Lecture 3: The Recursive Problem – Principle of Optimality and Dynamic Programming

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Problem SP suggests principle of optimality

- Consider the stationary version of the **sequence problem (SP)**:

$$V^*(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$
$$\text{s.t. } x_{t+1} \in \Gamma(x_t) \text{ for all } t \geq 0.$$

- Based on intuition, one could conjecture that:
    - The optimal path should satisfy:

    $$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*) \text{ for each } t.$$

    - Value function should satisfy **Functional/Bellman equation (FE)**:

    $$V^*(x^*) = \max_{y \in \Gamma(x^*)} F(x^*, y) + \beta V^*(y).$$

- Equivalence of **(SP)** and **(FE)**: **the Principle of Optimality (PO)**.
- How to solve **(FE)**: **Contraction Mapping Theory (CMT)**.

Outline

How could PO not apply?

It is useful to start by analyzing examples in which the principle of optimality doesn't apply – that is, a solution to (SP) does not solve (FE).

How could recursive optimization be different than sequential optimization?

When this happens, the problem is referred to as "dynamically inconsistent."

Outline

Hyperbolic discounting

- We typically work with exponential discounting:

$$U_t = u_t + \delta u_{t+1} + \delta^2 u_{t+2} + ...$$

  where δ (previously β) captures broad motives for caring less about later periods than earlier (what are some reasons?).

- This is ok if this is not the focus of our analysis (**simplicity**), but might be restrictive if our goal is to analyze certain motives.
  - For some psychological motives, exponential form is restrictive: Same trade-off for periods t vs $t+1$ and $t+20$ and $t+21$.

- What if we instead have **quasi-hyperbolic discounting**:

$$U_t = u_t + \beta\delta u_{t+1} + \beta\delta^2 u_{t+2} + ...$$

  - The new discount factor, $\beta < 1$, captures the **salience of now**.
  - Useful to capture myopia and self-control issues (Strotz-Laibson).
  - It also generates dynamic (time) inconsistency...

Problem: Dynamic inconsistency in preferences

- Exercise has benefit today -2. Has delayed benefit of 3.
- Suppose $\beta = 1/2$ and $\delta = 1$.
- Questions about the one-shot optimal plan (sequence problem):
  - Do you exercise today?
  - Do you plan to exercise tomorrow?
- Questions about the recursively optimal plan:
  - Do you exercise tomorrow?

Why does the PO not apply in this example?

Outline

Inflation-output trade-off

Consider a stylized model with inflation-output trade-off:

$$\max_{\pi_0, y_0, \pi_1, y_1} -\left(\pi_0^2 + (y_0 - y^*)^2\right) - \beta\left(\pi_1^2 + (y_1 - y^*)^2\right),$$

$$\text{s.t. } \pi_0 = E_0[\pi_1] + y_0, \text{ where } E_0[\pi_1] = \pi_1$$

$$\text{and } \pi_1 = y_1.$$

- y_t is the output gap from the flexible price level (normalized to 0).
- Constraint is a caricaturized NK Phillips curve: Inflation depends on y_t and **future inflation** expectations (through firms' price setting).
- Since there is no uncertainty, rational expectations imply $E_0[\pi_1] = \pi_1$.
- Policymaker chooses $\{\pi_t, y_t\}$ subject to the constraint.
- Dislikes π_t, and likes to keep y_t close to a target, y^*. Suppose target is strictly positive, $y^* > 0$ (e.g., political friction).
- Does the PO apply here? Let's start with recursive optimization...

Recursive problem in the future

- At date 1, the recursive problem is

$$\max_{\pi_1, y_1} -\left(\pi_1^2 + (y_1 - y^*)^2\right) \text{ s.t. } \pi_1 = y_1.$$

- The FOC for y_1 implies:

$$\underbrace{y_1}_{\text{marginal cost}} = \overbrace{(y^* - y_1)}^{\text{marginal benefit from overheating}}.$$

- The solution is

$$\pi_1^{\text{rec}} = y_1^{\text{rec}} = \frac{y^*}{2} > 0.$$

- The planner "overheats" the economy (induces a level of output that is above the flexible price level). She optimally trades off output and inflation.

