Autonomous Control Systems:

Notes AND Problem Packet

LACC 2019 Summer Program

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Day 1:

Contents:

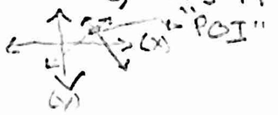
Day 1:

Part 1/6

Solving Systems of equations:

Key Ideas:

- A system of equations is a set of equations which, when manipulated, can produce "Points-of-Intersection". (POI)
- POI are of significant importance, as it tells where two lines "meet-up" and touch.



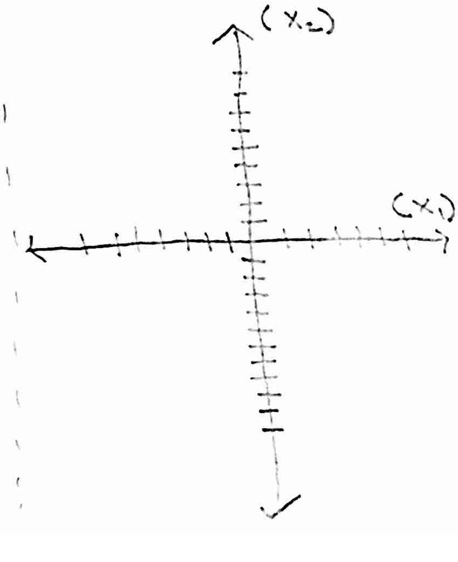
- Typical equations you'll see are "Linear" ⇒ $2x_1 + 3x_2 = 4$
- By manipulating equations, we can find where lines cross.

Example #1*) Consider

$$2x_1 + x_2 = 1 \quad (1)$$
$$-3x_1 + 4x_2 = 2 \quad (2)$$

a) Solve for intersection point of equations (1) & (2)

b) Plot Lines

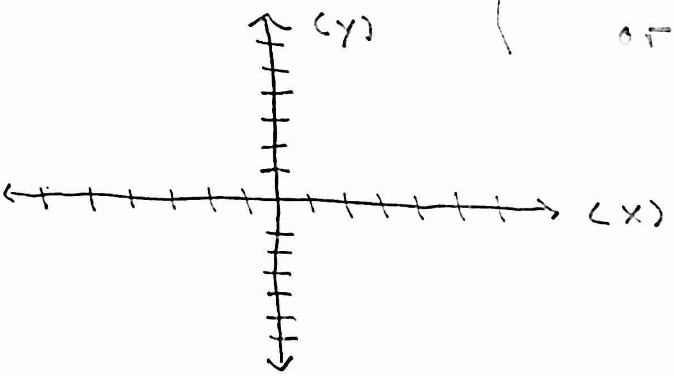


Example #2:

Consider
$$4x + 6y = 2 \quad (1)$$
$$2x + 3y = 1 \quad (2)$$

a) Try solving this system: Do you notice anything with respect to the coefficients?

b) Graph these two lines below: (Are these line parallel or perpendicular?)

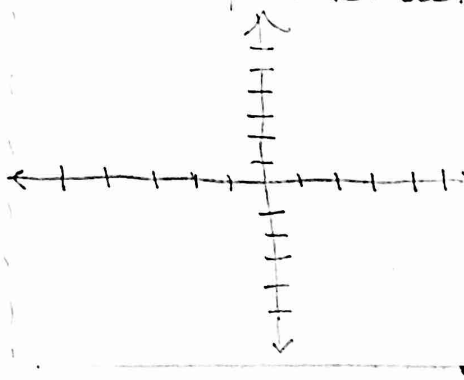


Example #3:

Consider
$$-2x_1 + 3x_2 = 2 \quad (1)$$
$$5x_1 - 2x_2 = 1 \quad (2)$$

a) solve the system ((1) and (2)) for the points of intersection.

b) Sketch quick plot below:



Part 1/6:

Solving Systems of equations with Matrices:

Key Ideas:

- We saw before we can solve "by-hand" simple systems, but this becomes rather burdensome with many equations....
  - → Consider if we were given 3 equations & 3 variables:

$$\begin{aligned} (1)\ 1 &= x_1 + 2x_2 + 3x_3 \\ (2)\ 2 &= 4x_1 + 5x_2 + 6x_3 \\ (3)\ 3 &= 7x_1 + 8x_2 + 9x_3 \end{aligned}$$

... Not too easy by hand computation...

- One of the central ideas of (linear) systems is to write the expressions in the form: $\boxed{\vec{y} = A\vec{x}}$
  - $\vec{y}$ is a "vector": a column of numbers: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \to \mathbb{R}^3$ (3 colums)
  - A is a "matrix": it has rows & colums: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \to \mathbb{R}^{2\times 2}$ (Two Cols, Two Rows)
  - $\vec{x}$ is a vector which contains our x_1, x_2, x_3.... variables
  - ⇒ Can solve for $\vec{x}$ with A.I*y command in python.
- Our goal is to write our system ((1),(2),(3)) (equations) as $\vec{y} = A\vec{x}$ and have a computer solve for us!!

Example #4:

$$\begin{aligned} (1)\ 1 &= x_1 + 2x_2 - 3x_3 \\ (2)\ 2 &= 4x_1 - 5x_2 + 6x_3 \\ (3)\ 3 &= -7x_1 + 8x_2 + 9x_3 \end{aligned}$$

(example from above)

a) what is y-vector?

b) what is A-matrix?

c) what is x-vector?

d) Case study: Forming $\vec{y} = A\vec{x}$ using python. Whats A.I*$\vec{y}$?

