Lecture 2: Variational Approach to the Sequence Problem

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Canonical sequence problem

- Recall the canonical problem written in terms of state variables:

$$\text{Problem (SP-N):} \qquad V^*(0, x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(t, x_t, x_{t+1})$$
$$\text{s.t. } x_{t+1} \in \Gamma(t, x_t) \text{ for all } t \geq 0.$$

- Recall also the cake-eating problem in canonical notation,

$$V^*(a_0) = \max_{\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(a_t - a_{t+1})$$
$$\text{s.t. } a_{t+1} \in [0, a_t] \text{ for all } t \geq 0.$$

Where we are going

Our goal is to "characterize" the solution to such problems:

- Find **sufficient** optimality conditions such that, if we find a path that satisfies these conditions, then we know it is the optimal path.

Outline

1. A general sufficiency theorem

2. Application I: Consumption-savings problem
   - FOCs & TVC
   - Characterizing the solution
   - Discussion: Permanent Income Hypothesis

3. Application II: Optimal growth in the neoclassical model

SP-N with minimal structure

Assumption

V.1. *For each t, $F(t, x, y)$ is differentiable in (x, y).*

- Notation: $F_x(t, x, y) = \partial F(t, x, y) / \partial x$ and $F_y(t, x, y) = ...$

Assumption

V.2. *For each t, $F_x(t, x, y) \geq 0$. In addition, $X \subset \mathbb{R}_+$.*

- Having more of the state variable is "good."

Assumption

V.3 (resp V.3$^\mathbf{S}$). *For each t, $F(t, x, y)$ is weakly (resp. strictly) concave (over the set of feasible (x, y)'s defined by $y \in \Gamma(t, x)$).*

- Key assumption. To bring insights from standard optimization.

FOCs: Euler equations

- Note that the sum in the objective function can be written as,

$$\sum_{\tilde{t}=0}^{t-1} \beta^{\tilde{t}} F\left(\tilde{t}, x_{\tilde{t}}, x_{\tilde{t}+1}\right) + \overbrace{\beta^t F\left(t, x_t, x_{t+1}\right) + \beta^{t+1} F\left(t+1, x_{t+1}, x_{t+2}\right)}^{\text{the (only) terms where } x_{t+1} \text{ shows up}} \\ + \sum_{\tilde{t}=t+2}^{\infty} \beta^{\tilde{t}} F\left(\tilde{t}, x_{\tilde{t}}, x_{\tilde{t}+1}\right)$$

- Thus, x_{t+1}^* should maximize $F\left(t, x_t^*, x_{t+1}\right) + \beta F\left(t+1, x_{t+1}, x_{t+2}^*\right)$.
  - "Variational approach": consider possible variations one at a time
- Taking the first order condition, we obtain the **Euler equation**:

$$F_y\left(t, x_t^*, x_{t+1}^*\right) + \beta F_x\left(t+1, x_{t+1}^*, x_{t+2}^*\right) = 0 \text{ for each } t. \tag{1}$$

  - Finite-horizon interpretation: for a T-period problem, these are the first $T-1$ FOCs

FOCs: Transversality condition

- The two period cake-eating version was $\frac{\beta}{a_1 - a_2} a_2 = 0$. Intuition?
  - In the canonical notation, this would be $-\beta F_y(1, x_1, x_2)\, x_2 = 0$.
- Consider the infinite horizon analogue of this condition:

$$\lim_{T \to \infty} -\beta^T F_y\left(T, x_T^*, x_{T+1}^*\right) x_{T+1}^* = 0. \tag{2}$$

- Using Euler equation, this can be equivalently written as:

$$\lim_{T \to \infty} \beta^{T+1} F_x\left(T+1, x_{T+1}^*, x_{T+2}^*\right) x_{T+1}^* = 0.$$

- Relabeling the index $T+1$ as T, we can also write this as,

$$\lim_{T \to \infty} \beta^T F_x\left(T, x_T^*, x_{T+1}^*\right) x_T^* = 0. \tag{3}$$

  - Finite-horizon interpretation: for a T-period problem, this is the last (i.e., the Tth) FOC

A sufficiency theorem

Theorem (Sufficiency)

Consider problem (SP) with assumptions V.1-V.3.

