ECON0108 2022-23 Part 1

Slides for Lecture 3

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Complete models

- First study some **complete models**.
  - Complete models admit only structures which deliver a **unique** value of outcomes given values of observed and unobserved exogenous variables.
- Examples -
  - single equation models with a single endogenous variable,
  - simultaneous equations models delivering unique solutions for endogenous variables,
  - models of strategic interaction with unique solution.
- We will consider,
  - recursive, "triangular" equation systems - popular in microeconometrics - control function methods.
  - closely related models incorporating conditional independence restrictions, and treatment effect models,

Triangular models

- Consider **complete** simultaneous equations models for **endogenous** outcomes X and Y with **triangular** structure

$$\begin{gathered} Y = h(X, U) \\ X = g(Z, V) \end{gathered}$$

"triangular" because endogenous Y does not appear in the equation for endogenous X.

  - In a linear simultaneous equations model with this recursive structure

$$\begin{aligned} Y &= \alpha + \beta X + U \\ X &= \gamma + \delta Z + V \end{aligned}$$

the matrix of coefficients on the endogenous variables is **triangular**.

$$\left[\begin{array}{cc} Y & X \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ -\beta & 1 \end{array} \right] = \left[\begin{array}{cc} 1 & Z \end{array} \right] \left[\begin{array}{cc} \alpha & \gamma \\ 0 & \delta \end{array} \right] + \left[\begin{array}{cc} U & V \end{array} \right]$$

Triangular models

- We will study models for **endogenous** outcomes X and Y with structural equations as follows:

  - parametric Gaussian model,

$$Y = \alpha_0 + \alpha_1 X + U$$

  - nonparametric, non-Gaussian additive latent variate model,

$$Y = h(X) + U$$

  - nonparametric, nonadditive latent variate model,

$$Y = h(X, U)$$

    with h monotone in scalar U

Triangular Gaussian model

- Y and X are generated by a triangular Gaussian model:

$$
\begin{aligned}
Y &= \alpha_0 + \alpha_1 X + U \\
X &= g(Z) + V
\end{aligned}
$$

$$
\left[\begin{array}{c} U \\ V \end{array}\right] \sim N\left(\left[\begin{array}{c} 0 \\ 0 \end{array}\right], \left[\begin{array}{cc} \sigma_{uu} & \sigma_{uv} \\ \sigma_{uv} & \sigma_{vv} \end{array}\right]\right) \qquad \left[\begin{array}{c} U \\ V \end{array}\right] \perp\!\!\!\perp Z
$$

Triangular Gaussian model

- Y and X are generated by a triangular Gaussian model:

$$\begin{aligned} Y &= \alpha_0 + \alpha_1 X + U \\ X &= g(Z) + V \end{aligned}$$

$$\begin{bmatrix} U \\ V \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{uu} & \sigma_{uv} \\ \sigma_{uv} & \sigma_{vv} \end{bmatrix} \right) \qquad \begin{bmatrix} U \\ V \end{bmatrix} \perp\!\!\!\perp Z$$

- Conditional distribution of U given V and Z

$$U|V = v, Z = z \sim N\left(\frac{\sigma_{uv}}{\sigma_{vv}} v, \sigma_{uu} - \frac{\sigma_{uv}^2}{\sigma_{vv}} \right)$$

Triangular Gaussian model

- Y and X are generated by a triangular Gaussian model:

$$\begin{aligned} Y &= \alpha_0 + \alpha_1 X + U \\ X &= g(Z) + V \end{aligned}$$

$$\begin{bmatrix} U \\ V \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{uu} & \sigma_{uv} \\ \sigma_{uv} & \sigma_{vv} \end{bmatrix} \right) \qquad \begin{bmatrix} U \\ V \end{bmatrix} \perp\!\!\!\perp Z$$

- Conditional distribution of U given V and Z

$$U | V = v, Z = z \sim N\left(\frac{\sigma_{uv}}{\sigma_{vv}} v, \sigma_{uu} - \frac{\sigma_{uv}^2}{\sigma_{vv}} \right)$$

- The expected value of Y given V and Z.

