Problem Set 1 Solution

October 26, 2022

Problem set 1

Exercise 1

(i): We know that preference relation is complete, thus if pairs (x, y) and (y, x) are not in $\succ$ we must have that $x \sim y$.
(ii): $\Rightarrow$: By definition $\succ$ is asymmetric. We need to show that it is negatively transitive. Suppose it is not. Then there exists x, y, z and $x \succ y$ such that none of $x \succ z$ or $z \succ y$ is true, which means we must have $z \succsim x$ and $y \succsim z$. By the transitivity of $\succsim$ we get $y \succsim x$, which is a contradiction.

$\Leftarrow$: Compete: for any x, y we have that either (i)$x \succ y$, (ii) $y \succ x$ or $\neg(x \succ y)$ *and* $\neg(y \succ x)$. Then (i) implies $x \succsim y$, (ii) implies $y \succsim x$ and (iii) implies $y \sim x$. Therefore we have completeness.
Transitive: Take $x \succsim y$ and $y \succsim z$. Suppose that $\neg(x \succsim z)$, which by completeness implies $z \succsim x$ and thus $z \succ x$. Then by negative transitivity we have $y \succ x$ or $z \succ y$, contradicting $xx \succsim y$ and $y \succsim z$.

Exercise 2

Suppose $x \in B \cap argmax_{\succsim} A$. then $x \succsim y$ for all $y \in A$ and $x \in B$. $B \subseteq A$ implies that $x \succsim y$ for all $y \in B$ as well. Thus $x \in argmax_{\succsim} B$.

Exercise 4

(i): $\Rightarrow$: Suppose property α holds. Consider $x \in B \cap C(A)$. $B \subseteq A$ implies that $x \in B \subseteq A \subseteq X$. Also it is clear that $x \in C(A)$. Thus by property α we have $x \in C(B)$.

$\Leftarrow$: Suppose $x \in B \subseteq A \subseteq X$ and $x \in C(A)$. Since $B \subseteq A$, we have $B \cap C(A) \subseteq C(B)$. It is clear that $x \in B \cap C(A)$. Thus $x \in C(B)$.

(ii): $\Rightarrow$: Fix a $x \in C(B)$. $C(A) \cap C(B) \neq \emptyset$ implies that there exists a y belonging to both $C(A)$ and $C(B)$. Thus we have $x, y \in C(B)$ and $y \in C(A)$. According to property β this implies $x \in C(A)$.

$\Leftarrow$: Suppose $B \subseteq A \subseteq X$ and there exists x, y such that $x, y \in C(B)$ and $y \in C(A)$. Then it is clear that $y \in C(A) \cap C(B) \neq \emptyset$. This implies that $C(B) \subseteq C(A)$. As a result $x \in C(A)$.

(iii): Suppose HARP holds. Fix x such that $x \in B \subseteq A \subseteq X$ and $x \in C(A)$. Consider $y \in C(B)$. Since $B \subseteq A$ we have $\{x, y\} \subseteq A \cap B = B$. By HARP we get $x \in C(B)$. Thus property α holds. Now consider x, y such that $x, y \in C(B)$ and $y \in C(A)$. Applying HARP directly we get $x \in C(A)$. So property β also holds.

Now suppose property α and property β hold. Fix $\{x, y\} \subseteq A \cap B$ such that $x \in C(A)$ and $y \in C(B)$. We know that $x \in A \cap B \subseteq A \subseteq X$. Property α implies that $x \in C(A \cap B)$. Similarly we have $y \in C(A \cap B)$. Since $x, y \in C(A \cap B)$ and $y \in C(B)$, property β implies $x \in C(B)$. By a symmetric argument we get $y \in C(A)$. Thus HARP holds.

Exercise 6

(i): Both α and β are satisfied.

(ii): No. Suppose the threshold utility $\bar{u}$ is zero. Then the consumer simply chooses the first book in set A according to S. As a result the bookseller cannot learn anything about consumer's preference $\succsim$.

(iii): Consumer's choice does not always manifest how they value different alternatives. Choices can also be affected by the environment, which in this case is the ordering S sat by the bookseller. As a result, the preference elicited by consumer's choices may not necessarily coincide with "true" preference of the consumer.

ECON0106: **Microeconomics**

Problem Set 1

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Due date: 10 October, 12:30

Question 1. 1. Choice, Preferences, Utility: Exercises 1 and 2

Question 2. 1. Choice, Preferences, Utility: Exercise 4

Question 3. 1. Choice, Preferences, Utility: Exercise 6

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Problem Set 2 Solution

November 3, 2022

Problem set 2

Exercise 1

$(i) \Rightarrow (ii)$: Fix $x \in X$, let's prove that $X_{x\succsim}$ is closed. Consider a convergent sequence $y_n \to y$, where $y_n \in X_{x\succsim}$ for all n. By continuity we have that $x \succsim y$, which means the limit is in $X_{x\succsim}$. Thus we have the closeness. By the same argument we can prove that $X_{\succsim x}$ is also closed.

$(ii) \Rightarrow (iii)$: One can see that $X_{x\succ}$ and $X_{\succ x}$ are the complements of $X_{\succsim x}$ and $X_{x\succsim}$ respectively. Thus $X_{x\succ}$ and $X_{\succ x}$ are open sets.

$(iii) \Rightarrow (iv)$: $X_{\succ y}$ is an open set, therefore exists a $\epsilon_1 > 0$ such that $B_{\epsilon_1}(x) \in X_{\succ y}$. Consider the intersection $B_{\epsilon_1}(x) \cap X_{x\succ}$. If this intersection is empty, $B_{\epsilon_1}(x) \in X_{\succsim x}$. Because $X_{x\succ}$ is open there is ϵ_2 such that $B_{\epsilon_2}(y) \in X_{x\succ}$. Let $\epsilon = min(\epsilon_1, \epsilon_2)$.

If the intersection is non-empty. Let z be a point in $B_{\epsilon_1}(x) \cap X_{x\succ}$, then $x \succ z \succ y$. Since $X_{z\succ}$ is also open, there is a ϵ_2 such that $B_{\epsilon_3}(y) \in X_{z\succ}$. Now notice that $X_{\succ z}$ is an open set as well. Thus there exists a ϵ_4 such that $B_{\epsilon_4}(x) \in X_{\succ z}$. Now let $\epsilon = min(\epsilon_3, \epsilon_4)$.

$(iv) \Rightarrow (i)$: Suppose $\{x_n\}$ and $\{y_n\}$ are two convergent sequences with limits x and y respectively. Also we assume that $x_n \succsim y_n$ for all n. We proceed by proof by contradiction. Assume that $y \succ x$. Then there is $\epsilon > 0$ such that $\forall x' \in B_\epsilon(x)$ and $\forall y' \in B_\epsilon(y)$, $y' \succ x'$. Since $x_n \to x$ and $y_n \to y$, there is a nature number N such that $x_n \in B_\epsilon(x)$ and $y_n \in B_\epsilon(y)$ for all $n > N$. This means that $y_n \succ x_n$ for all $n > N$, which is a contradiction.

Exercise 3

Let $\succsim$ be a convex preference relation. Consider the set $argmax_{\succsim} A$. For any $x, y \in argmax_{\succsim} A$ we must have $\lambda x + (1-\lambda)y \succsim y$ for all $\lambda \in [0,1]$ as $\succsim$ is convex and $x \succsim y$. Since $y \succsim y'$ for all $y' \in A$, by transitivity we must also have $\lambda x + (1-\lambda)y \succsim y'$ for all $y' \in A$. Therefore $\lambda x + (1-\lambda)y \in argmax_{\succsim} A$ for all $\lambda \in [0,1]$. Thus $argmax_{\succsim} A$ is convex.

Now suppose $\succsim$ is strictly convex. If there are multiple elements in $argmax_{\succsim} A$, say x, y. Since $x \neq y$ and $x \succsim y$, by strict convexity we have that $\lambda x + (1-\lambda)y \succ y$. Also $\lambda x + (1-\lambda)y \in A$ for all $\lambda \in [0,1]$ because A is convex. This is a contradiction since $y \succsim y'$ for all $y' \in A$.

Exercise 4

(1): Fix a $x \in X$. Let $min(x)$ be the smallest entry and $max(x)$ be the largest entry. Since k is a finite number, $min(x)$ and $max(x)$ are well defined. Let $\alpha = 2max(x)$ and $\alpha' = \frac{1}{2}min(x)$. Because $\succsim$ is a strongly monotone preference, we have $\alpha\mathbf{1} \succsim x \succsim \alpha'\mathbf{1}$. (Actually $\alpha\mathbf{1} \succ x \succ \alpha'\mathbf{1}$).

(2): Existence: We know from (1) that there are α and α' such that $\alpha\mathbf{1} \succ x \succ \alpha'\mathbf{1}$. Consider the line segment connecting $\alpha\mathbf{1}$ and $\alpha'\mathbf{1}$, which is defined as the set $\{(\lambda\alpha + (1-\lambda)\alpha')\mathbf{1} | \lambda \in [0,1]\}$.
Let $B = \{\lambda \in [0,1] | (\lambda\alpha + (1-\lambda)\alpha')\mathbf{1} \succ x\}$. Notice that $B \neq \emptyset$ as $0 \in B$. Let $b = sup\, B$ and $b' = b\alpha + (1-b)\alpha'$.

