ECON0108 2022-23 Part 1

Slides for Lecture 1

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Statistics and Econometrics

- We focus mainly on *structural* econometrics
  - An economic process generates data. We want to know about the *structure* of the process.
- *Statistical* analysis reveals features of the probability distributions of observed variables.
- *Structural econometric* analysis informs about the structure of the process that generated these distributions and data.
- A structural *model* is a collection of restrictions on the structure of a process.

An example of a process and model

- For example: consider a model of a fish market at a harbour.

- Data comprise:

  - measures of prices (P) at which fish are sold day by day,
  - measures of quantities of fish sold (Q),
  - measures of factors which influence buyers and sellers, e.g. weather, season, resources of buyers, prices of e.g. substitutes.

- The variables whose values are not affected by the operation of the market are *exogenous* variables, the others are *endogenous* variables.

An example of a process and model

- A structural model could posit the existence of, and place restrictions on:
  - a *demand* relationship delivering the amount of fish (Q^D) buyers would buy at each price,
  - a *supply* relationship delivering the amount of fish (Q^S) fishers catch given conditions at sea (Z),
  - a *market clearing* condition, e.g. price adjusts so that all fish are sold.

$$\begin{aligned} Q^D &= \alpha + \beta P + U \\ Q^S &= \gamma + \delta Z + V \end{aligned}$$

$$Q = Q^S = Q^D$$

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$$\begin{aligned} Q^D &= \alpha + \beta P + U \\ Q^S &= \gamma + \delta Z + V \end{aligned}$$

$$Q = Q^S = Q^D$$

- There will typically be some restriction on the dependence of U, and Z, e.g. $E[ZU] = 0$, or $\forall z$, $E[U|Z = z] = c$.

- Importantly in structural econometrics objects are endowed with contextual meaning.

Identification analysis

- Identification analysis tells us what can be known of the process from the data it yields under particular restrictions on the structure of the process and the manner of observation and measurement.

- In this part of the course we will learn how to answer the following questions.
  - What is the force of the various restrictions that make up a model?
  - What minimal restrictions are required for knowledge of particular features of a process?
  - What is the nature of the knowledge that can be obtained? Does a model point or set identify a structural feature?

- There are implications for design of measurement, surveys, experiments etc.

The fish market example - an instrumental variable model

- With P denoting price and Q denoting quantity sold and Z denoting sea conditions the model we specified had

$$\begin{aligned} Q &= \alpha + \beta P + U \\ Q &= \gamma + \delta Z + V \end{aligned}$$

now impose the restriction: $\forall z,\ E[U|Z = z] = c$ some constant.

- Under this conditional mean independence restriction:

$$\begin{aligned} E[Q|Z = z_1] &= \alpha + \beta E[P|Z = z_1] + c \\ E[Q|Z = z_2] &= \alpha + \beta E[P|Z = z_2] + c \end{aligned}$$

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- The model identifies the value of β if there are values $\{z_1, z_2\}$ in the support of Z such that

$$E[P|Z = z_1] \neq E[P|Z = z_2].$$

Then:

$$\beta = \frac{E[Q|Z = z_1] - E[Q|Z = z_2]}{E[P|Z = z_1] - E[P|Z = z_2]}$$

These 5 lectures

- Focus is almost entirely on identification in the context of structural econometric models
  - Economic theory has a role to play in model construction.
  - There is the possibility of measuring the magnitude of policy effects and understanding their genesis.
  - There is the possibility of transferring results from one environment to another.
- We will touch only briefly on "treatment effects" and "programme evaluation" models in which:
  - there is often a minor role for economics,
  - transferability to new environments can be problematic.

The topics we will study

1. Definitions: Hurwicz structures, structural functions, structural features, models, point and set identification of structural features.
2. From identification to estimation: analogue estimation.
3. Complete models - fully simultaneous, and recursive triangular parametric and nonparametric models.
4. Control function methods and conditional independence restrictions.
5. Incomplete models, instrumental variable models.
6. Incomplete models with discrete outcomes and/or high dimensional heterogeneity - partial identification.

Structures

- Let Y denote a list of endogenous variables, Z denotes a list of observed exogenous variables and U denote a list of unobserved variables.

- A structure comprises a coupled pair

$$S \equiv (h, F_{UZ})$$

  - F_{UZ} is the probability distribution of (U, Z),
  - h is a function $h : \mathcal{R}_{YZU} \rightarrow \mathbb{R}$ which specifies the combinations of values of Y, Z and U that can occur.

$$h(Y, Z, U) = 0$$

- The set

$$\mathcal{Y}(z, u) = \{y : h(y, z, u) = 0\}$$

  is the set of values of Y that the structure can deliver when $Z = z$ and $U = u$.

