Question 1: Nash Equilibrium with Stochastic Choice

(i) We have to show that

$$\sigma_i \text{ admits an APU representation } \implies \sigma_i \text{ satisfies Ordinal IIA.}$$

First, note that for any finite set S_i, $\Delta_i(S_i)$ is a compact and convex subset of the Euclidean space.
Second, the objective function

$$F\left(p_i(s_{i,1}), ..., p_i(s_{i,|S_i|}), \right) = \sum_{s_i \in S_i} p_i(s_i)u_i(s_i) - c_i\left(p_i(s_i)\right)$$

is continuous and strictly concave: the first term of the summand is linear while the second term is strictly concave since $c_i\left(p_i(s_i)\right)$ is strictly convex.
Hence

$$\arg\max_{p_i} F\left(p_i(s_{i,1}), ..., p_i(s_{i,|S_i|}), \right)$$

is a singleton and is characterised by the first order condition (since c_i is continuously differentiable):

$$u_i(s_i) - c_i'\left(\sigma_i(s_i|S_i)\right) = 0 \quad \forall s_i \in S_i.$$

Then for any $s_i, s_i' \in S_i \cap S_i'$, we have

$$c_i'\left(\sigma_i(s_i|S_i)\right) - c_i'\left(\sigma_i(s_i|S_i')\right) = c_i'\left(\sigma_i(s_i'|S_i)\right) - c_i'\left(\sigma_i(s_i'|S_i')\right) = 0$$

which implies

$$c_i'\left(\sigma_i(s_i|S_i)\right) - c_i'\left(\sigma_i(s_i'|S_i)\right) = c_i'\left(\sigma_i(s_i|S_i')\right) - c_i'\left(\sigma_i(s_i'|S_i')\right)$$

Composing by the exponential function, we have:

$$\frac{\exp\left(c_i'\left(\sigma_i(s_i|S_i)\right)\right)}{\exp\left(c_i'\left(\sigma_i(s_i'|S_i)\right)\right)} = \frac{\exp\left(c_i'\left(\sigma_i(s_i|S_i')\right)\right)}{\exp\left(c_i'\left(\sigma_i(s_i'|S_i')\right)\right)}$$

Then, defining $\phi := \exp \circ c_i'$ we have

$$\frac{\phi\left(\sigma_i(s_i|S_i)\right)}{\phi\left(\sigma_i(s_i'|S_i)\right)} = \frac{\phi\left(\sigma_i(s_i|S_i')\right)}{\phi\left(\sigma_i(s_i'|S_i')\right)}.$$

It just remains to show that ϕ satisfies the stated conditions.

- $\phi(0) = \lim_{x\to 0} \exp(c_i'(x)) = 0$ and it is clearly non-negative;
- ϕ is continuous since it is the composition of two continuous functions (c_i is continuously differentiable by assumption).
- Finally, since $c'(x)$ is strictly increasing since it is continuous and c is convex. Moreover, exp is also strictly increasing.

Hence we proved that σ_i satisfies Ordinal IIA.

(ii) Define the auxiliary game $\tilde{\Gamma} := \langle I, \Sigma, \tilde{u}\rangle$ where $\Sigma = \times_{i\in I}\Sigma_i$ and $\Sigma_i = \Delta(S_i)$. Moreover, Σ_i is a compact and convex subset of the Euclidean space $\mathbb{R}^{|S_i|}$ since S_i is assumed to be finite. The utility function $\tilde{u}_i$ is defined as

$$\tilde{u}_i := \sum_{s_i\in S_i}\sum_{s_{-i}\in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}) - \sum_{s_i\in S_i} c_i(\sigma_i(s_i)).$$

Clearly $\tilde{u}$ is continuous in σ (it is linear) and concave in σ_i by the convexity of c_i. Thus u_i is quasi-concave in σ_i. The requirements of Theorem 3, page 21 of the Game theory lecture notes by Kartik, are satisfied. There exists a PSNE of $\tilde{\Gamma}$.
Let such a PSNE be $p^* = (p_i^*)_{i\in I}$ i.e. $\forall i \in I$

$$p_i^* \in \arg\max_{p_i\in\Sigma_i} u_i(p_i, p_{-i}^*) - \sum_{s_i\in S_i} c_i\big(\sigma_i^*(s_i)\big).$$

Hence p^* is an NE with APU of the APU game.

(iii) What can we say about the set of Nash equilibrium with additive perturbed utility of when $u_i(1,1)$ increases?

To answer this question, we apply Corollary 2 from lecture notes 14. on Monotone Comparativie Statics in Games, which states the following:

Corollary 2: Let $\Gamma = \langle I, X, u^*\rangle$ and $\tilde{\Gamma} = \langle I, \tilde{X}, \tilde{u}^*\rangle$ be two normal-form games such that, for each $i \in I$,

(a) $X_i, \tilde{X}_i$ are compact, complete sublattices of an Euclidean space, such that $\tilde{X}_i \geq_{ss} X_i$,

(b) $u_i^*, \tilde{u}_i^*$ are continuous and quasisupermodular, and

(c) $\tilde{u}_i^*$ single-crossing dominates u_i^*

then

- the set of Nash equilibria of each game is a nonempty complete lattice,
- the largest (smallest) Nash equilibrium of $\tilde{\Gamma}$ is greater than the largest (smallest) Nash equilibrium of Γ.