Recursive problem now

- At date 0, the planner takes its future choices as given.
- The first step of the recursive problem is then given by,

$$\max_{\pi_0, y_0} -\left(\pi_0^2 + (y_0 - y^*)^2\right) - \beta\left((\pi_1^{\text{rec}})^2 + (y_1^{\text{rec}} - y^*)^2\right),$$

$$\text{s.t. } \pi_0 = \pi_1^{\text{rec}} + y_0$$

$$\text{and } \pi_1^{\text{rec}} = y_1^{\text{rec}} = y^*/2.$$

- Check that the solution to this is given by

$$y_0^{\text{rec}} = \frac{-y^*/2 + y^*}{2} < \frac{y^*}{2} \text{ and } \pi_0^{\text{rec}} = \frac{y^*}{2} + \frac{-y^*/2 + y^*}{2} > \frac{y^*}{2}.$$

Recursive problem now

- The planner overheats the economy a bit less because she already starts with a large level of baseline inflation to deal with.
- But the baseline inflation is created by the planner's own actions!
- This suggests there a potential remedy: Commit not to create inflation.
- And this suggests the one-shot optimization might be different...

One-shot problem: Solution is different

- The one-shot problem reduces to

$$\max_{y_0, y_1} -\left((y_1 + y_0)^2 + (y_0 - y^*)^2\right) - \beta\left(y_1^2 + (y_1 - y^*)^2\right).$$

- The FOC for y_1 implies,

$$\underbrace{(y_1 + y_0)}_{\text{current marginal cost}} + \underbrace{\beta y_1}_{\text{future marginal cost}} = \overbrace{\beta\left(y^* - y_1\right)}^{\text{marginal benefit from overheating}} .$$

- The first term raises MC relative to sequential, so the solution features,

$$y_1^{\text{one-shot}} < y_1^{\text{rec}} = \frac{y^*}{2}.$$

  - Exact solution is not important. Just note that it is different than the sequential solution—and the direction of the difference.

Dynamic inconsistency

- At date 0, the planner plans to overheat the economy less at date 1.
  - Intuitively, she recognizes π_1 also raises π_0, which is costly, so she perceives greater marginal cost from creating a future boom.
- At date 1, she would like to overheat more relative to the original plan.
  - Intuitively, π_0 is already set by the time we reach date 1. This reduces the planner's marginal cost and induces her to boom more.
- So there is dynamic inconsistency here even though the preferences look fine. What's the issue?

Dynamic inconsistency is caused by constraints

- Take the state variable as last period's inflation, $x_t = \pi_{t-1}$ (since the problem allows the current inflation to change within the period).
- If the Phillips Curve took the following form (which is also used),

$$\pi_0 = \overbrace{\pi_{-1}}^{\text{expectations at time 0 are backward looking and predetermined}} + y_0$$

then we wouldn't have time inconsistency. You could formulate this problem in our canonical notation. (Can you see how?)

Dynamic inconsistency is caused by constraints

- The problem emerges because the constraint has the form,

$$\pi_0 = \overbrace{\pi_1}^{\text{depends on future choices}} + y_0$$

- So we have a situation in which the state evolution is given by,

$$x_{t+1} = x_t + \tilde{g}\left(x_t, z_t, x_{t+1}, z_{t+1}, ..\right) \text{ as opposed to } x_{t+1} = x_t + \tilde{g}\left(x_t, z_t\right).$$

- Committing to future actions can be valuable since it affects the current outcomes via forward-looking constraints.

- This is a common issue in macro policy analysis because **economic agents' expectations** of future states—which depend on future policy choices—tend to affect current outcomes (Kydland-Prescott).

Need: Time consistency of *preferences and constraints*

- So we need time-consistency of preferences and constraints.
- Otherwise, there is (direct or indirect) benefit from **commitment.**
- Our canonical problem is time-consistent in both dimensions since:

$$\sup_{\{x_{t+1},z_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \tilde{F}(x_t, z_t)$$

$$\text{s.t. } z_t \in \tilde{\Gamma}(x_t),$$

$$x_{t+1} = x_t + \tilde{g}(x_t, z_t) \text{ for all } t \geq 0 \text{ and } x_0 \text{ given.}$$

- We could therefore hope for an equivalence of (SP) and (FE).

