Lecture 7: The Income Fluctuations Problem

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Outline

Consumption-savings with uncertain income

- Consider the following version of the consumption-savings problem,

$$\begin{aligned} & \max_{a_t, c_t} E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\left(z^t\right)\right), \\ \text{s.t.} \quad & c_t\left(z^t\right) + a_{t+1}\left(z^t\right) = (1+r)\, a_t\left(z^{t-1}\right) + w z_t, \\ & c_t\left(z^t\right) \geq 0 \text{ for each } t, z^t \text{ and } a_0 \text{ given,} \end{aligned}$$

where z_t is a random variable over $Z = \{\zeta^1, ..., \zeta^m\}$.

- Consumer is subject to **income shocks** (unemployment, health shocks, wage changes, demand/supply changes for self-employed...)

Incomplete markets

- The consumer also faces **incomplete markets**.
  - She can save via assets (e.g., deposits) that pay r regardless of the shock.
  - But she is not allowed to undertake investments that deliver different asset payoffs in different states.
- To see how this is restrictive, imagine the consumer could buy insurance at time t for adverse shocks to her income at time $t+1$. If the realized income is high, the consumer pays insurance fees. If the realized income is low, she receives payoff from the insurance company. We are not allowing for these types of arrangements.
  - Can actually prove that, with such arrangements, we are back to the baseline PIH.
- The lack of market-provided insurance will drive some of the results.

Borrowing constraint

As in the deterministic case, the problem is so far not well defined. We need to put a restriction on the assets that the consumer leaves behind:

1. As before, we could assume no borrowing,

   $$a_{t+1}\left(z^t\right) \geq 0 \text{ for each } t, z^t.$$

   Relatedly, we could allow a limited amount of borrowing,

   $$a_{t+1}\left(z^t\right) \geq -\phi,$$

   where ϕ denotes an exogenous borrowing limit.

Borrowing constraint

2. Alternatively, we could allow the consumer to borrow as long as the expected present value of her debt in the far future is zero,

$$\lim_{t \to \infty} E_0 \left[\left(\frac{1}{1+r} \right)^t a_{t+1} \left(z^t \right) \right] \geq 0.$$

3. Alternatively, we could impose an endogenous borrowing limit,

$$a_{t+1} \left(z^t \right) \geq -\frac{w\zeta^1}{r} = -\left(\frac{w\zeta^1}{1+r} + \frac{w\zeta^1}{(1+r)^2} + \ldots \right),$$

where $w\zeta^1$ denotes the lowest possible wage realization.

- If this borrowing limit were violated, the consumer would be unable to pay back her debt after a sufficiently long sequence of unlucky draws—the sequence in which she keeps receiving the lowest wage, $w\zeta^1$. This would allow for feasible paths that violate condition 2.

Our goal: The income fluctuations problem

- These variations are examples of the **income fluctuations problem**.
- Workhorse model in macro to analyze how consumption responds to income shocks, as well as how borrowing constraints/precautionary savings affect this response.
- Theoretical basis for recent empirical work in applied macro/micro.
  - Stimulus check payments Johnson-Parker-Souleles (2008)
  - Unemployment benefits Ganong et al. (2021)
- We will focus on a version with exogenous borrowing constraints and constant elasticity utility. These features make the precautionary savings force particularly strong.
- Before we attack the infinite horizon problem, it might be useful to understand the precautionary savings motive in a two period version.

Outline

1. The canonical consumption-savings problem

2. A two period consumption-savings problem

3. Solving the canonical problem
    - Recursive Formulation & Euler Equation
    - Partitioning the State Space - Buffer stock behavior
    - Solving the FE and finding the stationary distribution

A two-period version

- Consider the problem,

$$\max_{c_0, c_1, a_1 \text{ and } a_2 \geq 0} u(c_0) + E[u(c_1)],$$
$$\text{s.t.} \quad c_0 + a_1 = wz_0,$$
$$\text{and} \quad c_1 + a_2 = a_1 + wz_1.$$

- Suppose also $z_0 = 1$ and that $z_1 = 1/2$ (low income) with probability $1/2$ and $z_1 = 3/2$ (high income) with probability $1/2$.
- I also assumed $\beta = (1 + r) = 1$ so that, in a deterministic problem, the consumer would choose $c_0 = c_1$.
- We allow for borrowing, $a_1 < 0$, but require $a_2 \geq 0$ as before.
- How would you solve this problem?