1. *Suppose there is a feasible allocation* $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$, *which is interior (i.e.,* $x_{t+1}^* \in int\left(\Gamma\left(t, x_t^*\right)\right)$ *for each* t*), which yields finite payoff (i.e.,* $\sum_{t=0}^{\infty} \beta^t F\left(t, x_t^*, x_{t+1}^*\right) < \infty$*), and which satisfies Eqs.* (1) *and one of Eqs.* (2) *or* (3). *Then, the allocation is an optimum.*

2. *If furthermore assumption V.3*^{s} *holds, then the allocation is the unique optimum.*

- Finite payoff from candidate is a natural assumption.
    - It will typically hold if the problem is well formulated—otherwise, we can obtain infinite payoff and there is no optimum.
- For proof, see SLP Section 4.5.

Outline

Consumption-savings with infinite horizon

Consider a consumption-savings problem in infinite horizon:

$$
\begin{aligned}
& \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \\
\text{s.t.} \quad & a_{t+1} = a_t(1+r) + w - c_t \text{ for each } t, \\
& c_t \geq 0, a_{t+1} \geq 0 \text{ for each } t, \text{ and given } a_0 > 0.
\end{aligned}
$$

- Consumer with utility function with standard properties:
  - Continuously **differentiable, strictly increasing, strictly concave**.
- Faces constant wage, $w \geq 0$, and interest rate, $r > 0$.
  - We can allow for $r = 0$ when $w = 0$ (the cake eating problem).
  - We can also allow growing w—with more notational complexity.
- Maximizes consumption-savings path subject to no borrowing, $a_{t+1} \geq 0$.

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First order conditions

- The problem can be written as

$$\max_{\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(a_t\left(1+r\right)+w-a_{t+1}\right),$$
$$\text{s.t.} \quad a_{t+1} \in \left[0, a_t\left(1+r\right)+w\right] \text{ for each } t.$$

- Assume u such that the problem satisfies **assumptions V.1-V.3**$^{\text{s}}$.

First order conditions: Euler equation

The Euler equation (1) implies,

$$u'\left(a_t\left(1+r\right)+w-a_{t+1}\right)=\beta\left(1+r\right)u'\left(a_{t+1}\left(1+r\right)+w-a_{t+2}\right). \qquad (4)$$

- Equivalently,

$$\underbrace{u'\left(c_t\right)}_{\text{marginal utility at } t} = \beta\left(1+r\right)\underbrace{u'\left(c_{t+1}\right)}_{\text{marginal utility at } t+1},$$

  where c_t and c_{t+1} are derived from the budget constraint.

- So the equation relates the marginal utility in subsequent periods, adjusting for discounting, β, and the return on investment, $1+r$.

Special case: Isoelastic utility

- To see these trade-offs more clearly, consider a special case that is commonly used in macro, the isoleastic utility function:

$$u(c) = \frac{\varepsilon}{\varepsilon - 1}\left(c^{\frac{\varepsilon-1}{\varepsilon}} - 1\right).$$

- Here, ε is elasticity of intertemporal substitution (EIS).
    - Can also be written as $\frac{c^{1-\rho}-1}{1-\rho}$ with $\rho = 1/\varepsilon$ the relative risk aversion.
- The marginal utility is given by:

$$u'(c) = c^{-1/\varepsilon}.$$

- What happens as $\varepsilon \to \infty, \varepsilon \to 1$, and $\varepsilon \to 0$?

Euler equation with isoelastic utility

- For the special case, the Euler equation becomes:

$$\frac{c_{t+1}}{c_t} = \frac{a_{t+1}(1+r) + w - a_{t+2}}{a_t(1+r) + w - a_{t+1}} = (\beta(1+r))^{\varepsilon}. \tag{5}$$

- Note that consumption growth is positive, $c_{t+1}/c_t > 1$, iff:

$$\beta(1+r) > 1.$$

  - Reducing β (impatience) tends to decrease future consumption relative to current, and increasing r tends to do the opposite. Why?
  - $\beta(1+r)$ captures the net incentives to postpone consumption.

- ε determines how much consumption responds to incentives. E.g. consider the case $\beta(1+r) > 1$ so that consumption grows.
  - Higher EIS $\varepsilon \Longrightarrow$ Consumption grows faster. Why?
  - Lower EIS $\varepsilon \Longrightarrow$ Consumption grows, but more slowly.