Example # 5 : Consider the system:

(1) $8 = -2x_1 + 2x_2 + 7x_3$
(2) $-3 = 4x_1 + 5x_2 - x_3$
(3) $11 = 9x_1 + x_2 + 3x_3$

a) Find $\vec{y}$-vector (The constants vector)

b) Find A-Matrix (The coefficient matrix)

c) Find x-vector (The x-variables vector)

d) Case Study:
- Form equations in python.
- Run command A.I * $\vec{y}$, what is solution?

Part 1/6 Recap:
- We can solve systems of equations by-hand, or by using a computer for efficiency.
- Our goal is to write systems as $\vec{y} = A\vec{x}$
- We can solve for $\vec{x}$ with A.I * $\vec{y}$ in python.

Part 2/6

Formal Introduction to Vector and Matrix Addition, Subtraction, Multiplication and Inversion.

Vector Math Key Ideas:

- The easiest way to think about vectors and matrices are as "containers" of numbers with an associated "dimention."
- A vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is said to have "dimention" 3 since the container has 3 numerical entries.

In Math notation, we write the dimention as $\mathbb{R}^3$

- Vectors of the same dimention (# of elements) can be both added and subtracted:

$$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix} \quad ; \quad \begin{bmatrix}3\\0\end{bmatrix} - \begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$$

- To turn a vector on its side, we can "transpose" it:

$$\mathbb{R}^2 \rightarrow \begin{bmatrix}1\\2\end{bmatrix}^T \Rightarrow \underset{\uparrow\, \mathbb{R}^{1\times 2}}{\begin{bmatrix}1 & 2\end{bmatrix}} \quad ; \quad \text{if } \vec{x} = \begin{bmatrix}1\\2\end{bmatrix} \quad \vec{x}^T = \begin{bmatrix}1 & 2\end{bmatrix}$$

- Dimentionality is important, it tells us if vector multiplication, addition, subtraction is possible.

→ We represent dimentionality w/ $\mathbb{R}^{(rows) \times (columns)}$

→ $\mathbb{R}$ is Real #s

  - To multiply two vectors: $\begin{bmatrix}1\\2\end{bmatrix}$; $\begin{bmatrix}3\\4\end{bmatrix}$

→ 1) Transpose one of vectors:

→ $\begin{bmatrix}1\\2\end{bmatrix}^T = \begin{bmatrix}1 & 2\end{bmatrix}$

→ 2) Multiply numbers in sequential order, add results.

→ $\begin{bmatrix}1 & 2\end{bmatrix} \cdot \begin{bmatrix}3\\4\end{bmatrix} = (1\cdot 3) + (2 \cdot 4) = 3 + 8 = 11$

Lets try some examples to answer these questions:

Example #6

$\vec{V_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{V_2} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ $\vec{V_3} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $\vec{V_4} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

1) List the vectors dimentionality:

2) Transpose each vector, write new dimentionalities:

3) What is $\vec{V_1}^T \cdot \vec{V_2}$? What is $\vec{V_3}^T \cdot \vec{V_4}$?

4) What is $\vec{V_1} + \vec{V_2}$? What is $\vec{V_3} + \vec{V_4}$?

5) Can we multiply $\vec{V_1}$ and $\vec{V_3}$? If so, what is the result? Else, why not?

6) Can we add or subtract $\vec{V_2}$ and $\vec{V_4}$? Why or why not?

Part 2/6

Matrix Math Key Ideas:

Tips for adding, subtracting, multiplying matrices:

- We represent the number of elements of a matrix with symbology $\mathbb{R}^{(\text{rows}) \times (\text{columns})}$ → $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
  - We say mathematically: $A \in \mathbb{R}^{2 \times 2}$
  - We say in plain english: "A contains 2 rows by 2 colums of numbers"
- With matrices, we can <u>add</u>, <u>subtract</u>, <u>multiply</u>, <u>invert</u>

Consider: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ $A \in \mathbb{R}^{2 \times 2}$, $B \in \mathbb{R}^{2 \times 2}$

- To Add A + B:

Add elements in same positions:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

- To subtract A + B:

Subtract elements in same positions:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Note: For matrix addition / subtraction: <u>Dimentions</u> (rows/cols) <u>must be equal</u>!!

- To Multiply A & B, we multiply rows and colums, like the vector techique, per each row – column pair:

ie) $A \cdot B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (1 \cdot 5 + 2 \cdot 7) & (1 \cdot 6 + 2 \cdot 8) \\ (3 \cdot 5 + 4 \cdot 7) & (3 \cdot 6 + 4 \cdot 8) \end{bmatrix}$

multiply each row of <u>A</u> by each column of <u>B</u>.

group rows of first Matrix

group columns of second matrix

Note: For matrix multiplication to work:

"Inner dims must be equal" $A \to \mathbb{R}^{2 \times 2}$ $B \to \mathbb{R}^{2 \times 2}$ same #! ✓

Example #7

Do by hand, check with python.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

1) What is $A+B$?; What is $A-B$? Is this allowable

2) What is $A+C$? What is $B+D$? Is this allowable?

3) What is $A \cdot B$? What is $D \cdot C$? Is this allowable?

4) What is $A \cdot D$? What is $B \cdot C$? (Which is not allowable?)
What is $B \cdot C^T$?

