Biostat 216 Homework 6

Due Nov 11 Friday @ 11:59pm

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Eigenvalues and eigenvectors

- Q1. Diagonalize (show the steps to find eigenvalues and eigenvectors)
$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
and compute $\mathbf{X}\boldsymbol{\Lambda}^k\mathbf{X}^{-1}$ to prove the formula
$$\mathbf{A}^k = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}.$$

- Q2. Suppose the same $\mathbf{X}$ diagonalize both $\mathbf{A}$ and $\mathbf{B}$. That is they have the same eigenvectors in $\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}_1\mathbf{X}^{-1}$ and $\mathbf{B} = \mathbf{X}\boldsymbol{\Lambda}_2\mathbf{X}^{-1}$. Prove that $\mathbf{AB} = \mathbf{BA}$.

- Q3. Suppose the eigenvalues of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are $\lambda_1, \ldots, \lambda_n$. Show that $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$. Hint: λ_is are roots of the characteristic polynomial.

- Q4. Ture of false. For each statement, indicate it is true or false and gives a brief explanation.
    1. If the columns of $\mathbf{X}$ (eigenvectors of a square matrix $\mathbf{A}$) are linearly independent, then (a) $\mathbf{A}$ is invertible; (b) $\mathbf{A}$ is diagonalizable; (c) $\mathbf{X}$ is invertible; (d) $\mathbf{X}$ is diagonalizable.
    2. If the eigenvalues of $\mathbf{A}$ are 2, 2, 5 then the matrix is certainly (a) invertible; (b) diagonalizable.
    3. If the only eigenvectors of $\mathbf{A}$ are multiples of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, then the matrix has (a) no inverse; (b) a repeated eigenvalue; (c) no diagonalization $\mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$.

- Q5. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$. Show that $\mathbf{AB}$ and $\mathbf{BA}$ share the same non-zero eigenvalues. Hint: examine the eigen-equations.

- Q6. Find the eigenvalues of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A}+\mathbf{B}$:
$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}.$$
Are eigenvalues of $\mathbf{A}+\mathbf{B}$ equal to the sum of eigenvalues of $\mathbf{A}$ and $\mathbf{B}$?

- Q7. Suppose $\mathbf{A}$ has eigenvalues 0, 3, 5 with independent eigenvectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ respectively.
  1. Give a basis for $\mathcal{C}(\mathbf{A})$ and a basis for $\mathcal{N}(\mathbf{A})$.
  2. Find a particular solution to $\mathbf{A} = \mathbf{v} + \mathbf{w}$. Find all solutions.
  3. Show that the linear equation $\mathbf{A}\mathbf{x} = \mathbf{u}$ has no solution.

Positive definite matrices

- Q8. Suppose $\mathbf{C}$ is positive definite and $\mathbf{A}$ has independent columns. Apply the energy test to show that $\mathbf{A}'\mathbf{C}\mathbf{A}$ is positive definite.

- Q9. Show that the diagonal entries of a positive definite matrix are positive.

- Q10. Suppose $\mathbf{S}$ is positive definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ with corresponding orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.
  1. What are the eigenvalues of the matrix $\lambda_1\mathbf{I} - \mathbf{S}$? Is it positive semidefinite?
  2. How does it follow that $\lambda_1\mathbf{x}'\mathbf{x} \geq \mathbf{x}'\mathbf{S}\mathbf{x}$ for every $\mathbf{x}$?
  3. Draw the conclusion: The maximum value of the Rayleigh quotient
  $$R(\mathbf{x}) = \frac{\mathbf{x}'\mathbf{S}\mathbf{x}}{\mathbf{x}'\mathbf{x}}$$
  is λ_1.
  4. Show that the maximum value of the Rayleigh quotient subject to the condition $\mathbf{x} \perp \mathbf{u}_1$ is λ_2. Hint: expand $\mathbf{x}$ in eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.
  5. Show that the maximum value of the Rayleigh quotient subject to the conditions $\mathbf{x} \perp \mathbf{u}_1$ and $\mathbf{x} \perp \mathbf{u}_2$ is λ_3.
  6. What is the maximum value of $\frac{1}{2}\mathbf{x}'\mathbf{S}\mathbf{x}$ subject to the constraint $\mathbf{x}'\mathbf{x} = 1$. Hint: write down the Lagrangian and set its derivative to zero.