Stochastic Processes: Lecture 4

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Point processes

- A point process is a random list of points $T_i \in \mathcal{T} \subset \mathbb{R}^D$.

- The total number of points $N(\mathcal{T})$ may be fixed or random.

- For $A \subset \mathcal{T}$, let $N(A)$ be the total number of points in A:

$$N(A) = \sum_{i=1}^{N(\mathcal{T})} 1\{T_i \in A\}\,.$$

- We are interested in *non-explosive* point processes, for which

$$\Pr(N(A) < \infty) = 1 \quad \text{when} \quad \mathrm{vol}(A) < \infty\,.$$

Poisson processes

- The points $T_i \in \mathcal{T}$ follow a homogeneous poisson process with intensity $\lambda > 0$ if

$$N(A_j) \overset{\perp}{\sim} \mathrm{Pois}(\lambda \cdot \mathrm{vol}(A_j))$$

  for disjoint sets $A_j \subset \mathcal{T}$ that satisfy $\mathrm{vol}(A_j) < \infty$.

- Define $\lambda(t) : \mathcal{T} \to [0, \infty)$ so that

$$\int_A \lambda(t)dt < \infty \quad \text{whenever} \quad \mathrm{vol}(A) < \infty\,.$$

  Then for a non-homogeneous point process on $\mathcal{T}$ with intensity function $\lambda(t)$

$$N(A_j) \overset{\perp}{\sim} \mathrm{Pois}\left(\int_{A_j} \lambda(t)dt\right)$$

  for disjoint sets $A_j \subset \mathcal{T}$ that satisfy $\mathrm{vol}(A_j) < \infty$.

A sampling technique

Theorem 1
Let T_i be the points of a Poisson process on $\mathcal{T}$ with intensity function $\lambda(t) \geq 0$, where $\Lambda(\mathcal{T}) = \int_{\mathcal{T}} \lambda(t)dt$. Then T_i can be sampled by

1. *generating $N(\mathcal{T}) \sim Pois(\Lambda(\mathcal{T}))$ and*

2. *generating $N(\mathcal{T})$ independent T_i with probabilities*

$$Pr(T_i \in A) = \frac{1}{\Lambda(\mathcal{T})} \int_A \lambda(t)dt\,.$$

A sampling technique

Proof.

For $J \geq 1$, let $A_1, \ldots, A_J$ be disjoint subsets of $\mathcal{T}$ and define $A_0 = \{t \in \mathcal{T} | t \notin \cup_{j=1}^{J} A_j\}$. Let $n_j \geq 0$ for $j = 1, \ldots, J$. Let

$$\begin{aligned} P_* &= \Pr(N(A_1) = n_1, \ldots, N(A_J) = n_J) \\ &= \sum_{n_0=0}^{\infty} \Pr(N(A_0) = n_0, N(A_1) = n_1, \ldots, N(A_J) = n_J)\,. \end{aligned}$$

Set $n = n_0 + n_1 + \cdots + n_J$. Under this sampling scheme:

$$\begin{aligned} P_* &= \frac{n!}{n_0! n_1! \ldots n_J!} \sum_{n_0=0}^{\infty} \frac{e^{-\Lambda(\mathcal{T})} \Lambda(\mathcal{T})^n}{n!} \prod_{j=0}^{J} \left(\frac{\Lambda(A_j)}{\Lambda(\mathcal{T})} \right)^{n_j} \\ &= \sum_{n_0=0}^{\infty} \prod_{j=0}^{J} \frac{e^{-\Lambda(A_j)} \Lambda(A_j)^{n_j}}{n_j!} = \prod_{j=1}^{J} \frac{e^{-\Lambda(A_j)} \Lambda(A_j)^{n_j}}{n_j!}\,. \end{aligned}$$

□

A sampling technique

Corollary 1

Let T_i be the points of a homogeneous Poisson process on $\mathcal{T}$ with intensity $\lambda > 0$, where vol$(\mathcal{T}) < \infty$. Then we may sample the process by

1. *generating $N(\mathcal{T}) \sim Pois(\Lambda(\mathcal{T}))$ and*
2. *generating $T_i \overset{iid}{\sim} Uni(\mathcal{T})$, $i = 1, \ldots, N(\mathcal{T})$.*

Proof.