Example # 8:

1) Consider system :

$$-4 = 2x_1 + 3x_2 + x_3 \quad (1)$$
$$1 = 3x_1 - x_2 + 7x_3 \quad (2)$$
$$5 = 6x_1 + 11x_2 - 5x_3 \quad (3)$$

a) write system as $\vec{Y} = A\vec{x}$ below:

b) use numpy to find $\vec{x}$ (Point of intersection → POI) by using the matrix inverse property $A.I \cdot \vec{y} = \vec{x}$
$(A^{-1} \cdot y = x)$

2) Consider system

$$1 = x_1 + 2x_2 + 3x_3 \quad (1)$$
$$2 = 4x_1 + 5x_2 + 6x_3 \quad (2)$$
$$4 = 8x_1 + 10x_2 + 12x_3 \quad (3)$$

i) repeat steps above in problem 1. What is $\vec{x}$? What does numpy report for $\vec{x} = A.I\,\vec{x}$?
$(x = A^{-1}y)$

ii) Change the 10 coefficient on x_2 in eq. (3) to 11.
- Does system now have $\vec{X}$ solution? (solve for $\vec{x}$)
- Why do you think the inverse now exists?

Part 3/6:

Essential properties to know about Matrices:
(Dimention, Rank, Nullspace, Fundamental Theorum of Linear Algebra)

Key Ideas:

- The best part about linear systems is that almost all concepts can be taught in 2 Dimentions or 3: $\mathbb{R}^2$; $\mathbb{R}^3$
- As we saw in part 2/6, not all matrices have inverses.
- In fact, the following concepts explain why some systems dont have solutions for $\vec{x}$. (or exact solutions)

Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; we see A has dimentions 2×2

- We see the first column of A → $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ could be multiplied by 2 to acheive $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ (the same as column 2 of A)
- Fact: If a column of A can be scaled (multiplied) by some number "α", then the two columns of the matrix are linearly dependent.

→ so, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are linearly dependent since the first can be multiplied by factor of two to equal the 2nd column.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ → col 1 & col 2 are Lin. dependent

ie) 1 independent column
1 dependent column.

- Fact: If there is a linearly dependent column of A, the matrix inverse will not exist.

Numpy has tools to tell us if our A matrix has linearly dependent columns:

cont'd

• A particularly helpful function in numpy is the matrix_rank(A) function. The function will return how many linearly independent columns A has.

→ Rank = "Number of independent columns"

Fact: The dimention of A [dim(A)] minus the number of independent columns of A equals the number of linearly dependent columns.

→ In the math community, we refer to the number of independent columns as the Rank of A, and the number of dependent columns as the dimention of the nullspace of A: Nullspace reffered to as $\mathcal{N}(A)$

This property is known as the fundamental theorum of Linear Algebra: $\boxed{\dim(A) - \text{Rank}(A) = \dim(\mathcal{N}(A))}$

(With examples, becomes very easy)

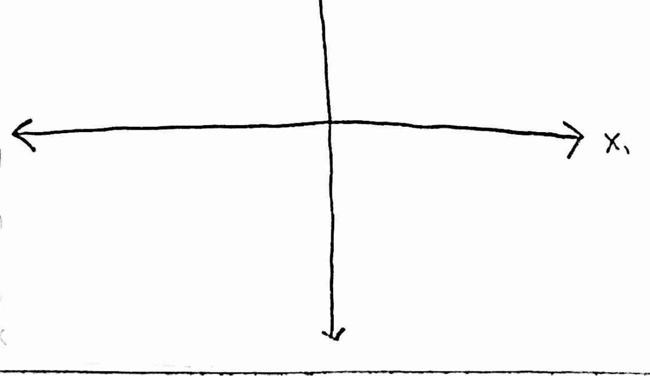
Example #9) re-consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

a) What's dimention of A?

b) Find rank of A. (with numpy's matrix_rank function)

c) Determine the dimention (# of vectors in nullspace)

d) Find a vector in nullspace of A. Graph vector columns of A and nullspace vector found on graph:



Example #10) consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

a) What's the dimention of A?

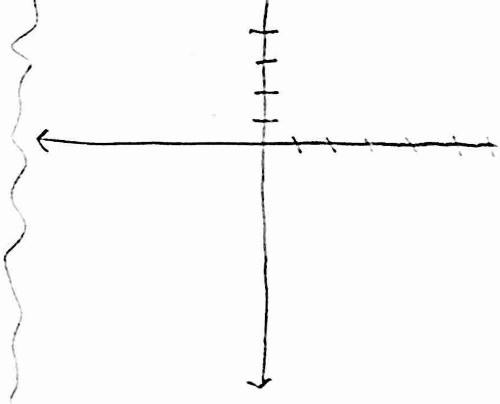
b) Find rank of A.

c) Determine the dimention of the nullspace of A.
(use fundamental theorum of Linear Algebra)

Example #11) consider $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

a) Find the dimention and rank of A.

b) Find the dimention of nullspace of A and a vector in the nullspace of A. Graph.



Part 4/6

• An Introduction to Motion and Kinematics:

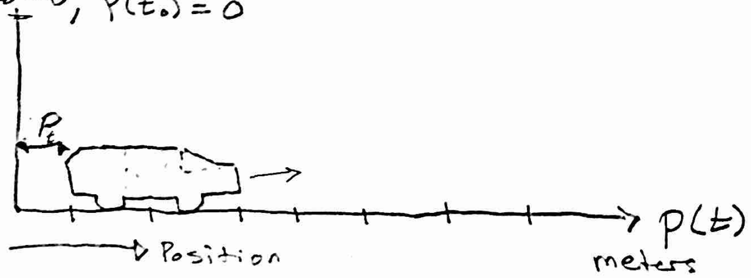
Key ideas for motion:

• An end goal of ours is to analyze and control a dynamical motion model of a car using linear algebra techniques we've learned...and some control theory.
(...Down the road)

• To do so, we need a light introduction on how position, velocity, and acceleration are related and can describe motion of cars, drones, rockets and much more.