Is $b'\mathbf{1} \in X_{x\succ}$? No: If $b'\mathbf{1} \in X_{x\succ}$, since $X_{x\succ}$ is open, there exists $\delta > 0$ such that $B_\delta(b'\mathbf{1}) \in X_{x\succ} \Rightarrow$ $[(b + \frac{1}{2}\delta)\alpha + (1 - (b + \frac{1}{2}\delta)\alpha']\mathbf{1} \in X_{x\succ} \Rightarrow b$ is not $sup\, B$.

Is $b'\mathbf{1} \in X_{\succ x}$? No: If $b'\mathbf{1} \in X_{\succ x}$, since $X_{x\succ}$ is open, there exists $\delta > 0$ such that $B_\delta(b'\mathbf{1}) \in X_{\succ x} \Rightarrow \forall \epsilon < \frac{1}{2}\delta,\ b - \epsilon \notin B$.
Then by completeness, we must have that $b'\mathbf{1} \sim x$. Then we just let $\beta_x = b'$

Uniqueness: Suppose we have $\beta_x \neq \beta_x^{'}$ such that $x \sim \beta_x\mathbf{1} \sim \beta_x^{'}\mathbf{1}$. We have either $\beta_x > \beta_x^{'}$ or $\beta_x < \beta_x^{'}$. However by the strong monotonicity both situations are impossible. Thus there is a unique β_x.

(3): Homogeneous of degree one: Implied by the homotheticity of the preference relation.

Continuity: Consider a sequence $x_n \to x$, we want to show that $u(x_n) \to u(x)$. Fix an $\epsilon > 0$, there is a natural number N such that $\forall n > N$, $|x_n - x| < \epsilon$. Because the preference relation is monotone, for all $n > N$ we have $\beta_{x_n} \in [\bar{\beta}, \underline{\beta}]$ (one can find $x' \gg x_n,\ x'' \ll x_n\ \forall n > N$). Now we truncate the sequence $\{x_n\}$ and only consider the part where $n > N$. From now on we call this new sequence $\{x_n\}$. It is cleat that the sequence $\{\beta_{x_n}\}$ is bounded.

Next we want to show that every convergent subsequence of $\{\beta_{x_n}\}$ converges to β_x. Suppose this is not true, which means there is a subsequence $\{\beta_{x_{m(n)}}\} \to \beta' \neq \beta_x$. There are two cases where $\beta' > \beta_x$ or $\beta' < \beta_x$. First consider the case $\beta' > \beta_x$. Strong monotonicity implies that $\beta'\mathbf{1} \succ \beta_x\mathbf{1}$. Let $\beta'' = \frac{\beta' + \beta_x}{2}$, then by strong monotonicty again we have $\beta'\mathbf{1} \succ \beta''\mathbf{1} \succ \beta_x\mathbf{1}$. Since $\{\beta_{x_{m(n)}}\} \to \beta' \neq \beta_x$, there is a M such that $\beta_{x_{m(n)}} > \beta''$ for all $m > M$, which means $x_{m(n)} \sim \beta_{x_{m(n)}}\mathbf{1} \succ \beta''\mathbf{1}$ for all $m > M$. The preference relation is continuous and $x_m(n) \to x$, thus we must have $\beta_x \sim x \succsim \beta''\mathbf{1}$, which is a contradiction. A symmetric argument can prove that $\beta' < \beta_x$ also leads to contradiction.

Q3

(i): Yes

(ii):
property 1: $x - x = 0$
property 2: $|x - y| = |y - x|$
property 3: Consider the sequence $(x_n, y_n) \to (x, y)$, such that $|x_n - y_n| < \epsilon$ for all n. We want to show that $|x - y| < \epsilon$. $|x - y| = |x - y - x_n + x_n - y_n + y_n| = |x - x_n + y_n - y + x_n - y_n| \leq |x - x_n| + |y - y_n| + |x_n - y_n| < \epsilon$ (triangular inequality) as n goes to infinity.
property 4: It is clear that $|x - y| \leq |z - w|$ if $z \geq x \geq y \geq w$.
property 5: $(x - 0.5\epsilon, x + 0.5\epsilon)$
property 6: $M(x) = x + \epsilon\ if\ x + \epsilon < 1$ otherwise $M(x) = 1$. $m(x) = x - \epsilon\ if\ x - \epsilon > 0$, otherwise $m(x) = 0$

(iii).1:

Let (x_n) be a sequence such that $x_0 = 1$ and $x_{n+1} = m(x_n)$. let $E(a) := \{b \in X : bSa\} = \{b \in X : aSb\}$. Note that as $E(a)$ is closed (by P-3, continuity of S) and bounded from below, $m(a) = \min E(a) \neq \emptyset$. This implies that the sequence defined above is well-defined.

Now we want to show that the sequence converges to zero. Suppose not. As $X = [0, 1]$ is closed, $x_n \to x \in X$ and, by assumption, $x > 0$. Moreover, the sequence is monotonically decreasing, which means $x_n \geq x$ for all n. By P-5, $\exists \epsilon' > 0$ such that $B(x, \epsilon') \subset E(x)$ and by P-4, $\forall y, z \in B(x, \epsilon')$, ySz. Now, by convergence of the sequence, $\exists N$ such that $\forall n \geq N$, $|x_n - x| < \epsilon'$, which implies that $x_n \in B(x, \epsilon')$ and therefore $m(x_n) \leq x - \epsilon' < x$, which leads to a contradiction.

Finally we show that $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $x_n = 0$. Note that $\exists N' \in \mathbb{N}$ such that $\forall n \geq N'$, $0 \in E(x_n)$. To see this, note that by P-5, $\exists \epsilon'' > 0$ such that $\forall x < \epsilon''$ we have $0Sx$. As $x_n \to 0$, $\exists$ such finite N. Then, as $0 \in E(x_{N'})$, then $0 = m(x_{N'}) = x_{N'+1}$. Set $N = N' + 1$.

(iii).2:

Now we want to Define G. Let $G(1) = 1$ and define G on $[x_1, x_0]$ as any strictly increasing, continuous function satisfying $G(x_1) = 1/\lambda$ and $G(x_0) = 1$. For $x \in [x_{n+1}, x_n]$, let $G(x) = G(M(x))/\lambda$. By P-6, G is strictly increasing. As $x_n = 0$ for some finite n, then we're done. Note that this implies that $G(0) > 0$.

Then we verify that (G, λ) represents S.

- $1/\lambda \leq G(a)/G(b) \leq \lambda \Rightarrow aSb$ for $a, b \in X$.

Without loss of generality, assume $a \leq b$. Then if $a < 1$, $1/\lambda \leq G(a)/G(b) \Leftrightarrow G(b)/\lambda \leq G(a) = G(M(a))/\lambda \Leftrightarrow G(b) \leq G(M(a))$. If $a = 1$, $G(b) \leq G(1) = G(M(a))$. As G is increasing by construction, $b \leq M(a)$. Then, $m(a) \leq a \leq b \leq M(a)$. By P-4, as $M(a)Sm(a)$, thus aSb.

- $1/\lambda \leq G(a)/G(b) \leq \lambda \Leftarrow aSb$ for $a, b \in X$.

Let without loss $a \leq b$. Then $m(a) \leq a \leq M(a)$. By definition of M, given $b \in E(a)$, $b \leq M(a) = \max E(a)$. Therefore, by P-4, $m(a) \leq a \leq b \leq M(a)$.
If $M(a) < 1$, then as G is increasing by construction,

$$G(M(a))/\lambda = G(a) \leq G(b) \leq G(M(a)) \Leftrightarrow$$
$$1/\lambda \leq (1/\lambda)G(M(a))/G(a) \leq G(b)/G(a) \leq G(M(a))/G(a) = \lambda$$

If $M(a) = 1 \Rightarrow M(b) = 1$ by P-4,

$$G(a)/\lambda \leq G(M(a))/\lambda \leq G(a) \leq G(b) \leq 1 \Leftrightarrow$$
$$1/\lambda \leq G(b)/G(a) \leq 1/G(a) \leq G(M(a))\lambda = G(1)\lambda = \lambda$$

(iii).3:

Note that $\frac{1}{\lambda} \leq \frac{G(a)}{G(b)} \leq \lambda \Leftrightarrow -\ln\lambda \leq \ln G(a) - \ln G(b) \leq \ln\lambda \Rightarrow |\ln G(a) - \ln G(b)| \leq \ln\lambda$. As $\lambda > 1$, $\ln\lambda > 0$. As $G : X \to (0, 1]$ by construction, $\ln G(x) \in \mathbb{R}\ \forall x \in X$. By construction, $\ln G$ is strictly increasing and well defined (as $G(x) > 0\ \forall x \in [0, 1]$, including $x = 0$). Just set $H = \ln G$ to be the desired function and $\epsilon = \ln\lambda$.

(iii).4:

We need two lemmas first:

Lemma 1 Let $f : X \to Y$ and $g : Y \to X$ be such that $g \circ f = \mathrm{id}_X$, the identity function on X. Then f is injective and g is surjective.

Proof of lemma 1:
g is surjective: $\forall x \in X$, $\exists y \in Y$ such that $g(y) = x$. Let $y = f(x)$. Then, $g(y) = g(f(x)) = x$.
f is injective: $\forall x, x' \in X$, such that $f(x) = f(x')$, then $x = x'$. As $x = g(f(x))$ and $x' = g(f(x'))$, by the fact that g is a function it follows that $x = x'$.

Lemma 2 Let X and Y be intervals on the real line. Let $f : X \to Y$ be bijective and strictly increasing. Then f is continuous.