First order conditions: Transversality condition

Let us now consider the transversality condition (2), which becomes,

$$\lim_{T \to \infty} \beta^T u' \left(a_T \left(1 + r\right) + w - a_{T+1} \right) a_{T+1} = 0. \tag{6}$$

- Equivalently,

$$\lim_{T \to \infty} \beta^T u' \left(c_T \right) a_{T+1} = 0.$$

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The lifetime budget constraint

- A useful approach is to aggregate budget constraints at individual times into a budget constraint that must hold over the lifetime.
- Recall the budget constraint at time t:

$$c_t + a_{t+1} = a_t\left(1+r\right) + w.$$

- Let's multiply with $\left(\frac{1}{1+r}\right)^t$ and sum over $t \in \{0, 1, .., T\}$. Then:

$$\left(\frac{1}{1+r}\right)^T a_{T+1} + \sum_{t=0}^{T} c_t \left(\frac{1}{1+r}\right)^t = a_0\left(1+r\right) + \sum_{t=0}^{T} w \left(\frac{1}{1+r}\right)^t. \quad (7)$$

The lifetime budget constraint

- As $T \to \infty$, we obtain **the lifetime budget constraint**:

$$\underbrace{\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T} a_{T+1}}_{\text{pdv of wealth left unconsumed}} + \underbrace{\sum_{t=0}^{\infty} c_t \left(\frac{1}{1+r}\right)^{t}}_{\text{pdv of lifetime consumption}} = \underbrace{a_0\,(1+r) + w + \frac{w}{r}}_{\text{pdv of lifetime wealth}}.$$

- For the cake eating ($r = 0, w = 0$), the sum of all cakes eaten and left unconsumed is equal to the initial size of the cake, a_0.

- More generally, the condition also accommodates $w \geq 0$ as well as $r \geq 0$ by evaluating things in present value at time 0.

- Note that the LBC is implied by the BC's at each time t. But it does not necessarily imply the BC's. We will bring back the BC's later to construct the path of assets consistent with our solution.

Characterizing the solution: TVC

- The lifetime budget constraint plus economic intuition suggests optimality requires an alternative transversality condition,

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T} a_{T+1}=0. \tag{8}$$

- But the math so far says the transversality condition is given by,

$$\lim_{T\to\infty}\beta^{T} u'\left(c_T\right) a_{T+1}=0.$$

- As it turns out, the two are the same!
  - Using the Euler equation, $u'\left(c_t\right)=\beta\left(1+r\right)u'\left(c_{t+1}\right)$, we have,

$$u'\left(c_T\right)=\frac{u'\left(c_0\right)}{\left(\beta\left(1+r\right)\right)^{T}}.$$

  - Plugging this into the transversality condition, we obtain (8).

Characterizing the solution: Solving the LBC

- Once we apply the transversality condition in (8), the lifetime budget constraint becomes:

$$\sum_{t=0}^{\infty} c_t \left(\frac{1}{1+r}\right)^t = \left(a_0 + \frac{w}{r}\right)(1+r). \tag{9}$$

- Recall that optimal consumption grows at constant rate, so that:

$$c_t = c_0 (1+g)^t \text{ for each } t.$$

- So Eq. (9) becomes one equation in one unknown c_0.
- This can be solved with some algebra (that involves geometric sums).

The solution: Optimal consumption path

- Solving the equation, we obtain the initial consumption level,

$$c_0 = (r - g)\left(a_0 + \frac{w}{r}\right).$$

- From here, we can solve for the complete path $\{c_t, a_{t+1}\}_{t=0}^{\infty}$.
- The solution for the consumption path comes from the Euler equation,

$$c_t = c_0 (1 + g)^t .$$

- We lost the assets, a_{t+1}, in the process of switching to the lifetime budget constraint. But we can recover them...

The solution: Optimal asset path

- Recall that assets satisfy the budget constraint

$$c_t + a_{t+1} = a_t(1+r) + w \text{ for each } t.$$

  - Starting with a_0 and using the expression for c_0 we can solve for a_1.
  - We can then use the expression for a_1, c_1 to solve for a_2, and so on.

- This method works in principle but it is algebraically intensive.
- Alternatively, we could use the observation that the future c_t's should have a similar functional form as c_0 in view of the self-similarity of the problem....

The solution: Optimal asset path

- Consider the same problem but starting at time t with assets a_t.
- Following the same steps as above, the solution to the sub-problem is,

$$c_t = (r - g)\left(a_t + \frac{w}{r}\right).$$

- This should coincide with the solution to the original problem.
  - We are already using the ideas from dynamic programming. For now, only to simplify the math. We will explore in greater detail next time.
- The solution we obtained earlier was given by, $c_t = c_0 (1 + g)^t$.
- Combining these, we could obtain a solution for a_t,

$$\begin{aligned} a_t &= \frac{c_t}{r - g} - \frac{w}{r} \\ &= \frac{c_0 (1 + g)^t}{r - g} - \frac{w}{r} \\ &= \left(a_0 + \frac{w}{r}\right)(1 + g)^t - \frac{w}{r}. \end{aligned}$$