Apply the Theorem with constant $\lambda(t)$. Then

$$\Pr(T_i \in A) = \frac{\lambda \int_A dt}{\lambda \int_{\mathcal{T}} dt} = \mathrm{vol}(A)/\mathrm{vol}(\mathcal{T})\,.$$

$\square$

Poisson processes on $[0, \infty)$

A Poisson process on $[0, \infty)$ can be represented by the counting function

$$N(t) = N([0, t]) = \sum_{i=1}^{\infty} 1\{T_i \leq t\}, \quad 0 \leq t < \infty .$$

The homogeneous Poisson process on $[0, \infty)$ is defined by these properties:

1. $N(0) = 0$;
2. for $0 \leq s < t$, $N(t) - N(s) \sim \text{Pois}(\lambda(t - s))$;
3. for $0 = t_0 < t_1 < \cdots < t_m$, $N(t_i) - N(t_{i-1})$ are independent.

Simulation methods

It can be shown that

$$T_i - T_{i-1} \sim \exp(\lambda)\,, \quad i \geq 1\,. \tag{1}$$

A heuristic argument says: under (1) and for some x,

$$\Pr(T_i - T_{i+1} > x) = e^{-\lambda x}\,,$$

but if $T_i - T_{i+1} \geq x$, then the interval $(T_{i-1}, T_{i-1} + x)$ has no events. Under the Poisson model, this probability is

$$f(0; \lambda x) = \frac{(\lambda x)^0 e^{-\lambda x}}{0!} = e^{-\lambda x}\,.$$

The exponential spacings method simulates a homogeneous Poisson process thus: setting $T_0 = 0$,

$$T_i = T_{i-1} + E_i\,, \quad E_i \overset{iid}{\sim} \exp(\lambda)\,, \quad i \geq 1\,.$$

Simulation methods

Following previous discussion, we can also simulate a homogeneous Poisson process on $[0, T]$ by

1. generating $N \sim \text{Pois}(\lambda T)$,
2. generating $S_i \overset{iid}{\sim} \text{Uni}([0, T]),\ i = 1, \ldots, N,$ and
3. setting $T_i = S_{(i)}$.

Non-homogeneous Poisson process on $[0, \infty)$

The non-homogeneous Poisson process on $[0, \infty)$ has these properties:

1. $N(0) = 0$;
2. for $0 \le s < t$, $N(t) - N(s) \sim \text{Pois}\Big(\int_s^t \lambda(x)dx\Big)$;
3. for $0 = t_0 < t_1 < \cdots < t_m$, $N(t_i) - N(t_{i-1})$ are independent.

The cumulative rate function is $\Lambda(t) = \int_0^t \lambda(x)dx$. Start by assuming $\lim_{t\to\infty} \Lambda(t) = \infty$ and $\lambda(t) > 0$, $\forall t$. Define variables $Y_i = \Lambda(T_i)$ and the counting function

$$N_y(t) = \sum_{i=1}^{\infty} 1\{Y_i \le t\} = \sum_{i=1}^{\infty} 1\{T_i \le \Lambda^{-1}(t)\} = N(\Lambda_{-1}(t)).$$

Non-homogeneous Poisson process on $[0, \infty)$

Define variables $Y_i = \Lambda(T_i)$ and the counting function

$$N_y(t) = \sum_{i=1}^{\infty} 1\{Y_i \leq t\} = \sum_{i=1}^{\infty} 1\{T_i \leq \Lambda^{-1}(t)\} = N(\Lambda_{-1}(t))\,.$$

Note that $N_y(0) = 0$ and

$$\begin{aligned} N_y(t) - N_y(s) &= N(\Lambda^{-1}(t)) - N(\Lambda^{-1}(s)) \sim \text{Pois}\left(\int_{\Lambda^{-1}(t)}^{\Lambda^{-1}(s)} \lambda(x)dx\right) \\ &= \text{Pois}\left(\Lambda(\Lambda^{-1}(t)) - \Lambda(\Lambda^{-1}(s))\right) = \text{Pois}(t-s)\,. \end{aligned}$$