Solution to two-period problem

- Substituting c_0 and c_1, and $a_2^* = 0$, the problem becomes,

$$\max_{a_1} u(w - a_1) + \frac{1}{2} u\left(a_1 + \frac{w}{2}\right) + \frac{1}{2} u\left(a_1 + \frac{3w}{2}\right).$$

- Taking the first order conditions, we have the Euler equation,

$$u'\left(\overbrace{w - a_1^*}^{c_0}\right) = \frac{1}{2} u'\left(\overbrace{a_1^* + \frac{w}{2}}^{c_1^L}\right) + \frac{1}{2} u'\left(\overbrace{a_1^* + \frac{3w}{2}}^{c_1^H}\right). \tag{1}$$

  - Euler equation equates expected marginal utility (since $\beta(1+r) = 1$).
  - Note how this equation pins down the level of savings a_1^*.

The solution with quadratic utility

- It is useful to review the solution with quadratic utility

$$u(c) = \phi c - \frac{c^2}{2},$$

(We take ϕ sufficiently large so that $u'(c) > 0$ over relevant range)

- This is a nice benchmark since it leads to linear marginal utility,

$$u'(c) = \phi - c.$$

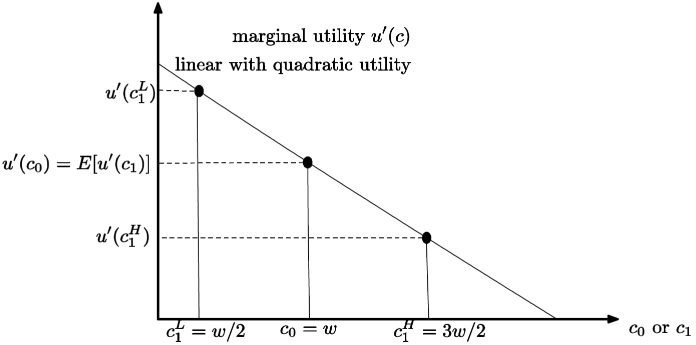
- Plugging this into the Euler equation, we obtain

$$c_0 = E[c_1] = \frac{1}{2} c_1^L + \frac{1}{2} c_1^H, \text{ which implies } a_1^* = 0.$$

- The consumer equates **expected consumption** across periods, very similar to what would obtain in a deterministic model.

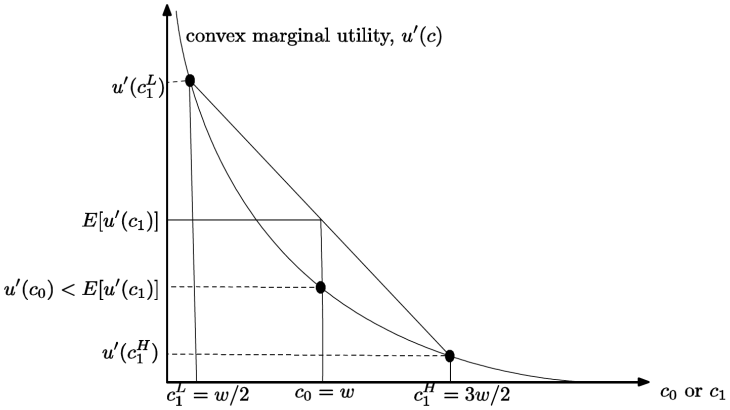
- In this example this is achieved by choosing $a_1^* = 0$....

The solution with quadratic utility



- This picture also suggests the quadratic utility is special.
- What if $u'(c)$ is convex (i.e., $u'\left(c_1^L\right)$ is much greater than $u'\left(c_1^H\right)$)?

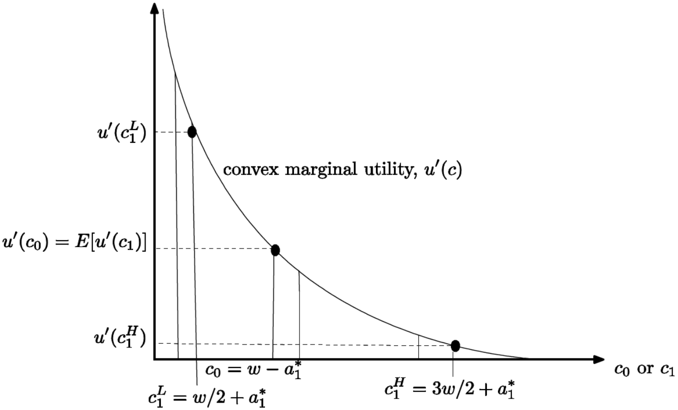
The solution with convex marginal utility



The solution with convex marginal utility

- When the marginal utility is convex, choosing $a_1^* = 0$ and consuming your wages in every state (which was optimal with quadratic utility) is no longer optimal.
- This violates the Euler equation since $u'(c_0) < E[u'(c_1)]$.
- As the picture suggests, $E[u'(c_1)]$ is higher because the marginal utility in the low income state is very high (which is intuitive).
- The actual solution to problem (1), that ensures $u'(c_0) = E[u'(c_1)]$, now features a higher level of savings, $a_1^* > 0$.
- This lowers $c_0 = w - a_1$ but raises $c_1^L = \frac{w}{2} + a_1$ (as well as c_1^H).