Proof of lemma 2:
Fix $x_0 \in X$. Let $\epsilon > 0$ and assume that ϵ is small such that $[f(x_0) - \epsilon, f(x_0) + \epsilon] \subseteq Y$.
As f is a bijection, $\exists x', x''$ such that $f(x') = f(x_0) - \epsilon$ and $f(x'') = f(x_0) + \epsilon$. where $x', x'' \neq x_0$.
As f is increasing, and $f(x') = f(x_0) - \epsilon < f(x_0)$, it must be that $x' < x_0$. By a symmetric argument, $x'' > x_0$.
Let $\delta = \min\{x_0 - x', x'' - x_0\} > 0$. Then, $\forall x \in (x_0 - \delta, x_0 + \delta) \subseteq (x', x'')$, it must be that $f(x') = f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon = f(x'')$, which implies continuity of f at x_0.
As x_0 was taken arbitrarily, we have that f is continuous.
If the intervals are closed, take $x_0 = \min X$. Then, for the same reasoning, $\forall x \in (x_0, x_0 + \delta) \subseteq (x_0, x'')$, $f(x_0) < f(x) < f(x_0) + \epsilon = f(x'')$. The proof for $x_0 = \max X$ is symmetric.

Now we can start to prove that M is continuous on $[0, m(1)]$.

1. $\forall x \in [M(0), m(1)]$, $\{x\} = E(m(x)) \cap E(M(x))$.
   Suppose not. Then $\exists y \in X$ such that $y \in E(m(x)) \cap E(M(x)) \backslash \{x\}$.
   Claim: $y \in [m(x), M(x)]$. Suppose not. (i) If $0 \leq y < m(x) < x < M(x)$, where the strict inequalities with respect to x follow from P-6. Again by P-6, as $M(y) \leq M(x) < 1$, then $M(y) < M(x)$. Then $M(x) \notin E(y)$ and by P-2 $y \notin E(M(x))$. The argument is symmetric for

$y > M(x)$.
Then $y \in [m(x), M(x)] \cap E(m(x)) \cap (E(M(x)) \backslash \{x\})$. Now by P-2 this means that $m(x), M(x) \in E(y)$, which then implies that $0 \leq m(y) \leq m(x) < x < M(x) \leq M(y) \leq 1$. If $1 \geq y > x \geq M(0)$, then $m(y) > m(x) \geq 0$ by P-6, which contradicts the above inequality. If $0 \leq y < x \leq m(1)$, then $M(y) < M(x) \leq 1$ by P-6, which contradicts the above inequality.

2. M and m are continuous on $[M(0), m(1)]$.
We know that $M(m(x)) = x = m(M(x))$ $\forall x \in [M(0), m(1)]$ by the previous steps. $M \circ m = m \circ M = id$ on $[M(0), m(1)]$. By the first lemma above, we have that these are bijections on $[M(0), m(1)]$. By the second and P-6, we have that M and m are continuous on $[M(0), m(1)]$.

3. M is continuous on $[0, M(0))$
First note that $\forall y \in [M(0), m(1)]$, $m(M(y)) = y \Leftrightarrow m^{-1}(y) = M(y)$, where the inverse exists because m is strictly increasing on this interval. This implies that M is also continuous.
Now take any $y \in [M(0), m(1)]$. $M(m(y)) = y \Leftrightarrow M^{-1}(y) = m(y)$. Then, $\forall x \in [0, M(0))$, $M^{-1}(M(x)) = m(M(x)) = x$. Moreover, $m(M(0)) = 0$.
Note m is continuous on $[M(0), m(1)]$, therefore m^{-1} is continuous on $[0, m(m(1))]$. As $\forall x \in [0, M(0))$, $M(x) = m^{-1}(x)$, then M is continuous on $[0, M(0))$.

Therefore we conclude that M is continuous on $[0, m(1)]$

(iii).5:
As M is continuous on $[0, m(1)]$, H is continuous on $[m(1), 1] = [x_1, x_0]$ by construction and $G(x) = G(M(x))/\lambda$ for $x \in [x_2, x_1)$ and $M(x) \in [x_1, x_0]$, then G is continuous on $[x_2, x_1)$. By induction suppose that G is continuous for $[x_n, 1]$. Then by the same argument, G is continuous on $[x_{n+1}, 1]$. As $\exists N \in \mathbb{N}$ such that $x_N = 0$, then by induction we have that G is continuous on $[0, 1]$. Therefore, H is continuous.

ECON0106: **Microeconomics**

Problem Set 2

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Due date: 17 October, 12:30

Question 1. 2. Structural Properties of Preferences and Utility Representations: Exercise 1

Question 2. 2. Structural Properties of Preferences and Utility Representations: Exercises 3 and 4

Question 3. There is an intuitive notion of what it means for an object x to be more similar to y than to z. This is an important notion to think about, e.g., whether choosing between more similar alternatives is easier or harder. Let us use what we've learn to model such an intuition.

For simplicity, let X be the unit interval, $[0,1]$. Let's define a binary relation S on X and consider the following properties:

Property 1: For all $x \in X$, xSx.

Property 2: For all $x, y \in X$, if xSy, then ySx.

Property 3: The graph of the relation S in $X \times X$ is a closed set (continuity).

Property 4: If $z \geq x \geq y \geq w$ and zSw, then xSy (betweenness).

Property 5: For any $x \in X$, there is an open interval around x such that xSy for all y in the interval.

Property 6: Let $M(x) := \max\{y | ySx\}$ and $m(x) := \min\{y | ySx\}$. Then M and m are nondecreasing functions, and strictly increasing whenever they don't have the values 0 or 1.

Question 3.(i) Do these properties capture your intuition about the concept of "approximately the same"?

Question 3.(ii) For $\epsilon > 0$, show that S_ϵ, defined by $xS_\epsilon y$ if $|x - y| \leq \epsilon$ satisfies all the the properties.

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Question 3.(iii) Now suppose S satisfy all the properties above and let $\epsilon > 0$. Show there is a strictly increasing and continuous function $H : X \to \mathbb{R}$ such that xSy if and only if $|H(x) - H(y)| \leq \epsilon$. **This part is hard.** Let's split the questions into multiple subquestions to guide you through the construction.

(iii).1 Let $\{x_n\}$ be a sequence such that $x_{n+1} = m(x_n)$ and $x_0 = 1$. Prove that this sequence is well-defined and there exists a natural number N such that $\forall n \geq N$, $x_n = 0$.

(iii).2 Given a real number $\lambda > 0$, construct a strictly increasing function G on $[0,1]$ such that $\frac{1}{\lambda} \leq \frac{G(x)}{G(y)} \leq \lambda$ iff xSy. (*Hint: You may want to use the sequence in the previous question and construct the function recursively. You could first define G on* $[x_1, x_0]$*, then extend it to the rest of* $[0,1]$.)

(iii).3 Given a real number $\epsilon > 0$. Construct a fuction H such that $|H(x) - H(y)| < \epsilon$ iff xSy and show that H is well-defined and strictly increasing.

(iii).4 Prove that $M(x)$ is continuous on the interval $[0, m(1)]$. (*Hint: if a function f is bijective and strictly increasing, f is continuous.*)

(iii).5 Prove that H is continuous on $[0,1]$.

Problem Set 3 Solution

November 4, 2022

Lecture 3

Exercise 2

(i) B is nonempty-valued because $0 \in B(p, w)$ for any (p, w).

(ii) B is closed-valued. Take (x^n) such that $x^n \to x$ and $x^n \in B(p, w)$. Then $\sum_i p_i x_i^n \leq w$. Because weak inequalities in $\mathbb{R}^k$ are preserved in the limit and $p \cdot x$ is a continuous function of x, we get $p \cdot x \leq w$.

B is bounded. By definition, $x \geq 0$ and is bounded above by say $(\max_i w/p_i)\mathbf{1}$.

Therefore, because $B(p, w) \subseteq \mathbb{R}^k$, by the Heine-Borel theorem, $B(p, w)$ is compact.

(iii) B is uhc at (p_0, w_0). First we want to show that there exists a converging subsequence for any (x^n) such that $x^n \in B(p^n, w^n)$ for some sequence $(p^n, w^n) \to (p_0, w_0)$.

To that end, we will show that the sequence is bounded and then appeal to the Bolzano-Weierstrass theorem. Because (w^n) and (p_i^n) are converging sequences (recall that a sequence in $(\mathbb{R}^k, d_2)$ converges iff it converges in each dimension), they have a bounded range, $\sup\{w^1, w^2, ...\} < B$ and $\inf\{p_i^1, p_i^2, ...\} > P_i$ for some $B > 0$ and $P_i \geq \min_j P_j$ for some $\delta > 0$.[1] For each n, $x_i^n \leq \max_i w^n/p_i^n \leq \max_i B/P_i$. Thus, (x^n) is bounded. By the Bolzano-Weierstrass theorem, this subsequence admits a converging subsequence.

Let x_0 be the limit of this subsequence and rename the elements so that $x^n \to x_0$. Again because weak inequalities in $\mathbb{R}^k$ are preserved in the limit and $p \cdot x$ is a continuous function of x, we get $p_0 \cdot x_0 \leq w_0$.

