Communication Equilibrium and General Mechanisms

January 14, 2023

Outline

- Correlated equilibrium
- Communication equilibrium
- Generalized principal-agent problems
- Examples: Auctions and bilateral trade

Readings

Game theory Myerson [1991]: Chapter 6

Myerson (1982): Optimal Coordination Mechanisms in generalized Principal-Agent Problems–Myerson [1982]

Myerson (1994): Communication, Correlated Equilibria and Incentive Compatibility Myerson [1994]

Contracts and Correlated Strategies

Assume first an extreme kind of implicit-coordination assumption: players can not only communicate, but they can also sign jointly binding contracts to coordinate their strategies

Complete information game $(N, (A_i)_i, (u_i)_i)$

Correlated strategy: $\mu \in \Delta(A_1 \times \cdots \times A_n)$ (this set includes the set of mixed strategy profiles $\Delta(A_1) \times \cdots \times \Delta(A_n)$)

Expected payoff to player i when μ is implemented:

$$U_i(\mu) = \sum_{a \in A} \mu(a) u_i(a)$$

Allocation payoff: $U(\mu) = (U_i(\mu))_{i \in N}$

A **contract** is

$$\tau = (\tau_C)_{C \subseteq N} \in \prod_{C \subseteq N} \Delta(A_C)$$

If players in C sign the contract, then τ_C is the correlated strategy that is implemented

For allocation payoffs in the (closed and convex) set

$$\{U(\mu) : \mu \in \Delta(A)\} \subset \mathbb{R}^N$$

there is a contract that yields this allocation if all players sign the contract

However, in any singing contract game, not all such contract could actually be signed by everyone

Player i will always reject a contract that gives him a payoff below what he could get against the worst (for him) correlated strategy the other players can use against him

Player i must get at least the **minmax value of i**:

$$v_i := \min_{\mu_{-i} \in \Delta(A_{-i})} \left(\max_{a_i} \sum_{a_{-i}} \mu_{-i}(a_{-i}) u_i(a_i, a_{-i}) \right)$$

μ_{-i} that achieves the minimum above is the minmax correlated strategy against i
von Neumann and Morgenstern maxmin theorem tells us that

$$v_i = \max_{\alpha_i \in \Delta(A_i)} \left(\min_{a_{-i} \in A_{-i}} \sum_{a_i} \alpha_i(a_i) u_i(a_i, a_{-i}) \right)$$

That is, player i has a strategy that guarantees him an expected payoff that is above v_i no matter what the other players may do

Definition

A correlated strategy μ is **individually rational** for the set of all players if

$$U_i(\mu) \geq v_i \quad \forall\, i \in N$$

These are the **participation constraints** or individual-rationality constraints

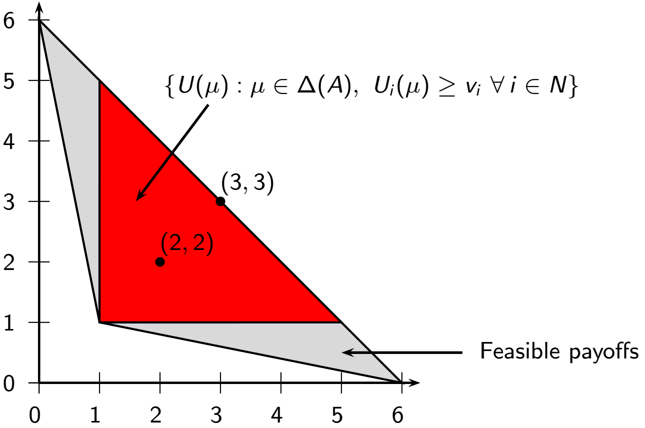
$$\{\mu : U_i(\mu) \geq v_i \quad \forall\, i \in N\}$$

is the set of allocations that can be achieved in equilibria of the implicit contract-signing game, when every player has the option to sign or to sign nothing and choose an action in A_i

Example: Prisoner dilemma

	C	D
C	$(2,2)$	$(0,6)$
D	$(6,0)$	$(1,1)$

$v_1 = v_2 = 1$



The following contract implements $(2,2)$: "play C if both sign the contract, play D if the other player does not sign"
(sign, sign) is indeed a Nash equilibrium

	C	D	sign
C	$(2,2)$	$(0,6)$	$(0,6)$
D	$(6,0)$	$(1,1)$	$(1,1)$
sign	$(6,0)$	$(1,1)$	$(2,2)$

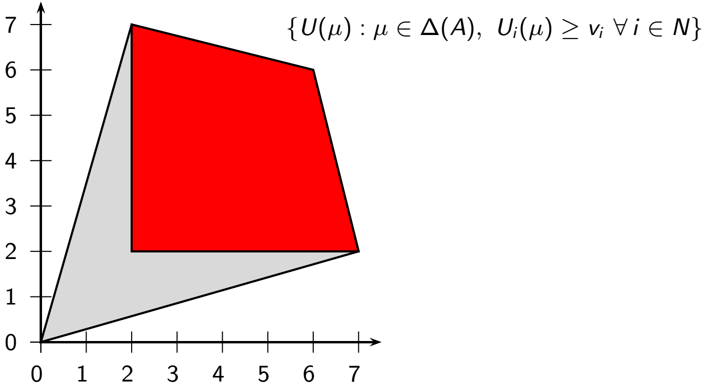
Better contract implementing $(3,3)$: "flip a coin to play (C,D) or (D,C) if both sign the contract, play D if the other player does not sign"

	C	D	sign
C	$(2,2)$	$(0,6)$	$(0,6)$
D	$(6,0)$	$(1,1)$	$(1,1)$
sign	$(6,0)$	$(1,1)$	$(3,3)$

Example: Chicken game

	C	D
C	$(2,7)$	$(6,6)$
D	$(0,0)$	$(7,2)$

$v_1 = v_2 = 2$



What is the set of all equilibrium payoffs that can be achieved in a normal form game when we allow any form of preplay communication (including possibly mediated communication), but without any contract or commitment?