The solution with convex marginal utility



- Note that the solution now features $c_0 < E[c_1]$. Consume relatively less now and "save for a rainy day"—the L state of date 1.

Convex marginal utility = prudence

- So convex marginal utility generates precautionary savings.
- Note that the condition for the convexity of the marginal utility is,

$$\frac{d^2 u'(c)}{dc^2} > 0, \text{ or equivalently, } u'''(c) > 0.$$

- The condition depends on the third derivative. Related to but not the same as risk aversion, which comes from concavity, $u''(c) < 0$.
- The convex marginal utility condition is also known as "prudence."
    - Interpretation: more risk → more savings. That doesn't follow from risk aversion alone.
- Most utility functions in macro satisfy the prudence condition.

Constant elasticity utility features prudence

- We work with the constant elasticity utility, which we denote by

$$u(c) = \frac{c^{1-\rho} - 1}{1-\rho}.$$

  - This is the same as $\frac{\varepsilon}{\varepsilon-1}\left(c^{\frac{\varepsilon-1}{\varepsilon}} - 1\right)$ after substituting $\rho = 1/\varepsilon$.
  - We work with the new notation since risk now plays a more central role in this context and ρ captures the attitude towards risk.
  - ρ captures the coefficient of relative risk aversion (CRRA). So the utility function is also known as the CRRA utility.

- Check that CRRA utility satisfies the prudence condition since,

$$u'''(c) = \rho(\rho+1)\, c^{-(\rho+2)} > 0.$$

  - Note: higher ρ implies higher prudence (as well as higher risk aversion).
  - Note: $u'''(0) = \infty$ so prudence is especially strong for low levels of c.

Outline

The canonical income fluctuations problem

The problem we want to solve is,

$$\max_{a_t, c_t} E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\left(z^t\right)\right), \text{ where } u(c) = \frac{c^{1-\rho}-1}{1-\rho}$$

$$\text{s.t.} \quad c_t\left(z^t\right) + a_{t+1}\left(z^t\right) = (1+r)\, a_t\left(z^{t-1}\right) + w z_t,$$

$$c_t\left(z^t\right) \geq 0, \;\; a_t\left(z^t\right) \geq 0 \text{ for each } t, z^t \text{ and } a_0 \text{ given.}$$

- Note: we assumed no borrowing for simplicity. But the case with an exogenous borrowing limit, $a_t\left(z^t\right) \geq -\phi$, is similar.

The canonical income fluctuations problem

- For now, suppose z_t is iid over $Z = \{\zeta^1, ..., \zeta^m\}$ with $\zeta^1 > 0$.
  - Normalize $E[z_t] = 1$ so w is the average wage.
- We also assume $\beta(1+r) < 1$. The consumer is sufficiently impatient that, absent uncertainty, she would choose a declining consumption path and would eventually deplete her assets and hit the borrowing constraint.
- With uncertainty, there is a second force—precautionary savings—that pushes in the opposite direction.
- So there is a tension. The optimal policy will balance these two forces and resolve the tension.

The canonical income fluctuations problem

- This problem is similar to the canonical income fluctuations problem analyzed e.g. in Deaton (ECMA, 1990) and Aiyagari (QJE, 1994).
- The analytical characterization is not straightforward. So we will be more informal (for the proofs, see the above papers and the references therein, + more recent work by Ben Moll).
    - Aside: when the income fluctuations problem is embedded into a GE framework, the resulting model has a particular structure—it's a mean-field game. Lots of interesting recent progress in that area.
- But the numerical solution is straightforward.
    - State-of-the-art algorithm: Carroll's endogenous gridpoint method.
    - We'll see this later.

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Bounding the problem

- The tension is between $\beta(1+r) < 1$ and the precautionary savings.
- For high levels of c, the precautionary motive is weak so impatience prevails.
- Thus, we conjecture (and "verify" later) that assets can be bounded from above by some $\bar{a}$. We can then analyze the problem using dynamic programming techniques...