$$E[Y | V = v, Z = z] = \alpha_0 + \alpha_1 \left(g(z) + v \right) + \frac{\sigma_{uv}}{\sigma_{vv}} v$$

Given $Z = z$, there is $V = v$ if and only if $X = x \equiv g(z) + v$, so:

$$E[Y | X = x, Z = z] = \alpha_0 + \alpha_1 x + \frac{\sigma_{uv}}{\sigma_{vv}} \left(x - g(z) \right)$$

Triangular Gaussian model

- We have shown structures admitted by this model have:

$$E[Y|X=x,Z=z]=\alpha_0+\alpha_1 x+\frac{\sigma_{uv}}{\sigma_{vv}}(x-g(z)) \qquad E[X|Z=z]=g(z)$$

so

$$\begin{aligned}E[Y|X=x,Z=z] &= \alpha_0+\alpha_1 x+\frac{\sigma_{uv}}{\sigma_{vv}}(x-E[X|Z=z]) \\ &= \alpha_0+\left(\alpha_1+\frac{\sigma_{uv}}{\sigma_{vv}}\right)x-\frac{\sigma_{uv}}{\sigma_{vv}}E[X|Z=z]\end{aligned}$$

Triangular Gaussian model

- We have shown structures admitted by this model have:

$$E[Y|X=x,Z=z] = \alpha_0 + \alpha_1 x + \frac{\sigma_{uv}}{\sigma_{vv}}(x - g(z)) \qquad E[X|Z=z] = g(z)$$

so

$$\begin{aligned} E[Y|X=x,Z=z] &= \alpha_0 + \alpha_1 x + \frac{\sigma_{uv}}{\sigma_{vv}}(x - E[X|Z=z]) \\ &= \alpha_0 + \left(\alpha_1 + \frac{\sigma_{uv}}{\sigma_{vv}}\right)x - \frac{\sigma_{uv}}{\sigma_{vv}}E[X|Z=z] \end{aligned}$$

- Consider two values z' and z'' and define

$$x' = E[X|Z=z'] \qquad x'' = E[X|Z=z'']$$

and note that

$$\begin{aligned} E[Y|X &= x', Z = z'] = \alpha_0 + \alpha_1 x' \\ E[Y|X &= x'', Z = z''] = \alpha_0 + \alpha_1 x'' \end{aligned}$$

and so

$$\alpha_1 = \frac{E[Y|X=x'',Z=z''] - E[Y|X=x',Z=z']}{E[X|Z=z''] - E[X|Z=z']}$$

Triangular Gaussian model: analogue estimation

- Structures admitted by this model have:

$$\begin{aligned} E[Y|X=x,Z=z] &= \alpha_0 + \alpha_1 x + \frac{\sigma_{uv}}{\sigma_{vv}}\left(x - E[X|Z=z]\right) \\ &= \alpha_0 + \left(\alpha_1 + \frac{\sigma_{uv}}{\sigma_{vv}}\right)x - \frac{\sigma_{uv}}{\sigma_{vv}}E[X|Z=z] \end{aligned}$$

- Analogue estimation can be done by:
  - calculating an estimate $E[\widehat{X|Z=}z]$ then
  - estimating e.g. by OLS the coefficients in

$$Y_i = \alpha_0 + \left(\alpha_1 + \frac{\sigma_{uv}}{\sigma_{vv}}\right)X_i - \frac{\sigma_{uv}}{\sigma_{vv}}E[\widehat{X|Z=}Z_i] + \varepsilon_i, \quad i \in \{1,\dots,N\}$$

or

$$Y_i = \alpha_0 + \alpha_1 X_i + \frac{\sigma_{uv}}{\sigma_{vv}}\left(X_i - E[\widehat{X|Z=}Z_i]\right) + \varepsilon_i, \quad i \in \{1,\dots,N\}$$

Additive latent variable model

- This model imposes the restrictions:
  - X and Y are generated by

$$\begin{aligned} Y &= h(X) + U \\ X &= g(Z) + V \end{aligned}$$

  - and

$$E[U|V = v, Z = z] = c(v) \qquad E[V|Z = z] = 0$$

  - for non-Gaussian models $c(\cdot)$ is generally nonlinear.