We will show that, even when one contracting party is not able to observe whether the other party took certain actions, mediation can enhance contractual coordination and mitigate moral hazard

First observation: players are able to achieve the convex hull of the set of Nash equilibrium payoffs, by using jointly controlled lotteries, or simply by letting a mediator publicly reveal the realization of a random device

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For example, tossing a fair coin allows to achieve the outcome $\mu = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ with payoffs $(\frac{9}{2}, \frac{9}{2})$ in the chicken game:

	a	b
a	$(2,7)$	$(6,6)$
b	$(0,0)$	$(7,2)$

This plan of actions is not binding, but is self-enforcing

It differs from every Nash equilibrium without communication

NE are (a,a), (b,b) and $(\frac{2}{3}a, \frac{1}{3}a)$, with payoffs $(2,7)$, $(7,2)$, $(\frac{14}{3}, \frac{14}{3})$.

communication games

a **game is with communication**, when players can communicate with each other and with outside mediators, and that players and mediators have implicit opportunities to observe random variables that have objective probability distributions about which everyone agrees.

consider a game in normal form:

$$\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N}), A = \times_{i \in N} A_i$$

game with mediator: mediator (privately) recommends an action to each player

it is common knowledge that the mediator will determine his recommendations according to the probability distribution μ in $\Delta(A)$

- $\mu(a)$ denotes the probability that any given strategy profile $a = (a_i)i \in N$ would be recommended by the mediator

- expected payoff to players i under this correlated strategy μ–if everyone obeys the recommendations–is

$$U_i(\mu) = \sum_{a \in A} \mu(a) u_i(a)$$

- μ is a correlated equilibrium of Γ iff

$$\begin{aligned} U_i(\mu) &\geq \sum_{a \in A} \mu(a) u_i(a_{-i}, \delta_i(a_i)), \qquad (1) \\ \text{for all } i &\in N \text{ and for all } \delta_i : A_i \to A_i \end{aligned}$$

Definition

If μ satisfies these inequalities and $\mu \in \Delta(A)$ we say that μ is a **correlated equilibrium** of Γ.

the condition above is equivalent to the following system of inequalities **problem set!**:

$$\sum_{a_{-i} \in A_{-i} \mu(a)} \mu(a) \left[u_i(a) - u_i(a_{-i}, e_i) \right] \quad \geq \quad 0 \tag{2}$$

$$\text{for all } i \quad \in \quad N, \; a_i \in A_i, e_i \in A_i.$$

inequalities (1) and (2) are called **strategic incentive constraints:**

They represent the mathematical inequalities that a correlated strategy must satisfy to guarantee that all players could rationally obey the mediator's recommendations.

a vector $\mu \in \mathbb{R}^N$ is a correlated equilibrium iff it satisfies the **strategic incentive constraints** and the following **probability constraints**:

$$\sum_{e \in A} \mu(e) = 1 \text{ and } \mu(a) \geq 0 \text{ for all } a \in A$$

set of correlated equilibria: properties

- for any finite game in strategic form the set of correlated equilibria is a compact and convex set
- can be characterized by a finite collection of linear inequalities.
- if we want to find the correlated equilibrium that maximizes the sum of players expected payoffs in Γ we have the problem of maximizing a linear objective:

$$\max_{\mu} \sum_{i \in N} U_i(\mu)$$

subject to linear constraints

  - strategic incentive constraints
  - probability constraints

why mediated communication systems?

why can we simply focus attention on mediated communication systems in which it is rational for all players to obey the mediator?

because such communication systems can simulate any equilibrium of any system that can be generated from any given strategic form game by adding any communication system–they are **canonical**

this is known as the **'revelation principle' for normal-form games**

we investigate this next

a communication system has **inputs** and **outputs**

let R_i denote the set of all strategies that player i could use to determine the reports that he sends out, into the communication system, and let M_i denote the set of all messages that player i could receive from the communication system

for any $r = (r_i)_{i \in N}$ in $R = \times_{i \in N} R_i$ and any $m = (m_i)_{i \in N}$ in $M = \times_{i \in N} M_i$, let $v(m \,|\, r)$ denote the conditional probability that m would be the messages received by various players if each player i were sending reports according to r_i

Definition

a **communication system** is a function $v : R \to \Delta(M)$

communication strategies

given such a communication system, the set of pure communication strategies that player i can use for determining the report that he sends out and the action in A_i that he ultimately implements (as a function of the messages that he receives) is

$$B_i = \{(r_i, \delta_i) \,|\, r_i \in R_i \ , \ \delta_i : M_i \to A_i\}$$

player $i's$ expected payoff depends on the communication strategies of all players according to the function $\bar{u}_i$

$$\bar{u}_i((r_j, \delta_j)_{j \in N}) = \sum_{m \in M} v(m \,|\, r\,) u_i\,((\delta_j(m_j))_{j \in N})\,.$$

a communication system $v : R \rightarrow \Delta(M)$ generates a communication game Γ_v, where

$\Gamma_v = (N, (B_i)_{i \in N}, (\bar{u}_i)_{i \in N})$.