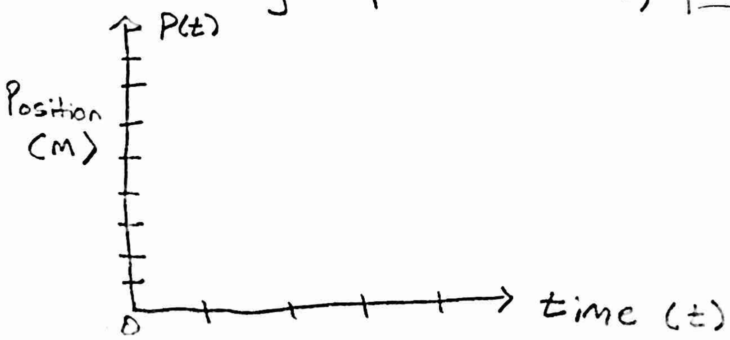
• Suppose a car is driving along a straight road and has the following position equation $p(t)$, with t representing the elapsed time from start: $p(t) = t^2$

Starts at $t_0 = 0$, $p(t_0) = 0$



• Motion is best seen graphed out, plot position vs time below:

$$p(t) = t^2$$



- To find velocity (speed), we'll introduce a "rate-of-change" technique.

- FACT: Velocity x Time = Distance → $V \times T = D$

  Velocity → "Distance per unit time" is → 35 MPH

- Can write this velocity expression as

  d_1 = First distance point

  d_2 = Second distance point

  t_1 = time at d_1

  t_2 = time at d_2

  V_{avg} = Average velocity (speed + direction) over $t_1 \to t_2$

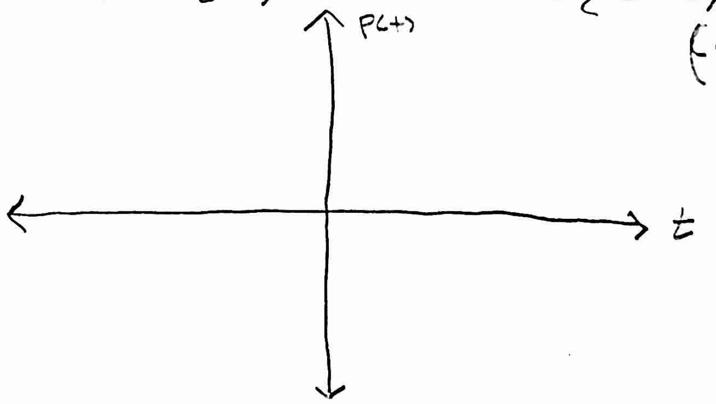
Average Velocity Formula

$$V_{average} = \frac{d_2 - d_1}{t_2 - t_1} = \frac{\Delta d}{\Delta t}$$

Δ represents "change" between two points, whether it be time, position, etc.

Example # 12: Suppose we have the position function $p(t) = \frac{1}{2}t^2$ with t = time.

a) Plot the position $p(t) = \frac{1}{2}t^2$, for times $\{t = 0, \frac{1}{2}, 1, 2, 3\}$ (or more if you like)



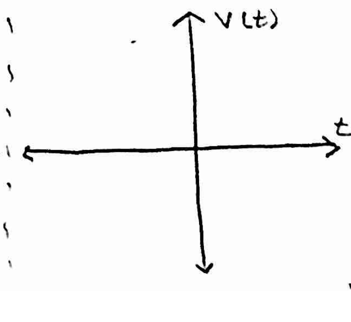
b) Find the average velocities between: $t = [0, 1]$, → V_{0-1}

$t = [1, 2]$, V_{1-2}

& $t = [2, 3]$ V_{2-3}

c) Do the velocities increase or decrease?

↳ Plot: →



Part 4/6

- Using a calculus technique ("Derivatives"), we can manipulate the previous example's position function to find an exact function describing the velocity of our car, as seen next:

Key Ideas for Motion Continued:

- We mentioned before that position, velocity, and acceleration are all related to one another.

$p(t)$ = position $v(t)$ = velocity $a(t)$ = acceleration

- We can use a basic calculus technique to link these concepts together, the Derivative: Symbology is $\frac{d(x)}{dt}$ or $\dot{x}(t)$, $x \to$ Function of time

★ Suppose $p(t) = \frac{1}{2}t^2$, then

"$\frac{dp(t)}{dt}$" $= V(t) = \frac{1}{2} \cdot 2 \cdot t^{2-1}$ (bring down) (Subtract Power by 1)

$\boxed{\text{same meaning}}$

"$\dot{p}$" $= V(t) = 1 \cdot t^1 = t$

To find velocity $V(t)$, we take t's power ie, "2" in this case, bring it down, and multiply it to t's coefficient ie "1/2" here. Then, we subtract 1 from the power ie "2"

→ Steps For Derivative of $\alpha \cdot t^{n}$ (n → nth Power, some number)

1) Lower power of $t \to (n)$
2) Multiply α and n
3) Subtract 1 from power of $t \xrightarrow{ie} n-1$

Example # 13

consider → $p(t) = 2 \cdot t^3$

- Find $v(t)$, Plot both $p(t)$ and $v(t)$ in Python,
- Can you also find acceleration $a(t) = \frac{dv(t)}{dt}$ and Plot?