Payoff-relevant state: cash-in-hand

- As in the NGM, the payoff state variable in this problem is (a_t, z_t).
- We could reduce the dimension further with a simplifying observation.
- Let us define **cash-in-hand** as the total cash available (to consume),

$$x_t = (1 + r)\, a_t + w z_t.$$

  - Note that a_t or z_t enter the budget constraint only through x_t.
  - Since z_t also do not affect the new wage draw (iid assumption), we can define the value function over a single dimension, $V(x)$.
  - Intuitively, the source of the cash – whether it is previous savings or current wages – doesn't matter.
    Aside: why does this not work if z is not iid?

The Bellman equation in terms of cash-in-hand

- The Bellman equation can then be written as,

$$
\begin{aligned}
V(x) \quad &= \quad \max_{a^{next},c} u(c) + \beta \sum_{z^{next} \in Z} V\left(x^{next}\right) \pi\left(z^{next}\right) \\
\text{subject to} \quad & \quad a^{next} + c = x \text{ with } c \geq 0, a^{next} \geq 0 \\
\text{and} \quad & \quad x^{next} = (1+r)\, a^{next} + w z^{next}
\end{aligned}
$$

- Equivalently, using more compact notation, we have

$$
V(x) = \max_{a^{next} \in [0,x]} u\left(x - a^{next}\right) + \beta \sum_{z^{next} \in Z} V\left((1+r)\, a^{next} + w z^{next}\right) \pi(z^{next}). \tag{2}
$$

Solving the Bellman equation

- Note also that $a \leq \overline{a}$, implies $x \leq \overline{x} = (1+r)\,\overline{a} + w\zeta^m$.
- So the cash-in-hand also lies in a bounded set, $X = [0, \overline{x}]$.
- So we are looking for $V \in B\left([0, \overline{x}]\right)$ that satisfies Eq. (2).
- The usual steps then imply there is a bounded, cts, strictly concave, strictly increasing and differentiable (for $x > 0$) value function $V = V^*$ that solves (2).

Envelope condition

- Define the policy functions with,

$$A(x) = a^{next} \text{ and } C(x) = x - a^{next}.$$

- Applying the Envelope Theorem to problem (2), we obtain,

$$V'(x) = u'(x - A(x)) = u'(C(x)). \tag{3}$$

The Euler equation

- The FOC for the Bellman equation is then given by,

$$\frac{u'(x - A(x))}{\beta(1+r)} \begin{cases} = E\left[V'\left((1+r)A(x) + wz^{next}\right)\right] & \text{if } A(x) > 0, \\ \geq E\left[V'\left((1+r)A(x) + wz^{next}\right)\right] & \text{if } A(x) = 0. \end{cases} \tag{4}$$

- Substituting Eq. (3), this also implies the Euler "equation",

$$\frac{u'(C(x))}{\beta(1+r)} \begin{cases} = E\left[u'\left(C\left(x^{next}\right)\right)\right] & \text{if } A(x) > 0, \\ \geq E\left[u'\left(C\left(x^{next}\right)\right)\right] & \text{if } A(x) = 0. \end{cases}$$

- Note there is no guarantee that $A(x) > 0$. Consumer can run into the borrowing constraint, $A(x) = 0$, in which case the Euler equation holds as inequality. In fact, we will see this sometimes happens.

Outline

When does the constraint bind?

- We further claim that the borrowing constraint binds, that is, the consumer chooses $A(x) = 0$, for sufficiently low levels of x.
- To see this, consider $x = w\zeta^1$, which obtains when the assets are zero and the wage realization is the lowest level (rock bottom).
- For this level of x, things can only improve:

$$x^{next} = (1+r)\,A(x) + wz^{next} \geq x = w\zeta^1, \text{ regardless of } z^{next}.$$

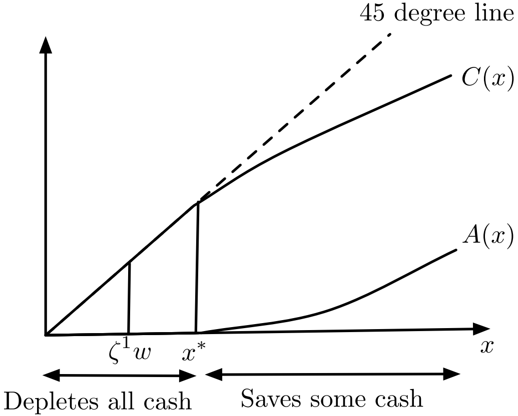
- Since $\beta(1+r) < 1$ and $C(x)$ is increasing, this implies,

$$u'(C(x)) > \beta(1+r)\,E\left[u'\left(C\left(x^{next}\right)\right)\right].$$

- So the optimality condition (4) holds as inequality and $A(x) = 0$.

