Biostat 216 Homework 4

Due Oct 30 @ 11:59pm

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Q1.

Show the following claims.

1. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two vector spaces of same order, then their **intersection** $\mathcal{S}_1 \cap \mathcal{S}_2$ is a vector space.

2. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two vector spaces of same order, then their **union** $\mathcal{S}_1 \cup \mathcal{S}_2$ is not necessarily a vector space.

3. The **span** of a set of $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$, defined as the set of all possible linear combinations of $\mathbf{x}_i$s

$$\text{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\} = \left\{ \sum_{i=1}^{k} \alpha_i \mathbf{x}_i : \alpha_i \in \mathbb{R} \right\},$$

   is a vector space in $\mathbb{R}^n$.

4. The null space of an matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a vector space.

Q2.

Let

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

1. Find the matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ containing independent columns of $\mathbf{A}_1$ and $\mathbf{A}_2$.

2. Find a rank factorization $\mathbf{A} = \mathbf{C}\mathbf{R}$ of each of $\mathbf{A}_1$ and $\mathbf{A}_2$.

3. Produce a basis for the column spaces of $\mathbf{A}_1$ and $\mathbf{A}_2$. What are the dimensions of those column spaces (the number of independent vectors)? What are the ranks of $\mathbf{A}_1$ and $\mathbf{A}_2$? How many independent rows in $\mathbf{A}_1$ and $\mathbf{A}_2$?

Q3.

1. Show that an **orthocomplement set** of a set $\mathcal{X}$ (not necessarily a subspace) in a vector space $\mathcal{V} \subseteq \mathbb{R}^m$

$$\mathcal{X}^\perp = \{\mathbf{u} \in \mathcal{V} : \langle \mathbf{x}, \mathbf{u} \rangle = 0 \text{ for all } \mathbf{x} \in \mathcal{X}\}$$

   is a vector space.

2. Show that, if $\mathcal{S}$ is a subspace of a vector space $\mathcal{V} \subseteq \mathbb{R}^m$, then $\mathcal{S} = (\mathcal{S}^\perp)^\perp$.

Q4.

In this exercise, we show the fact

$$\text{rank}(\mathbf{A} + \mathbf{B}) \le \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$$

for any two matrices $\mathbf{A}$ and $\mathbf{B}$ of same size in steps.

1. Show that the sum of two vector spaces $\mathcal{S}_1$ and $\mathcal{S}_2$ of same order

$$\mathcal{S}_1 + \mathcal{S}_2 = \{\mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in \mathcal{S}_1, \mathbf{x}_2 \in \mathcal{S}_2\}$$

   is a vector space.

2. Show that $\dim(\mathcal{S}_1 + \mathcal{S}_2) \le \dim(\mathcal{S}_1) + \dim(\mathcal{S}_2)$.

3. Show that $\mathcal{C}(\mathbf{A} + \mathbf{B}) \subseteq \mathcal{C}(\mathbf{A}) + \mathcal{C}(\mathbf{B})$.

4. Conclude that $\text{rank}(\mathbf{A} + \mathbf{B}) \le \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B})$.

Q5. Fundamental theorem of ranks

1. Show that $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}'\mathbf{A})$ and $\operatorname{rank}(\mathbf{A}') = \operatorname{rank}(\mathbf{A}\mathbf{A}')$. Hint: we did it in class, using rank-nullity theorem.

2. Show that $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}')$ using part 1. Hint: we showed this remarkable result using rank factorization in class. This question asks you to show it using part 1.

Q6.

1. If $\mathbf{A}$ and $\mathbf{B}$ are two matrices with the same number of rows, then

$$\mathcal{C}((\mathbf{A} \ \ \mathbf{B})) = \mathcal{C}(\mathbf{A}) + \mathcal{C}(\mathbf{B}).$$

2. If $\mathbf{A}$ and $\mathbf{C}$ are two matrices with the same number of columns, then

$$\mathcal{R}\left(\begin{pmatrix} \mathbf{A} \\ \mathbf{C} \end{pmatrix}\right) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{C})$$

   and

$$\mathcal{N}\left(\begin{pmatrix} \mathbf{A} \\ \mathbf{C} \end{pmatrix}\right) = \mathcal{N}(\mathbf{A}) \cap \mathcal{N}(\mathbf{C}).$$