Example #14

consider → $p(t) = -\frac{1}{2}t^2 + t + 4t^3$

- Repeat steps in #13

Example #15

consider → $p(t) = 3t^3 - 2t^2 + 5$

- Repeat steps in #13

Part 4/6:

- We see from #13 – #15 that taking a derivative of $p(t)$ (with respect to t) gives us the velocity function $v(t)$

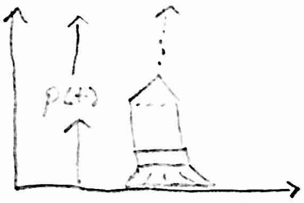
Fact: The derivative of $v(t)$ is acceleration, $a(t)$

- We can write this as $\frac{dv(t)}{dt} = a(t)$ & $\frac{dp(t)}{dt} = v(t)$
- We can also express derivatives as $v(t) = \dot{p}(t) = \frac{dp(t)}{dt}$

(The symbology is confusing, but a few more examples will help)

$a(t) = \dot{v}(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left(\frac{dp}{dt}\right)$

Example #16: A rocket is designed to launch from a launch station vertically following the position function



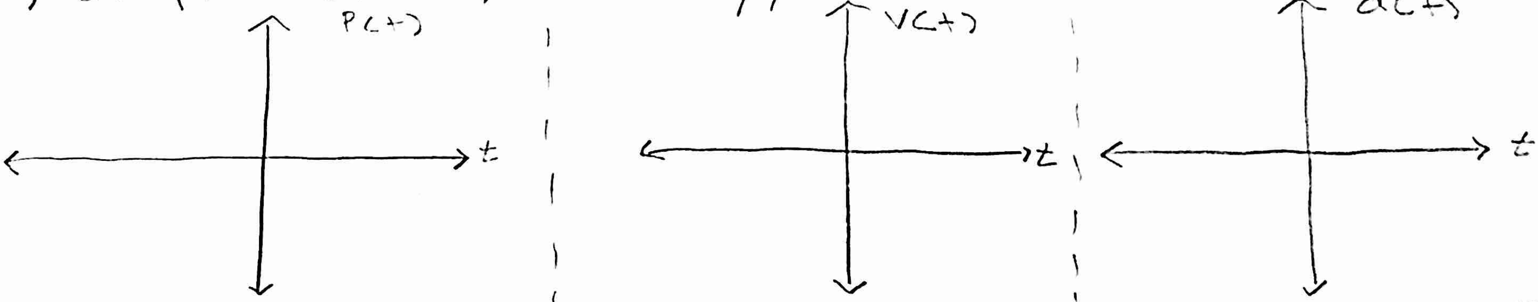
$p(t) = 1/3 \cdot t^2 + 4$

a) What is the velocity of the rocket at $t = 3$ seconds?

b) What is the acceleration of the rocket at $t = 2$ sec?

<u>Plot Motion Equations:</u>

c) Graph Position, velocity, acceleration functions:



Part 4/6:

Key Equations of Motion (Kinematic Relationships)

- distance (synonomous to position) is $d(t)$; Initial position is d_0
- Velocity is $V(t)$; Initial velocity is V_0 (units m/s)
- acceleration is $a(t)$; acceleration is usually constant: $a(t) = a_0$ (units m/s^2)

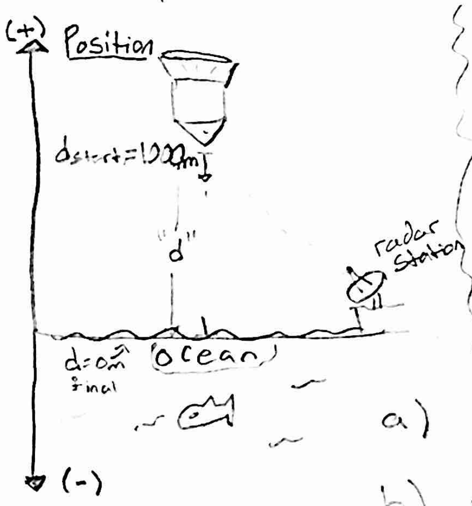
Eq1) Distance Equation: $d(t) = \frac{1}{2} \cdot a_0 \cdot t^2 + V_0 t + d_0$ · t represents elapsed time from start

Eq 2) Distance Equation: $d(t_2) = \dfrac{V_2^2(t_2) - V_1^2(t_1)}{2a_0}$ · t_1, t_2 are time instances

These equations assume constant acceleration (a_0), and are very helpful when considering "free falling" objects: (only gravity)

Setup!

Example #17:



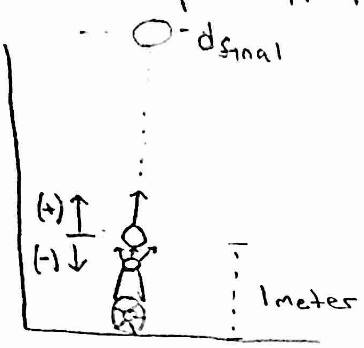
- Your rocket from #16 is now heading back to earth. Ground station reads the rocket is at $d_0 = +1000m$ from sea-level. The rocket is reported to have a velocity of $V_0 = 0 m/s$ (its just turned around and free falling.) Since the rocket is in free-fall, acceleration is constant and equals gravity $\Rightarrow -9.81 m/s^2$

a) At what time will the rocket hit the ocean? ie $d = 0m$

b) If $V_0 = -30 m/s$ @ t_0 (experiment start time → 0s), find collision time as previously in ⓐ.