- The set

$$\mathcal{U}(y, z) = \{u : h(y, z, u) = 0\}$$

  is the set of values of U that can deliver the value y of Y when $Z = z$.

Complete and incomplete structures

- The set
$$\mathcal{Y}(z, u) = \{y : h(y, z, u) = 0\}$$
is the set of values of Y that the structure can deliver when $Z = z$ and $U = u$.

- For example in a linear model with structures such that $Y = Z'\beta + U$
$$h(y, z, u) = y - z'\beta - u$$
$$\mathcal{Y}(z, u) = \{z'\beta + u\}$$

- When $\mathcal{Y}(z, u)$ is *singleton* for all z and u, S is a *complete* structure.

- Here $U(y, z) = \{y - z'\beta\}$ is singleton. But not when Y is discrete while U is continuous.

Complete and incomplete structures

- Consider a single equation model with two endogenous explanatory variables Y_1 and Y_2 and,

$$Y_1 = Z'\beta + \alpha Y_2 + U$$

- Let $y = (y_1, y_2)$. There is:

$$h(y, z, u) = y_1 - z'\beta - \alpha y_2 - u$$

$$\mathcal{Y}(z, u) = \{y : y_1 - z'\beta - \alpha y_2 - u = 0\}$$

which is not singleton. This is an *incomplete* structure.

Complete and incomplete structures

- Consider a single equation model with two endogenous explanatory variables Y_1 and Y_2 and,
$$Y_1 = Z'\beta + \alpha Y_2 + U$$
- Let $y = (y_1, y_2)$. There is:
$$h(y, z, u) = y_1 - z'\beta - \alpha y_2 - u$$
$$\mathcal{Y}(z, u) = \{y : y_1 - z'\beta - \alpha y_2 - u = 0\}$$
which is not singleton. This is an *incomplete* structure.

- Models which only admit complete structures are called *complete models*. We will first study complete models.

- Another example. The "triangular" model admitting structures such that
$$\begin{aligned} Y_1 &= Z'\beta_1 + \alpha Y_2 + U_1 \\ Y_2 &= Z'\beta_2 + U_2 \end{aligned}$$
is complete. What is the function h here?

Complete structures

- In a complete structure, $S \equiv \{h, F_{UZ}\}$, values of endogenous outcomes Y are *uniquely* determined by a structural function

$$h(y, z, u) = 0$$

given values z and u of observed Z and unobserved U.

- A complete structure S generates a **single** distribution F^S_{YZ} (or $F^S_{Y|Z}$).

- Structures S' and S'' such that $F^{S'}_{YZ} = F^{S''}_{YZ}$ are **observationally equivalent.**

Complete structures

- A **model** comprises restrictions on admissible structures.

- Let $\mathcal{M}_\Gamma$ be the set of admissible structures defined by a model, Γ. Consider $S^0 \in \mathcal{M}_\Gamma$.
  - The model Γ **identifies** S^0 if there is no $S^* \in \mathcal{M}_\Gamma$ such that $F_{YZ}^{S^*} = F_{YZ}^{S^0}$.
  - Γ is **uniformly** identifying if it identifies all $S \in \mathcal{M}_\Gamma$.

Complete structures

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  - The model Γ **identifies** S^0 if there is no $S^* \in \mathcal{M}_\Gamma$ such that $F_{YZ}^{S^*} = F_{YZ}^{S^0}$.
  - Γ is **uniformly** identifying if it identifies all $S \in \mathcal{M}_\Gamma$.

- Consider a **feature** of a structure, $\theta(S)$ and a model Γ.

- Γ **point** identifies $\theta(S^0)$ if $\theta(S)$ is constant across all structures admitted by Γ and observationally equivalent to S^0.

  - Uniform point identification if Γ point identifies $\theta(S)$ for all $S \in \mathcal{M}_\Gamma$.

Identification of structural features

- If there exists a functional $\mathcal{G}(F_{YZ})$ such that for all $S^* \in M_\Gamma$,

$$\theta(S^*) = a \Longrightarrow \mathcal{G}(F^*_{YZ}) = a$$

where F^*_{YZ} is the distribution delivered by S^*, then Γ uniformly point identifies θ.