In our case, the two games are given by $\Gamma = \langle I, X, u\rangle$ and $\tilde{\Gamma} = \langle I, X, \tilde{u}\rangle$, where $\tilde{\Gamma}$ is the game with the increased payoff for player i when $(1,1)$ is played. X_i is the set of mixed strategies for player i. In our case, a mixed strategy is completely characterized by a number in $[0,1]$, hence $\sigma_i \in X_i = [0,1]$, the probability with which action 1 is played. Clearly, X_i compact. Moreover, it is also a complete sublattice of $\mathbb{R}$. Finally, $X_i \geq_{ss} X_i$ holds trivially. Thus, we have that the first set of conditions on the set of strategies are fulfilled.

Denote the strategies of i and j by $p = \sigma_i$ and $q = \sigma_j$. Then, we can write u_i^*, $\tilde{u}_i^*$ as follows:

$$\begin{aligned} u_i^*(p,q) = {} & pq\, u_i(1,1) + (1-p)q\, u_i(0,1) + p(1-q)\, u_i(1,0) + \\ & (1-p)(1-q)\, u_i(0,0) - (c_i(p) + c_i(1-p)) \end{aligned} \tag{1}$$

$$\begin{aligned} \tilde{u}_i^*(p,q) = {} & pq\, \tilde{u}_i(1,1) + (1-p)q\, \tilde{u}_i(0,1) + p(1-q)\, \tilde{u}_i(1,0) + \\ & (1-p)(1-q)\, \tilde{u}_i(0,0) - (c_i(p) + c_i(1-p)) \end{aligned} \tag{2}$$

Thus, we also have that u_i^* and $\tilde{u}_i^*$ are linear in σ[1] and thus continuous.

[1] This follows from the fact that u_i^* and $\tilde{u}_i^*$ are linear in p and q.

For quasimodularity, we use the equivalence $f \in C^2$ in $y \in Y$, then f is supermodular in y if and only if $\frac{\partial^2}{\partial y_i \partial y_j} f \geq 0, \forall i \neq j$ and the fact that supermodularity implies quasisupermodularity. In particular,

$$\frac{\partial^2 u_i^*(p,q)}{\partial y_i \partial y_j} = \underbrace{u_i(1,1) - u_i(0,1)}_{>0} + \underbrace{u_i(0,0) - u_i(1,0)}_{>0}$$

Note that the inequalities above come from the matching assumptions in the question. Thus, u_i^* is supermodular $\implies$ u_i^* is quasisupermodular. And following the same logic, $\tilde{u}_i^*$ is quasisupermodular.

Lastly, we need to show that $\tilde{u}_i^*$ single-crossing dominates u_i^*. Note that $\tilde{u}_i$ differs from u_i in that $\tilde{u}_i(1,1) > u_i(1,1)$. Then using (1) and (2), we can rewrite $\tilde{u}_i^*(p,q) = u_i^*(p,q) - pq\, u_i(1,1) + pq\, \tilde{u}_i(1,1)$. Then for any given (p', q') s.t. $(p', q') \geq (p,q)$,

$$\begin{aligned}
\tilde{u}_i^*(p', q') - \tilde{u}_i^*(p,q) &= u_i^*(p', q') - p'q'\, u_i(1,1) + p'q'\, \tilde{u}_i(1,1) - u_i^*(p,q) + pq\, u_i(1,1) - pq\, \tilde{u}_i(1,1) \\
&= u_i^*(p', q') - u_i^*(p,q) + \left(\underbrace{\tilde{u}_i(1,1) - u_i(1,1)}_{>0} \right) \underbrace{(p'q' - pq)}_{\geq 0} \qquad (3)
\end{aligned}$$

Clearly, $\tilde{u}_i^*(p', q') - \tilde{u}_i^*(p,q) > (\geq) 0$ if $u_i^*(p', q') - u_i^*(p,q) > (\geq) 0$ and we have shown that $\tilde{u}_i^*$ single-crossing dominates u_i^*. Therefore, we fulfill all the requirements of Corollary 2 and thus can apply it to obtain that the largest (smallest) Nash equilibrium of $\tilde{\Gamma}$ is greater than the largest (smallest) Nash equilibrium of Γ. Thus, we have that set of NE of the game $\tilde{\Gamma}$ weak set dominates the set of NE of the game Γ.

(iv) First, note that since $c_i : [0,1] \longrightarrow \mathbb{R}$ is continuous, $c_i([0,1])$ is a compact subset of $\mathbb{R}$ which implies that there is an $A, B \in \mathbb{R}$ such that $A \leq c_i(x) \leq B$ for any $x \in [0,1]$ and any $i \in I$.