Basic Analysis I

Introduction to Real Analysis, Volume I

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May 16, 2022
(version 5.6)

Typeset in LaTeX.

During the writing of this book, the author was in part supported by NSF grants DMS-0900885 and DMS-1362337.

The date is the main identifier of version. The major version / edition number is raised only if there have been substantial changes. Each volume has its own version number. Edition number started at 4, that is, version 4.0, as it was not kept track of before.

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The LaTeX source for the book is available for possible modification and customization at github: https://github.com/jirilebl/ra

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Introduction

0.1 About this book

This first volume is a one semester course in basic analysis. Together with the second volume it is a year-long course. It started its life as my lecture notes for teaching Math 444 at the University of Illinois at Urbana-Champaign (UIUC) in Fall semester 2009. Later I added the metric space chapter to teach Math 521 at University of Wisconsin–Madison (UW). Volume II was added to teach Math 4143/4153 at Oklahoma State University (OSU). A prerequisite for these courses is usually a basic proof course, using for example [H], [F], or [DW].

It should be possible to use the book for both a basic course for students who do not necessarily wish to go to graduate school (such as UIUC 444), but also as a more advanced one-semester course that also covers topics such as metric spaces (such as UW 521). Here are my suggestions for what to cover in a semester course. For a slower course such as UIUC 444:

§0.3, §1.1–§1.4, §2.1–§2.5, §3.1–§3.4, §4.1–§4.2, §5.1–§5.3, §6.1–§6.3

For a more rigorous course covering metric spaces that runs quite a bit faster (such as UW 521):

§0.3, §1.1–§1.4, §2.1–§2.5, §3.1–§3.4, §4.1–§4.2, §5.1–§5.3, §6.1–§6.2, §7.1–§7.6

It should also be possible to run a faster course without metric spaces covering all sections of chapters 0 through 6. The approximate number of lectures given in the section notes through chapter 6 are a very rough estimate and were designed for the slower course. The first few chapters of the book can be used in an introductory proofs course as is done, for example, at Iowa State University Math 201, where this book is used in conjunction with Hammack's Book of Proof [H].

With volume II one can run a year-long course that also covers multivariable topics. It may make sense in this case to cover most of the first volume in the first semester while leaving metric spaces for the beginning of the second semester.

The book normally used for the class at UIUC is Bartle and Sherbert, *Introduction to Real Analysis* third edition [BS]. The structure of the beginning of the book somewhat follows the standard syllabus of UIUC Math 444 and therefore has some similarities with [BS]. A major difference is that we define the Riemann integral using Darboux sums and not tagged partitions. The Darboux approach is far more appropriate for a course of this level.

Our approach allows us to fit a course such as UIUC 444 within a semester and still spend some time on the interchange of limits and end with Picard's theorem on the existence and uniqueness of solutions of ordinary differential equations. This theorem is a wonderful example that uses many results proved in the book. For more advanced students, material may be covered faster so that we arrive at metric spaces and prove Picard's theorem using the fixed point theorem as is usual.

Other excellent books exist. My favorite is Rudin's excellent *Principles of Mathematical Analysis* [R2] or, as it is commonly and lovingly called, *baby Rudin* (to distinguish it from his other great analysis textbook, *big Rudin*). I took a lot of inspiration and ideas from Rudin. However, Rudin is a bit more advanced and ambitious than this present course. For those that wish to continue mathematics, Rudin is a fine investment. An inexpensive and somewhat simpler alternative to Rudin is Rosenlicht's *Introduction to Analysis* [R1]. There is also the freely downloadable *Introduction to Real Analysis* by William Trench [T].

A note about the style of some of the proofs: Many proofs traditionally done by contradiction, I prefer to do by a direct proof or by contrapositive. While the book does include proofs by contradiction, I only do so when the contrapositive statement seemed too awkward, or when contradiction follows rather quickly. In my opinion, contradiction is more likely to get beginning students into trouble, as we are talking about objects that do not exist.