Part 4/6

Example #18)



A cannon tests its launch strength by blasting a cannon ball vertically upwards. The designers of the cannon report to us that the ball exits the cannon at $V_0 = 150\ m/s$ at an initial height of $d_0 = 1$ meter. Gravity acts on the ball, starting at $t_0 = 0_s$: gravity $= a_0 = -9.81\ m/s^2$

a) At what distance does the ball reach its (final) terminal height?

b) At what time does the ball reach its terminal height?

c) At what time does the ball reach 1 meter again?

d) What is the ball's velocity at 1 meter during free-fall (going down)?

e) Use python to plot the position, velocity, and acceleration of the cannonball over its flight time.

Part 5/6:

Introduction to Physics and Forces

- So far, we have solved linear systems of equations (Part 1/2) analyzed matrix properties (Part 3), took derivatives of position $p(t)$ to get velocity and acceleration and also applied motion equations to real world examples (Part 4) (Kinematics)

- Now, learning the basics of physics ⇒ (forces + free body diagrams) will complete the necessary toolset to tackle simple control theory problems.

Key Ideas for Forces:

- In the field of physics, there are three laws the great "Sir Isaac Newton" proposed: These are called "Newtons laws of Motion"
- Not suprisingly, force has units called "Newtons" → $\frac{kg \cdot m}{s^2}$
- The second law is of particular interest to us:

Newton's 2nd Law: "The sum of forces on an object equals the mass of the object times its current acceleration"

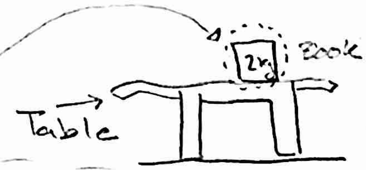
ie: $$\sum_{i=0}^{N} F_i = m \cdot a_{net}(t)$$

m → mass of rigid body object

$a_{net}(t)$ → resulting acceleration of object.

$\sum$ → Sum of external forces on object.

→ Suppose a mass (Book) of two Kilograms rests on a table

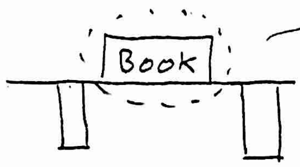


Key idea

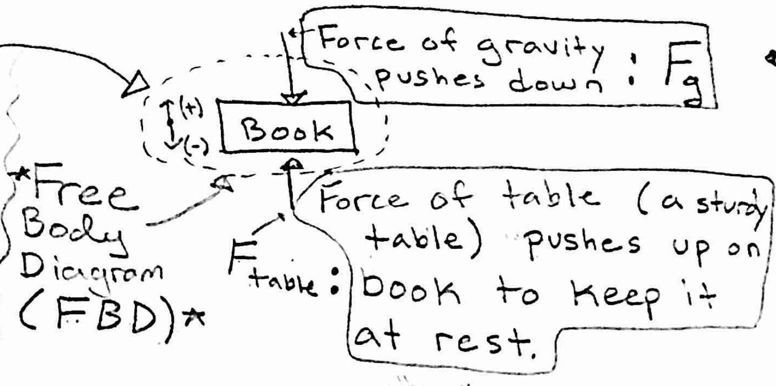
* We can isolate any object of interest and analyze forces on it!

- We know intuitively gravity exerts force down on the book
- We know intuitively the table exerts equal force (up) on book

Part 5/6 ; Key ideas for Forces, cont'd:



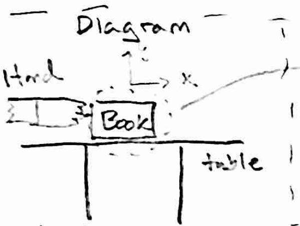
$F_g = m_{book} \cdot g$



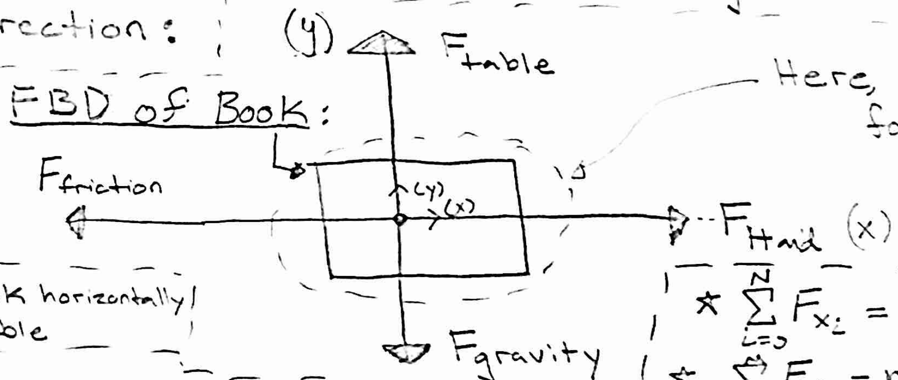
← This practice of isolating objects and examining the forces acting on the object is called a "Free Body Diagram"