Γ_v is the appropriate game in strategic form to describe the structure of decision-making and payoffs when the game Γ has been transformed by allowing the players to communicate through the communication system v before choosing their ultimate payoff-relevant actions.

to characterize rational behavior in the game with communication, we should look among equilibria of Γ_v

Proposition (revelation principle for strategic form games)

any equilibrium of Γ_v is equivalent to a correlated equilibrium of Γ as defined by the strategic incentive constraints (2)

remarks

for any communication system v, there may be many equilibria of the communication game Γ_v, and these equilibria may be equivalent to different correlated equilibria

- adding a communication system does not eliminate any of the equilibria of the original game
- the **set of correlated equilibria**:
  - characterizes the union of the sets of equilibria of all communication games that can be generated from Γ
  - it has a simple and tractable mathematical structure, because it is closed and convex and is characterized by a finite system of linear inequalities
  - may be strictly larger than the convex hull of the set of NE

summary The revelation principle for strategic-form games asserted that any equilibrium of any communication system can be simulated by a communication system in which the only communication is from a central mediator to the players, without any communication from the players to the mediator.

Bayesian games with communication

in Bayesian Games with communication, there may be a need for players to **talk** as well as to **listen** in mediated communication systems

$\Gamma^b = (N, (A_i)_{i\in N}, (T_i)_{i\in N}, (p_i)_{i\in N}, (u_i)_{i\in N})$, be a finite Bayesian game with incomplete information.

consider mediated communication systems of the following form:

1. first, each player is asked to report his type confidentially to the mediator;
2. then–after getting these reports–mediator **confidentially recommends an action to each player**

Definition (mediation plans/ mechanisms)

A function $\mu : T \to \Delta(A)$ is called a mediation plan or mechanism for the game Γ^b with communication.

1. for any $a \in A$ and any $t \in T$, let $\mu(a\,|\,t\,)$ denote the conditional probability that the mediator would recommend to each player i that he should use action a_i, given report vector t
2. $\mu(a\,|\,t\,)$ must satisfy probability constraints: $\sum_{a \in A} \mu(a\,|\,t\,) = 1$ and $\mu(a\,|\,t\,) \geq 0$ for all $a \in A$ and $t \in T$.
3. if every player reports his type honestly to the mediator and obeys the recommendations of the mediator, then the expected utility for type t_i of player i from the plan μ is

$$U_i(\mu\,|\,t_i\,) = \sum_{t_{-i} \in T_{-i}} \sum_{a \in A} p_i(t_{-i}\,|\,t_i\,)\mu(a\,|\,t\,)u_i(a, t),$$

where $T_{-i} = \times_{j \in} T_j$ and $t = (t_{-i}, t_i)$

Bayesian communication games

A mediation plan μ induces a communication game Γ_μ^b in which each player must select his type report and his plan for choosing an action in A_i as a function of the mediators recommendation. Formally, Γ_μ^b is itself a Bayesian game of the form:

$$\Gamma_\mu^b = (N, (B_i)_{i\in N}, (T_i)_{i\in N}, (\dot{p_i})_{i\in N}, (\bar{u}_i)_{i\in N}),$$

where, for each player i, $B_i = \{(s_i, \delta_i) \,|\, s_i \in T_i, \delta_i : A_i \to A_i\}$

and $\bar{u}_i : (\times_{i\in N} B_j) \times T \to \mathbb{R}$ is defined by the equation

$$\bar{u}_i((s_j, \delta_j)_{j\in N}, t) = \sum_{a\in A} \mu(a \,|\, s_{j\, j\in N}) u_i((\delta_j(a_j))_{j\in N}, t)$$

Suppose, that the type of player i were t_i, but that he used the strategy (s_i, δ_i) in the communication game Γ^b_μ. Then if all other players are honest and obedient to the mediator, $i's$ expected utility payoff would be

$$U_i^*(\mu, \delta_i, s_i \,|t_i\,) = \sum_{t_{-i} \in T_{-i}} \sum_{a \in A} p_i(t_{-i} \,|t_i\,)\mu(a \,|t_{-i}, s_i\,)u_i((a_{-i}, \delta_i(a_i)), t)$$

We say that a mediation μ is **incentive compatible** iff it is a Bayesian equilibrium for all players to report their types honestly and obey the mediator's recommendations when he uses the mediation plan μ.

incentive compatible mediation plans μ **is incentive compatible iff it satisfies the following general incentive constraints:**

$$U_i(\mu \,|t_i\,) \geq U_i^*(\mu, \delta_i, s_i \,|t_i\,), \forall i \in N, \forall t_i, s_i \in T_i \textbf{ and } \delta_i : A_i \to A_i$$

revelation principle for general Bayesian games

revelation principle for general Bayesian games

Given any general communication system and any Bayesian Equilibrium of the induced communication game, there exists an equivalent incentive-compatible mediation plan, in which every type of every player gets the same expected utility as in the given Bayesian equilibrium of the induced communication game.

If a player's type set consists trivially of only one possible type, so that the Bayesian game is essentially equivalent to a strategic form game, then an incentive compatible mechanism is a correlated equilibrium.

Communication equilibrium (or generalized correlated equilibrium)

incentive compatible mechanisms are a generalization of correlated equilibria to the case of games with incomplete information.

set of IC mediation plans

the set of incentive compatible mediation plans is a closed convex set, characterized by a finite system of inequalities that are linear in μ.

- the need to give players an incentive to report their information honestly may be called **adverse selection**.
- the need to give players an incentive to implement their recommended actions maybe **called moral hazard**.

general IC

the incentive constraints are a general mathematical characterization of the effect of adverse selection and moral hazard in Bayesian games.

