# ECON0106: **Microeconomics**

## **Take-Home Exam**

Duarte Gonçalves*

University College London

Due date: 18 December, 9:59

You are *encouraged* to work with your classmates. However, take note that this is an individual examination: the work that you submit has to be of your own authorship. You are encouraged to collaborate with your colleagues in working out any given part of any of the questions, but you are expected to work on, and have thought through every single part of all the questions.

You **cannot** share .tex files nor portions of code that relate to any answers: you must type up and turn-in your own answers. All work is to be typed in LaTeX and you will need to submit both a .pdf file and the .tex files along all other raw files needed to compile the .pdf, if any (e.g. .bib files, images, preambles, etc).

You are not allowed to collaborate with anyone outside this class or to post questions regarding class material in any forum; please respect it. If you use any source, please do make sure you cite it correctly. For every question, you need to list all colleagues with whom you worked in developing your answer. You will need to provide complete answers.

The exam has three questions. You will have 24 hours to complete the exam; you have until the 18th December, 9:59am London time to submit your answers. However, I advise against leaving it for last minute to avoid technological hickups: late submissions will not be accepted.

The exam has a lot of text and seems long, but most of the questions do not require very long answers and can be done quickly. A quick resolution of the exam would take about 2-2.5 hours (plus typing). You should try to work out as much as you can, but no more than 8 hours.

Good luck!

* Department of Economics, University College London; duarte.goncalves@ucl.ac.uk. Please do not share these notes with anyone outside of this class.

Question 1. (Nash Equilibrium with Stochastic Choice)[1]

Let's take a look at an individual decision-making characterization of stochastic choice.

(i) Let $A_i$ be a set of alternatives and $\mathscr{S}_i$ the set of all finite menus or subsets of $A_i$, i.e. $\mathscr{S}_i := \{S_i \subseteq A_i \mid |S_i| < \infty\}$. A **stochastic choice rule** $\sigma_i$ is a mapping that associates each menu $S_i \in \mathscr{S}_i$ with a probability distribution over $S_i$; let's write $\sigma_i(s_i \mid S_i)$ the probability with which decision-maker $i$ chooses alternative $s_i$ when faced with menu $S_i$, where $\sigma_i(\cdot \mid S_i) \in \Delta(S_i)$.

A useful characterization of the stochastic choice rule $\sigma_i$ is to express it as the outcome of a maximization procedure. We will say that $\sigma_i$ admits an **additive perturbed utility (APU) representation** if there is a utility function $u_i : A_i \to \mathbb{R}$ and a cost function $c_i : [0,1] \to \mathbb{R} \cup \{\infty\}$ such that, for every $S_i \in \mathscr{S}_i$,

$$\sigma_i(\cdot \mid S_i) = \arg\max_{p_i \in \Delta(S_i)} \sum_{s_i \in S_i} p_i(s_i) u_i(s_i) - c_i(p_i(s_i))$$

with $c_i$ strictly convex, continuously differentiable on $(0,1)$, with $\lim_{p \downarrow 0} c_i'(p) = -\infty$. This is a fairly general and powerful characterization of stochastic choice with many interesting applications and interpretations (uncertainty, random utility, etc.).

It turns out that an *equivalent* characterization of APU is a generalization of Luce's IIA: $\sigma_i$ satisfies **ordinal independence of irrelevant alternatives (Ordinal IIA)** if there is a continuous and monotone $\phi : [0,1] \to \mathbb{R}_+ \cup \{\infty\}$ with $\phi(0) = 0$ such that

$$\frac{\phi(\sigma_i(s_i \mid S_i))}{\phi(\sigma_i(s_i' \mid S_i))} = \frac{\phi(\sigma_i(s_i \mid S_i'))}{\phi(\sigma_i(s_i' \mid S_i'))}$$

for all $S_i, S_i' \in \mathscr{S}_i$ and $s_i, s_i' \in S_i \cap S_i'$.

Show that if $\sigma_i$ admits an APU representation, then it satisfies Ordinal IIA. (You only need to show one direction of the equivalence.)

We've discussed several reasons why individual choices may be random (mistakes, new information arrives, temporary fluctuations of preferences, etc). But why would this not be true also in strategic settings? Let's define a new solution concept based on a model of stochastic choice, Nash equilibrium with APU.

[1] This question is partly based on the paper 'Stochastic Choice and Revealed Perturbed Utility', by Fudenberg, Iijima, and Strzalecki — `http://dx.doi.org/10.3982/ECTA12660`.

Let $\Gamma = \langle I, S, u \rangle$ be a finite normal-form game and $c = (c_i)_{i \in I}$ be such that for each $i \in I$, $c_i : [0,1] \to \mathbb{R}$ is continuous, strictly convex, and continuously differentiable on $(0,1)$, with $\lim_{p \downarrow 0} c_i'(p) = -\infty$. We call $G = \langle \Gamma, c \rangle$ an APU game.

We will say that $\sigma$ is a Nash equilibrium with additive perturbed utility of the APU game $G = \langle \Gamma, c \rangle$ if for every player $i$, $\sigma_i \in \arg\max_{\sigma_i' \in \Sigma_i} u_i(\sigma_i', \sigma_{-i}) - \sum_{s_i \in S_i} c_i(\sigma_i'(s_i))$.[2]

(ii) Show that every APU game has a Nash equilibrium with additive perturbed utility.