- Applying Newtons 2nd law: $\overset{\text{"sum"}}{\sum} \text{Forces} = F_{table} - F_{grav} = m_{book} \cdot a_{net}$
  - → Know that book is at rest ; $a_{net} = 0$ (Book not moving)
  - → Therefore, the force of the table matches the force of gravity on book: $F_{table} - F_{gravity} = m_{book} \cdot \overset{a_{net}}{0} = 0$

- Fact: We can have forces on an object in more than 1 direction:



- Hand pushing on book horizontally on flat table

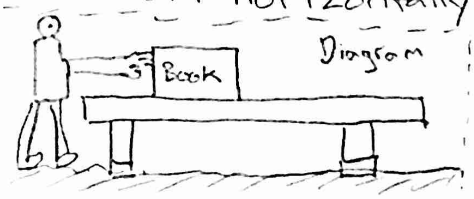


Here, we have forces in x and y

$$\star \sum_{i=0}^{N} F_{x_i} = m \cdot a_{net_x}$$
$$\star \sum_{i=0} F_{y_i} = m \cdot a_{net_y}$$
Key Force Equations

- We'll solve several examples to drill the idea of FBD's and how forces affect the acceleration of an object.

Example # 19) Take the example of the hand pushing a book horizontally on flat table:



$F_{applied} = 5N$

$m_{book} = 2\ kg$

$g = -9.81\ m/s^2$

$F_{friction} = \mu \cdot F_N$, $\mu = .2$

F_N = "Normal Force" (Table force up)

μ = friction coefficient $= 0.3$

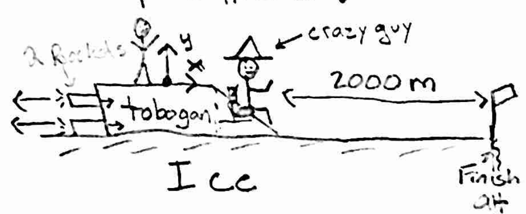
Goal is to:

1) Draw FBD of book
2) Sum forces in x & y using Newtons 2nd Law.
3) Find a_{net_x}, a_{net_y}

Cont'd next Page.

Example # 19 cont'd:

Example #20:



$m_{combo} = 120$ kg
$\mu = 0.2$
$g = -9.81\ m/s^2$

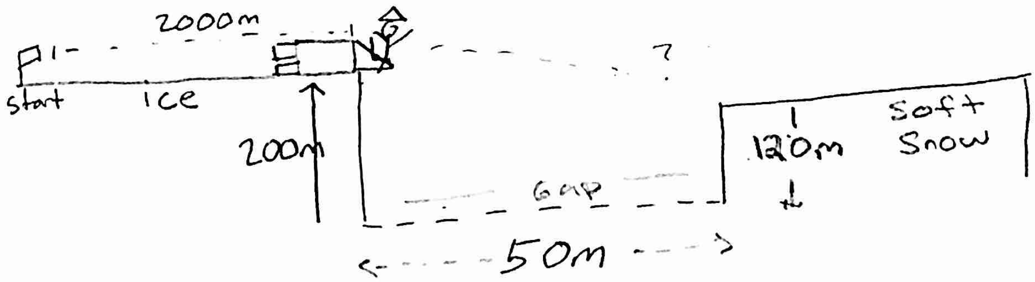
- Rick and Morty strap 2 booster rockets to their tobogan sled and set up on a frozen lake. Each booster rocket delivers 150N of forwards force on the sled. The ice has a coefficient of friction $\mu = 0.2$. The total weight of the scientist + boosters + tobogan is 120 Kg.

a) Rick wants to figure out what will be his <u>net acceleration rates in x & y</u>. (Draw FBD and Sum Forces → In x & y)

(Motion and Kinematics)

b) Morty is nervous. He wants to know how long it will take him to hit the finish line 2000m away.

Example #20 cont'd:

c) 

Unfortunately, our friend rick (somehow) did not realize there was a ledge at the end of the lake. We need to figure out whether he will clear the gap to the other side.

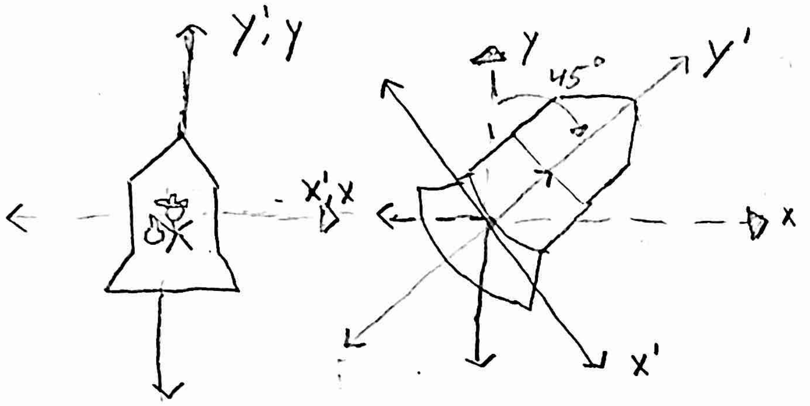
i) Find the scientists speed (velocity) after their 2000m ride. (Assume thrusters turned off at cliff's start)

ii) Will Rick and Marty clear the 50m (horizontal) gap?

iii) Bonus: How big of a gap can the scientists clear at the launch speed?