A communication equilibrium [Myerson, 1982] of a Bayesian game is a Nash equilibrium of some preplay and interim communication extension of the game

- The communication system should possibly include a mediator who can send outputs but also receive **inputs** from the players (two-way communication)
- A communication equilibrium outcome is a function $\mu : T \to \Delta(A)$

The function $\mu : T \to \Delta(A)$ is also called a mechanism or mediation plan

Summary

A **canonical communication equilibrium** of a Bayesian game

$$\langle N,\, p,\, (T_i)_i,\, (A_i)_i,\, (u_i)_i \rangle$$

is a Nash equilibrium of the one-stage communication extension of the game in which each player

- first, truthfully reveals his type to the mediator (adverse selection)
- then, follows the recommendation of action of the mediator (moral hazard)

i.e. for all $i \in N$, $t_i \in T_i$, $s_i \in T_i$ and $\delta_i : A_i \to A_i$,

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) \sum_{a \in A} \mu(a \mid t) u_i(a, t) \geq$$

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) \sum_{a \in A} \mu(a \mid s_i, t_{-i}) u_i(\delta_i(a_i), a_{-i}, t)$$

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Omniscient mediators

Notice that for an uninformed player (or a player with no incentive constraints) the previous inequality can be rewritten as (problem set!)

$$\sum_{t_{-i}\in T_{-i}} p(t) \sum_{a\in A} \mu(a \mid t) u_i(a,t) \geq \sum_{t_{-i}\in T_{-i}} p(t) \sum_{a\in A} \mu(a \mid t) u_i(\delta_i(a_i), a_{-i}, t)$$

$$\sum_{a\in A} \sum_{t_{-i}\in T_{-i}} p(t)\mu(a \mid t) u_i(a,t) \geq \sum_{a\in A} \sum_{t_{-i}\in T_{-i}} p(t)\mu(a \mid t) u_i(\delta_i(a_i), a_{-i}, t)$$

$$\sum_{a_{-i}\in A_{-i}} \sum_{t_{-i}\in T_{-i}} p(t)\mu(a \mid t) u_i(a,t) \geq \sum_{a_{-i}\in A_{-i}} \sum_{t_{-i}\in T_{-i}} p(t)\mu(a \mid t) u_i(a_i', a_{-i}, t) \quad \forall\, a, a_i'$$

Remember that for later when we will talk about **information design** and **Bayes-Correlated equilibrium**

Revelation Principle for Bayesian Games: The set of communication equilibrium outcomes coincides with the set of canonical communication equilibrium outcomes

How to generate a communication equilibrium without a mediator?

Again, usually impossible with two players with the Nash equilibrium, but possible under some assumptions with the correlated equilibrium (of a cheap talk extension of the game); see, e.g., Forges [1988], Vida [2007]

Again, possible with 3, 4, 5 or more players depending on the assumptions: see Forges [1990], Lehrer and Sorin [1997], Ben-Porath [2003], Gerardi [2004], Vida [2007]

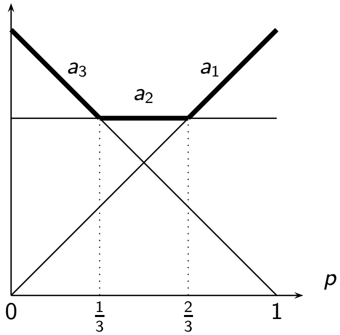
2-player game in which only player 1 (the sender) has information T, and only player 2 has decisions A
$\mu : T \to \Delta(A)$ is a communication equilibrium iff

$$\sum_{a \in A} \mu(a \mid t) u_S(a, t) \geq \sum_{a \in A} \mu(a \mid s) u_S(a, t), \quad \forall s, t \in T$$

$$\sum_{t \in T} \Pr_\mu(t \mid a) u_R(a, t) \geq \sum_{t \in T} \Pr_\mu(t \mid a) u_R(a', t), \quad \forall a \in [\mu],\ a' \in A$$

Example. Face-to-face (even multistage) communication cannot matter in the following game:

	a_1	a_2	a_3
t_1	$3, 3$	$1, 2$	$0, 0$
t_2	$2, 0$	$3, 2$	$1, 3$



Expected payoffs (fine lines) and best reply expected payoffs (bold lines) for the receiver
But mediated or noisy communication allows some (Pareto improving) information transmission

For example, when $\Pr(t_1) = 1/2$

$$\mu(t_1) = \frac{1}{2}a_1 + \frac{1}{2}a_2 \quad \text{and} \quad \mu(t_2) = a_2$$

is a Pareto improving communication equilibrium

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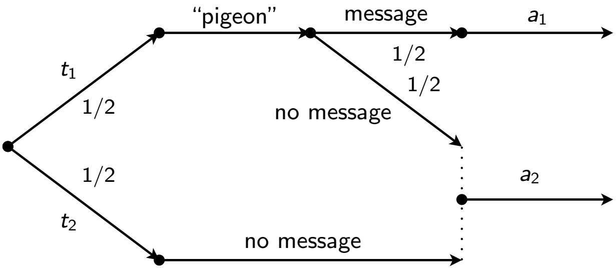
is a Pareto improving communication equilibrium

Notice that this communication equilibrium can be achieved with the following simple *noisy communication channel*

Player 1 can send a carrier pigeon, who arrives to player 2 with probability 1/2

If he sends it only when his type is t_1 we get:



Generalized Principal-Agent Problems

We now generalize further the previous setting by considering a mediator who also has the power to design the game that player must play.