(iii) Let $\Gamma$ be such that (a) $I = \{i, j\}$; (b) for $i \in I$, $S_i = \{0, 1\}$, and (c) for $s_j = s_i \neq s_i'$ assume $u_i(s_i, s_j) > u_i(s_i', s_j)$, i.e. both players want to match the opponent's action. Fix $c$. What can we say about the set of Nash equilibrium with additive perturbed utility of when $u_i(1,1)$ increases?

(iv) Let $\Gamma$ be a finite normal-form game and fix $c$. Suppose $\{\lambda^n\}_n$ is such that $\lambda^n > 0$ and $\lambda^n \to 0$, and $\{\sigma^n\}_n$ a sequence of mixed strategy profiles such that, for each $n$, $\sigma^n$ is a Nash equilibria with additive perturbed utility of the extended game $G^n = \langle \Gamma, c^n \rangle$, where for each $i \in I$, $c_i^n := \lambda^n c_i$.

Suppose that $\sigma^n \to \sigma$. Is $\sigma$ a Nash equilibrium of the underlying normal-form game $\Gamma$? If so, prove it. If not, provide a counterexample.

Question 2. (Kill the Unicorns)[3]

Let $I$ denote the finite set of bidders. Each bidder $i$ can bid $s_i \in S_i = \tilde{S} = \{s^1, \ldots, s^k\} \subset \mathbb{N}_0$, where $\tilde{S}$ is finite; $v > 0$ denotes the (known) common value of the good.

The payoffs are given as follows: Let $T(s) = \{i \in I \mid s_i = s_j \text{ for some } j \neq i\}$. For a given strategy profile $s$ and for every player $i \in I$, if $i \in T(s)$ or $s_i < \max_{j \notin T(s), j \neq i} s_j$, then player $i$ gets $-s_i$, and, if otherwise, player $i$ gets $v - s_i$.

This is an interesting modification of the all pay auction: all pay their bids, and if a player bids the highest, that player gets the good, which is commonly valued at $v > 0$. However, if there is a tie, both bids get 'cancelled' (i.e. players still pay their bid, but neither gets the good), and the second highest value bid is considered. Again, if there is a single player bidding that value then the player gets the good; if there are multiple players bidding that value, all their bids get 'cancelled' (but still paid), and we then consider the third highest value; etc.

[2] Where, as usual in game theory, $u_i(\sigma_i, \sigma_{-i}) = \sum_{(s_j)_{j \in I} = s \in S} (\prod_{j \in I} \sigma_j(s_j)) u_i(s_i, s_{-i})$.

[3] This is the name of the board game that inspired this question. The board game actually has sequential auctions and a special cards and whatnot, but let's analyze the core of the economics behind it.

(i) Let $|I| = 3$, $\tilde{S} = \{s^1, s^2\}$ and $v > s^2 > s^1 \geq 0$. Solve for all symmetric Nash equilibria.

(ii) Suppose an auctioneer is selling the the item according to the auction setting described in (i). Compute the auctioneer's expected revenue from (i) (for all symmetric Nash equilibria) assuming $s^k = k$, for $k = 1, 2$.

(iii) Now let's compare this to the expected revenue of the auctioneer under the standard all-pay auction. As before, let $|I| = 3$, $\tilde{S} = \{s^1, s^2\}$, and $v > s^2 > s^1 \geq 0$; $v$ is commonly known, as before. Differently from before, the players bidding highest get the good, with uniform tie-breaking (no bid 'cancelling'). If another player bids strictly more, they don't get the good. In all cases, they always pay their bid. Solve for all symmetric Nash equilibria assuming $s^k = k$, $k = 1, 2$.

Compute the auctioneer's expected revenue (for all symmetric Nash equilibria) and compare it to what you obtained in the previous (sub)question.

(iv) Assume $|I| = 3$, $\tilde{S} = \{1, 2\}$, and $v > 2$. In which auction setup, (i) or (iii), are bidders better off? (Note: it may depend on the particular symmetric Nash equilibrium.)

(v) Suppose that the auctioneer values the item at $\tilde{v}$. Are there values of $\tilde{v}$ for which the auctioneer is better off with the auction setting described in (i) than in the one in (iii)? Comment.

(vi) Let's try to tackle a slightly more general version of the auction setup in (i).

Let $|I| = n \geq 2$, $\tilde{S} = \{s^1, s^2, ..., s^k\}$ and $v > s^k > s^{k-1} > ... > s^2 > s^1 \geq 0$. Solve for a fully mixed symmetric Nash equilibrium. You are not required to provide a closed-form solution, but do provide the best characterization of the all equilibria you can.

Question 3. (Bargaining with Asymmetric Information)

A buyer is bargaining to procure from a supplier a good that is worth £1 to that buyer. The supplier can produce the good at cost $c$, which is drawn uniformly from $[0, 1]$. The realized cost is known to the supplier at the beginning of the game, but not to the buyer, who only knows the distribution of the cost. There are two periods of bargaining. The buyer makes a take-it-or-leave-it offer of a purchase price at the start of each period that the supplier accepts or rejects. The game ends when an offer is accepted or after two periods, whichever comes first. Both players discount period 2 payoffs with a common discount factor of $\delta \in (0, 1)$. The supplier incurs the cost $c$ only when the agreement is reached. If an offer of price $p$ is accepted in period $t \in \{1, 2\}$, the buyer's payoff is

$\delta^{t-1}(1-p)$ and the seller's $\delta^{t-1}(p-c)$. If the games ends and no offer is accepted, then both players get a payoff of zero.

Characterize the pure strategy weak Perfect Bayesian equilibria in which the buyer's offer in the second period is weakly higher than the offer in the first period, and the supplier plays a threshold acceptance strategy.