx := \lim_{\tilde{a} \searrow a} \lim_{\tilde{b} \nearrow b} \int_{\tilde{a}}^{\tilde{b}} f(x) dx$$

## 6.10 Euler Gamma Function

**Def.** For  $s > 0$ :

$$\Gamma(s) := \int_0^{\infty} e^{-x} x^{s-1} dx.$$

The gamma function interpolates the function  $n \mapsto (n-1)!$ . It converges for all  $s > 0$ .