Finally the increments of $N_y(t)$ are increments of $N(\Lambda^{-1}(t))$. Independence of the latter implies independence for the former. Therefore,

$$Y_i = \Lambda(T_i) \sim PP(1)\,.$$

More exponential spacings

We have shown $Y_i = \Lambda(T_i) \sim PP(1)$. Setting $Y_0 = T_0 = 0$, we can therefore simulate T_i thus:

$$
\begin{aligned}
&Y_i = Y_{i-1} + E_i\,, \quad E_i \overset{iid}{\sim} \exp(1)\,, \quad i \geq 1\,,\\
&T_i = \Lambda^{-1}(Y_i) = \Lambda^{-1}(\Lambda(T_{i-1}) + E_i)\,.
\end{aligned}
$$

Comments:

- If $\lim_{t\to\infty} \Lambda(t) = \Lambda_0$, then $\Lambda^{-1}(y)$ does not exists for $y > \Lambda_0$. If $\Lambda(T_i) + E_i > \Lambda_0$, then there is no T_{i+1} and the process stops.
- The algorithm is convenient when Λ and Λ^{-1} are available in closed form.
- The algorithm works even when Λ takes finite jumps or is constant on some intervals by taking

$$
\Lambda^{-1}(y) = \inf\{t \geq 0 | \Lambda(t) \geq y\}\,.
$$

Thinning (rejection sampling for point processes)

Let $\widetilde{\lambda}(t) \geq \lambda(t)$ and assume we can sample from a Poisson process on $\mathcal{T}$ with $\widetilde{\lambda}$ for intensity function. The following algorithm generates $(T_1, \ldots, T_N) \sim NHPP(\mathcal{T}, \lambda)$:

1. Generate $(\widetilde{T}_1, \ldots, \widetilde{T}_{\widetilde{N}})) \sim NHPP(\mathcal{T}, \widetilde{\lambda})$;
2. if $\widetilde{N} > 0$, then for $i \in \{1, \ldots, \widetilde{N}\}$:
   2.1 draw $u_i \sim \text{Uni}(0, 1)$;
   2.2 if $u_i < \rho(\widetilde{T}_i) = \lambda(\widetilde{T}_i)/\widetilde{\lambda}(\widetilde{T}_i)$, then $\widetilde{T}_i \in \{T_1, \ldots, T_N\}$.

Why thinning works

Let $N(A)$ be the number of points T_i in a set A and $\widetilde{N}(A)$ be the analogue for points $\widetilde{T}_i$. Note that $\widetilde{N}(A) \sim \text{Pois}(\int_A \widetilde{\lambda}(t)dt)$. Then the probability a point in $\widetilde{T}_i \in A$ is accepted is

$$\rho(A) = \frac{\int_A \rho(t)\widetilde{\lambda}(t)dt}{\int_A \widetilde{\lambda}(t)dt} = \frac{\int_A \lambda(t)dt}{\int_A \widetilde{\lambda}(t)dt}\,.$$

It holds that $N(A)|\widetilde{N}(A) \sim \text{binom}(\widetilde{N}(A), \rho(A))$. Marginalizing over $\widetilde{N}(A)$ gives

$$N(A) \sim \text{Pois}\left(\rho(A)\int_A \widetilde{\lambda}(t)dt\right) = \text{Pois}\left(\int_A \lambda(t)dt\right)\,.$$

Independence of N on non-overlapping sets is inherited from $\widetilde{N}$. Therefore $(T_1, \ldots, T_N) \sim NHPP(\mathcal{T}, \lambda)$.

The temporal Hawkes process

The temporal Hawkes process is a non-homogeneous Poisson process on $[0, \infty)$ with (conditional) intensity function given by

$$\lambda(t|T_k < t) = \lambda_0 + \sum_{T_K < t} g(t - T_k)\,,$$

where $g > 0$ is a non-increasing *triggering function*. Because the intensity increases after an observation, the Hawkes process is referred to as *self-exciting* and is useful for modeling contagion.