I try to avoid unnecessary formalism where it is unhelpful. Furthermore, the proofs and the language get slightly less formal as we progress through the book, as more and more details are left out to avoid clutter.

As a general rule, I use $:=$ instead of $=$ to define an object rather than to simply show equality. I use this symbol rather more liberally than is usual for emphasis. I use it even when the context is "local," that is, I may simply define a function $f(x) := x^2$ for a single exercise or example.

Finally, I would like to acknowledge Jana Maříková, Glen Pugh, Paul Vojta, Frank Beatrous, Sönmez Şahutoğlu, Jim Brandt, Kenji Kozai, Arthur Busch, Anton Petrunin, Mark Meilstrup, Harold P. Boas, Atilla Yılmaz, Thomas Mahoney, Scott Armstrong, and Paul Sacks, Matthias Weber, Manuele Santoprete, Robert Niemeyer, Amanullah Nabavi, for teaching with the book and giving me lots of useful feedback. Frank Beatrous wrote the University of Pittsburgh version extensions, which served as inspiration for many more recent additions. I would also like to thank Dan Stoneham, Jeremy Sutter, Eliya Gwetta, Daniel Pimentel-Alarcón, Steve Hoerning, Yi Zhang, Nicole Caviris, Kristopher Lee, Baoyue Bi, Hannah Lund, Trevor Mannella, Mitchel Meyer, Gregory Beauregard, Chase Meadors, Andreas Giannopoulos, Nick Nelsen, Ru Wang, Trevor Fancher, Brandon Tague, Wang KP, an anonymous reader or two, and in general all the students in my classes for suggestions and finding errors and typos.

0.2 About analysis

Analysis is the branch of mathematics that deals with inequalities and limits. The present course deals with the most basic concepts in analysis. The goal of the course is to acquaint the reader with rigorous proofs in analysis and also to set a firm foundation for calculus of one variable (and several variables if volume II is also considered).

Calculus has prepared you, the student, for using mathematics without telling you why what you learned is true. To use, or teach, mathematics effectively, you cannot simply know *what* is true, you must know *why* it is true. This course shows you *why* calculus is true. It is here to give you a good understanding of the concept of a limit, the derivative, and the integral.

Let us use an analogy. An auto mechanic that has learned to change the oil, fix broken headlights, and charge the battery, will only be able to do those simple tasks. He will be unable to work independently to diagnose and fix problems. A high school teacher that does not understand the definition of the Riemann integral or the derivative may not be able to properly answer all the students' questions. To this day I remember several nonsensical statements I heard from my calculus teacher in high school, who simply did not understand the concept of the limit, though he could "do" the problems in the textbook.

We start with a discussion of the real number system, most importantly its completeness property, which is the basis for all that comes after. We then discuss the simplest form of a limit, the limit of a sequence. Afterwards, we study functions of one variable, continuity, and the derivative. Next, we define the Riemann integral and prove the fundamental theorem of calculus. We discuss sequences of functions and the interchange of limits. Finally, we give an introduction to metric spaces.

Let us give the most important difference between analysis and algebra. In algebra, we prove equalities directly; we prove that an object, a number perhaps, is equal to another object. In analysis, we usually prove inequalities, and we prove those inequalities by estimating. To illustrate the point, consider the following statement.

Let x be a real number. If $x < \varepsilon$ is true for all real numbers $\varepsilon > 0$, then $x \leq 0$.

This statement is the general idea of what we do in analysis. Suppose next we really wish to prove the equality $x = 0$. In analysis, we prove two inequalities: $x \leq 0$ and $x \geq 0$. To prove the inequality $x \leq 0$, we prove $x < \varepsilon$ for all positive ε. To prove the inequality $x \geq 0$, we prove $x > -\varepsilon$ for all positive ε.

The term *real analysis* is a little bit of a misnomer. I prefer to use simply *analysis*. The other type of analysis, *complex analysis*, really builds up on the present material, rather than being distinct. Furthermore, a more advanced course on real analysis would talk about complex numbers often. I suspect the nomenclature is historical baggage.