- *Proof*. If $\mathcal{G}$ exists and S' and S'' have $\theta = a'$ and a'', then

$$\mathcal{G}(F^{'}_{YZ}) = a' \qquad \mathcal{G}(F''_{YZ}) = a''$$

and if S' and S'' are observationally equivalent then

$$F^{'}_{YZ} = F''_{YZ} \Longrightarrow a' = a''.$$

Identification of structural features

- If there exists a functional $\mathcal{G}(F_{YZ})$ such that for all $S^* \in M_\Gamma$,

$$\theta(S^*) = a \Longrightarrow \mathcal{G}(F^*_{YZ}) = a$$

where F^*_{YZ} is the distribution delivered by S^*, then Γ uniformly point identifies θ.

- *Proof*. If $\mathcal{G}$ exists and S' and S'' have $\theta = a'$ and a'', then

$$\mathcal{G}(F'_{YZ}) = a' \qquad \mathcal{G}(F''_{YZ}) = a''$$

and if S' and S'' are observationally equivalent then

$$F'_{YZ} = F''_{YZ} \Longrightarrow a' = a''.$$

- We have shown: If there exists a functional $\mathcal{G}(F_{YZ})$ such that for all $S^* \in M_\Gamma$,

$$\theta(S^*) = a \Longrightarrow \mathcal{G}(F^*_{YZ}) = a$$

then model Γ uniformly point identifies θ because each collection of observationally equivalent structures has a common value of θ.

Identification of structural features

- If there exists a functional $\mathcal{G}(F_{YZ})$ such that for all $S^* \in M_\Gamma$,

$$\theta(S^*) = a \Longrightarrow \mathcal{G}(F^*_{YZ}) = a$$

  then model Γ uniformly point identifies θ.

  - **Analogue estimation**: $\hat{\theta} = \mathcal{G}(\hat{F}_{YZ})$. Put the hat on.
  - **Overidentification** when there is more than one distinct functional $\mathcal{G}$.

- When there is overidentification of a structural feature there is more than one way to estimate it.

  - Sufficiently different estimates suggest the model employed is *misspecified*.
  - A model is misspecified relative to a distribution F_{YZ} if it admits no structure which delivers that distribution.

Example: measurement error

- A linear model with explanatory variable $\tilde{X}$ measured with error W so $X = \tilde{X} + W$ is observed.

$$Y = \alpha + \beta\tilde{X} + U \qquad X = \tilde{X} + W$$

giving

$$Y = \alpha + \beta X + (U - \beta W)$$

- Suppose there are variables Z such that:

$$E[U - \beta W | Z = z] = c \text{ a constant}$$

- The model identifies β if there are values $\{z_1, z_2\}$ in the support of Z such that:

$$E[X|Z = z_1] \neq E[X|Z = z_2]$$

in which case:

$$\beta = \frac{E[Y|Z = z_1] - E[Y|Z = z_2]}{E[X|Z = z_1] - E[X|Z = z_2]} = \mathcal{G}(F_{YZ})$$

Example: the fish market model

- Let Y_1 be the quantity of fish sold, Y_2 the price and scalar Z is an exogenous measure of sea conditions..

- Consider a fish market model which admits structures with

$$\begin{aligned} Y_1 &= \alpha + \beta Y_2 + U \\ Y_1 &= \gamma + \delta Z + V \end{aligned}$$

  and the distribution of U restricted so that $E[ZU] = 0$. Normalise $E[U] = 0$.

  - There is $Cov(Z, U) = E[ZU] - E[Z]E[U] = 0$.

- Write the first structural equation as follows.

$$Y_1 = \begin{bmatrix} 1 & Y_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + U$$

Example: the fish market model

- Let Y_1 be the quantity of fish sold, Y_2 the price and scalar Z is an exogenous measure of sea conditions

$$Y_1 = \begin{bmatrix} 1 & Y_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + U$$

- Premultiply by $\begin{bmatrix} 1 \\ Z \end{bmatrix}$

$$\begin{bmatrix} 1 \\ Z \end{bmatrix} Y_1 = \begin{bmatrix} 1 \\ Z \end{bmatrix} \begin{bmatrix} 1 & Y_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 1 \\ Z \end{bmatrix} U$$

take expectations

$$\begin{bmatrix} E[Y_1] \\ E[ZY_1] \end{bmatrix} = \begin{bmatrix} 1 & E[Y_2] \\ E[Z] & E[ZY_2] \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

solve

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & E[Y_2] \\ E[Z] & E[ZY_2] \end{bmatrix}^{-1} \begin{bmatrix} E[Y_1] \\ E[ZY_1] \end{bmatrix} = \mathcal{G}(F_{YZ})$$

provided $Cov[Z, Y_2] \neq 0$.

Analogue estimation

- The linear model for **scalar** Y and $k \times 1$ vector X

$$Y = X'\beta + U \qquad\qquad E[XU] = 0$$

  - Note: Y is scalar, X is a vector. These are *random variables* NOT data.
  - The model specifies a relationship amongst variables and restricts the distribution of unobservable variables.