$\Rightarrow$ At rock bottom, don't save. In fact, would borrow if possible.

The shape of the policy functions



- Since $A(x)$ is weakly increasing, we conclude that there exists $x^* \geq w\zeta^1$ s.t. borrowing constraint binds iff $x < x^*$.

Deterministic analogue as policy limit

- This analysis helps to understand the policy functions to some extent—we've split the state space into a saving and no-saving region. Let's now further analyze the saving region. For this further help comes from the deterministic analogue of the problem.
- Suppose there is no uncertainty, $Z = \{1\}$, and the consumer is **allowed to borrow** (subject to $\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t a_{t+1} = 0$).
- This is a slight variation of the problem from Lecture 2, with solution:
  - Consumption "grows" (more accurately, shrinks) at rate

$$1 + g = (\beta (1 + r))^{\varepsilon} < 1 \text{ (where } \varepsilon = 1/\rho).$$

  - The level of consumption is given by,

$$c_0 = (r - g) \left(a_0 + \frac{w}{r}\right).$$

Deterministic analogue as policy limit

- After substituting $x = (1+r)\,a_0 + w$, the last equality implies,

$$c_0 = \frac{r-g}{1+r}\left(x + \frac{w}{r}\right)$$

- So the policy functions in terms of cash-in-hand are given by,

$$C^{\text{det}}(x) = \frac{r-g}{1+r}\left(x + \frac{w}{r}\right) \text{ and thus, } A^{\text{det}}(x) = x - C^{\text{det}}(x)\,.$$

- Using intuition, we could use these policy functions (which belong to a different problem) to bound the policy functions in our problem...

Deterministic analogue as policy limit

- Based on our earlier two-period analysis, we can conjecture that:

    1. In view of the precautionary savings motive (which is strengthened by the borrowing constraints), our consumer will choose,

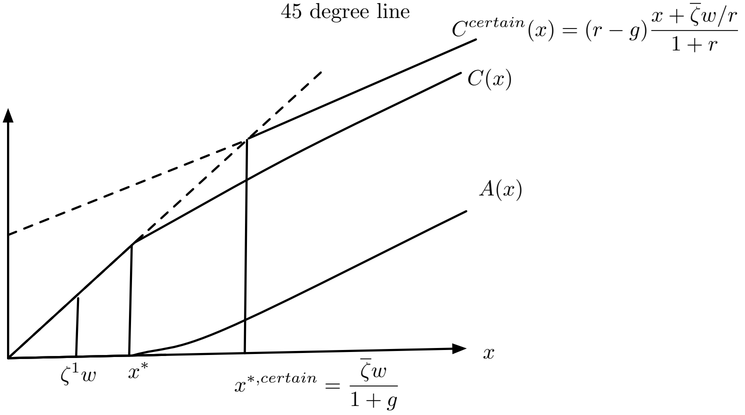
    $$C(x) < C^{\text{det}}(x) \text{ and } A(x) > A^{\text{det}}(x) \text{ for each } x.$$

    2. As $x \to \infty$, the precautionary motive (as well as the risk aversion with respect to additive risks) disappears and the policy functions have the same slopes as the deterministic case. That is, we will have

    $$\lim_{x\to\infty} \frac{dC(x)}{dx} = \frac{r-g}{1+r}$$

        - Intuition: For large x and c, the agent is not worried about uncertainty, so she spends or saves the marginal cash as in the deterministic case.

The shape of the policy functions



- Why is this useful?
  - Next step: compare with savings that would keep x fixed on average
  - Will allow us to partition state space into different regions

Dynamics of cash-in-hand

- Return again to the dynamics of cash-in-hand,

$$x_{t+1} = (1+r)(x_t - C(x_t)) + wz^{next}.$$

- Taking expectations and using $E[z^{next}] = 1$, we have,

$$E[x_{t+1}] = (1+r)(x_t - C(x_t)) + w.$$

- Use this to define a consumption level that would keep cash on hand fixed "on average":