Outline

Back to our canonical problem

$$
\begin{aligned}
V^*(x_0) \quad &= \quad \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
& \quad\quad \text{s.t. } x_{t+1} \in \Gamma(x_t) \text{ for all } t \geq 0.
\end{aligned}
$$

- Next: "the equivalence" of the one-shot and the recursive problems.
- There are two sides to this argument:
    1. Optimization at one-go implies recursive optimization (the PO),
    2. Recursive optimization implies optimization at one-go (converse).
- Theorems from SLP for reference, but we will not go into detail here.

Outline

Formalizing the Principle of Optimality

- Recall notation: V^* and $\{x_t^*\}_{t=0}^{\infty}$ are the solution to (SP).

Theorem (SLP 4.2)

The function V^* solves the Bellman or Functional Equation (FE):

$$V^*(x_0) = \max_{x_1 \in \Gamma(x)} F(x_0, x_1) + \beta V^*(x_1)$$

Theorem (SLP 4.4)

An optimal plan, $\{x_t^*\}_{t=0}^{\infty}$, satisfies the principle of optimality (PO):

$$V^*(x_t^*) = F\left(x_t^*, x_{t+1}^*\right) + \beta V^*\left(x_{t+1}^*\right) \text{ for each } t.$$

- See SLP for the proof. Key idea: optimization for entire path $\Rightarrow$ optimization today given optimal behavior tomorrow

Principle of Optimality: the NGM

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(f\left(k_t\right) - k_{t+1}\right), \quad k_{t+1} \in \left[0, f\left(k_t\right)\right] \text{ for each } t, \quad k_0 \text{ given.}$$

$$\begin{aligned}
V^*(k_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \\
&= \max_{0 \le k_1 \le f(k_0)} \left\{ \max_{\{k_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \right\} \\
&= \max_{0 \le k_1 \le f(k_0)} \left\{ \max_{\{k_{t+2}\}_{t=0}^{\infty}} \left[u(f(k_0) - k_1) + \beta \sum_{t=0}^{\infty} \beta^t u(f(k_{t+1}) - k_{t+2}) \right] \right\} \\
&= \max_{0 \le k_1 \le f(k_0)} \left\{ u(f(k_0) - k_1) + \beta \max_{\{k_{t+2}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_{t+1}) - k_{t+2}) \right\} \\
&= \max_{0 \le k_1 \le f(k_0)} \left\{ u(f(k_0) - k_1) + \beta V^*(k_1) \right\} \\
&\implies V^* \text{ solves the Bellman equation!}
\end{aligned}$$

Is the converse true?

- It would be nice if the converse was also true, so if we find a function that solves (FE) or a plan that satisfies (PO) then we solve (SP).
    - But this is not always the case with $T = \infty$ as the following shows.

Outline

A pathological example

- Consider the consumption-savings problem with linear utility:

$$V^*(a_0) = \max_{\{c_t, a_{t+1}\}} \sum_{t=0}^{\infty} \beta^t (a_t(1+r) - a_{t+1}),$$
$$\text{s.t.} \quad a_{t+1} \geq 0, \text{ and } a_0 \text{ given.}$$

- Suppose $\beta(1+r) = 1$. Indifferent between consuming and saving.
    - So there are many optimal paths (what are those?) and $V^*(a_0) = a_0$.
- Now consider the always-save path: $\tilde{a}_t = \tilde{a}_0 (1+r)^{t+1}$ for each t:
    - This satisfies the PO since $V^*(\tilde{a}_t) = \frac{1}{1+r} V^*(\tilde{a}_{t+1})$.
    - But yields value of 0 so clearly not optimal.
- You can check that this path violates the Transversality Condition of the variational approach.

Ruling out the pathological case

- The example pathological case reflects nondiminishing importance of the distant future:

$$\lim_{T\to\infty} \beta^T V^*(x_T) \neq 0 \text{ for some feasible (or candidate) paths.}$$

- Intuitively, the issue is that recursive optimization does not capture "deviations at infinity."
  - If infinite future is unimportant, the issue can be safely ignored.
  - Otherwise, paths that violate the TVC might seem recursively optimal.
- As long as the distant future is unimportant, the converse also holds...