Example #21

Part 5/6



Jeff Bezos is using his private rocket to get to the moon. He orients the rocket 45° clockwise and applies a Force of 80,000 N through the " on board thrusters. His rocket weighs 4000 Kg.

a) Draw free body diagram of rocket tilted at 45°. (Hint: use trigonometry to find Forces in x, y)

b) Solve for net accelerations using the fbd technique
→ ie, Find a_{net_x} and a_{Net_y} with newtons 2nd law,

An Introduction to Modeling
The "OPEN LOOP" Control Problem:

Key Ideas for setting up the "open-Loop" system of equations:

- Knowing how to set-up a free-body diagram (FBD) gives us the tools to setup our control problems.

- In this section, we will revist the example problems of part 5 (#'s 17-21) and put them in the form we need to add <u>feedback</u>, so the object will optimally move as we wish.

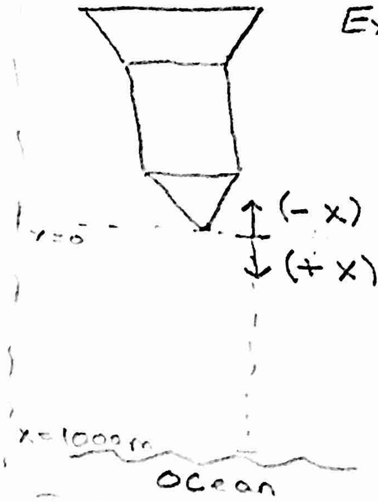
- Another word(s) for "Open-Loop" is "Finish the physics problem"! An open-loop system simply refers to the system of equations produced when solving an FBD for a_{net_x}, a_{net_y}.

- We'll see our knowledge of <u>basic derivatives</u>, <u>FBD's</u> and <u>Linear Algebra</u> is all we need to start simple control problems.

Suppose we considered the free-falling rocket in example #17.

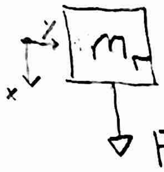
- We know in our example #17 that only <u>gravity</u> exerted a force on the rocket → (ignoring wind drag)

Example 17 revisited



- Our FBD would look like

FBD Rocket



$F_{gravity}$

→ so newtons 2nd Law says:

$$\sum_{i=0}^{n} F_i = m_{rocket} \times a_{net_x}$$

$$\underbrace{(m_{rocket} \cdot g)}_{\text{"}F_g\text{"}} = \underbrace{m_{rocket} \cdot a_{net}}_{\text{resulting acceleration on rocket } (a_{net})}$$

The goal here is to build a "Linear system model" of the rocket's FBD

- We see $a_{net_x} = g = 9.81\,m/s^2$

- we know $\frac{dx(t)}{dt} = v(t)$ $\quad \frac{dv(t)}{dt} = a(t)$ w/ $x \rightarrow$ Position, $v \rightarrow$ Velocity, $a \rightarrow$ accel

- Can use alternative symbology: $\dot{x} = v$, $\dot{v} = a$, $a = \ddot{x}$

- remembering two derivatives of position is acceleration, ie: $\ddot{x}(t) = a(t) = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right)$, we can build a <u>system</u>

If $\ddot{x} = a_{net_x} = g = 9.81$, know $\dot{x} = v(t)$

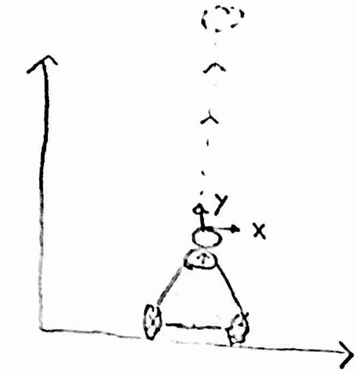
Can group expressions of velocity and acceleration w/ vectors and matrices

$$\underbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}_{\substack{\text{derivative of} \\ \text{Position and velocity} \\ \dot{\vec{x}}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{"A"}} \underbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}_{\substack{\text{Position and} \\ \text{Velocity} \\ \vec{x}}} + \underbrace{\begin{bmatrix} 0 \\ 9.81 \end{bmatrix}}_{\substack{\text{constant input} \\ \text{force vector "B"}}} \cdot u$$

Linear Model: $\boxed{\dot{\vec{x}} = A\vec{x} + Bu}$

u → a potential control (later) [u=1 Now.]

Example # 22



$F_{grav} = m_{ball} \cdot g$

$g = -9.81 \, m/s^2$

$\beta = .1$

Using: $y(t)$, $v(t) \rightarrow \dot{y}(t)$

$a(t) \rightarrow \dot{v}(t) \rightarrow \ddot{y}(t)$

for pos, vel, acc variables.

our cannon-engineer wants to build a Linear-System-Model of his cannon ball. He knows he can use motion (Kinematic) equations, but also wants to account for air drag on the launched cannon-ball. The force of air-drag is proportional to the current speed $(\vec{v})$ the cannon ball is traveling: $F_{drag} = \beta \cdot \vec{V}_y(t)$. The team decides they want you to work the physics Equations / FBD to produce a system-model: $\dot{x} = Ax + Bu$ (open-loop control model)

Example #23:

Tesla wants a dynamic model of its model 3 for highway travel. They'd like a simple (2nd order) model for tracking the vehicle's position and velocity as a user presses the gas pedal and break.

- The car weighs 1000 kg and has an effective highway drag of $B = 50 \rightarrow F_{drag} = B \cdot V(t)$, and is dependent on the cars velocity.

a) Draw an FBD of the car on a highway:
(Model both the break (F_b) and gas-pedal F_p

b) Write down the state space: $\dot{x} = Ax + Bu$
(Hint: $\vec{u}$ has two inputs (break and pedal), so B has two columns)