Biostat 216 Homework 5

Due Nov 15 @ 11:59pm

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1 Q1

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Show the following properties of generalized inverses.

1. For any generalized inverse $\mathbf{A}^-$, $\text{rank}(\mathbf{A}^-) \geq \text{rank}(\mathbf{A})$.
2. $\text{rank}(\mathbf{A}^+) = \text{rank}(\mathbf{A})$.
3. $(\mathbf{A}^-)'$ is a generalized inverse of $\mathbf{A}'$.
4. $(\mathbf{A}^+)' = (\mathbf{A}')^+$.

2 Q2 Householder algorithm for QR factorization

Let $\mathbf{v} \in \mathbb{R}^n$. Define the **Householder reflection matrix**

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2} = \mathbf{I} - 2\mathbf{u}\mathbf{u}',$$

where $\mathbf{u}$ is the unit vector $\mathbf{v}/\|\mathbf{v}\|$.

1. Show that $\mathbf{H}$ is symmetric and orthogonal.
2. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ such that $\|\mathbf{a}\| = \|\mathbf{b}\|$. Find a Householder matrix such that $\mathbf{H}\mathbf{a} = \mathbf{b}$.
3. Let $\mathbf{a} \in \mathbb{R}^n$ be a non-zero vector. Find a Householder matrix such that
$$\mathbf{H}\mathbf{a} = \begin{pmatrix} \|\mathbf{a}\| \\ \mathbf{0}_{n-1} \end{pmatrix}.$$
4. Let $\mathbf{a} \in \mathbb{R}^n$. Find a Householder matrix such that
$$\mathbf{H}\mathbf{a} = \begin{pmatrix} a_1 \\ \|\mathbf{a}_{2:n}\| \\ \mathbf{0}_{n-2} \end{pmatrix}.$$

5. Let $\mathbf{A} \in \mathbb{R}^{n \times p}$. Describe how to find a sequence of Householder matrices $\mathbf{H}_1, \ldots, \mathbf{H}_p$ such that

$$\mathbf{H}_p \mathbf{H}_{p-1} \cdots \mathbf{H}_1 \mathbf{A} = \mathbf{R},$$

   where $\mathbf{R} \in \mathbb{R}^{n \times p}$ is an upper triangular matrix.

6. Describe how this generates a full QR decomposition of matrix $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q} \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $\mathbf{R}$ is upper triangular.

3 Q3

1. For any $\mathbf{X} \in \mathbb{R}^{n \times p}$ and $\mathbf{y} \in \mathbb{R}^n$, show that the normal equation

$$\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{y}$$

   always has at least one solution.

2. Show that $(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}'$ is a generalized inverse of $\mathbf{A}$.

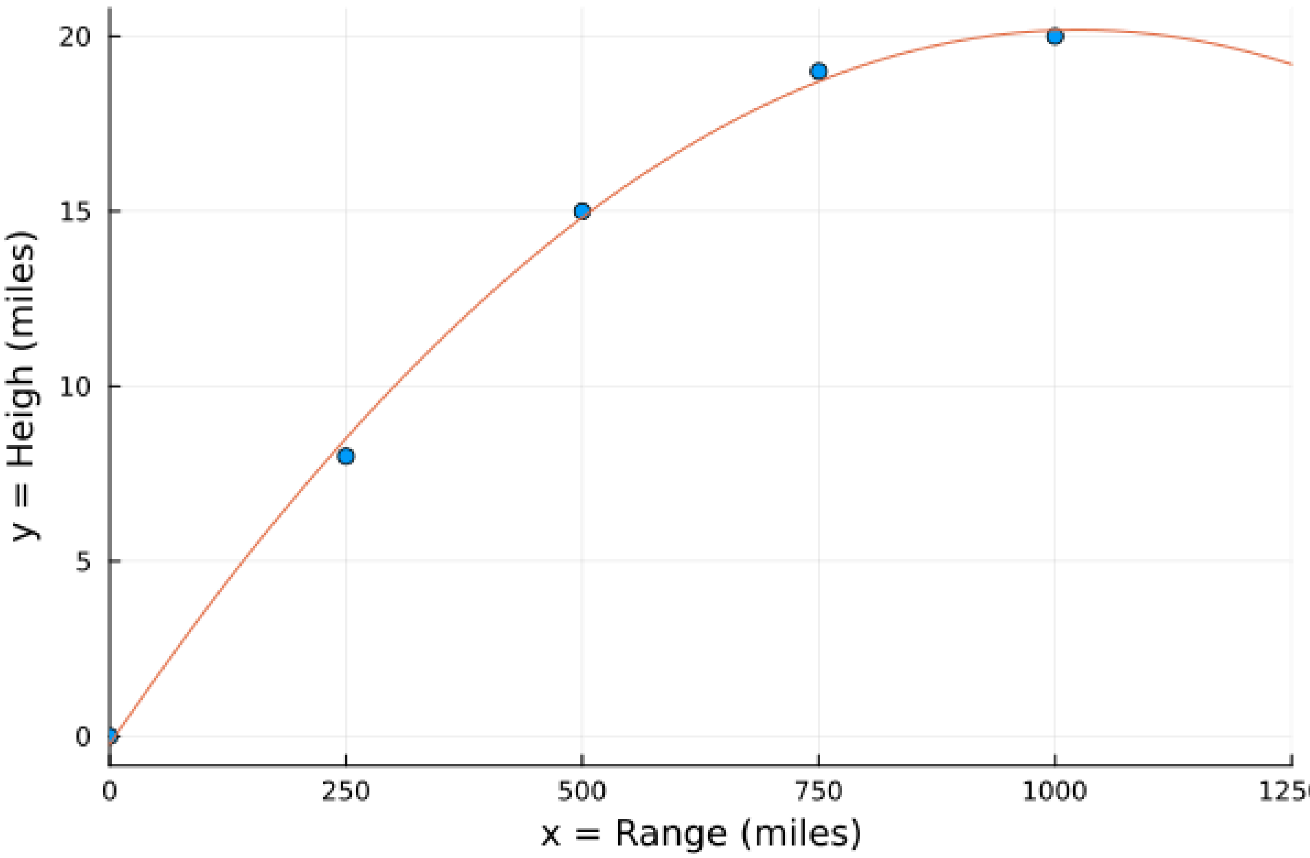
3. $\mathbf{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}'$ is the orthogonal projector onto $\mathcal{C}(\mathbf{A})$.

4 Q4 Missile tracking

A missile is fired from enemy territory, and its position in flight is observed by radar tracking devices at the following positions.

x=Position down range (1000 miles)	0	0.25	0.5	0.75	1
y=Height (1000 miles)	0	0.008	0.015	0.019	0.020

Suppose that intelligence sources indicate that enemy missiles are programmed to follow a parabolic flight path: $y = f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$. Where is the missile expected to land? Hint: You can find the solution using computer program. For example, in Julia, least squares solution is obtained by command `A \ b`.



5 BV exercises

12.2, 12.4, 12.8