Biostat 216 Homework 6+7

Due Dec 3 Friday @ 11:59pm

Submit a PDF (scanned/photographed from handwritten solutions, or converted from RMarkdown or Jupyter Notebook) to Gracescope on CCLE.

Eigenvalues and eigenvectors

- Q1. Diagonalize (show the steps to find eigenvalues and eigenvectors)
  $$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
  and compute $\mathbf{X}\mathbf{\Lambda}^k\mathbf{X}^{-1}$ to prove the formula
  $$\mathbf{A}^k = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}.$$

- Q2. Suppose the same $\mathbf{X}$ diagonalize both $\mathbf{A}$ and $\mathbf{B}$. They have the same eigenvectors in $\mathbf{A} = \mathbf{X}\mathbf{\Lambda}_1\mathbf{X}^{-1}$ and $\mathbf{B} = \mathbf{X}\mathbf{\Lambda}_2\mathbf{X}^{-1}$. Prove that $\mathbf{AB} = \mathbf{BA}$.

- Q3. Suppose the eigenvalues of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are $\lambda_1, \ldots, \lambda_n$. Show that $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$. Hint: λ_is are roots of the characteristic polynomial.

- Q4. Ture of false. For each statement, indicate it is true or false and gives a brief explanation.

  1. If the columns of $\mathbf{X}$ (eigenvectors of a square matrix $\mathbf{A}$) are linearly independent, then (a) $\mathbf{A}$ is invertible; (b) $\mathbf{A}$ is diagonalizable; (c) $\mathbf{X}$ is invertible; (d) $\mathbf{X}$ is diagonalizable.
  2. If the eigenvalues of $\mathbf{A}$ are 2, 2, 5 then the matrix is certainly (a) invertible; (b) diagonalizable.
  3. If the only eigenvectors of $\mathbf{A}$ are multiples of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, then the matrix has (a) no inverse; (b) a repeated eigenvalue; (c) no diagonalization $\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$.

- Q5. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$. Show that $\mathbf{AB}$ and $\mathbf{BA}$ share the same non-zero eigenvalues. Hint: examine the eigen-equations.

Positive definite matrices

- Q6. Suppose $\mathbf{C}$ is positive definite and $\mathbf{A}$ has independent columns. Apply the energy test to show that $\mathbf{A}'\mathbf{C}\mathbf{A}$ is positive definite.

- Q7. Suppose $\mathbf{S}$ is positive definite with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ with corresponding orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

  1. What are the eigenvalues of the matrix $\lambda_1 \mathbf{I} - \mathbf{S}$? Is it positive semidefinite?
  2. How does it follow that $\lambda_1 \mathbf{x}'\mathbf{x} \geq \mathbf{x}'\mathbf{S}\mathbf{x}$ for every $\mathbf{x}$?
  3. Draw the conclusion: The maximum value of the Rayleigh quotient
  $$R(\mathbf{x}) = \frac{\mathbf{x}'\mathbf{S}\mathbf{x}}{\mathbf{x}'\mathbf{x}}$$
  is λ_1.
  4. Show that the maximum value of the Rayleigh quotient subject to the condition $\mathbf{x} \perp \mathbf{u}_1$ is λ_2. Hint: expand $\mathbf{x}$ in eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.
  5. Show that the maximum value of the Rayleigh quotient subject to the conditions $\mathbf{x} \perp \mathbf{u}_1$ and $\mathbf{x} \perp \mathbf{u}_2$ is λ_3.
  6. What is the maximum value of $\frac{1}{2}\mathbf{x}'\mathbf{S}\mathbf{x}$ subject to the constraint $\mathbf{x}'\mathbf{x} = 1$. Hint: write down the Lagrangian and set its derivative to zero.
  7. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ with largest singular value σ_1. Find the maximum value of $\frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}$.
  8. Let $\mathbf{B}$ be a submatrix of $\mathbf{A} \in \mathbb{R}^{m \times n}$. Show that the largest singular value of $\mathbf{B}$ is always less than the largest singular value of $\mathbf{A}$. Hint: use Q7.7.