We call this individual the **principal** (player 0), and the other players the **agents** $(i = 1, \ldots n)$

The principal can be thought as a regulator, contract enforcer or social planner who can commit to some action in A_0 (and can still help players to communicate information and coordinate their actions)

We follow here Myerson [1982]

New element: contractible actions: A_0

Examples of actions $a_0 \in A_0$:

- a probability of trade and a transfer (price) from the buyer (agent 2) to the seller (agent 1)
- winner of an auction, who pays what
- a matching of agents (students to schools, patients to organ donators, workers to firms), and possibly a transfer
- level of public good investments, contributions of each agent
- a winning candidate in an election

As before, the principal faces two constraining factors

- Each agent i has some private information ($t_i \in T_i$) that the principal does not observe (adverse selection)
- Each agent i has a private decision domain (A_i) that the principal cannot control (moral hazard)

The generalized principal-agent problem is a Bayesian game as before, augmented with A_0 and the principal's preference:

$$\langle N,\ p,\ (T_i)_{i \in N},\ (A_i)_{i \in N},\ (u_i)_{i \in N}, \boldsymbol{A_0}, \boldsymbol{u_0} \rangle$$

- $N = \{1, \ldots, n\}$: agents
- $A = A_0 \times A_1 \times \cdots \times A_n$: allocations / action profiles
- $u_i : A \times T \to \mathbb{R}$ and $u_0 : A \times T \to \mathbb{R}$: agents and principal utilities

Remark: This formulation also applies to conventional principal-agent problems. For example, if an agent chooses an effort level which affects a quantity of output and the principal can observe the quantity of output before paying the agent, then the principal's decision a_0 is itself a function mapping observed outputs to salary levels for the agent, and so A_0 is a set of functions from $\mathbb{R}$ to $\mathbb{R}$.

Exactly as in the case of correlated and communication equilibria, **the revelation principle applies:** instead of considering arbitrary communication systems with arbitrary Nash equilibria, we can focus on direct mechanisms

$$\mu : T \to \Delta(A)$$

together with honest and obedient strategies (two simplifications: class of communication system and strategies)

Definition

A mechanism / allocation $\mu : T \to \Delta(A)$ is **(Bayesian) incentive compatible** if for all $i \in N$, $t_i \in T_i$, $s_i \in T_i$ and $\delta_i : A_i \to A_i$,

$$U_i(\mu \mid t_i) := \sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) \sum_{a \in A} \mu(a \mid t) u_i(a, t) \geq$$

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) \sum_{a \in A} \mu(a \mid s_i, t_{-i}) u_i(\delta_i(a_i), a_{-i}, t)$$

An incentive compatible allocation is also called a feasible allocation

Notice that when $|A_0| = 1$, an incentive compatible allocation is a communication equilibrium; if in addition $|T_i| = 1$ for every i, then an incentive compatible allocation is a correlated equilibrium

If the only action choice of each agent is to participate or not ($A_i = \{Yes, No\}$), and if payoff is constant (so zero) if some agent i does not participate, then incentive-compatibility can be decomposed into the usual **incentive and participation constraints**:

$$U_i(\mu \mid t_i) := \sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) \sum_{a_0 \in A_0} \mu(a_0 \mid t) u_i(a_0, t) \geq$$

$$U_i(\mu, s_i \mid t_i) := \sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) \sum_{a_0 \in A_0} \mu(a \mid s_i, t_{-i}) u_i(a_0, t)$$

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i} \mid t_i) \sum_{a_0 \in A_0} \mu(a_0 \mid t) u_i(a_0, t) \geq 0$$

Problem of the principal: Find an incentive compatible allocation $\mu : T \to \Delta(A)$ in order to maximize his expected utility

$$U_0(\mu) := \sum_{t \in T} p(t) \sum_{a \in A} \mu(a \mid t) u_0(a, t)$$

If T and A are finite, then the number of mappings $\delta_i : A_i \to A_i$ is also finite. The incentive compatible condition is characterized by finitely many linear inequalities (linear in μ), the objective of the principal is also linear in μ, so the problem of the principal is a *linear programming problem*

multi-stage games with communication

- consider the following 2-stage game:
- stage 1:Player 1 must choose either a_1 or b_1
  - if he chooses a_1 then the game ends and the payoffs to players 1 and 2 are $(3,3)$.
  - if he chooses b_1 then there is a second stage of the game
- stage 2: each player i must choose either x_i or y_i, and the payoffs to players 1 and 2 depend on their second-stage moves as follows:

	x_2	y_2
x_1	$7,1$	$0,0$
y_1	$0,0$	$1,7$

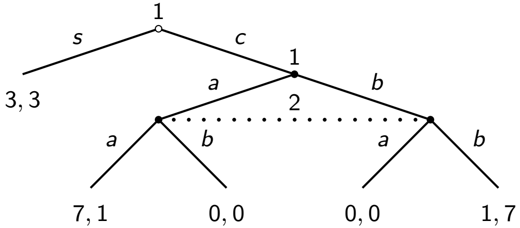
the normal form representation of this game in strategic form may be written as follows:

	x_2	y_2
a_1x_1	$3, 3$	$3, 3$
a_1y_1	$3, 3$	$3, 3$
b_1x_1	$7, 1$	$0, 0$
b_1y_1	$0, 0$	$1, 7$

easy to see that b_1y_1 will not be chosen if the two-stage game becomes with communication the b_1y_1 may arise as an outcome of the game using the following mediation plan:

- at stage 1 the mediator recommends that player 1 should choose b_1
- at stage 2 with probability 0.5 the mediator recommends moves x_1 and x_2, and with probability 0.5 the mediator recommends moves y_1 and y_2.
- either player will be able to gain by unilaterally disobeying the mediator at stage 2.

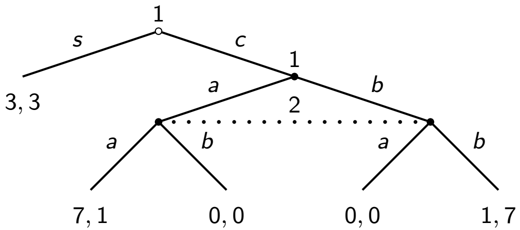
Communication in multistage games



Strategy *cb* is strictly dominated for player 1, so can we say that there is zero probability of player 1 choosing c at the first stage and b at the second stage, when communication is possible?