The solution: Summary

- So we have a candidate solution written in closed form,

$$\begin{aligned} c_0 &= (r-g)\left(a_0 + \frac{w}{r}\right) \\ c_t &= c_0 (1+g)^t \\ a_t &= \frac{c_0 (1+g)^t}{r-g} - \frac{w}{r}, \text{ where } 1+g = (\beta(1+r))^{\varepsilon}. \end{aligned} \tag{10}$$

- So far, this is only a candidate. To apply the sufficiency theorem, we also need to check the solution is interior and yields finite value.
    - In particular: we ignored $a_t \geq 0$. Is it violated?
    - Check both interiority and finite value requirement as homework (PS1)

The solution: Applying the sufficiency theorem

Result

Consider the path, $\{c_t, a_{t+1}\}_{t=0}^{\infty}$, characterized above. If the parameters satisfy ...., then the above path is the unique optimum.

- The problem satisfies assumptions V.1-V.3$^{\mathrm{S}}$.
- The path satisfies Euler and transversality by construction.
- The path is interior and yields finite value under the conditions.
- Thus, the result follows from the Sufficiency Theorem.

Outline

Comparative statics: Transitory income

Consider the solution,

$$c_0 = (r - g)\left(a_0 + \frac{w}{r}\right),$$

together with $c_t = c_0\,(1+g)^t$, where $1 + g = (\beta\,(1+r))^{\varepsilon}$.

- Suppose we increase transitory income, a_0, but leave w unchanged.
  - You can imagine a one-time cash windfall (examples?) or a one-time increase in the wage without any change in future income prospects.
- How much does the consumer react to a unit increase in a_0?
  - This is typically very small for commonly used values of r and g.

Comparative statics: Permanent income

- Suppose we increase the permanent wage, w, but leave a_0 unchanged.
  - Can imagine a structural change that permanently raises the wages for this worker (e.g., an increase in skill premium for skilled workers).
- How much does the consumer react to a unit increase in w?
- This is typically much larger than the effect of a_0.
  - In fact, if $g = 0$, the pass through on consumption is one-to-one.

Permanent Income Hypothesis (PIH)

- The special case $g = 0$ summarizes the story (that applies more generally):

$$c_0 = ra_0 + w.$$

  1. Consumption reacts very little to the temporary component of income.
  2. But it reacts considerably (one-to-one) to the permanent income.

- Why? What is the intuition for the result? What is the model saying?
- This is **the permanent income hypothesis** of Friedman (1957).
  Closely related to **the life cycle hypothesis** of Modigliani-Brumberg (1954).

PIH generated considerable empirical research

- The PIH generated a lot of empirical research.
- There is some evidence that supports the qualitative predictions.
  - Consumption seems more reactive to permanent changes in income/taxes.
  - Households seem to be inclined to borrow at young age, when permanent income is typically higher than transitory income.
  - They seem to save and plan for retirement at middle age, when permanent income is typically lower than transitory income.

Violations of the PIH

But the literature has also uncovered major violations of the PIH.

1. For many households, consumption seems to react to transitory income by a much larger magnitude than predicted by the theory.
    - See, e.g., Parker, Souleles, Johnson, McClelland (AER, 2013), "Consumer Spending and the Economic Stimulus Payments of 2008."
2. It also seems to react to permanent income much less than one-to-one.
    - See, e.g., the job market paper of Ludwig Straub.

These are sometimes referred to as respectively "excess sensitivity" and "excess smoothness" puzzles.

Outline

Neoclassical growth model

Next consider the neoclassical growth model:

- Infinite horizon $t \in \{0, 1, ..\}$ and single consumption good.
- Two factors: Capital and labor.
- For simplicity, we assume labor is fixed and we normalize it to 1.
- Capital can be produced (investment) and it depreciates at rate δ:

$$k_{t+1} = (1 - \delta) k_t + f(k_t) - c_t. \tag{11}$$

  - Here $f(k_t)$ denotes the "neoclassical" per-labor production function.