$$C^{steady}(x) \equiv \frac{rx_t + w}{1+r}$$

- Then we have that

$$E[x_{t+1}] < x_t \text{ iff } C(x_t) > C^{steady}(x)$$

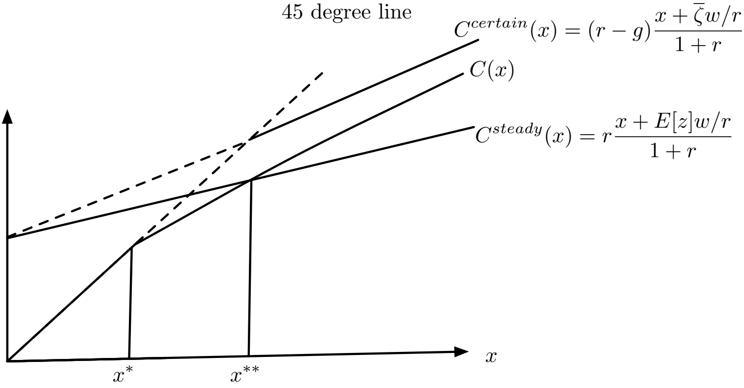
Dynamics of cash-in-hand

- Now we can return to our earlier results about the consumption function converging to that of the analogous deterministic problem.
- Crucially, our second observation there implies that the $C(x)$ and $C^{steady}(x)$ curves *must* cross. Why?
  - $C^{steady}(x)$ is the optimal policy of the deterministic problem with $\beta(1+r) = 1$ and $g = 0$.
  - Since we have $\beta(1+r) < 1$ and $g < 0$ for the limiting deterministic problem, we also have,

$$\lim_{x \to \infty} \frac{dC(x)}{dx} = \frac{r-g}{1+r} > \frac{r}{1+r} = \frac{dC^{steady}(x)}{dx}.$$

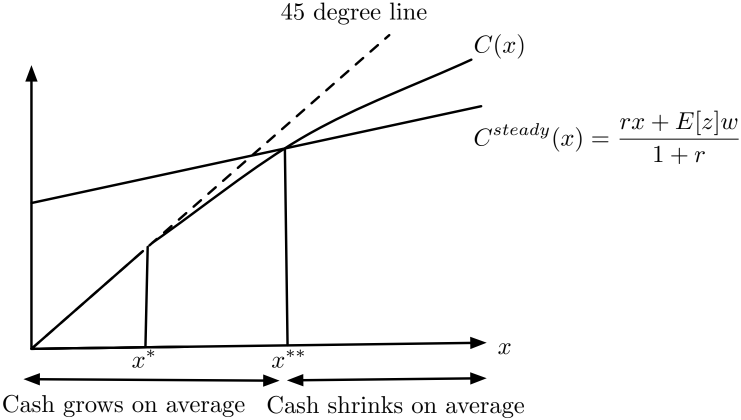
  - Thus $C(x)$ has a slope of 1 at x^* and then decreases towards a slope of $\frac{r-g}{1+r} > \frac{r}{1+r}$, so it's always steeper than the "steady" line

Dynamics of cash-in-hand



- Let x^{**} denote the point at which $C(x)$ intersects $C^{steady}(x)$.

Dynamics of cash-in-hand



- Note that x is bounded "on average." (In fact, we can also verify our conjecture that x remains in $[0, \overline{x}]$ for all shocks.)

The buffer-stock behavior

- We thus have the following partition of the state space:

   1. If $x \leq x^*$, then the consumer is constrained. **Depletes cash**.
   2. If $x \in [x^*, x^{**}]$, then the consumer saves some for a rainy day.

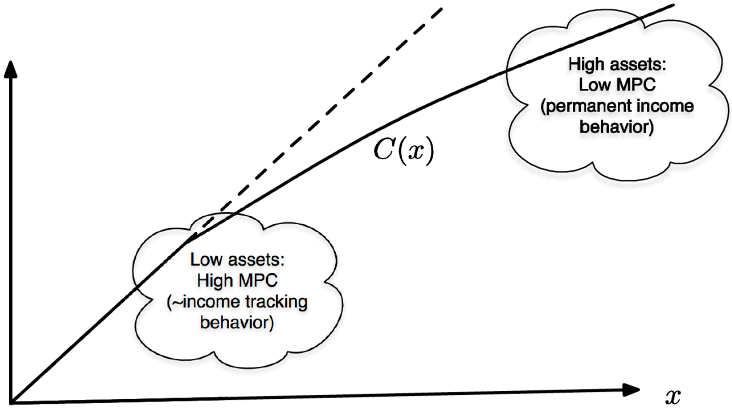
      In either case, the consumer **accumulates cash/buffer**.

   3. If $x \geq x^{**}$, the consumer saves some but not enough to prevent cash from declining in expectation. **Decumulates the buffer**.