Exponential decay

A common choice for g is the exponential decay function:

$$\lambda(t|T_k < t) = \lambda_0 + \alpha \sum_{T_K < t} e^{-\beta(t - T_k)} .$$

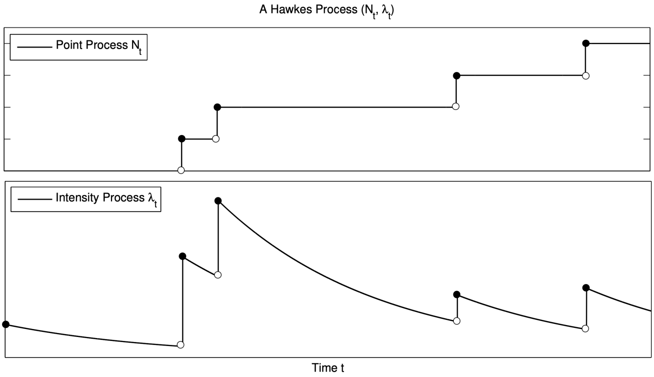


Figure 1: A Hawkes Process with Exponential Decaying Intensity (N_t, λ_t)

Dassios and Zhao, 2013

Exponential decay

A common choice for g is the exponential decay function:

$$\lambda(t|T_k < t) = \lambda_0 + \alpha \sum_{T_K < t} e^{-\beta(t - T_k)} \, .$$

Exponential decay has pros and cons:

- Pros: exponential decay has computational benefits. Process simulation and likelihood computations are both $O(N)$.
- Cons: the exponential rate of decay may be to fast for certain applications, precluding long-term dependencies.

We will return to linear-time computing for the exponential triggering kernel later.

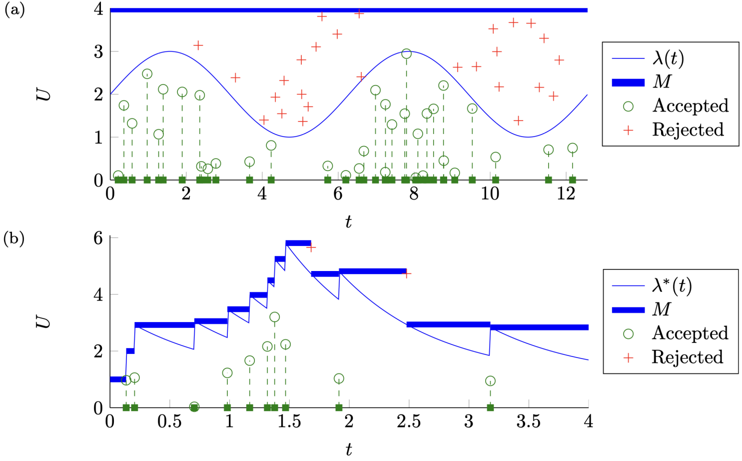
Ogata's modified thinning algorithm

Algorithm 2 Generate a Hawkes process by thinning.

procedure HAWKESBYTHINNING(T, $\lambda^*(\cdot)$)
    require: $\lambda^*(\cdot)$ non-increasing in periods of no arrivals.
    $\varepsilon \leftarrow 10^{-10}$ (some tiny value > 0).
    $P \leftarrow []$, $t \leftarrow 0$.
    while $t < T$ **do**
        Find new upper bound:
            $M \leftarrow \lambda^*(t + \varepsilon)$.
        Generate next candidate point:
            $E \leftarrow \mathrm{Exp}(M)$, $t \leftarrow t + E$.
        Keep it with some probability:
            $U \leftarrow \mathrm{Unif}(0, M)$.
        if $t < T$ **and** $U \leq \lambda^*(t)$ **then**
            $P \leftarrow [P, t]$.
        end if
    end while
    return P
end procedure

Laub et al., 2015

Ogata's modified thinning algorithm



Laub et al., 2015