Biostat 216 Homework 5

Due Nov 19 Friday @ 11:59pm

Submit a PDF (scanned/photographed from handwritten solutions, or converted from RMarkdown or Jupyter Notebook) to Gracescope on CCLE.

- Q1. Find the three projectors in the first picture in this [lecture note (https://ucla-biostat216-2021fall.github.io/slides/07-proj/07-proj.html)](https://ucla-biostat216-2021fall.github.io/slides/07-proj/07-proj.html).
  - $\mathbf{P}_1$ projects into $\text{span}(\{\mathbf{1}_2\})$ along $\text{span}\left(\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}\right)$.
  - $\mathbf{P}_2$ projects into $\text{span}(\{\mathbf{1}_2\})$ along $\text{span}\left(\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}\right)$.
  - $\mathbf{P}_3$ is the orthogonal projection into $\text{span}(\{\mathbf{1}_2\})$.

- Q2. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Prove that $\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Give an example that it is not necessary true if $\mathbf{A}$ is not symmetric.

- Q3. Find the orthogonal projection of the point $\mathbf{1}_3$ into the plane spanned by the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$.

- Q4. Matrices that satisfy $\mathbf{A}'\mathbf{A} = \mathbf{A}\mathbf{A}'$ are said to be *normal*. Give an example of asymmetric (not symmetric), normal matrix. If $\mathbf{A}$ is normal, then prove that every vector in $\mathcal{C}(\mathbf{A})$ is orthogonal to every vector in $\mathcal{N}(\mathbf{A})$.

- Q5. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, show that $(\mathbf{x} + \mathbf{y}) \perp (\mathbf{x} - \mathbf{y})$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.

- Q6. Let $\mathbf{A}$ be a symmetric matrix. Show that the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to $\mathcal{N}(\mathbf{A})$.

- Q7. Find an orthonormal basis for each of the four fundamental subspaces of
$$\mathbf{A} = \begin{pmatrix} 2 & -2 & -5 & -3 & -1 & 2 \\ 2 & -1 & -3 & 2 & 3 & 2 \\ 4 & -1 & -4 & 10 & 11 & 4 \\ 0 & 1 & 2 & 5 & 4 & 0 \end{pmatrix}.$$
Hint: Gram-Schmidt algorithm can be helpful.

- Q8. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ be two orthogonal matrices. Show that $\mathbf{AB}$ is an orthogonal matrix. Construct an example to show that $\mathbf{A} + \mathbf{B}$ need not be orthogonal.

- Q9. Determinant.
    - (1). Find the determinant of the following two matrices without doing any computations:
    $$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
    - (2). Let $\mathbf{A} \in \mathbb{R}^{5 \times 5}$ with $\det(\mathbf{A}) = -3$. Find $\det(\mathbf{A}^3)$, $\det(\mathbf{A}^{-1})$, and $\det(2\mathbf{A})$.
    - (3). Find the determinant of the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 6 \end{pmatrix}$. Hint: find the row and column permutations that make $\mathbf{A}$ triangular; then use product rule.

- Q10. Let
$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
    - (1). Find eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{A}^{-1}$. Do they have same eigenvectors? What's the relationship between their eigenvalues?
    - (2). Find eigenvalues of $\mathbf{B}$ and $\mathbf{A} + \mathbf{B}$. Are eigenvalues of $\mathbf{A} + \mathbf{B}$ equal to eigenvalues of $\mathbf{A}$ plus eigenvalues of $\mathbf{B}$?
    - (3). Find eigenvalues of $\mathbf{AB}$ and $\mathbf{BA}$. Are the eigenvalues of $\mathbf{AB}$ equal to eigenvalues of $\mathbf{A}$ times eigenvalues of $\mathbf{B}$? Are the eigenvalues of $\mathbf{AB}$ equal to eigenvalues of $\mathbf{BA}$?