

Biostat 216 Homework 6+7

Due Dec 3 Friday @ 11:59pm

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Eigenvalues and eigenvectors

- Q1. Diagonalize (show the steps to find eigenvalues and eigenvectors)

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and compute $\mathbf{X}\mathbf{\Lambda}^k\mathbf{X}^{-1}$ to prove the formula

$$\mathbf{A}^k = \frac{1}{2} \begin{pmatrix} 1 + 3^k & 1 - 3^k \\ 1 - 3^k & 1 + 3^k \end{pmatrix}.$$

- Q2. Suppose the same \mathbf{X} diagonalize both \mathbf{A} and \mathbf{B} . They have the same eigenvectors in $\mathbf{A} = \mathbf{X}\mathbf{\Lambda}_1\mathbf{X}^{-1}$ and $\mathbf{B} = \mathbf{X}\mathbf{\Lambda}_2\mathbf{X}^{-1}$. Prove that $\mathbf{AB} = \mathbf{BA}$.
- Q3. Suppose the eigenvalues of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are $\lambda_1, \dots, \lambda_n$. Show that $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.
Hint: λ_i s are roots of the characteristic polynomial.
- Q4. True or false. For each statement, indicate it is true or false and gives a brief explanation.
 - If the columns of \mathbf{X} (eigenvectors of a square matrix \mathbf{A}) are linearly independent, then (a) \mathbf{A} is invertible; (b) \mathbf{A} is diagonalizable; (c) \mathbf{X} is invertible; (d) \mathbf{X} is diagonalizable.
 - If the eigenvalues of \mathbf{A} are 2, 2, 5 then the matrix is certainly (a) invertible; (b) diagonalizable.
 - If the only eigenvectors of \mathbf{A} are multiples of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, then the matrix has (a) no inverse; (b) a repeated eigenvalue; (c) no diagonalization $\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$.
- Q5. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$. Show that \mathbf{AB} and \mathbf{BA} share the same eigenvalues. Hint: examine the eigen-equations.

Positive definite matrices

- Q6. Suppose \mathbf{C} is positive definite and \mathbf{A} has independent columns. Apply the energy test to show that $\mathbf{A}'\mathbf{C}\mathbf{A}$ is positive definite.
- Q7. Suppose \mathbf{S} is positive definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ with corresponding orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.
 1. What are the eigenvalues of the matrix $\lambda_1 \mathbf{I} - \mathbf{S}$? Is it positive semidefinite?
 2. How does it follow that $\lambda_1 \mathbf{x}'\mathbf{x} \geq \mathbf{x}'\mathbf{S}\mathbf{x}$ for every \mathbf{x} ?
 3. Draw the conclusion: The maximum value of the Rayleigh quotient

$$R(\mathbf{x}) = \frac{\mathbf{x}'\mathbf{S}\mathbf{x}}{\mathbf{x}'\mathbf{x}}$$

is λ_1 .

4. Show that the maximum value of the Rayleigh quotient subject to the condition $\mathbf{x} \perp \mathbf{u}_1$ is λ_2 . Hint: expand \mathbf{x} in eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$.
5. Show that the maximum value of the Rayleigh quotient subject to the conditions $\mathbf{x} \perp \mathbf{u}_1$ and $\mathbf{x} \perp \mathbf{u}_2$ is λ_3 .
6. What is the maximum value of $\frac{1}{2}\mathbf{x}'\mathbf{S}\mathbf{x}$ subject to the constraint $\mathbf{x}'\mathbf{x} = 1$. Hint: write down the Lagrangian and set its derivative to zero.
7. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ with largest singular value σ_1 . Find the maximum value of $\frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}$.
8. Let \mathbf{B} be a submatrix of $\mathbf{A} \in \mathbb{R}^{m \times n}$. Show that the largest singular value of \mathbf{B} is always less than the largest singular value of \mathbf{A} . Hint: use Q7.7.

SVD

- Q8. Find the closest rank-1 approximation (Frobenius norm or L2 norm) to these matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

- Q9. Moore-Penrose inverse from SVD.