The buffer-stock behavior

- This is known as **the buffer-stock behavior**.
  - Simple/intuitive behavior. The consumer is saving some amount—despite her impatience—to weather future income shocks.
- Conceptually, the buffer-stock behavior is the model's answer to the tension we laid out at the beginning of the analysis.
- Impatience tends to deplete the buffer, but precautionary savings tends to replenish the buffer for low levels of cash-in-hand.

Key implication: Declining MPCs



- A recent literature finds evidence consistent with this prediction.

Outline

Discrete-State Markov Process for Income

- Finite number of income realizations: $y \in \{y_1, \cdots, y_J\}$
- $\mathbf{P}$ is Markov transition matrix where
  - (j, j')th element of $\mathbf{P}$ is $\Pr(y_{t+1} = y_{j'} | y_t = y_j) = p_{jj'}$
  - $\forall\ j, j' \quad p_{jj'} \in [0, 1]$
  - $\forall j, \quad \sum_{j'=1}^{J} p_{jj'} = 1$

Discrete-State Markov Process for Income

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  - $\forall\ j, j' \quad p_{jj'} \in [0, 1]$
  - $\forall j, \quad \sum_{j'=1}^{J} p_{jj'} = 1$
- Stationary distribution is vector π with elements π_j
  - solves
    $$\pi = \mathbf{P}^\top \pi, \qquad \mathbf{P}^\top = \text{transpose of } \mathbf{P}$$
    (Eigenvalue problem = same form as $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ with $\lambda = 1$; Equivalently row vector $\tilde{\pi}$ s.t. $\tilde{\pi} = \tilde{\pi}\mathbf{P}$)
  - easy method for finding π in practice: take N large, some π_0
    $$\pi \approx (\mathbf{P}^\top)^N \pi_0$$
  - Logic: $\pi_{t+1} = \mathbf{P}^\top \pi_t$ and hence $\pi \approx \pi_N = (\mathbf{P}^\top)^N \pi_0$

Bellman Eq w Discrete-State Markov Process

$$V(a, y_j) = \max_{c,a'} u(c) + \beta \sum_{j'=1}^{J} V(a', y_{j'})\, p_{jj'}$$

subject to

$$c + a' \leq y_j + (1+r)\, a$$
$$a' \geq \underline{a}$$

- Euler Equation is

$$u'(c(a, y_j)) \geq \beta (1+r) \sum_{j'=1}^{J} u'(c(a, y_{j'}))\, p_{jj'}$$

- Solution is set of J functions $c(a, y_j)$

Value Function Iteration

- Step 1: Discretized asset space $\mathcal{A} = \{a_1, a_2, \ldots, a_N\}$. Set $a_1 = \underline{a}$
- Step 2: Guess initial $V_0(a, y_j)$. Reasonable first guess is

$$V_0(a, y) = \sum_{t=0}^{\infty} \beta^t u(ra + y) = \frac{u(ra + y)}{1 - \beta}$$

- Step 3: Set $\ell = 1$. Loop over all $a_i \in \mathcal{A}$ and solve

$$a'_{\ell+1}(a_i, y_j) = \arg\max_{a' \in \mathcal{A}} u(y_j + (1 + r) a_i - a') + \beta \sum_{j'=1}^{J} V_\ell(a', y_{j'}) p_{jj'}$$

$$V_{\ell+1}(a_i, y_j) = \max_{a' \in \mathcal{A}} u(y_j + (1 + r) a_i - a') + \beta \sum_{j'=1}^{J} V_\ell(a', y_{j'}) p_{jj'}$$

$$= u\left(y_j + (1 + r) a_i - a'_{\ell+1}(a_i, y_j)\right) + \beta \sum_{j'=1}^{J} V_\ell\left(a'_{\ell+1}(a_i, y_j), y_{j'}\right) p_{jj'}$$

Value Function Iteration

- Step 4: Check for convergence $\epsilon_\ell < \bar{\epsilon}$

$$\epsilon_\ell = \max_{i,j} \left| V_{\ell+1}\left(a_i, y_j\right) - V_\ell\left(a_i, y_j\right) \right|$$

  - If $\epsilon_\ell \geq \bar{\epsilon}$, go to Step 2 with $\ell := \ell + 1$
  - If $\epsilon_\ell < \bar{\epsilon}$, then

- Step 5: Extract optimal policy functions
  - $a'(a, y) = a_{\ell+1}(a, y)$
  - $V(a, y) = V_{\ell+1}(a, y)$
  - $c(a, y) = y_i + (1 + r)a - a'(a, y)$