(iv) Let U open such that $U \cap B(p_0, w_0) \neq \emptyset$. Take some $x \in U \cap B(p_0, w_0)$ and $x \neq \mathbf{0}$. Because U is open, there exists some $\delta > 0$ and $i \in [n]$ such that $y_i = x_i - \delta > 0$ and $y_j = x_j$ for $j \neq i$ and $y \in U$. We also have $p_0 \cdot y < p_0 \cdot x \leq w_0$. Let $\tilde{\delta} = w_0 - p_0 \cdot y$. Choose $\epsilon > 0$ such that $\epsilon \mathbf{1} \cdot y + \epsilon < \tilde{\delta}$. Take any $(p, w) \in B_\epsilon(p_0, w_0)$. Then, $p < p_0 + \epsilon \mathbf{1}$ and $w > w_0 - \epsilon$. Because of the choice of ϵ, $p \cdot y < (p_0 + \epsilon \mathbf{1}) \cdot y < w_0 - \epsilon < w$ (using that $y \geq 0$). Therefore $y \in U \cap B(p, w)$ for all $(p, w) \in B_\epsilon(p_0, w_0)$.

If $x = \mathbf{0}$, then $x \in B(p, w)$ for all (p, w) and $\mathbf{0} \in U$. Thus $x \in U \cap B(p, w)$ for all $(p, w) \in B_\epsilon(p_0, w_0)$ for any ϵ.

(v) A continuous utility representation follows from Debreu's theorem and the fact $\mathbb{Q}^k$ is a countable order-dense in $\mathbb{R}^k$ when $\succeq$ is continuous.

(vi) All the conditions of Berge's theorem are met: the objective function is constant in (p, w) and continuous in x, hence continuous in $(x, (p, w))$, and the constraint is non-empty, compact-valued and continuous (both lhc and uhc) at all $(p, w) \in \mathbb{R}^k_{++} \times \mathbb{R}_+$. Therefore, $v(p, w)$ is continuous and $x(p, w)$ uhc at all (p, w).

[1] The value $\min_i P_i$ is bounded away from 0 because $p_0 > 0$ and k is finite.

Exercise 5

We first observe that continuity of $\succeq$ and of the utility representation u imply that $x(p,w)$ and $h(p,u)$ are nonempty and Lemma 2 applies. Local non-satiation implies the Walras' law.

(i): Suppose $x \in h(p, v(p,w))$ then by Lemma 2, $u(x) = v(p,w)$; Suppose $x \notin x(p,w)$. Then $p \cdot x > w$ (otherwise it is feasible and attain $v(p,w)$). There is $x' \in x(p,w)$ such that $u(x') = v(p,w)$ and $p \cdot x' = w$ (Walras' law). This contradicts that $x \in h(p, v(p,w))$. Thus $h(p, v(p,w)) \subseteq x(p,w)$

Suppose $x \in x(p,w)$ but $x \notin h(p, v(p,w))$. Then there exists $x' \in h(p, v(p,w))$ such that $u(x') \geq u(x) = v(p,w)$ and $p \cdot x' < p \cdot x = w$ by Walras' law. Then we can find $\epsilon > 0$ such that $B_\epsilon(x') \subseteq B(p,w)$. By local non-satiation, there is $x'' \in B_\epsilon(x')$ such that $x'' \succ x' \succeq x$ and x'' is feasible. That contradicts $x \in x(p,w)$ and thus $x(p,w) \subseteq h(p, v(p,w))$.

Take $x \in h(p, v(p,w)$. Then $p \cdot x = e(p, v(p,w)) = w$ where the last part follows from $x \in x(p,w)$ and Walras' law.

(ii): Take $x \in h(p,u)$ and suppose $x \notin x(p, e(p,u))$. By Lemma 2, we have $u(x) = u$ and by definition $p \cdot x = e(p,u)$. Now take $x' \in x(p, e(u,p))$ (from non-emptiness). Because $x \notin x(p, e(p,u))$, $u(x') > u$, thus $p \cdot x' \geq e(u,p)$ and by feasibility $p \cdot x' \leq e(u,p)$. Therefore $p \cdot x' = e(p,u)$. But then by continuity u, we can find $\lambda \in [0,1)$ such that $u(\lambda x') \geq u$ and $p \cdot \lambda x' < e(p,u)$. This contradicts that $x \in h(p,u)$ as $\lambda x'$ has higher utility and lower expenditure. Thus $h(p,u) \subseteq x(p, e(p,u))$.

Take $x \in x(p, e(p,u))$ then by Walras' law, $p \cdot x = e(p,u)$. Suppose $x \notin h(p,u)$. This mean that $u(x) < u$. Then let $x' \in h(p,u)$. We get $p \cdot x' = e(p,u)$ and $u(x') \geq u$. Then x' is feasible and $u(x') > u(x)$. A contradiction that $x \in x(p, e(p,u))$. Therefore $x \in h(p,u)$.

Finally, by Lemma 2, $x \in h(p,u) \Rightarrow u(x) = u$ and $x \in x(p, e(u,p))$ implies $u(x) = v(p, e(u,p))$ by definition. Therefore, $u = v(p, e(u,p))$.

Lecture 4

Exercise 2

(i) The functions f and g are supermodular. For any $x, x' \in X$,

$$
\begin{aligned}
& f(x \vee x') - f(x') \geq f(x) - f(x \wedge x') \\
& g(x \vee x') - g(x') \geq g(x) - g(x \wedge x') \\
& \Rightarrow \alpha(f(x \vee x') - f(x')) + \beta(g(x \vee x') - g(x')) \geq \alpha(f(x) - f(x \wedge x')) + \beta(g(x) - g(x \wedge x'))
\end{aligned}
$$

for any $\alpha, \beta \geq 0$.Therefore, $(\alpha f + \beta g)(x \vee x') - (\alpha f + \beta g)(x') \geq (\alpha f + \beta g)(x) - (\alpha f + \beta g)(x \wedge x')$.

(ii) Take $x, x' \in X$ such that $f(x) - f(x \wedge x') \geq 0$. Then, $(g \circ f)(x) - (g \circ f)(x \wedge x') \geq 0$ because g is strictly increasing. Then because $(g \circ f)$ is supermodular, $(g \circ f)(x \vee x') - (g \circ f)(x') \geq (g \circ f)(x) - (g \circ f)(x \wedge x') \geq 0$. Then using that g is strictly increasing, we get $f(x \vee x') - f(x') \geq 0$

To prove strict inequality, observe that $f(x) - f(x \wedge x') > 0$ implies that $(g \circ f)(x) - (g \circ f)(x \wedge x') > 0$ because g is *strictly* increasing. Then the same reasoning applies.

(iii) Given that we have only defined partial derivatives for $\mathbb{R}^n$, we assume that $Y \subseteq \mathbb{R}^n$.

First we show that $f : \mathbb{R}^n \to \mathbb{R}$ is supermodular iff it is pairwise supermodular: for all i, j, $i \neq j$, the restriction $f(\cdot, y_{-i-j}) : \mathbb{R}^2 \to \mathbb{R}$ is supermodular.

The $\Rightarrow$ is immediate from the definition.

Let's prove $\Leftarrow$. First, let's prove that if f has pairwise supermodularity and $x_j \geq y_j$ for each j, then

$$
f(x_i, x_{-i}) - f(y_i, x_{-i}) \geq f(x_i, y_{-i}) - f(y_i, y_{-i})
$$

Take, $x, y \in \mathbb{R}^n$ with $x_j \geq y_j$ for each j. WLOG thake $i = 1$.

$$\begin{aligned} f(x_1, x_{-1}) - f(y_1, x_{-1}) &= \sum_{j\neq 1} f(x_1, x_2, ..., x_j, y_{j+1}, ...y_n) - f(x_1, x_2, ..., x_{j-1}, y_j, ..., y_n) \\ &\geq \sum_{j\neq 1} f(y_1, x_2, ..., x_j, y_{j+1}, ..., y_n) - f(y_1, x_2, ..., x_{j-1}, y_j, ..., y_n) \\ &= f(y_1, x_{-1}) - f(y_1, y_{-1}) \end{aligned}$$

where the inequality follows from pairwise supermodularity on (x_1, y_j) and (y_1, x_j). After rearrangement the inequality we just got implies that

$$f(x_1, x_{-1}) - f(y_1, x_{-1}) \geq f(x_1, y_{-1}) - f(y_1, y_{-1})$$

Now take any $x, y \in \mathbb{R}^n$.

$$\begin{aligned} f(x \vee y) - f(y) &= \sum_i f(x_1 \vee y_1, ..., x_i \vee y_i, y_{i+1}, ...y_n) - f(x_1 \vee y_1, ..., x_{i-1} \vee y_{i-1}, y_i, ...y_n) \\ &\geq \sum_i f(x_1 \vee y_1, ..., x_{i-1} \vee y_{i-1}, x_i, y_{i+1}, ...y_n) - f(x_1 \vee y_1, ..., x_{i-1} \vee y_{i-1}, x_i \wedge y_i, y_{i+1}, ...y_n) \\ &\geq \sum_i f(x_1, ..., x_{i-1}, x_i, x_{i+1} \wedge y_{i+1}, ..., x_n \wedge y_n) - f(x_1, ..., x_{i-1}, x_i \wedge y_i, x_{i+1} \wedge y_{i+1}, ..., x_n \wedge y_n) \\ &= f(x) - f(x \wedge y) \end{aligned}$$

where the first inequality follows from parirwise supermodularity on (x_i, z) and (y_i, z) where z is any entry in the vectors above and the second from applying our above result on pairwise supermodularity.