1. With singular value decomposition $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}'$, verify that

$$\mathbf{X}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}' = \mathbf{V}_r\mathbf{\Sigma}_r^{-1}\mathbf{U}_r' = \sum_{i=1}^r \sigma_i^{-1} \mathbf{v}_i \mathbf{u}_i',$$

where $\mathbf{\Sigma}^+ = \text{diag}(\sigma_1^{-1}, \dots, \sigma_r^{-1}, 0, \dots, 0)$ and $r = \text{rank}(\mathbf{X})$, satisfies the four properties of the [Moore-Penrose inverse](https://ucla-biostat216-2021fall.github.io/slides/06-matinv/06-matinv.html#Generalized-inverse-and-Moore-Penrose-inverse-(optional)) ([https://ucla-biostat216-2021fall.github.io/slides/06-matinv/06-matinv.html#Generalized-inverse-and-Moore-Penrose-inverse-\(optional\)](https://ucla-biostat216-2021fall.github.io/slides/06-matinv/06-matinv.html#Generalized-inverse-and-Moore-Penrose-inverse-(optional))).

2. Show that $\text{rank}(\mathbf{X}^+) = \text{rank}(\mathbf{X})$.
3. Show that $\mathbf{X}^+\mathbf{X}$ is the orthogonal projection into $\mathcal{C}(\mathbf{X}')$ and $\mathbf{X}\mathbf{X}^+$ is the orthogonal projection into $\mathcal{C}(\mathbf{X})$.
4. Show that $\boldsymbol{\beta}^+ = \mathbf{X}^+\mathbf{y}$ is a minimizer of the least squares criterion $f(\boldsymbol{\beta}) = \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|^2$. Hint: check $\boldsymbol{\beta}^+$ satisfies the normal equation $\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{y}$.
5. Show that $\boldsymbol{\beta}^+ \in \mathcal{C}(\mathbf{X}')$.
6. Show that if another $\boldsymbol{\beta}^*$ minimizes $f(\boldsymbol{\beta})$, then $\|\boldsymbol{\beta}^*\| \geq \|\boldsymbol{\beta}^+\|$. This says that $\boldsymbol{\beta}^+ = \mathbf{X}^+\mathbf{y}$ is the least squares solution with smallest L2 norm. Hint: since both $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^+$ satisfy the normal equation, $\mathbf{X}'\mathbf{X}\boldsymbol{\beta}^* = \mathbf{X}'\mathbf{X}\boldsymbol{\beta}^+$ and deduce that $\boldsymbol{\beta}^* - \boldsymbol{\beta}^+ \in \mathcal{N}(\mathbf{X})$.

Optimization and multivariate calculus

- Q10.

1. Explain why the intersection $K_1 \cap K_2$ of two convex sets is a convex set.
2. Prove that the maximum F_3 of two convex functions F_1 and F_2 is a convex function. Hint: What is the set above the graph of F_3 ?

- Q11. Show that these functions are convex:

1. Entropy $x \log x$.
2. $\log(e^x + e^y)$.
3. ℓ_p norm $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p}$, $p \geq 1$.
4. $\lambda_{\max}(\mathbf{S})$ as a function of the symmetric matrix \mathbf{S} . Hint: Q7.6 and Q10.2.

- Q12. Minimize $f(x_1, x_2) = \frac{1}{2}\mathbf{x}'\mathbf{S}\mathbf{x} = \frac{1}{2}x_1^2 + 2x_2^2$ subject to the constraint $\mathbf{A}'\mathbf{x} = x_1 + 3x_2 = b$.

1. What is the Lagrangian $L(\mathbf{x}, \lambda)$ for this problem.
2. What are the three equations "derivative of L=zero"?
3. Solve these equations to find $\mathbf{x}^* = (x_1^*, x_2^*)$ and the multiplier λ^* .
4. Verify that the derivative of the minimum cost is $\partial f^*/\partial b = -\lambda^*$.