- Consumption function restricted to implied grid so not very accurate

Stationary Distribution via Simulation

- Step 1: Set seed of random number generator
- Step 2: Initialize array to hold consumption c_{it} and assets a_{it} for large number I of individuals and time periods T
- Step 3: Loop over agents i, draw y_{i0} from stationary distribution. Set $a_{i0} = 0$
- Step 4: Loop over all time periods t. Use policy function $a'(a, y)$ to compute next period assets $a_{i,t+1}$ for each agent. Use budget constraint to get implied c_{it}. Draw $y_{i,t+1}$ using Markov chain P.
- Step 5: Compute mean asset holdings as

$$A_t = \frac{1}{I} \sum_{i=1}^{I} a_{it}$$

  and check that A_t has converged
- Code: see 2nd part of `vfi_IID.m`

Stationary Distribution via Transition Matrix

- Simulation often bad idea bc slow and introduces numerical error
- Recall: stationary distribution π of income process y solves

$$\pi = \mathbf{P}^\top \pi \qquad \text{or} \qquad \pi \approx (\mathbf{P}^\top)^N \pi_0 \quad \text{for large } N$$

- Idea of method 2: form big transition matrix of joint (a, y) process, let's call it $\mathbf{B}$, and use same strategy
- Step 1: Fix point in grid (a_i, y_j). For all possible grid points $a_{i'}, y_{j'}$ (important: all $a_{i'}$ forced to be on grid $\mathcal{A} = \{a_1, ..., a_N\}$) compute

$$\Pr(a_{t+1} = a_{i'}, y_{t+1} = y_{j'} | a_t = a_i, y_t = y_j)$$

- Can do this by interpolation of policy function $a'(a_i, y_j)$
- Step 2: Stack! 1. Stack grids for a (dim $= N$) and y (dim $= J$) into large $K = N \times J$ grid. Stack Pr's into big matrix $K \times K$ matrix $\mathbf{B}$
- Step 3: Stat dist g, a $K \times 1$ vector w entries $g(a_i, y_j)$, solves

$$g = \mathbf{B}^\top g \qquad \text{or} \qquad g \approx (\mathbf{B}^\top)^N g_0 \quad \text{for large } N$$

Something useful to think about

- We solved for wealth dist of economy with large number of people (say simulation with $N = 100,000$ to approximate continuum)

- How many Bellman equations did we solve?

- Why?

Endogenous grid point method

- Step 1: Construct grid $\mathcal{A}$ and set $a_1 = \underline{a}$
- Step 2: Set $\ell = 0$. Guess initial $c_0(a_i, y_j)$. A good first guess is

$$c_0(a_i, y_j) = ra + y$$

- Step 3: Construct expected marginal utility of consumption by adding across income states (note that $p_{jj'}$ determined by y_j)

$$\mathrm{EMUC}_\ell(a'_i, y_j) = \sum_{j'=1}^{J} u'(c_\ell(a'_i, y_{j'}))\, p_{jj'}$$

- Step 4: Use Euler equation at equality to get MUC today and c, a

$$\begin{aligned} \mathrm{MUC}_\ell(a'_i, y_j) &= \beta R \times \mathrm{EMUC}_\ell(a'_i, y_j) \\ \implies c_\ell(a'_i, y_j) &= u'^{-1}(\mathrm{MUC}_\ell(a'_i, y_j)) \\ a_\ell(a'_i, y_j) &= \frac{c_\ell(a'_i, y_j) + a'_i - y_j}{1 + r} \end{aligned}$$

  - This is the opposite of what we want! So, invert...

Endogenous Grid Method

- Step 5: Invert $a_\ell(a'_i, y_j) \implies a'(a, y_j)$ on an endogenous grid (a) Interpolate on $\mathcal{A}$ to get $a_{\ell+1}(a_i, y_i)$. Use BC to calculate $c_{\ell+1}$
- Step 6: Deal with borrowing constraints: define $a^*(y_j) = a_\ell(\underline{a}, y_j)$. Then for $a_i < a^*(y_j)$, $a_i \in \mathcal{A}$

$$
\begin{aligned}
a_{\ell+1}(a_i, y_j) &:= \underline{a} \\
c_{\ell+1}(a_i, y_j) &:= (1+r)\, a_i + y_j - \underline{a}
\end{aligned}
$$

- Step 7: Stop if $\epsilon_\ell < \bar{\epsilon}$ and return policy functions, where

$$
\epsilon_\ell = \max_{i,j} \left| c_{\ell+1}(a_i, y_j) - c_\ell(a_i, y_j) \right|
$$

If $\epsilon_\ell \geq \bar{\epsilon}$, go to Step 3 with $\ell := \ell + 1$