Principle of optimality: Converse results

Theorem (SLP 4.3)

Suppose V solves (FE) and that for any feasible plan $\{x_t\}_{t=0}^{\infty}$:

$$\lim_{T\to\infty} \beta^T V(x_T) = 0. \tag{1}$$

Then, V solves (SP); that is, $V = V^*$.

Theorem (SLP 4.5)

Suppose the feasible plan, $\{\tilde{x}_t^*\}_{t=0}^{\infty}$, satisfies (PO) and that

$$\lim_{T\to\infty} \beta^T V^*(\tilde{x}_T^*) = 0. \tag{2}$$

Then, $\{\tilde{x}_t^*\}_{t=0}^{\infty}$ is an optimum for (SP).

- See SLP for proofs. Intuition: optimization today given optimization tomorrow does not necessarily deal with value "escaping at infinity".

Intuition via sketch of proof

For any plan,

$$\begin{aligned}
V(k_0) &\geq u(f(k_0) - k_1) + \beta V(k_1) \\
&\geq u(f(k_0) - k_1) + \beta u(f(k_1) - k_2) + \beta^2 V(k_2) \\
&\dots \\
&\geq \sum_{t=0}^{T-1} \beta^t u(f(k_t) - k_{t+1}) + \beta^T V(k_T)
\end{aligned}$$

with $\lim_{t\to\infty} \beta^T V(k_T) = 0$, we have global optimality with infinite horizon.

Discussion of the converse results

- Given the equivalence, our next challenge is to characterize the solution of (FE) (and so equivalently that of (SP)).
- For this we will assume F (or the state variable) is bounded.
    - That's strictly speaking overkill.
    - But note that it makes the previous pathology irrelevant for us, since a bounded F will also result in a bounded V^* (as this is a discounted sum of F's), which in turn will satisfy condition (2).
    - Relatedly, once we assume bounded F, our solution method to (FE) will produce bounded V. Since this satisfies (1), we will invoke the converse result to argue $V = V^*$.
    - The takeaway message now is that you can safely ignore the pathology: For our problems, (PO) as well as its converse will hold.

Outline

Outline

Bellman equation

- The Bellman equation (FE) for the general problem is:

$$V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\}.$$

- e.g. in the neoclassical growth model:

$$V(k) = \max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta V(k')\}$$

- Known: functions u and f. Unknown: function V. We want to find V.
    - why is knowing V so useful?
- The key observation is that this equation can be equivalently written as a **fixed point problem** over the space of value functions.
- Sounds mysterious. But it is intuitive: start by thinking about the V on the LHS and the V on the RHS as two different functions, and the $\max\{u(\cdot) + \beta\cdot\}$ as being an operator that links them...

Bellman operator

- Pick some function $V_0(k)$ and plug in on the RHS. Define new function $V_1(k)$ as:
$$V_1(k) = \max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta V_0(k')\}$$
- If $V_0 = V_1$, we are done (guess and verify method). If not, use V_1 on the RHS and define V_2. After $n+1$ times:
$$V_{n+1}(k) = \max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta V_n(k')\}$$
- Can think of the RHS as applying an operator T to function V_n:
$$V_{n+1} = TV_n$$
T is a mapping from a set of functions onto itself called a **Bellman operator**.
- Solving the Bellman equation $\iff$ finding a fixed point of mapping T:
$$V_\infty = TV_\infty$$

A map of the theory – for details see SLP

1. Does a fixed point exist?
2. Is it unique?
3. How to find it? Characterize it?

- Contraction Mapping Theorem: if operator T is a *contraction mapping*, then
  - it has a unique fixed point
  - it converges from anywhere (from any initial guess V_0)
- Blackwell Sufficiency Theorem: Let B be the space of bounded functions. $T : B \to B$ is a contraction mapping if it satisfies *monotonicity* and *discounting*.
  - the two conditions are generally satisfied in econ applications
  - we ensure functions are bounded by restricting to bounded pay-offs
- Properties of V^*: continuous, monotonic, concave
- Benveniste-Scheinkman: V^* is differentiable.

Outline

How about the optimal plan(s)?