Now, W.O.L.G we assume that there are two vectors $y' = (y'_i, y'_j, y_{-ij})$ and $y'' = (y''_i, y''_j, y_{-ij})$ such that $y'_i \leq y''_i$ and $y'_j \leq y''_j$,

$$\begin{aligned} & f(y''_i, y''_j, y_{-ij}) - f(y'_i, y''_j, y_{-ij}) \geq f(y''_i, y'_j, y_{-ij}) - f(y'_i, y'_j, y_{-ij}) \\ \iff & \int_{y'_i}^{y''_i} \frac{\partial}{\partial y_i} f(y_i, y''_j, y_{-ij}) dy_i \geq \int_{y'_i}^{y''_i} \frac{\partial}{\partial y_i} f(y_i, y'_j, y_{-ij}) dy_i \\ \iff & 0 \leq \int_{y'_i}^{y''_i} \frac{\partial}{\partial y_i} f(y_i, y''_j, y_{-ij}) - f(y_i, y'_j, y_{-ij}) dy_i = \int_{y'_i}^{y''_i} \int_{y'_j}^{y''_j} \frac{\partial^2}{\partial y_i \partial y_j} f(y_i, y_j, y_{-ij}) dy_j dy_i \end{aligned}$$

From the above, it is immediate that if $\frac{\partial^2}{\partial y_i \partial y_j} f(y_i, y_j, y_{-ij}) \geq 0$ for any i, j, then f is pairwise supermodular.

To see the converse holds, suppose — for the purpose of contradiction — that at some $\hat{y}$, $\frac{\partial^2}{\partial y_i \partial y_j} f(y_i, y_j, y_{-ij})|_{y=\hat{y}} < 0$. Then, in an open ball of radius $\epsilon > 0$ around $\hat{y}$, the cross-partial derivative is still strictly negative. That is, $\forall y : ||y - \hat{y}||_2 < \epsilon$, $\frac{\partial^2}{\partial y_i \partial y_j} f(y_i, y_j, y_{-ij}) < 0$.

Then for any $y_i \in [\hat{y}_i - \epsilon/2, \hat{y}_i + \epsilon/2]$ $y_j \in [\hat{y}_j - \epsilon/2, \hat{y}_j + \epsilon/2]$, $d((y_i, y_j, \hat{y}_{-ij}), \hat{y}) = \sqrt{(y_i - \hat{y}_i)^2 + (y_j - \hat{y}_j)^2} \leq \sqrt{\epsilon^2/2} < \epsilon$. Let $y'_\ell = \hat{y}_\ell - \epsilon/2$, $y''_\ell = \hat{y}_\ell + \epsilon/2$ for $\ell = i, j$. We then obtain that

$$\begin{aligned} & 0 > \int_{y'_i}^{y''_i} \int_{y'_j}^{y''_j} \frac{\partial^2}{\partial y_i \partial y_j} f(y_i, y_j, y_{-ij}) dy_j dy_i \\ \iff & f(y''_i, y''_j, y_{-ij}) - f(y'_i, y''_j, y_{-ij}) < f(y''_i, y'_j, y_{-ij}) - f(y'_i, y'_j, y_{-ij}). \end{aligned}$$

(iv) Suppose not, that is, $\exists x, x' \in X$ such that $g(x \vee x') + g(x \wedge x') < g(x) + g(x')$. Then, by definition of g, there are $y, y' \in Y$ such that $g(x \vee x') + g(x \wedge x') < f(x, y) + f(x', y')$. And again, by definition of g, $g(x \vee x') + g(x \wedge x') \geq f(x \vee x', y \vee y') + f(x \wedge x', y \wedge y')$. Hence, if g is not supermodular, there are $(x, y), (x', y') \in X \times Y$ such that $f(x \vee x', y \vee y') + f(x \wedge x', y \wedge y') < f(x, y) + f(x', y')$, implying that f is not supermodular.

Exercise 4

(i): Suppose both $x_1, x_2 \in x^*(e)$. For all $\lambda \in [0,1]$ we have $U(e-(\lambda x_1+(1-\lambda)x_2), f(\lambda x_1+(1-\lambda)x_2)) \geq U(e-(\lambda x_1+(1-\lambda)x_2), \lambda f(x_1)+(1-\lambda)f(x_2)) = U((\lambda+1-\lambda)e-(\lambda x_1+(1-\lambda)x_2), \lambda f(x_1)+(1-\lambda)f(x_2)) = U(\lambda(e-x_1)+(1-\lambda)(e-x_2), \lambda f(x_1)+(1-\lambda)f(x_2)) \geq min\{U(e-x_1, f(x_1)), U(e-x_2, f(x_2))\}$, where the first inequality follows from that f is concave and U is increasing in its second argument and the last inequality holds as U is quasi-concave. Thus $\lambda x_1 + (1-\lambda x_2) \in x^*(e)$.

(ii): Suppose $x_1, x_2 \in x^*(e)$ and $x_1 \neq x_2$. We have $\lambda \in [0,1]$ we have $U(e-(\lambda x_1+(1-\lambda)x_2), f(\lambda x_1+(1-\lambda)x_2)) > U(e-(\lambda x_1+(1-\lambda)x_2), \lambda f(x_1)+(1-\lambda)f(x_2)) = U((\lambda+1-\lambda)e-(\lambda x_1+(1-\lambda)x_2), \lambda f(x_1)+(1-\lambda)f(x_2)) = U(\lambda(e-x_1)+(1-\lambda)(e-x_2), \lambda f(x_1)+(1-\lambda)f(x_2)) \geq min\{U(e-x_1, f(x_1)), U(e-x_2, f(x_2))\}$. In this case the first inequality becomes strict since f is strict concave and U is strictly increasing. Thus we have $U(e-(\lambda x_1+(1-\lambda)x_2), f(\lambda x_1+(1-\lambda)x_2)) > U(e-x_1, f(x_1)), U(e-x_2, f(x_2))$, which is a contradiction. Thus $x_1 = x_2$.

(iii): All we want to do is applying corollary 3, whcih means we want to check all the conditions in corollary are satisfied. Set dominance and quasi-supermodular are trivially met. So we focus on single-crossing dominance.

Let $V(x,e) = U(e-x, f(x))$ be the utility function. Also let $x_1 > x_2$ and $e_1 > e_2$. Suppose that $U(e_2-x_1, f(x_1)) \geq U(e_2-x_2, f(x_2))$, we want to show that $U(e_1-x_1, f(x_1)) \geq U(e_1-x_2, f(x_2))$. Consider the following integral:

$$\int_{e_1}^{e_2}\int_{x_2}^{x_1} \frac{\partial^2}{\partial e \partial x} V(x,e)dxde = \int_{e_1}^{e_2} (\frac{\partial V}{\partial e}|_{x_1} - \frac{\partial V}{\partial e}|_{x_2})de \tag{1}$$

$$= [V(x_1,e_1) - V(x_2,e_1)] - [V(x_1,e_2) - V(x_2,e_2)] \tag{2}$$

By chain rule we get $\frac{\partial^2}{\partial e \partial x}V(x,e) = -\frac{\partial^2 U}{\partial^2 g} + \frac{\partial^2 U}{\partial g \partial f}\frac{\partial f}{\partial x} \geq 0$, which means the integral in (1) is non-negative. Thus the difference in (2) is also non-negative, which implies the single-crossing dominance.

ECON0106: **Microeconomics**

Problem Set 3

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University College London

Due date: 24 October, 12:30

Question 1. 3. Optimal Choice and Consumer Theory: Exercise 2

Question 2. 3. Optimal Choice and Consumer Theory: Exercise 5

Question 3. 4. Monotone Comparative Statics of Individual Choices: Exercise 2

Question 4. 4. Monotone Comparative Statics of Individual Choices: Exercise 4

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Problem Set 4 Solution

November 4, 2022

Lecture 5

Exercise 1

(i) (a) Archi does not imply vNM continuity nor independence.

Let $X = \{0, 1\}$. Abusing notation, identify $p \in \Delta X$ with $Pr[X = 1]$. Define the preferences as follow:

$$\begin{aligned} &p, q \notin \mathbb{Q} : && p \succeq q \Leftrightarrow p \geq p' \\ &p, q \in \mathbb{Q} : && p \sim q \\ &p \notin \mathbb{Q}, q \in \mathbb{Q} : && p \succ q \end{aligned}$$

Archi: Let $p \succ p' \succ p''$. Two cases: $p, p', p'' \notin \mathbb{Q}$ and $p > p' > p''$ or $p'' \in \mathbb{Q}$ and $p, p' \notin \mathbb{Q}$ and $p > p'$.

We are going to use the property that for any open interval (a, b), there exists $x \in \mathbb{Q} \cap (a, b)$ and $y \in (a, b) \setminus \mathbb{Q}$.

First case, one can find $\alpha p + (1 - \alpha)p'' \in (p', p) \setminus \mathbb{Q}$ and $\beta p + (1 - \beta)p'' \in (p'', p) \cap \mathbb{Q}$.

Second case, if $p'' > p$ then one can find $\alpha p + (1 - \alpha)p'' \in (p, p'') \setminus \mathbb{Q}$ and $\beta p + (1 - \beta)p'' \in (p, p'') \cap \mathbb{Q}$.

If $p'' < p$ then one can find $\alpha p + (1 - \alpha)p'' \in (p', p) \setminus \mathbb{Q}$ and $\beta p + (1 - \beta)p'' \in (p'', p) \cap \mathbb{Q}$.