Analogue estimation

- The linear model for **scalar** Y and $k \times 1$ vector X

$$Y = X'\beta + U \qquad\qquad E[XU] = 0$$

- There is $XY = XX'\beta + XU$ so, if expectations exist, $E[XY] = E[XX']\beta$ and if $\operatorname{rank}(XX') = k$

$$\beta = E[XX']^{-1}E[XY] = \mathcal{G}(F_{YX})$$

- **Estimation**: Let x_i be a realisation of X. Let y_i be a realisation of Y. Put the hat on: analogue estimators:

$$\hat{\beta} = \left(\widehat{E[XX']}\right)^{-1}\widehat{E[XY]}$$

Analogue estimation : OLS

- **Estimation**: Let x_i be a realisation of k-element vector X. Let y_i be a realisation of Y. There is the analogue estimator:

$$\hat{\beta} = \left(\widehat{E[XX']}\right)^{-1}\widehat{E[XY]}$$

$$\widehat{E[XX']} = \frac{1}{n}\sum_{i=1}^{n} x_i x_i' \qquad \widehat{E[XY]} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$$

- Define arrays of realisations:

$$\underset{n\times k}{X_n} \equiv \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \qquad \underset{n\times 1}{Y_n} \equiv \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

so there is:

$$\frac{1}{n}\sum_{i=1}^{n} x_i x_i' = \frac{1}{n}X_n'X_n \qquad \frac{1}{n}\sum_{i=1}^{n} x_i y_i = \frac{1}{n}X_n'Y_n$$

$$\hat{\beta} = \left(X_n'X_n\right)^{-1}X_n'Y_n$$

Analogue estimation: a linear IV model

- The linear model for scalar Y and $k \times 1$ vector X and r element vector Z

$$Y = X'\beta + U \qquad E[ZU] = 0$$

- There is the implication (if expectations exist)

$$E[ZY] = E[ZX']\beta$$

and so for *any* $r \times r$ matrix Ω

$$\Omega E[ZY] = \Omega E[ZX']\beta$$

$$E[XZ']\Omega E[ZY] = E[XZ']\Omega E[ZX']\beta$$

- If the inverse exists there is the point identifying correspondence:

$$\beta = \left(E[XZ']\Omega E[ZX']\right)^{-1} E[XZ']\Omega E[ZY]$$

- Choosing $\Omega = E[ZZ']^{-1}$ gives

$$\beta = \left(E[XZ']E[ZZ']^{-1}E[ZX']\right)^{-1} E[XZ']E[ZZ']^{-1}E[ZY]$$

Analogue estimation: a linear IV model

- The linear model for scalar Y and $k \times 1$ vector X and r element vector Z

$$Y = X'\beta + U \qquad E[ZU] = 0$$

- If the inverse exists there is the point identifying correspondence:

$$\beta = \left(E[XZ']\Omega E[ZX']\right)^{-1} E[XZ']\Omega E[ZY]$$

  and if $r > k$ there are many choices for Ω and so overidentification

- But if $r = k$ then $E[XZ']$ etc are square and if inverses exist.

$$\beta = E[ZX']^{-1}\Omega^{-1}E[XZ']^{-1}E[XZ']\Omega E[ZY]$$

  and

$$\beta = E[ZX']^{-1}E[ZY]$$

The Generalised IV estimator

- Analogue estimator

$$\hat{\beta} = \left(\widehat{E[XZ']} \left(\widehat{E[ZZ']} \right)^{-1} \widehat{E[ZX']} \right)^{-1} \widehat{E[XZ']} \left(\widehat{E[ZZ']} \right)^{-1} \widehat{E[ZY]}$$

- Define arrays:

$$\underset{n \times k}{X_n} \equiv \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} \qquad \underset{n \times r}{Z_n} \equiv \begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{bmatrix} \qquad \underset{n \times 1}{Y_n} \equiv \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- There is:

$$\frac{1}{n} \sum_{i=1}^{n} x_i x_i' = \frac{1}{n} X_n' X_n \qquad \frac{1}{n} \sum_{i=1}^{n} z_i z_i' = \frac{1}{n} Z_n' Z_n \qquad \frac{1}{n} \sum_{i=1}^{n} x_i y_i = \frac{1}{n} X_n' Y_n$$

and so the GIV estimator

$$\hat{\beta} = \left((X_n' Z_n) \left(Z_n' Z_n \right)^{-1} (Z_n' X_n) \right)^{-1} (X_n' Z_n) \left(Z_n' Z_n \right)^{-1} X_n' Y_n$$