- So far, we focused on the value function, $V^*(x)$, but we didn't say much about the optimal plan/sequence, $\{x_t^*\}_{t=0}^{\infty}$.
  - Aside: V^* is unique. But is the optimal plan unique? **Yes**, when V^* strictly concave. True when F strictly concave and Γ is a convex set.
- From the (PO), we know x_{t+1}^* should solve (FE) starting with x_t^*.
- So consider the solution to the (FE) starting with some $x \in X$:

$$g(x) = \arg\max_{y \in \Gamma(x)} F(x, y) + \beta V^*(y). \tag{3}$$

  - The time invariant function $y = g(x)$ is called the **policy function**.
  - It tells you what to do now (y) given where you are (x).
  - Clearly, starting at any $x_0 \in X$, the policy function pins down the sequence $\{x_t^*\}_{t=0}^{\infty}$.

Outline

The FOCs and the envelope condition

- Recall the Bellman Equation:

$$V(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta V(y)\}.$$

- Since V differentiable, the necessary FOC is

$$F_2(x,y) + \beta V'(y) = 0 \tag{4}$$

- Note that the policy function $g(x)$ satisfies (why?)

$$V(x) = F(x, g(x)) + \beta V(g(x)). \tag{5}$$

- Differentiate (5) with respect to x:

$$V'(x) = F_1(x,y) + F_2(x,y)g'(x) + \beta V'(y)g'(x)$$

- This and (4) give the Benveniste Scheinkman (1979) formula

$$V'(x) = F_1(x,y) \tag{6}$$

The FOCs and the envelope condition

- Eq (6) is also known as the envelope condition

$$V'(x) = F_1(x, y)$$

- Plugging back to the FOC we get

$$F_2(x, y) + \beta F_1(x, y) = 0$$

- Looks familiar? Euler equation!
- All roads lead to Rome, but now we got there with additional structure.
    - For instance, we also have the value function, $V(x)$, which we didn't have before.

Outline

Main four findings from the theory

Under the relevant regularity conditions:

1. The Bellman equation has a unique strictly concave solution.
2. This solution is approached in the limit as $j \to \infty$ by iterations on

   $$V_{j+1}(x) = \max_{x'}\{F(x, x') + \beta V_j(x')\}$$

   starting from any bounded and continuous initial V_0.
3. There is a unique and time-invariant optimal policy of the form $x' = g(x)$ where g is chosen to maximize the RHS of the Bellman equation.
4. Off corners, the limiting value function V is differentiable.

Steps in analysis of the Bellman equation

To derive the Euler equation:

1. Write down the Bellman equation in terms of x and y.
2. Take the FOC w.r.t. y.
3. Apply the Benveniste-Scheinkman formula: $v'(x) = F_x(x, y)$
4. One step forward on the B-S formula
5. Plug back into the FOC

Let's apply this to a consumption-saving problem.

Consumption-saving problem

To derive the Euler equation:

1. Write down the Bellman equation in terms of x and y.

$$v(a) = \max_{a'} u((1+r)a - a') + \beta v(a')$$

2. Take the FOC w.r.t. a'.

$$u'(c) = \beta \frac{dv}{da}(a')$$

3. Apply the Benveniste-Scheinkman formula:

$$\frac{dv}{da}(a) = (1+r)u'(c)$$

4. One step forward on the B-S formula

$$\frac{dv}{da}(a') = (1+r)u'(c')$$

5. Plug back into the FOC

$$u'(c) = (1+r)\beta u'(c')$$

Outline

Solving the Bellman equation in practice

- The theory suggests three practical methods of solving (FE):
  1. Guess and verify
     - Often not feasible
  2. Value function iteration
     - Good convergence properties, but slow
  3. Policy function iteration
     - Faster than VFI

Brock-Mirman (1972)

- Consider a neoclassical growth model with $u(c) = \log c$ and $\delta = 1$:

$$\begin{aligned} v^*(k_0) &= \max_{\{k_{t+1}, c_t\}_{t=0}^{\infty}} \beta^t \log c_t \\ &\quad \text{s.t. } k_{t+1} + c_t = A k_t^{\alpha}. \end{aligned}$$

- Bellman equation is

$$v(k) = \max_{k'} \log(Ak^{\alpha} - k') + \beta v(k').$$

- It is possible to solve this model with pencil and paper using any of the three methods.
- Let's start with guess and verify.