Let us get on with the show…

0.3 Basic set theory

Note: 1–3 lectures (some material can be skipped, covered lightly, or left as reading)

Before we start talking about analysis, we need to fix some language. Modern* analysis uses the language of sets, and therefore that is where we start. We talk about sets in a rather informal way, using the so-called "naïve set theory." Do not worry, that is what majority of mathematicians use, and it is hard to get into trouble. The reader has hopefully seen the very basics of set theory and proof writing before, and this section should be a quick refresher.

0.3.1 Sets

Definition 0.3.1. A *set* is a collection of objects called *elements* or *members*. A set with no objects is called the *empty set* and is denoted by $\emptyset$ (or sometimes by $\{\}$).

Think of a set as a club with a certain membership. For example, the students who play chess are members of the chess club. The same student can be a member of many different clubs. However, do not take the analogy too far. A set is only defined by the members that form the set; two sets that have the same members are the same set.

Most of the time we will consider sets of numbers. For example, the set

$$S := \{0,1,2\}$$

is the set containing the three elements 0, 1, and 2. By ":=", we mean we are defining what S is, rather than just showing equality. We write

$$1 \in S$$

to denote that the number 1 belongs to the set S. That is, 1 is a member of S. At times we want to say that two elements are in a set S, so we write "$1,2 \in S$" as a shorthand for "$1 \in S$ and $2 \in S$."

Similarly, we write

$$7 \notin S$$

to denote that the number 7 is not in S. That is, 7 is not a member of S.

The elements of all sets under consideration come from some set we call the *universe*. For simplicity, we often consider the universe to be the set that contains only the elements we are interested in. The universe is generally understood from context and is not explicitly mentioned. In this course, our universe will most often be the set of real numbers.

While the elements of a set are often numbers, other objects, such as other sets, can be elements of a set. A set may also contain some of the same elements as another set. For example,

$$T := \{0,2\}$$

contains the numbers 0 and 2. In this case all elements of T also belong to S. We write $T \subset S$. See Figure 1 for a diagram.

*The term "modern" refers to late 19th century up to the present.

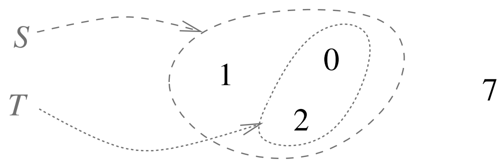


Figure 1: A diagram of the example sets S and its subset T.

Definition 0.3.2.

(i) A set A is a *subset* of a set B if $x \in A$ implies $x \in B$, and we write $A \subset B$. That is, all members of A are also members of B. At times we write $B \supset A$ to mean the same thing.

(ii) Two sets A and B are *equal* if $A \subset B$ and $B \subset A$. We write $A = B$. That is, A and B contain exactly the same elements. If it is not true that A and B are equal, then we write $A \neq B$.

(iii) A set A is a *proper subset* of B if $A \subset B$ and $A \neq B$. We write $A \subsetneq B$.

For the example S and T defined above, $T \subset S$, but $T \neq S$. So T is a proper subset of S. If $A = B$, then A and B are simply two names for the same exact set.

To define sets, one often uses the *set building notation*,

$$\bigl\{x \in A : P(x)\bigr\}.$$

This notation refers to a subset of the set A containing all elements of A that satisfy the property $P(x)$. Using $S = \{0,1,2\}$ as above, $\{x \in S : x \neq 2\}$ is the set $\{0,1\}$. The notation is sometimes abbreviated as $\bigl\{x : P(x)\bigr\}$, that is, A is not mentioned when understood from context. Furthermore, $x \in A$ is sometimes replaced with a formula to make the notation easier to read.

Example 0.3.3: The following are sets including the standard notations.

(i) The set of *natural numbers*, $\mathbb{N} := \{1,2,3,\ldots\}$.

(ii) The set of *integers*, $\mathbb{Z} := \{0,-1,1,-2,2,\ldots\}$.