Example: identification in a treatment effects model

- Let $D \in \{0,1\}$ indicate whether (1) or not (0) an unemployed person enrolls in a training programme.
- Let $U_0 \in \{0,1\}$ if there is (1) or is not (0) return to work within 1 year if **not** on a training programme ($D = 0$).
- Let $U_1 \in \{0,1\}$ if there is (1) or is not (0) return to work within 1 year if **on** a training programme ($D = 1$).
- We observe (Y, D) where

$$Y = DU_1 + (1 - D)U_0$$

Example: identification in a treatment effects model

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- We observe (Y, D) where

$$Y = DU_1 + (1 - D)U_0$$

- Define $u = (u_0, u_1)$ and $U = (U_0, U_1)$. A **structure** is (h, G_{UD}) where

$$h((y, d), (u_0, u_1)) = y - du_1 - (1 - d)u_0$$

  and G_{UD} is the joint distribution of binary U and binary D.
- The **structural feature** of interest here is the average treatment effect

$$\mu = E[U_1] - E[U_0]$$

Example: identification in a treatment effect model with RCT restrictions

- Recall

$$Y = DU_1 + (1 - D)U_0$$

- In a randomised control trial there is the *additional* restriction

$$\forall i \in \{0,1\} \quad E[U_i|D=0] = E[U_i|D=1]$$

- Recall $\lambda = P[D=1]$. There is

$$\mu = E[U_1|D=1] \times \lambda + E[U_1|D=0] \times (1-\lambda) \\ - (E[U_0|D=1] \times \lambda + E[U_0|D=0] \times (1-\lambda))$$

and with the RCT restriction

$$\begin{aligned} \mu &= E[U_1|D=1] - E[U_0|D=0] \\ &= E[Y|D=1] - E[Y|D=0] \\ &= \mathcal{G}(F_{YD}) \end{aligned}$$

and the model point identifies the ATE, μ.

Example: set identification in a treatment effects model, no RCT restriction

- Consider a structure with $P[D = 1] = \lambda$. There is

$$\begin{aligned}\mu = E[Y|D = 1] \times \lambda + E[U_1|D = 0] \times (1 - \lambda) \\ - (E[U_0|D = 1] \times \lambda + E[Y|D = 0] \times (1 - \lambda))\end{aligned}$$

  where the blue coloured terms can be determined given knowledge of F_{YD}.

- All structures with a particular value of $E[Y|D = 1]$, $E[Y|D = 0]$ and λ are observationally equivalent since these three quantities completely determine F_{YD}.

Example: set identification in a treatment effects model, no RCT restriction

- Consider a structure with $P[D=1]=\lambda$. There is

$$\mu = E[Y|D=1]\times\lambda + E[U_1|D=0]\times(1-\lambda) \\ -\left(E[U_0|D=1]\times\lambda + E[Y|D=0]\times(1-\lambda)\right)$$

- All structures with a particular value of $E[Y|D=1]$, $E[Y|D=0]$ and λ are observationally equivalent (OE) since these three quantities completely determine F_{YD}.

- The **largest** value attained by μ amongst these OE structures is got by setting $E[U_1|D=0]=1$ and $E[U_0|D=1]=0$ therefore

$$\mu \leq E[Y|D=1]\times\lambda + (1-\lambda) - E[Y|D=0]\times(1-\lambda)$$

- The **smallest** value attained by μ amongst these OE structures is got by setting $E[U_1|D=0]=0$ and $E[U_0|D=1]=1$ therefore

$$\mu \geq E[Y|D=1]\times\lambda - E[Y|D=0]\times(1-\lambda) - \lambda$$

Example: set identification in a treatment effects model

- Accordingly μ is **set identified** with

$$
\begin{aligned}
&E[Y|D=1]\times\lambda - E[Y|D=0]\times(1-\lambda)-\lambda \\
&\qquad\leq \mu \leq \\
&\qquad E[Y|D=1]\times\lambda + (1-\lambda) - E[Y|D=0]\times(1-\lambda)
\end{aligned}
$$

  where $\lambda = P[D=1]$.

- This can be expressed as follows.

$$
\begin{aligned}
&(E[Y|D=1]-E[Y|D=0]) \\
&\qquad -((1-\lambda)\times E[Y|D=1]+\lambda\times(1-E[Y|D=0])) \\
&\qquad\leq \mu \leq \\
&\qquad (E[Y|D=1]-E[Y|D=0]) \\
&\qquad +((1-\lambda)\times(1-E[Y|D=1])+\lambda\times E[Y|D=0])
\end{aligned}
$$