NO. If the mediator recommends c, and *after* c the he recommends (a, a) and (b, b) with probability 1/2 (payoff $(4, 4)$), nobody has an incentive to deviate

See Myerson [1986, Ecta] for communication equilibria in multistage games, with communication at every stage, and an appropriate revelation principle



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See Myerson [1986, Ecta] for communication equilibria in multistage games, with communication at every stage, and an appropriate revelation principle the key idea to this mediation plan is that player 1 must not learn whether x_1 or y_1 will be recommended to him at stage 2 until after it is too late to go back and choose a_1.

when we study multi-stage games with communication, we should take into account of the possibility of communication at each stage of the game

for multistage games, the revelation principle asserts that any equilibrium of any communication game that can be induced by adding a communication structure is equivalent to some mediation plan of the following form:

multi-stage games with communication: revelation principle

'canonical communication protocol'

- at the beginning of each stage, the players confidentially report their new information to the mediator
- then the mediator determines the recommended actions for the players at this stage, as a function of all reports received at this and all earlier stages, by applying some randomly selected feedback rule
- then the mediator confidentially tells each player the action that is recommended for him at this stage; and (assuming that all players know the probability distribution that the mediator used to select the feedback rule)
- it is an equilibrium of the induced communication game for all players to always report their information honestly and choose their actions obediently as the mediator recommends

Recently: actually mediator should be able to 'tremble' **?**

the Case of multiple principals

Suppose that there are K principals, numbered 1,...,K with k denoting a typical principal. Each principal k has n_k agents, numbered 1 to n_k with i denoting a typical agent. The sets of agents of each principal are disjoint, so that each agent belongs to at least one principal. Principal k together with his agents may be referred to corporation k. Let T_i^k and C_i^k denote the set of types and decisions for the $i'th$ agent belonging to principal k. Let C^0 denote the decision domain controlled by principal k.

Let $C^k = C_0^k \times C_1^k \times ... \times C_{n^k}^k$, $T^k = T_0^k \times T_1^k \times ... \times T_{n^k}^k$

and $C = C^1 \times C^2 \times ... \times C^K$; $T = T^1 \times T^2 \times ... \times T^K$

$u_0^k : C \times T \to \mathbb{R}$ denotes the utility of the k^{th} principal and

$u_i^k : C \times T \to \mathbb{R}$ denotes the utility of the i^{th} agent

A direct mechanism or communication scheme for the k^{th} principal is given by $\mu^k : C^k \times T^k \to \mathbb{R}$ where μ^k is the probability that the principal will use decision c_0^k and recommend c_i^k to his i^{th} agent, if t^k is the vector of reports that he receives from his n_k agents.μ^k must satisfy

Feasibility constraints:

$\mu^k(c^k \mid t^k) \geq 0$ and $\sum_{e^k \in C^k} \pi^k(e^k \mid t^k) = 1$ for all $c^k \in C^k$ and $t^k \in T^k$.

Now because the payoffs for the agents of one principal may depend on the decisions of other principals and their agents, we cannot define incentive compatibility for any one principal in isolation.

A mechanism μ^k for corporation k is incentive compatible in the context of mechanisms $(\mu^1, ..., \mu^k)$ iff it is an equilibrium for all k agents to be honest and obedient. Formally, μ^k is incentive compatible for corporation k in the context of $(\mu^1, ..., \mu^N)$ iff

$$
\begin{aligned}
& \sum_{\substack{t \in T \\ t_i^k = \tau_i^k}} \sum_{c \in C} p(t) \mu(d \,|\, t) u_i^k(c, t) \\
\geq \quad & \sum_{\substack{t \in T \\ t_i^k = \tau_i^k}} \sum_{c \in C} p(t) \mu^{-k}(c^{-k} \,\middle|\, t^{-k}) \mu^k(c^k \,\middle|\, t_{-i}^k, \hat{\tau}_i^K) \\
& \cdot u_i^k((c_{-k,i}, \delta_i^k(c_i^k)), t)
\end{aligned}
$$

$$
\begin{aligned}
\text{for all } i \quad & \in \quad \{1, ..., n^k\}, \text{ for all } \tau_i^k \in T_i^k, \\
\text{for all } \delta_i^k \quad & : \quad C_i^k \to C_i^k \text{ where} \\
\mu(c \,|\, t) \quad & = \quad \Pi_{j=1}^K \mu^j(c^j \,\middle|\, t^j) \\
\text{and } \mu^{-k}(\delta^{-k} \,\middle|\, t^{-k}) \quad & = \quad \Pi_{j \neq k} \mu^j(c^j \,\middle|\, t^j)
\end{aligned}
$$

revelation principle

When all agents are honest and obedient, principal $k's$ expected utility from the mechanisms $(\mu^1, ..., \mu^K)$ is

$$U_0^k(\mu^1, ..., \mu^K) = \sum_{t \in T} \sum_{c \in C} p(t) \left(\Pi_{j=1}^K \mu^j(c^j \,\middle|\, t^j) \right) u_0^k(c, t).$$

equilibria with mechanisms

- revelation principal can be straightforwardly established in this framework
- suppose that $(\mu^1, ..., \mu^{k-1}, \mu^{k+1}, ..., \mu^K)$ characterizes the planned behavior in corporations other than k, then the optimal mechanism for the principal k should be maximizing the expected utility subject to the IC constraints
- again a linear program, but now principal's k optimal incentive compatible mechanism depends on the mechanisms chosen by the other principals.
- assuming that the principals act non-cooperatively in setting up their respective coordination mechanisms we define:

Definition

$(\mu^1, ..., \mu^K)$ to be a **principal's equilibrium** iff, for each principal k, μ^k maximizes $k's$ expected utility subject to feasibility and incentive compatibility constraints.