Neoclassical production function

- Consider a continuous and twice differentiable function, $F\left(k_t, l_t\right)$.
- We say F is a neoclassical production function if it satisfies:
  1. Constant returns to scale, or linear homogeneity:
  $$F(\gamma k_t, \gamma l_t) = \gamma F(k_t, l_t), \quad \text{for each } \gamma \geq 0.$$
  2. Positive and diminishing marginal products for each $k_{t,} l_t \geq 0$:
  $$\frac{\partial F(k_t, l_t)}{\partial k_t} > 0, \quad \frac{\partial F(k_t, l_t)}{\partial l_t} > 0,$$
  $$\frac{\partial^2 F(k_t, l_t)}{\partial k_t^2} < 0, \quad \frac{\partial^2 F(k_t, l_t)}{\partial l_t^2} < 0.$$
  3. Inada conditions:
  $$\lim_{k_t \to 0} \frac{\partial F(k_t, l_t)}{\partial k_t} = \lim_{l_t \to 0} \frac{\partial F(k_t, l_t)}{\partial l_t} = \infty,$$
  $$\lim_{k_t \to \infty} \frac{\partial F(k_t, l_t)}{\partial k_t} = \lim_{l_t \to \infty} \frac{\partial F(k_t, l_t)}{\partial l_t} = 0.$$

Neoclassical per-labor production function

- We also define the per-labor production function as

$$f(k_t) = F(k_t, 1).$$

- Constant returns to scale assumption implies

$$F(k_t, l_t) = l_t f(k_t).$$

- So the general behavior of F is summarized by, $f(k_t) \equiv F(k_t, 1)$.
- Note that $f(k_t)$ is neoclassical if it satisfies $f'(k_t) > 0, f''(k_t) < 0$ for each $k_t \geq 0$, and $\lim_{k \to 0} f'(k) = \infty, \lim_{k \to \infty} f'(k) = 0$.
- A function that satisfies these conditions is $f(k) = k^\alpha$ for $\alpha \in (0, 1)$.

Neoclassical per-labor production function

- We also define the total resources available after production and depreciation as,

$$\phi(k_t) = f(k_t) + (1-\delta)\, k_t.$$

- With this definition, the resource constraint (11) becomes,

$$c_t + k_{t+1} = \phi(k_t).$$

- So this helps us to economize on notation.

Optimal growth problem

- Representative household with standard preferences $\sum_{t=0}^{\infty} \beta^t u(c_t)$.
    - $u(c)$ has the usual properties: $u'(c) > 0, u''(c) < 0$ for each $c > 0$.
    - We also assume the Inada condition: $\lim_{c \to 0} u'(c) = \infty$.
- Consider a (hypothetical) planner that solves:

$$
\begin{aligned}
& \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & c_t + k_{t+1} = \phi(k_t) \text{ for each } t, \\
& c_t, k_{t+1} \geq 0, \text{ and given } k_0.
\end{aligned}
$$

- This is known as the optimal (neoclassical) growth problem.

FOCs for the optimal growth problem

- Problem can be written as:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(\phi\left(k_t\right) - k_{t+1}\right)$$

$$k_{t+1} \in \left[0, \phi\left(k_t\right)\right] \text{ for each } t.$$

- Check that **assumptions V.1-V.3**$^{\mathrm{S}}$ are satisfied.
- The Euler equation is:

$$u'\left(\phi\left(k_t\right) - k_{t+1}\right) = \beta \phi'\left(k_{t+1}\right) u'\left(\phi\left(k_{t+1}\right) - k_{t+2}\right) \text{ for each } t.$$

- The transversality condition is:

$$\lim_{T \to \infty} \beta^T u'\left(\phi\left(k_T\right) - k_{T+1}\right) k_{T+1} = 0.$$

Steady-state and optimality

- Consider a steady-state, $\{k_t = k^*\}_{t=0}^{\infty}$, that satisfies the Euler equation.
- The steady-state is unique (why?) and characterized by:

$$\begin{aligned} \beta\phi'(k^*) &= 1, \\ \text{equivalently, } \beta\left(f'(k^*) + 1 - \delta\right) &= 1. \end{aligned}$$

- Check that the steady-state also satisfies the TVC. Then:

Result

If $k_0 = k^*$, then the unique optimum features $k_t = k^*$ for each t.

Transitional dynamics and optimality

- Now suppose we start with some $k_0 \neq k^*$. Then:

Claim

Given any $k_0 > 0$, there exists a unique path, $\{k_{t+1}\}_{t=0}^{\infty}$, such that the Euler equation holds for each t and $\lim_{t\to\infty} k_t = k^*$. If $k_0 < k^*$, then k_{t+1} is monotonically increasing. If $k_0 > k^*$, then k_t is monotonically decreasing.

- Can you establish this graphically?

Taking Stock

Taking stock: A general sufficiency theorem

A sufficiency theorem using a standard/variational approach:

- Exploits concavity of the objective function.
- Applies under relatively weak assumptions:
  - Straightforward to state the conditions. The real challenge is to find the path.
  - This can be done in some economic applications.
- Applies for nonstationary as well as stationary problems.
- But does not quite exploit the recursive structure for the latter.

Next lecture(s): Dynamic programming approach for stationary problems.