- The linear model with $h(x) = x'\alpha$ and the semiparametric **index** model with $h(x) = \tilde{h}(x'\alpha)$ are restricted versions of this model.

Additive latent variable model

- Model:

$$\begin{aligned} Y &= h(X) + U \\ X &= g(Z) + V \end{aligned}$$

$$E[U|V = v, Z = z] = c(v) \qquad E[V|Z = z] = 0$$

- Condition on $V = v$, $Z = z$.

$$E[Y|V = v, Z = z] = h(g(z) + v) + c(v)$$

Additive latent variable model

- Model:

$$\begin{aligned} Y &= h(X) + U \\ X &= g(Z) + V \end{aligned}$$

$$E[U|V = v, Z = z] = c(v) \qquad E[V|Z = z] = 0$$

- Condition on $V = v$, $Z = z$.

$$E[Y|V = v, Z = z] = h(g(z) + v) + c(v)$$

- Given $Z = z$, there is $V = v$ if and only if $X = x \equiv g(z) + v$, so:

$$E[Y|X = x, Z = z] = h(x) + c(x - g(z))$$

- $E[X|Z = z] = g(z)$ so $g(z)$ is identified if there is sufficient variation in z.

Additive latent variable model: partial derivatives

- Condition on $X = x$, $Z = z$.

$$E[Y|X = x, Z = z] = h(x) + c(x - g(z))$$

$$E[X|Z = z] = g(z)$$

- Consider partial derivatives with respect to x and z

$$\nabla_x E[Y|X = x, Z = z] = \nabla_x h(x) + c'(x - g(z))$$

$$\nabla_z E[Y|X = x, Z = z] = \nabla_z h(x) - c'(x - g(z))\nabla_z g(z)$$

$$\nabla_z E[X|Z = z] = \nabla_z g(z)$$

Additive latent variable model: partial derivatives

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$$\nabla_z E[X|Z = z] = \nabla_z g(z)$$

- Since $\nabla_z h(x) = 0$ (exclusion restriction), if $\nabla_z g(z) \neq 0$

$$\nabla_x E[Y|X = x, Z = z] + \frac{\nabla_z E[Y|X = x, Z = z]}{\nabla_z E[X|Z = z]} = \nabla_x h(x)$$

Additive latent variable model: analogue estimation

- Model:

$$\begin{aligned} Y &= h(X) + U \\ X &= g(Z) + V \end{aligned}$$

$$E[U|V = v, Z = z] = c(v) \qquad E[V|Z = z] = 0$$

- In the distributions generated by structures admitted by this model:

$$E[Y|X = x, Z = z] = h(x) + c(x - g(z))$$

- Estimate the function: $E[X|Z = z]$ more or less flexibly.

- Estimate the functions $h(\cdot)$ and $c(\cdot)$

$$Y_i = h(X_i) + c(X_i - \widehat{E[X|Z = Z_i]}) \qquad i \in \{1, \ldots, N\}$$

Non-additive models

- Consider identification in models admitting structures such that

$$\begin{aligned} Y &= h(X, U) \\ X &= g(Z, V) \end{aligned}$$

with:

- endogenous (Y, X), exogenous Z and unobserved scalar U and V,
- h weakly increasing in scalar U,
- g strictly increasing in scalar V,
- (U, V) and Z independently distributed.

$$(U, V) \perp\!\!\!\perp Z.$$

Quantiles: Definitions

- *Definition*: Random variable A with **distribution** function:

$$F_A(a) \equiv \Pr[A \leq a]$$

has **quantile** function

$$Q_A(p) \equiv \inf\{a : F_A(a) \geq p\}$$

- When A is **continuously** distributed

$$p = F_A(Q_A(p)) \qquad a = Q_A(F_A(a))$$

but not when A is **discrete**.

Quantiles: Distribution functions are uniformly distributed

- Random variable A has **distribution** and quantile function:

$$F_A(a) \equiv \Pr[A \leq a] \qquad Q_A(p) \equiv \inf\{a : F_A(a) \geq p\}$$

- *Definition*: $U \in [0,1]$ has a uniform distribution, $\mathit{Unif}(0,1)$ if $F_U(u) = u$. There is: $Q_U(p) = p$.