(iii) The set of *rational numbers*, $\mathbb{Q} := \bigl\{\frac{m}{n} : m,n \in \mathbb{Z} \text{ and } n \neq 0\bigr\}$.

(iv) The set of even natural numbers, $\{2m : m \in \mathbb{N}\}$.

(v) The set of real numbers, $\mathbb{R}$.

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

We create new sets out of old ones by applying some natural operations.

Definition 0.3.4.

(i) A *union* of two sets A and B is defined as

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

(ii) An *intersection* of two sets A and B is defined as

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

(iii) A *complement of B relative to A* (or *set-theoretic difference* of A and B) is defined as

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}.$$

(iv) We say *complement* of B and write B^c instead of $A \setminus B$ if the set A is either the entire universe or if it is the obvious set containing B, and is understood from context.

(v) We say sets A and B are *disjoint* if $A \cap B = \emptyset$.

The notation B^c may be a little vague at this point. If the set B is a subset of the real numbers $\mathbb{R}$, then B^c means $\mathbb{R} \setminus B$. If B is naturally a subset of the natural numbers, then B^c is $\mathbb{N} \setminus B$. If ambiguity can arise, we use the set difference notation $A \setminus B$.

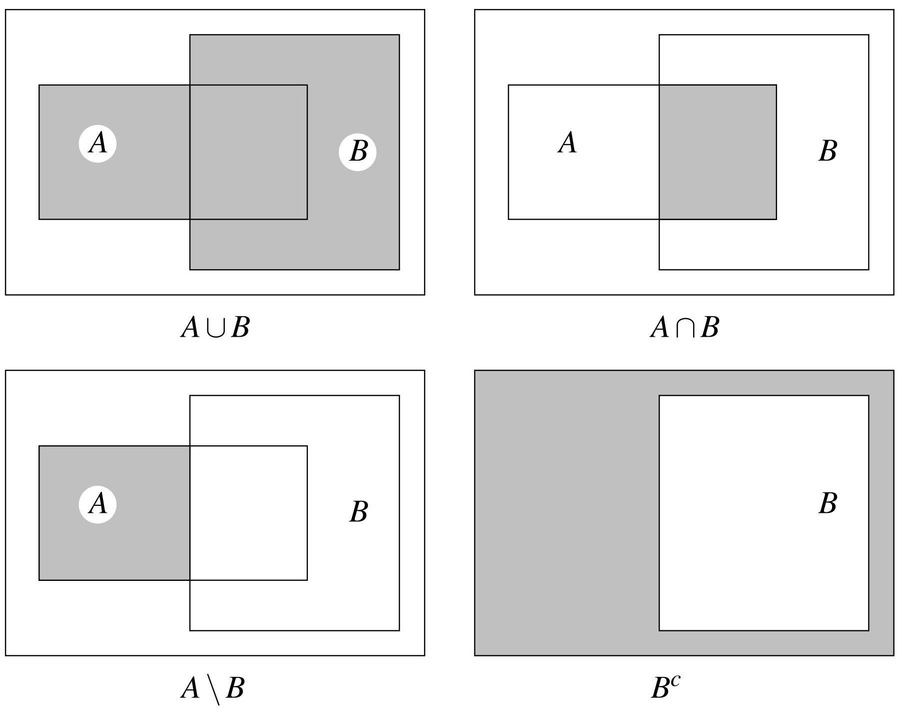


Figure 2: Venn diagrams of set operations, the result of the operation is shaded.

We illustrate the operations on the *Venn diagrams* in Figure 2. Let us now establish one of most basic theorems about sets and logic.

Theorem 0.3.5 (DeMorgan)**.** *Let A, B, C be sets. Then*

$$(B \cup C)^c = B^c \cap C^c,$$
$$(B \cap C)^c = B^c \cup C^c,$$

or, more generally,

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C),$$
$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Proof. The first statement is proved by the second statement if we assume the set A is our "universe."