Proposition

Principals' Equilibria do not always exist.

An example where equilibrium fails to exist:

$u_0^k \backslash u_1^k$	$t_1^k = \alpha$	$t_1^k = \beta$
$c_0^k = A$	$6, 1$	$0, z^k$
$c_0^k = B$	$0, z^k$	$6, 1$
$c_0^k = C$	$5, 0$	$5, 0$

Here the term z^k is determined by the other principals actions as follows:

for $k = 1$, $z^1 = 2$ if principal 2 chooses A or B

$= 1$ if principal 2 chooses C

for $k = 2$, $z^2 = 2$, if principal 1 chooses C

$= 1$, if principal 1 chooses A or B.

If principal 2 chooses A or B, principal 1 will choose C

If principal 1 chooses C, then principal 2 will choose C

If principal 2 chooses C then principal 1 will want to choose A or B....

why

existence fails because the set of incentive compatible mechanisms for principal k varies upper-hemicontinuously in other μ^j, not continuously as is required by the existence theorem of Debreu.

Example: Designing an Auction to Maximize Revenue

this is a summary
Based on a particular case of Myerson [1981].

A seller (the principal, player 0) wants to sell a single good which has a constant value for him (normalized to 0)

The set of agents is the set of bidders $N = \{1, \ldots, n\}$

The value estimate of bidder i is privately known: $t_i \in T_i = [a_i, b_i]$, with continuous probability density function f_i, and cumulative distribution function

$$F_i(t_i) = \int_{a_i}^{t_i} f_i(s_i)\, ds_i$$

Bidders' valuations are assumed to be stochastically independent random variables

Designing an auction is equivalent to choosing a mechanism $\mu = (p, x) : T \to A$, where $A = \mathbb{R}^n \times \mathbb{R}^n$

$p_i(t) \geq 0$ is the probability that bidder i gets the good, with $\sum_i p_i(t) \leq 1$ for every $t \in T$

$x_i(t)$ is the expected transfer (price) from bidder i to the seller

Assuming risk neutrality, the interim expected utility of bidder i given the mechanism (p, x) is

$$U_i(p, x, t_i) = \int_{T_{-i}} (t_i p_i(t) - x_i(t)) f_{-i}(t_{-i}) \, dt_{-i}$$

The expected utility of the seller given the mechanism (p, x) is

$$U_0(p, x) = \int_T \sum_i x_i(t) f(t) \, dt$$

The auction maximizing the revenue for the seller therefore maximizes

$$U_0(p,x) = \int_T \sum_i x_i(t) f(t)\, dt \qquad \text{(seller's exp. revenue)}$$

under the bidders' incentive contraints: for every $i \in N$, and $t_i, s_i \in T_i$,

$$\begin{aligned} U_i(p,x,t_i) &= \int_{T_{-i}} (t_i p_i(t) - x_i(t)) f_{-i}(t_{-i})\, dt_{-i} \\ &\geq \int_{T_{-i}} (t_i p_i(s_i, t_{-i}) - x_i(s_i, t_{-i})) f_{-i}(t_{-i})\, dt_{-i} \end{aligned} \qquad \text{(IC)}$$

and the bidders' participation contraints: for every $i \in N$, and $t_i \in T_i$,

$$U_i(p,x,t_i) \geq 0 \qquad \text{(PC)}$$

Doing some rewriting it can be shown that the expected revenue at any feasible mechanism is

$$\int_T \sum_{i \in N} \underbrace{\left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right)}_{J_i(t_i)} p_i(t) f(t) dt - \sum_i U_i(p, x, a_i)$$

$J_i(t_i)$ is the *virtual valuation* of bidder i, which is the valuation of bidder i minus a term related to his information rent

Proposition (The revenue equivalence theorem)

The seller's expected revenue from a feasible auction mechanism is completely determined by the probability assignment function and the rents $U_i(p, x, a_i)$ of the lowest type bidders

Implications: if we know who gets the object in each possible situation (as specified by p) and how much expected utility each bidder would get if his value estimate were at its lowest possible level a_i, then the seller's expected revenue does not depend on the payment function x

Hence: the seller gets the same expected revenue in any auction in which (i) the object always goes to the bidder with the highest valuation, and (ii) every bidder expects zero utility if his valuation is at its lowest possible level.

If bidders are symmetric, then the first-price, second-price, Dutch, English, first-price all pay, ... auctions raise the same expected revenue for the seller.

The seller optimum is obtained with the transfers

$$x_i(t) = p_i(t)t_i - \int_{a_i}^{t_i} p_i(s_i, t_{-i})ds_i$$

and the assignment p that maximizes

$$\int_T \sum_{i \in N} \left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) p_i(t)f(t)dt$$

under a monotonicity contraint of the interim probability that bidder i gets the good:

$$s_i \leq t_i \qquad \int_{T_{-i}} p_i(s_i, t{-}i)f_{-i}(t_{-i})dt_{-i} \leq \int_{T_{-i}} p_i(t_i, t{-}i)f_{-i}(t_{-i})dt_{-i}$$

Assumption (Regularity)