Not vNM continuous. Take $p'' \in \mathbb{Q}$ and $p, p' \notin \mathbb{Q}$ such that $p'' > p > p'$. Then for any $\gamma \in [0, 1]$, $\gamma p + (1 - \gamma)p'' > p'$. If the LHS is in $\mathbb{Q}$, then $\gamma p + (1 - \gamma)p'' \prec p'$ and if not then $\gamma p + (1 - \gamma)p'' \succ p'$.

Not independent. Take $p, p' \notin \mathbb{Q}$ and $p > p'$. Then $p \succ p'$ but we can find α such that $\alpha p + (1 - \alpha)p' \in \mathbb{Q}$ and therefore $\alpha p + (1 - \alpha)p' \prec p'$.

vNM continuity does not imply Archi or independence.

Let $x = \{0, 1, 2\}$. Define the preferences as follows

$$\delta_2 \succ \delta_1 \succ \delta_0$$
$$\text{for any } p \neq \delta_x,\ p \sim \delta_1$$

vNM continuity: $\delta_2 \succ \delta_1 \succ \delta_0$, any $\alpha\delta_2 + (1 - \alpha)\delta_0 \sim \delta_1$.

There is no $p \preceq \delta_0$ or $p \succeq \delta_2$. So the other possibilities are

$$\delta_2 \succ q \succeq p$$
$$p \succeq q \succ \delta_0$$
$$p \succeq q \succeq r$$

where $p, q, r \in \Delta X \setminus \{\delta_2, \delta_0\}$. Any mixture makes the DM indifferent with q.

Not Archi: The only p, p', p'' such that $p \succ p' \succ p''$ are $\delta_2 \succ \delta_1 \succ \delta_0$. But any mixture $\alpha p + (1 - \alpha)p'' \sim \delta_1$. Therefore the Archimedean property cannot hold.

Not independent. Again take $\delta_2 \succ \delta_1$. For any $\alpha \in [0, 1)$, $\alpha\delta_2 + (1 - \alpha)\delta_1 \sim \delta_1$.

Independence does not imply vNM continuity or Archi:

Let $X = \{0, 1, 2\}$. Define the preferences as follow

$$p \succ p' \Leftrightarrow p(1) > p'(1) \text{ or } p(1) = p'(1) \text{ and } p(2) \geq p'(2)$$

Satisfies independence: This follows from the fact that for any p'' and α,

$$p(x) \geq (>) p'(x) \Rightarrow \alpha p(x) + (1-\alpha)p''(x) \geq (>) \alpha p'(x) + (1-\alpha)p''(x)$$

Not vNM continuous or Archi. Take p, p', p'' such that

$$p(1) > p'(1) = p''(2) \text{ and } p'(2) > p''(2)$$

Therefore $p \succ p' \succ p''$ but for all $\alpha \in [0,1]$, $\alpha p + (1-\alpha)p'' \succ p'$.

(b) First note that using independence, for any $p \succ p'$ and $1 \geq \alpha > \beta \geq 0$, $\alpha p + (1-\alpha)p' \succ \beta p + (1-\beta)p'$. I will call this result the monotonicity lemma.
By independence,

$$\left(\frac{\alpha-\beta}{1-\beta}\right) p + \left[1 - \left(\frac{\alpha-\beta}{1-\beta}\right)\right] p' \succ \left(\frac{\alpha-\beta}{1-\beta}\right) p' + \left[1 - \left(\frac{\alpha-\beta}{1-\beta}\right)\right] p' = p'.$$

Then, again by independence,

$$\begin{aligned} \alpha p + (1-\alpha)p' &= \beta p + (1-\beta)\left[\left(\frac{\alpha-\beta}{1-\beta}\right) p + \left[1 - \left(\frac{\alpha-\beta}{1-\beta}\right)\right] p'\right] \\ &\succ \beta p + (1-\beta)\left[\left(\frac{\alpha-\beta}{1-\beta}\right) p' + \left[1 - \left(\frac{\alpha-\beta}{1-\beta}\right)\right] p'\right] = \beta p + (1-\beta)p' \end{aligned}$$

Now take $p \succeq p' \succeq p''$. If $p \sim p'$ or $p' \sim p''$, then we can take $\gamma = 1$ or 0. Assume $p \succ p' \succ p''$. Let $A = \{\alpha \in [0,1] : \alpha p + (1-\alpha)p'' \succeq p'\}$ and $a = \inf A$. By archimedean property, there is $\beta \in (0,1)$ such that $\beta p + (1-\beta)p'' \prec p'$ and by the monotonicity lemma this holds for all $b \leq \beta$. Therefore $a > 0$.
Suppose that $ap + (1-a)p'' \succ p' \succ p''$. Then by the Archimedean property, there is $\alpha \in (0,1)$ such that

$$\alpha(ap + (1-a)p'') + (1-\alpha)p'' \succ p'$$

Because $\alpha a < a$ ($a \neq 0$), then $a \neq \inf A$.
Suppose now that $ap + (1-a)p'' \prec p' \prec p$. Then again using the Archimedean property, there is $\beta \in (0,1)$:

$$\beta(ap + (1-a)p'') + (1-\beta)p \prec p'$$

Then $\beta a + (1-\beta) > a$ which contradicts that a is greatest lower bound.
Therefore $ap + (1-a)p'' \sim p'$

(c) Take $p \succ p' \succ p''$. By vNM continuity there is $\gamma \in (0,1)$ such that $\gamma p + (1-\gamma)p'' \sim p'$ ($\gamma = 1$ or 0 are excluded because $p \succ p' \succ p''$). Then by the monotonicity lemma, for $\alpha \in (\gamma, 1)$ and $\beta \in (0, \gamma)$,

$$\alpha p + (1-\alpha)p'' \succ p' \succ \beta p + (1-\alpha)p''$$

(ii) Assuming X is finite, this exactly corresponds to the first steps of vNM representation theorem. Note that we also only need independence.

(iii) (a) It implies both and is implied by neither.
To see it implies the Archimedean property, take any $p, p', p'' \in \Delta(X)$ such that $p \succ p' \succ p''$. By continuity, $\exists \epsilon > 0$ such that $\forall q \in B_\epsilon(p), q'' \in B_\epsilon(p'')$, $q \succ p' \succ p''$. As $\Delta(X)$ is convex, there $\exists \alpha \in (0,1)$ such that $\alpha p + (1-\alpha)p'' \in B_\epsilon(p)$ and $(1-\alpha)p + \alpha p'' \in B_\epsilon(p'')$.
To see it implies vNM continuity, use the fact that $\Delta(X)_{\succ x}$ and $\Delta(X)_{x \succ}$ are open and disjoint.
A counterexample to show it is not implied by either:
Fix $x^* \in X$.
Let $\succsim$ be such that $\forall p, p' \in \Delta(X)$, $p \sim p'$ if both $p(x^*), p'(x^*) \in \mathbb{Q}$ or if both $p(x^*), p'(x^*) \notin \mathbb{Q}$, and $p \succ p'$ if $p(x^*) \in \mathbb{Q}$ and $p'(x^*) \notin \mathbb{Q}$.
$\succsim$ satisfies both the Archimedean property and vNM continuity, but it is not continuous.

(b) Neither. Counterexample to "implies":
Fix $x^* \in X$. Let $\succsim$ be represented by $U(p) := (p(x^*) - 1/2)^2$. Let p, p', p'' be such that $p(x^*) = 1/4$ and $p'(x^*) = 1/2$, $p''(x^*) = 1$, $\alpha = 1/2$.
Counterexample to "is implied by": Fix $x^* \in X$. Let $\succsim$ be such that $\delta_{x^*} \succ p \sim p'$ for all $p, p,' \in \Delta(X) \setminus \{\delta_{x^*}\}$. Satisfies independence but fails continuity as $\Delta(X)_{x^* \succsim}$ is not closed.

(c) As independence and vNM continuity are equivalent to there being an expected utility representation, and such representation is continuous (linear) in p, then continuity is implied by independence and vNM continuity.

Exercise 3

(i): From the functional form of the utility function we know that x_1 and x_2 are identical in terms of the utility created for the agent. Thus, to maximise the utility the agent always purchase the good with lower price.

The ex-ante utility in regime 1: When price is $(1, 3)$, the agent spends all her budget on x_1. When price is $(3, 1)$ she spend all her money on x_2. Thus the expected utility is $\alpha f(\frac{w}{1}) + (1-\alpha) f(\frac{w}{1}) = f(w)$.

The ex-ante utility in regime 2: The expected price is $\alpha(1, 3) + (1-\alpha)(3, 1) = (\alpha + (1-\alpha)3, 3\alpha + (1-\alpha)) = (3-2\alpha, 2\alpha+1)$. Again the agent spends all her money on the cheaper good, thus the utility is just $f(\frac{w}{p_w})$, where $p_m = min\{3 - 2\alpha, 2\alpha + 1\}$.

Since $0 < \alpha < 1$ we have $p_m = min\{3 - 2\alpha, 2\alpha + 1\} > 1$, which means $f(\frac{w}{p_w}) < f(w)$.

(ii):No, consider this utility function: $u = x_2$.

The ex-ante utility in regime 1: The agent only buy good 2, so the expected utility is $\alpha \frac{w}{3} + (1-\alpha) w = \frac{\alpha}{3} w$

The ex-ante utility in regime 2: Again the agent only buy good 2, so the utility is $\frac{w}{2\alpha+1}$.

It is easy to see that the utility in regime 2 is always higher.