SVD

- Q8. Find the closest rank-1 approximation (Frobenius norm or L2 norm) to these matrices
$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- Q9. Moore-Penrose inverse from SVD.
  1. With singular value decomposition $\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}'$, verify that
  $$\mathbf{X}^{+} = \mathbf{V}\boldsymbol{\Sigma}^{+}\mathbf{U}^{T} = \mathbf{V}_r\boldsymbol{\Sigma}_r^{-1}\mathbf{U}_r' = \sum_{i=1}^{r} \sigma_i^{-1}\mathbf{v}_i\mathbf{u}_i',$$
  where $\boldsymbol{\Sigma}^{+} = \mathrm{diag}(\sigma_1^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots, 0)$ and $r = \mathrm{rank}(\mathbf{X})$, satifies the four properties of the [Moore-Penrose inverse (https://ucla-biostat216-2021fall.github.io/slides/06-matinv/06-matinv.html#Generalized-inverse-and-Moore-Penrose-inverse-(optional))](https://ucla-biostat216-2021fall.github.io/slides/06-matinv/06-matinv.html#Generalized-inverse-and-Moore-Penrose-inverse-(optional)).
  2. Show that $\mathrm{rank}(\mathbf{X}^{+}) = \mathrm{rank}(\mathbf{X})$.
  3. Show that $\mathbf{X}^{+}\mathbf{X}$ is the orthogonal projection into $\mathcal{C}(\mathbf{X}')$ and $\mathbf{X}\mathbf{X}^{+}$ is the orthogonal projection into $\mathcal{C}(\mathbf{X})$.
  4. Show that $\boldsymbol{\beta}^{+} = \mathbf{X}^{+}\mathbf{y}$ is a minimizer of the least squares criterion $f(\boldsymbol{\beta}) = \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|^2$. Hint: check $\boldsymbol{\beta}^{+}$ satisfies the normal equation $\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \mathbf{X}'\mathbf{y}$.
  5. Show that $\boldsymbol{\beta}^{+} \in \mathcal{C}(\mathbf{X}')$.
  6. Show that if another $\boldsymbol{\beta}^{\star}$ minimizes $f(\boldsymbol{\beta})$, then $\|\boldsymbol{\beta}^{\star}\| \geq \|\boldsymbol{\beta}^{+}\|$. This says that $\boldsymbol{\beta}^{+} = \mathbf{X}^{+}\mathbf{y}$ is the least squares solution with smallest L2 norm. Hint: since both $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^{+}$ satisfy the normal equation, $\mathbf{X}'\mathbf{X}\boldsymbol{\beta}^{\star} = \mathbf{X}'\mathbf{X}\boldsymbol{\beta}^{+}$ and deduce that $\boldsymbol{\beta}^{\star} - \boldsymbol{\beta}^{+} \in \mathcal{N}(\mathbf{X})$.

Optimization and multivariate calculus

- Q10.
  1. Explain why the intersection $K_1 \cap K_2$ of two convex sets is a convex set.
  2. Prove that the maximum F_3 of two convex functions F_1 and F_2 is a convex function. Hint: What is the set above the graph of F_3?

- Q11. Show that these functions are convex:
  1. Entropy $x \log x$.
  2. $\log(e^x + e^y)$.
  3. ℓ_p norm $\|\mathbf{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p}, p \geq 1$.
  4. $\lambda_{\max}(\mathbf{S})$ as a function of the symmetric matrix $\mathbf{S}$. Hint: Q7.6 and Q10.2.

- Q12. Minimize $f(x_1, x_2) = \frac{1}{2}\mathbf{x}'\mathbf{S}\mathbf{x} = \frac{1}{2}x_1^2 + 2x_2^2$ subject to the constraint $\mathbf{A}'\mathbf{x} = x_1 + 3x_2 = b$.

  1. What is the Langrangian $L(\mathbf{x}, \lambda)$ for this problem.
  2. What are the three equations "derivative of L=zero"?
  3. Solve these equations to find $\mathbf{x}^{\star} = (x_1^{\star}, x_2^{\star})$ and the multiplier $\lambda^{\star}$.
  4. Verify that the derivative of the minimum cost is $\partial f^{\star}/\partial b = -\lambda^{\star}$.