For every $i \in N$ the virtual valuation of bidder i,

$$J_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \tag{3}$$

is strictly increasing in t_i

Under regularity, the point-wise maximization of

$$\int_T \sum_{i \in N} \left(t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) p_i(t) f(t) dt$$

i.e., the maximization of $\sum_{i \in N} \left(t_i - \frac{1-F_i(t_i)}{f_i(t_i)} \right) p_i(t)$ for every t satisfies monotonicity

Hence, the optimal auction is to assign the good to bidder i with the highest virtual valuation $J_i(t_i)$ if $J_i(t_i) > 0$, and to keep the good otherwise

When all bidders are symmetric, bidder j with the highest type $\max_i t_i$ therefore gets the good if $\max_i t_i > c^{-1}(0)$, and pays

Remarks:

1. While we originally only required incentive compatibility, it turns out that the optimal auction can be implemented in (weakly) dominant strategies
2. One drawback of the optimal auction (compared to standard auction) is that it requires the seller to know the distributions of bidders' valuations F_i
3. The optimal auction is not efficient: sometimes the bidder with the highest valuation does not get the good because (i) the seller sometimes keep the good even if his valuation is 0 and some bidders have positive valuation (ii) even some bidder gets the good, if bidders are asymmetric, the bidder i with the highest virtual valuation $J_i(t_i)$ may not be the bidder with the highest valuation
4. Optimal auctions mechanisms can also be found without regularity using "ironing" [Myerson, 1981, Section 6]

Example.

$T_i = [0, 100]$, uniform distribution $f_i(t_i) = 1/100$

Virtual valuation:

$$J_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} = t_i - \frac{1 - \frac{t_i}{100}}{\frac{1}{100}} = 2t_i - 100$$

Reserve price:

$$J_i^{-1}(0) = 50$$

Hence, the seller risks a probability $(1/2)^n$ of keeping the object

When $n = 1$ we find back the monopoly price p^m that maximizes

$$p^m D(p^m) = p^m(1 - F_i(p^m)) = p^m(1 - \frac{p^m}{100}) = \frac{1}{100} p^m(100 - p^m)$$

Discrimination against bidders who are more likely to have higher valuations:

$T_i = [0, b_i]$, uniform distribution $f_i(t_i) = 1/b_i$

Virtual valuation:

$$J_i(t_i) = t_i - \frac{1 - F_i(t_i)}{f_i(t_i)} = t_i - \frac{1 - \frac{t_i}{b_i}}{\frac{1}{b_i}} = 2t_i - b_i$$

j with $b_j > b_i$ may not win the object even if $t_j > t_i$, when $2t_j - b_j < 2t_i - b_i$

Example: Ex-Post Inefficiency in Bilateral Trade

Based on Myerson and Satterthwaite [1983]

Set of agents: a seller a buyer $N = \{S, B\}$

The seller owns a single indivisible good

The seller's valuation is $t_S \in [a_S, b_S]$, distribution F_S

The buyer's valuation is $t_B \in [a_B, b_B]$, distribution F_B

Agents' valuations are assumed to be stochastically independent

Designing a trading mechanism is equivalent to choosing a mechanism $\mu = (p, x) : T \to A$, where $A = [0, 1] \times \mathbb{R}$
$p(t_S, t_B)$ is the probability of trade (that the good is transferred to the buyer)
$x(t_S, t_B)$ is the expected transfert from the buyer to the seller
For $i, j \in \{S, B\}$, $i \neq j$, let
Let $\bar{p}_i(t_i) = \int_{t_j} p(t) f_j(t_j) dt_j$ be the interim expected probability of trade for type t_i, and
$\bar{x}_i(t_i) = \int_{t_j} x(t) f_j(t_j) dt_j$ the interim expected transfert for type t_i
Assuming risk neutrality, the interim expected utilities of the seller and buyer given the mechanism (p, x) are

$$U_S(t_S) = \bar{x}_S(t_S) - t_S \bar{p}_S(t_S)$$

$$U_B(t_B) = t_B \bar{p}(t_B) - \bar{x}_B(t_B)$$

The trading allocation $(p, x) : T \to [0, 1] \times \mathbb{R}$ is **feasible** if it satisfies incentive and participation constraints for every (t_S, t_B):

$$U_S(t_S) \geq \bar{x}_S(s_S) - t_S\bar{p}_S(s_S), \quad \forall\, s_S \in T_S$$

$$U_B(t_B) \geq t_B\bar{p}(s_B) - \bar{x}_B(s_B), \quad \forall\, s_B \in T_B$$

$$U_S(t_S) \geq 0 \quad U_B(t_B) \geq 0$$

The trading allocation $(p, x) : T \to [0, 1] \times \mathbb{R}$ is **ex-post efficient** if

$$p(t_S, t_B) = \begin{cases} 1 & \text{if } t_S < t_B \\ 0 & \text{if } t_S > t_B \end{cases}$$

Theorem (Myerson and Satterthwaite, 1983)

If the seller's valuation is distributed with positive probability density over the interval $[a_S, b_S]$, and the buyer's valuation is distributed with positive probability density over the interval $[a_B, b_B]$, and if the interiors of these intervals have a nonempty intersection, then no incentive-compatible individually rational trading mechanism can be ex post efficient.

To show that result they rewrite necessary and sufficient conditions for incentive compatibility (as in auctions) and show that incentive compatibility with an ex-post efficient allocation implies (when the intervals intersect, i.e., $a_B < b_S$ and $a_S < b_B$)

$$U_S(b_S) + U_B(a_B) = -\int_{a_B}^{b_S} (1 - F_B(t))F_S(t)\, dt < 0$$

so individual rationality fails

$\int_{a_B}^{b_S}(1 - F_B(t))F_S(t)\, dt$ is the smallest subsidy required from an outside party to create a Bayesian incentive-compatible mechanism which is both ex post efficient and individually rational

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