BM (1972): guess and verify

- Make a guess that
$$v(k) = E + F \log k.$$
- E and F are coefficients to be determined. We have:
$$E + F \log k = \max_{k'} \log(Ak^\alpha - k') + \beta(E + F \log k')$$
- The FOC is $\frac{1}{Ak^\alpha - k'} = \frac{\beta F}{k'}$ which gives the policy function
$$k' = \frac{\beta F}{1 + \beta F} A k^\alpha$$
- Plugging this back into the Bellman equation gives
$$E + F \log k = \log\left(Ak^\alpha \frac{1}{1 + \beta F}\right) + \beta\left(E + F \log \frac{\beta F}{1 + \beta F} A k^\alpha\right).$$
- Solving for E and F gives the result. In particular:
$$k' = \alpha\beta A k^\alpha.$$
- Save and consume a constant fraction of output.

BM (1972): VFI

- Start with a "bad" guess $v_0(k) = 0$ (thus solve a one period problem).
- Solution: $c = Ak^\alpha$, plugging back into the Bellman equation:

$$v_1(k) = \log A + \alpha \log k$$

- Maximization in the second step gives $c = \frac{1}{1+\beta\alpha} Ak^\alpha$, $k' = \frac{\beta\alpha}{1+\beta\alpha} Ak^\alpha$ and

$$v_2(k) = constant + \alpha(1 + \alpha\beta) \log k$$

- Continuing, we get a geometric series recursion, which finally yields the answer (as before).
- Note that doing the first couple of steps can give us a hint as to the good guess for the guess and verify method!
- Outside of very special cases we must rely on numerical solutions...

Value Function Iteration

- Easiest method to numerically solve Bellman equation for $V(a)$
- Guess value function on RHS of Bellman equation then maximize to get value function on LHS
- Update guess and iterate to convergence right until convergence
- Contraction Mapping Theorem: guaranteed to converge if $\beta < 1$
- Simplest (and slowest)
- Let's see how it works in a deterministic income fluctuations problem

Saving Problem with Deterministic Income

- Assume that income is deterministic and constant $y_t = y$

$$\max_{\{a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$
$$c_t + a_{t+1} \leq y + Ra_t$$
$$a_{t+1} \geq \underline{a}$$

- Recursive formulation of household problem i.e. Bellman equation

$$V(a) = \max_{c,a'} \; u(c) + \beta V(a') \quad \text{s.t.}$$
$$c + a' \leq y + Ra$$
$$a' \geq \underline{a}$$

- Solution is
  - Value function: $V(a)$
  - Policy functions: $c(a), \; a'(a)$

Value Function Iteration

- Step 1: Discretized asset space $\mathcal{A} = \{a_1, a_2, \ldots, a_N\}$. Set $a_1 = \underline{a}$
- Step 2: Guess initial $V_0(a)$. Good guess is

$$V_0(a) = \sum_{t=0}^{\infty} \beta^t u(ra + y) = \frac{u(ra + y)}{1 - \beta}$$

- Step 3: Loop over all $\mathcal{A}$ and solve

$$\begin{aligned}
a'_{\ell+1}(a_i) &= \arg\max_{a' \in \mathcal{A}} u(y + (1 + r) a_i - a') + \beta V_\ell(a') \\
V_{\ell+1}(a_i) &= \max_{a' \in \mathcal{A}} u(y + (1 + r) a_i - a') + \beta V_\ell(a') \\
&= u\left(y + (1 + r) a_i - a'_{\ell+1}(a_i)\right) + \beta V_\ell\left(a'_{\ell+1}(a_i)\right)
\end{aligned}$$

Value Function Iteration

- Step 4: Check for convergence $\epsilon_\ell < \bar{\epsilon}$

$$\epsilon_\ell = \max_i |V_{\ell+1}(a_i) - V_\ell(a_i)|$$

  - if $\epsilon_\ell \geq \bar{\epsilon}$, go to Step 2 with $\ell := \ell + 1$
  - If $\epsilon_\ell < \bar{\epsilon}$, then
- Step 5: Extract optimal policy functions
  - $a'(a) = a_{\ell+1}(a)$
  - $V(a) = V_{\ell+1}(a)$
  - $c(a) = y + (1+r)a - a'(a)$
- Consumption function restricted to implied grid so not very accurate.