Lecture 6

Exercise 3

1. Because u is strictly increasing, if there are s, s' such that $u(s) = \mathbb{E}[u(\tilde{x})] = u(s')$ then $s = s'$.

   Similarly if we have $b > b'$, then because u is strictly increasing, $u(x - b) < u(x - b')$ for all realisation x of $\tilde{x}$. Therefore $\mathbb{E}[u(\tilde{x} - b)] < \mathbb{E}[u(\tilde{x} - b')]$ and there is a unique b such that $\mathbb{E}[u(\tilde{x} - b)] = u(0)$.

2. By definition, $c(F, \succsim_0) = s$ and $c(F - b, \succsim_0) = 0$. From Lemma 1, $c(F, \succsim_{-b}) = c(F - b, \succsim_0) + b$ and from Theorem 3, $c(F, \succsim_{-b}) = c(F, \succsim_0)$. Hence, $s = b$.

3. We know that $c(F, \succsim_0) = s$ and $c(F - b, \succsim_0) = 0$. From Lemma 1, $c(F, \succsim_{-b}) = c(F - b, \succsim_0) + b = b$. From Theorem 3, $b \geq 0$ iff $c(F, \succsim_{-b}) \leq c(F, \succsim_0)$ Hence, $b \leq s iff b \geq 0$.

4. (a) Let $\tilde{x} \sim F$. From Lemma 1, $c(F, \succsim_c) = c(F + c, \succsim_0) - c = s_y - c$. From Theorem 3, $c(F, \succsim_c) = c(F, \succsim_0) = s$. Then, $s = s_y - c$.

   (b) Suppose $c \geq 0$ (the argument is symmetric if otherwise). From Theorem 3, $c(F, \succsim_c) \geq c(F, \succsim_0)$ and thus $s_y - c \geq s$.

Exercise 5

1. (a) Take preferences such that they are represented by the utility function $u(z) := (1 + exp(-z))^{-1}$. For $x = 1, y = 1/2$, we get a contradiction.

   (b) Suppose the agent is an expected utility maximiser. Then $F \succeq \delta_{\mu_F}$ and $G^{(2)} \succeq \delta_{\mu_G^{(2)}}$ imply

$$1/2u(x) + 1/2u(-y) \geq u(\frac{x-y}{2})$$

$$\text{and } 1/4u(x) + 1/2u(\frac{x-y}{2}) + 1/4u(-y) \geq u(\frac{x-y}{2})$$

These two inequalities are equivalent and therefore the claim is correct.

2. Because the agent is risk averse and expected utility maximiser, a utility function represents his preferences iff u is concave.

   (a) Let $u(z) := -\exp(-(1 - exp(-\gamma z)))$ and $x = 2, y = 1$. For $\gamma = .24$, $-1.002 \approx \mathbb{E}_{F^{(2)}}[u] < -1 = u(0) < \mathbb{E}_F[u] = -.997$. Therefore the claim is incorrect.

   (b) I will show $\mathbb{E}_F[u] \leq \mathbb{E}_{G^{(2)}}[u]$,

$$1/2u(x) + 1/2u(-y) \leq 1/4u(x) + 1/2u(\frac{x-y}{2}) + 1/4u(-y)$$
$$\Leftrightarrow 1/2u(x) + 1/2u(-y) \leq u(\frac{x-y}{2})$$

   which holds by concavity of u. Therefore $\mathbb{E}_F[u] \geq u(0)$ implies $\mathbb{E}_{G^{(2)}}[u] \geq u(0)$. The claim is correct.

   Note that F is a mean-preserving spread of $G^{(2)}$ and therefore $F \succeq_{SOSD} G^{(2)}$. By definition of SOSD and because u is concave and strictly increasing, the claims follows.

   (c) Let $u(z) := 1 - exp(-\gamma z)$. Note that we can express z as $z = k(x + y) - ny$, where k is distributed according to a binomial distribution with parameters $(n, 1/2)$.

   Let$x = 2, y = 1$. Then, $z = 3k - n$ and

$$\mathbb{E}[u(3k - n)] = \mathbb{E}[1 - exp(-\gamma(3k - n))] = 1 - 2^{-n}\exp(\gamma n)(1 + \exp(-3\gamma))^n.$$

   Moreover, $\frac{d}{dn}E[u(3k - n)] < 0$ for any $\gamma > -\ln 2 + \ln(1 + \sqrt{5})$ and any $n \geq 1$. And, for $n = 1$, $\mathbb{E}[u(3k - n)] < 0$ for any $\gamma > -\ln 2 + \ln(1 + \sqrt{5})$. Thus, we found a counterexample.

   (d) Note that we can express z as $z = (kx - (n - k)y)/n$ where k is distributed according to a binomial distribution with parameters $(n, 1/2)$. As $\mathbb{V}(z) = (x+y)^2/n^2\mathbb{V}(k) = [(x+y)/2]^2/n$, and so the variance is decreasing and converging to zero as n increases.

   This means that the probability of $k/n \in [1/2 - 1/n, 1/2 + 1/n]$ converges to one as n grows. Moreover, as u is twice continuously differentiable, we can write

$$u(\frac{k}{n(x+y)} - y) = u(1/2(x - y)) + u'(1/2(x - y))|k/n - 1/2||x + y| + o(|k/n - 1/2|)$$

   Then, take any n even

$$\mathbb{E}_{G^{(n)}}[u] = P(\frac{k}{n} \in [1/2 - \frac{1}{n}, 1/2 + \frac{1}{n}])[u(\frac{x-y}{2}) + u'(\frac{x-y}{2})|\frac{k}{n} - 1/2||x+y| + o(|\frac{k}{n} - 1/2|)]$$
$$+ P(\frac{k}{n} \notin [1/2 - 1/n, 1/2 + 1/n])E[u|\frac{k}{n} \notin [1/2 - 1/n, 1/2 + 1/n]] \to u(\frac{x-y}{2}).$$

   As u is strictly increasing, $u((x - y)/2) > u(0)$. And then, for any fixed $\succsim$, there is an N such that for any $n > N, G^{(n)} \succ \delta_0$.

ECON0106: **Microeconomics**

Problem Set 4

Duarte Gonçalves*

University College London

Due date: 31 October, 12:30

Question 1. 5. Expected Utility: Exercise 1

Question 2. 5. Expected Utility: Exercise 3

Question 3. 6. Risk Attitudes: Exercise 3

Question 4. 6. Risk Attitudes: Exercise 5

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Problem Set 5 Solution

November 20, 2022

Lecture 7

Exercise 1

1.(i): Given a distribution F it is clear that $F(x) \geq F(x)$ and $F(x) \leq F(x)$, which means $F(x) \sim F(x)$. Thus the relation is reflexive.

1.(ii): Let F, H, G be three distribution. If $E_F(u) \geq E_G(u)$ and $E_G(u) \geq E_H(u)$ for all non-decreasing function u, we must have $E_F(u) \geq E_H(u)$ for all such u. Thus the relation is transitive.

1.(iii): Let F and G be two distributions. Suppose that $F(x) \geq G(x)$ and $F(x) \leq G(x)$ for all x, then we must have $F(x) = G(x)$ for all x, namely $F = G$. Thus the relation is anti-symmetric.

1.(iv): Counter example: Consider the following two discrete distributions over $[0,1]$: $P_F(x = 0) = 1/3, P_F(x = 1/2) = 1/3, P_F(x = 1) = 1/3$ and $P_G(x = 0) = 1/2, P_G(x = 1/2) = 1/10, P_G(x = 1) = 2/5$. Then it is clear that $F(0) = 1/3 < G(0) = 1/2$ but $F(1/2) = 2/3 > G(1/2) = 3/5$, which means we cannot compare F and G. Thus the relation is not complete.

2: It is a partial order

3: It is indeed a lattice. For two arbitrary distributions F and G in $\Delta([0,1])$ we define the following two functions: $H_{F,G}(x) = max\{F(x), G(x)\}$ and $L_{F,G}(x) = min\{F(x), G(x)\}$. I claim that both $H_{F,G}$ and $L_{F,G}$ are probability distributions over $[0,1]$. Let's consider $H_{F,G}$ first. $H_{F,G}(1) = max\{F(1), G(1)\} = max\{1, 1\} = 1$ and as F and G are increasing functions $H_{F,G}$ is also increasing. Finally since both F and G are right continuous $H_{F,G}$ is also right continuous. Hence $H_{F,G}$ is a well-defined distribution. Similarly we can show that $L_{F,G}$ is also an well defined distribution.

It is clear that if there is a distribution Q such that $Q(x) \leq F(x)$ and $Q(x) \leq G(x)$ for all $x \in [0,1]$, then $Q(x) \leq L_{F,G}$ for all x. Similarly if $Q(x) \geq F(x)$ and $Q(x) \geq G(x)$ for all $x \in [0,1]$ we must have $Q(x) \geq H_{F,G}$ for all x. Thus $L_{F,G} = F \vee G$ and $H_{F,G} = F \wedge G$.

It is also a complete lattice. Let S be a subset of $\Delta[0,1]$. We define $H_S(x) = sup\{F(x)|F \in S\}$ and $L_S(x) = inf\{F(x)|F \in S\}$. By a similar argument we could show that both L_S and H_S are well-defined distribution over $[0,1]$. By theorem 1 it is clear that L_S is the supremum and H_S is the infimum.

Exercise 2

(i): This follows from the fact that expectation is linear with respect to probability.