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- For **continuous** A,

$$B = F_A(A) \sim Unif(0,1)$$

  This because

$$\Pr[A \leq \alpha] \equiv F_A(\alpha)$$

$$\{a : a \leq \alpha\} = \{a : F_A(a) \leq F_A(\alpha)\}$$

$$\Pr[F_A(A) \leq F_A(\alpha)] = F_A(\alpha)$$

  So B is $Unif(0,1)$ because

$$\Pr[B \leq b] = b$$

Quantiles: Random variables as transformations of uniforms

- Random variable A has **distribution** and quantile function:

$$F_A(a) \equiv \Pr[A \leq a] \qquad Q_A(p) \equiv \inf\{a : F_A(a) \geq p\}.$$

- Definition: $U \in [0,1]$ has a uniform distribution, $Unif(0,1)$ if $F_U(u) = u$ then $Q_U(p) = p$.

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which follows from

$$F_A(A) = U \Rightarrow Q_A(U) = A$$

Quantiles: Random variables as transformations of uniforms

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- For continuous A,

$$B = F_A(A) \sim Unif(0,1).$$

  which follows from

$$F_A(A) = U \Rightarrow Q_A(U) = A$$

- For discrete and continuous A, $Q_A(U)$ has the same distribution as A.

$$Q_A(U) = A$$

Quantiles: Equivariance under monotone transformation

- Let random variable A have quantile function $Q_A(\tau)$.

$$P[A \leq Q_A(\tau)] = \tau$$

- Let random variable $B = f(A)$ where $f(\cdot)$ is monotone increasing.

- Since

$$\{a : a \leq Q_A(\tau)\} = \{a : f(a) \leq f(Q_A(\tau))\}$$

there is

$$\tau = P[A \leq Q_A(\tau)] = P[f(A) \leq f(Q_A(\tau))]$$

so

$$Q_{f(A)}(\tau) = f(Q_A(\tau)).$$

Quantiles: Normalization

- Continuous Y, scalar continuous U, h strictly monotonic (increasing):

$$Y = h(X, U)$$

- Can normalize $U \sim Unif(0,1)$ under monotonicity - free choice of units of measurement.

- Consider for continuous W

$$Y = h^*(X, W) = h^*(X, Q_W(F_W(W))) = h(X, U)$$

where

$$U \equiv F_W(W) \sim Unif(0,1)$$

and

$$h(X, \cdot) \equiv h^*(X, Q_W(\cdot))$$

Non-additive latent variable model

- *Structural functions*: X and Y are generated by

$$Y = h(X, U) \qquad X = g(Z, V)$$

  - h weakly monotonic (increasing) in U - allows discrete Y
  - g **strictly** monotonic (increasing) in V- requires continuous X.

- *Unobservables*: (U, V) independent of Z.

$$(U, V) \perp\!\!\!\perp Z$$

- Can normalize $V \sim Unif(0, 1)$ - and then

$$g(z, v) = Q_{X|Z}(v|z)$$

Non-additive latent variable model

- If (U, V) and Z are independent then U is independent of Z given V, $U \perp\!\!\!\perp Z|V$.

- **Proof**:

$$
\begin{aligned}
P[U \in \mathcal{U} \wedge Z \in \mathcal{Z}|V \in \mathcal{V}] &= \frac{P[U \in \mathcal{U} \wedge Z \in \mathcal{Z} \wedge V \in \mathcal{V}]}{P[V \in \mathcal{V}]} \\
&= \frac{P[U \in \mathcal{U} \wedge V \in \mathcal{V}]}{P[V \in \mathcal{V}]} \times P[Z \in \mathcal{Z}] \\
&= P[U \in \mathcal{U}|V \in \mathcal{V}] \times P[Z \in \mathcal{Z}|V \in \mathcal{V}]
\end{aligned}
$$

- We use a consequence of this shortly, namely

$$
\forall \tau, v, z \quad Q_{U|VZ}(\tau|v, z) = Q_{U|V}(\tau|v)
$$

Non-additive latent variable model

- Structural functions: X and Y are generated by

$$Y = h(X, U) \qquad X = g(Z, V) \qquad (U, V) \perp\!\!\!\perp Z$$

  h weakly monotonic (increasing) in U - g strictly monotonic (increasing) in V.