(ii): $G = (F + \omega)$ implies that $G(x) = F(x - \omega)$. Cumulative density function being increasing implies $F(x - \omega) \leq F(x)$ for all x, which means $G(x) \leq F(x)$ for all x. Thus G dominates F.

Exercise 4

(i) Denote $f(\theta|\sum_i x_i)$ the posterior density given $\sum_i x_i$. Using Bayes' rule, we get $f(\theta|\sum_i x_i) = \frac{f(\theta)\theta^{\sum_i x_i}(1-\theta)^{m-\sum_i x_i}}{\int f(\theta)\theta^{\sum_i x_i}(1-\theta)^{m-\sum_i x_i}d\theta}$ and $f(\theta|\sum_i x_i') = \frac{f(\theta)\theta^{\sum_i x_i'}(1-\theta)^{n-\sum_i x_i'}}{\int f(\theta)\theta^{\sum_i x_i'}(1-\theta)^{n-\sum_i x_i'}d\theta}$. Taking the ratio, we get

$$\theta^{\sum_i x_i - \sum_i x_i'}(1-\theta)^{m-n+\sum_i x_i' - \sum_i x_i} \frac{\int f(\theta)\theta^{\sum_i x_i'}(1-\theta)^{n-\sum_i x_i'}d\theta}{\int f(\theta)\theta^{\sum_i x_i}(1-\theta)^{m-\sum_i x_i}d\theta}$$

Because $\sum_i x_i - \sum_i x_i' \geq 0$ and $n \geq m$, this is strictly increasing in θ and therefore $\theta|x_1, ..., x_m \geq_{MLRP} \theta|x_1', ..., x_n'$. This in turn implies $\theta|x_1, ..., x_m \geq_{FOSD} \theta|x_1', ..., x_n'$ and thus we get $E[\theta|x_1, ..., x_m] \geq E[\theta|x_1', ..., x_n']$.

(ii) Taking the ratio $\frac{f(\theta|x_1,...,x_n)}{g(\theta|x_1,...,x_n)}$, we get

$$\theta^{\sum_i x_i - \sum_i x_i}(1-\theta)^{n-n+\sum_i x_i - \sum_i x_i} \frac{f(\theta)}{g(\theta)} \frac{\int g(\theta)\theta^{\sum_i x_i}(1-\theta)^{n-\sum_i x_i}d\theta}{\int f(\theta)\theta^{\sum_i x_i}(1-\theta)^{n-\sum_i x_i}d\theta}$$

This is nondecreasing as $f(\theta)/g(\theta)$ is nondecreasing. Therefore $f|x_1, ..., x_n \geq_{MLRP} g|x_1, ..., x_n$ and $E_F[\theta|x_1, ..., x_n] \geq E_G[\theta|x_1, ..., x_n]$.

Exercise 5

I will drop the superscript n when there is no ambiguity.

Nonnegative follows from $x_i - x_{i-1} = \frac{1}{n}(\overline{x} - \underline{x}) > 0$ and u being nondecreasing.

Nonincreasing: $c_{i+1} - c_i = \frac{u(x_{i+1}) - u(x_i) - u(x_i) + u(x_{i-1})}{\frac{1}{n}(\overline{x} - \underline{x})} \leq 0$. Rearranging we get $1/2u(x_{i+1}) + 1/2u(x_{i-1}) \leq u(x_i)$. Note that $x_i = 1/2x_{i+1} + 1/2x_{i-1}$ therefore this inequality holds by concavity of u.

Exercise 6

(i) We want to show that for all $x \in [\underline{x}, \overline{x}]$, the set of supergradient $\partial u^n(x) \neq \emptyset$.

First we consider the case where $x = x_i^n$ for some i. Since u is concave, we know from theorem 2 in lecture 3 that $\partial u(x) \neq \emptyset$ for all x. Let $c_{x_i^n} \in \partial u(x_i^n)$, then I claim $c_{x_i^n} \in \partial u^n(x_i^n)$. By definition of supergradient we have $u(x) \leq u(x_i^n) + c_{x_i^n}(x_i^n - x)$ for all $x \in [\underline{x}, \overline{x}]$. Moreover by construction $u^n(x) \leq u(x)$ for all x and $u^n(x_i^n) = u(x_i^n)$ for all $i = 1, 2 \ldots n$. Therefore $u^n(x) \leq u(x) \leq u(x_i^n) + c_{x_i^n}(x_i^n - x) = u^n(x_i^n) + c_{x_i^n}(x_i^n - x)$, which implies $u^n(x) \leq u^n(x_i^n) + c_{x_i^n}(x_i^n - x)$ for all x. Thus $c_{x_i^n} \in \partial u^n(x_i^n)$ holds.

Then we consider the case where $x \in [x_i^n, x_{i+1}^n]$ for some i. By construction u^n is linear on $[x_i^n, x_{i+1}^n]$ and the gradient at x is c_i^n. I claim that $c_i^n \in \partial u^n(x)$. Let $y \in [x_j^n, x_{j+1}^n]$ where $j \geq i$. From exercise 5 we know that $c_i^n \geq c_j^n$, which implies that $u^n(x) + c_i^n(y - x) \geq u^n(x) + \frac{u^n(y) - u^n(x)}{y - x}(y - x) = u^n(y)$. Thus $u^n(y) \leq u^n(x) + c_i^n(y - x)$ for all $y \geq x$. A similar argument shows that $u^n(y) \leq u^n(x) + c_i^n(y - x)$ for all $y \leq x$. Thus $c_i^n \in \partial u^n(x)$ holds.

Therefore $\partial u^n(x) \neq \emptyset$ for all $x \in [\underline{x}, \overline{x}]$. By theorem 2 in lecture 3 u^n is concave.

(ii) Suppose $i > k$, this means $x_k^n \leq y \leq x_{k+1}^n < x_i^n \leq x \leq x_{i+1}^n$. Since u^n is concave, we have

$$\begin{aligned}
u^n(x_{k+1}^n) &\geq u^n(y) + \frac{u^n(x) - u^n(y)}{x - y}(x_{k+1}^n - y) \\
\frac{u^n(x_{k+1}^n) - u^n(y)}{x_{k+1}^n - y} &\geq \frac{u^n(x) - u^n(y)}{x - y} \\
c_k^n &\geq \frac{u^n(x) - u^n(y)}{x - y} \\
u^n(y) + c_k^n(x - y) &\geq u^n(x)
\end{aligned}$$

Also by the concavity of u^n, we have the following:

$$\begin{aligned}
u^n(x_i^n) &\geq u^n(x) - \frac{u^n(x)-u^n(y)}{x-y}(x-x_i^n) \\
\frac{u^n(x_i^n)-u^n(x)}{x-x_i^n} &\geq -\frac{u^n(x)-u^n(y)}{x-y} \\
c_i^n &\leq \frac{u^n(x)-u^n(y)}{x-y} \\
u^n(y)+c_i^n(x-y) \leq u^n(x)
\end{aligned}$$

By the similar argument we can show that the same result holds when $i<k$. When $i=k$ it is clear that we have equality as u^n is linear on $[x_i^n, x_{i+1}^n]$. Therefore the desired inequality holds.

(iii) Since c_i^n is non-increasing, we have the following:

$$\begin{aligned}
&c_i^n \leq c_1^n \\
\Rightarrow &\frac{u(x_{i+1}^n)-u(x_i^n)}{x_{i+1}^n-x_i^n} \leq \frac{u(x_1^n)-u(\underline{x})}{x_1^n-\underline{x}} \\
\Rightarrow &u(x_{i+1}^n)-u(x_i^n) \leq u(x_1^n)-u(\underline{x}) \\
\Rightarrow &u(x)-u(x_i^n) \leq u(x_1^n)-u(\underline{x})
\end{aligned}$$

Exercise 7

(i) Take $x \in [x_{i-1}, x_i]$. Then

$$\begin{aligned}
\tilde{u}^n(x) = u(\overline{x}) + \sum_{j=i}^{n} d_j(x-x_j) \\
&= u^n(x_n) + \big(\sum_{j=i}^{n} c_j - \sum_{j=i}^{n-1} c_{j+1}\big)x - \sum_{j=i}^{n} c_j x_j + \sum_{j=i}^{n-1} c_{j+1}x_j \\
&= u^n(x_n) + c_i x - c_i x_i - \sum_{j=i+1}^{n} c_j(x_j - x_{j-1}) \\
&= u^n(x_n) + c_i x - c_i x_i - \sum_{j=i+1}^{n} u^n(x_j) - u^n(x_{j-1}) = u^n(x_i) + c_i(x-x_i) = u^n(x)
\end{aligned}$$

(ii) Because $\int u_a(x)dF \geq \int u_a(x)dG$ for all $a \in [\underline{x}, \overline{x}]$, then for any n, $\int u(\overline{x}) + \sum_i d_i^n u_{x_i^n}(x)dF \geq \int u(\overline{x}) + \sum_i d_i^n u_{x_i^n}(x)dG$ as $x_i^n \in [\underline{x}, \overline{x}]$ and $d_i^n \geq 0$. Therefore $E_F[u^n] \geq E_G[u^n]$.

ECON0106: **Microeconomics**

Problem Set 5

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University College London

Due date: 14 November, 12:30

Question 1. 7. Stochastic Orders: Exercises 1 and 2

Question 2. 7. Stochastic Orders: Exercise 4

Question 3. 7. Stochastic Orders: Exercises 5, 6, and 7 (proof of Theorem 3)

Question 4. 7. Stochastic Orders: Exercise 8

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