- Express Y in terms of V and Z

$$Y = h(g(Z, V), U)$$

$$Q_{Y|VZ}(\tau|v, z) = h(g(z, v), Q_{U|V}(\tau|v))$$

Non-additive latent variable model

- Structural functions: X and Y are generated by

$$Y = h(X, U) \qquad X = g(Z, V) \qquad (U, V) \perp\!\!\!\perp Z$$

h weakly monotonic (increasing) in U - g strictly monotonic (increasing) in V.

- Express Y in terms of V and Z

$$Y = h(g(Z, V), U)$$

$$Q_{Y|VZ}(\tau|v, z) = h(g(z, v), Q_{U|V}(\tau|v))$$

- Since $(V = v \wedge Z = z) \Leftrightarrow (X = x \equiv g(z, v) \wedge Z = z)$

$$Q_{Y|XZ}(\tau|x, z) = h(x, \color{red}{Q_{U|V}(\tau|v)})$$

$$x \equiv g(z, v) = Q_{X|Z}(v|z)$$

so

$$Q_{Y|XZ}(\tau|Q_{X|Z}(v|z), z) = h(x, Q_{U|V}(\tau|v))$$

Non-additive latent variable model: partial differences

- Structural functions: X and Y are generated by

$$Y = h(X, U) \qquad X = g(Z, V) \qquad (U, V) \perp\!\!\!\perp Z.$$

We have shown h weakly increasing in U and g strictly increasing in V implies

$$Q_{Y|XZ}(\tau | Q_{X|Z}(v|z), z) = h(x, Q_{U|V}(\tau|v))$$

Non-additive latent variable model: partial differences

- Structural functions: X and Y are generated by

$$Y = h(X, U) \qquad X = g(Z, V) \qquad (U, V) \perp\!\!\!\perp Z.$$

We have shown h weakly increasing in U and g strictly increasing in V implies

$$Q_{Y|XZ}(\tau | Q_{X|Z}(v|z), z) = h(x, Q_{U|V}(\tau|v))$$

- Consider $\{z', z''\}$ in support of Z. Define

$$x' \equiv Q_{X|Z}(v|z') \qquad x'' \equiv Q_{X|Z}(v|z'')$$

- Then

$$h(x', Q_{U|V}(\tau|v)) - h(x'', Q_{U|V}(\tau|v))$$

is identified by

$$Q_{Y|XZ}(\tau|Q_{X|Z}(v|z'), z') - Q_{Y|XZ}(\tau|Q_{X|Z}(v|z''), z'')$$

Angrist-Krueger QJE (1991)

1930-39 cohort: W: log wage, S: years of schooling, B: quarter of birth

$$\begin{aligned} W &= h(S, U) \\ S &= g(B, V) \end{aligned}$$

Quantiles of years of schooling (S) by quarter of birth

$$Q_{S|B}(v|b) \qquad b \in \{1, 2, 3, 4\}$$

$v =$	.1	.2	.3	.4	.5
$b = 1$	8.42	10.58	11.66	11.93	12.20
$b = 2$	8.48	10.64	11.67	11.95	12.22
$b = 3$	8.66	10.95	11.71	11.95	12.24
$b = 4$	8.75	11.06	11.71	11.98	12.25

Angrist-Krueger QJE (1991)

Estimated returns to schooling for median earner

$$\frac{Q_{W|SB}(.5|Q_{S|B}(v|b'),b') - Q_{W|SB}(.5|Q_{S|B}(v|b''),b'')}{Q_{S|B}(v|b') - Q_{S|B}(v|b'')}$$

$v =$	.1	.2	.3	.4	.5
$b' = 1$ $b'' = 2$	.065 (.121)	.012 (.090)	$-.508$ (.393)	$-.210$ (.218)	$-.176$ (.228)
$b' = 1$ $b'' = 3$	.070 (.030)	.045 (.014)	.048 (.095)	.064 (.106)	.060 (.111)
$b' = 1$ $b'' = 4$	.065 (.022)	.060 (.011)	.083 (.079)	.068 (.083)	.043 (.089)