Let us prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. Remember the definition of equality of sets. First, we must show that if $x \in A \setminus (B \cup C)$, then $x \in (A \setminus B) \cap (A \setminus C)$. Second, we must also show that if $x \in (A \setminus B) \cap (A \setminus C)$, then $x \in A \setminus (B \cup C)$.

So let us assume $x \in A \setminus (B \cup C)$. Then x is in A, but not in B nor C. Hence x is in A and not in B, that is, $x \in A \setminus B$. Similarly $x \in A \setminus C$. Thus $x \in (A \setminus B) \cap (A \setminus C)$.

On the other hand suppose $x \in (A \setminus B) \cap (A \setminus C)$. In particular, $x \in (A \setminus B)$, so $x \in A$ and $x \notin B$. Also as $x \in (A \setminus C)$, then $x \notin C$. Hence $x \in A \setminus (B \cup C)$.

The proof of the other equality is left as an exercise. □

The result above we called a *Theorem*, while most results we call a *Proposition*, and a few we call a *Lemma* (a result leading to another result) or *Corollary* (a quick consequence of the preceding result). Do not read too much into the naming. Some of it is traditional, some of it is stylistic choice. It is not necessarily true that a *Theorem* is always "more important" than a *Proposition* or a *Lemma*.

We will also need to intersect or union several sets at once. If there are only finitely many, then we simply apply the union or intersection operation several times. However, suppose we have an infinite collection of sets (a set of sets) $\{A_1, A_2, A_3, \ldots\}$. We define

$$\bigcup_{n=1}^{\infty} A_n := \{x : x \in A_n \text{ for some } n \in \mathbb{N}\},$$
$$\bigcap_{n=1}^{\infty} A_n := \{x : x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

We can also have sets indexed by two natural numbers. For example, we can have the set of sets $\{A_{1,1}, A_{1,2}, A_{2,1}, A_{1,3}, A_{2,2}, A_{3,1}, \ldots\}$. Then we write

$$\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m} = \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} A_{n,m} \right).$$

And similarly with intersections.

It is not hard to see that we can take the unions in any order. However, switching the order of unions and intersections is not generally permitted without proof. For instance,

$$\bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \{k \in \mathbb{N} : mk < n\} = \bigcup_{n=1}^{\infty} \emptyset = \emptyset.$$

However,

$$\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{k \in \mathbb{N} : mk < n\} = \bigcap_{m=1}^{\infty} \mathbb{N} = \mathbb{N}.$$

Sometimes, the index set is not the natural numbers. In such a case we require a more general notation. Suppose I is some set and for each $\lambda \in I$, there is a set A_λ. Then we define

$$\bigcup_{\lambda \in I} A_\lambda := \{x : x \in A_\lambda \text{ for some } \lambda \in I\}, \qquad \bigcap_{\lambda \in I} A_\lambda := \{x : x \in A_\lambda \text{ for all } \lambda \in I\}.$$

0.3.2 Induction

When a statement includes an arbitrary natural number, a common method of proof is the principle of induction. We start with the set of natural numbers $\mathbb{N} = \{1,2,3,\dots\}$, and we give them their natural ordering, that is, $1 < 2 < 3 < 4 < \cdots$. By $S \subset \mathbb{N}$ having a *least element*, we mean that there exists an $x \in S$, such that for every $y \in S$, we have $x \leq y$.

The natural numbers $\mathbb{N}$ ordered in the natural way possess the so-called *well ordering property*. We take this property as an axiom; we simply assume it is true.

Well ordering property of $\mathbb{N}$. *Every nonempty subset of $\mathbb{N}$ has a least (smallest) element.*

The *principle of induction* is the following theorem, which is in a sense* equivalent to the well ordering property of the natural numbers.

Theorem 0.3.6 (Principle of induction)**.** *Let $P(n)$ be a statement depending on a natural number n. Suppose that*

(i) (basis statement) $P(1)$ *is true.*

(ii) (induction step) *If $P(n)$ is true, then $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let S be the set of natural numbers m for which $P(m)$ is not true. Suppose for contradiction that S is nonempty. Then S has a least element by the well ordering property. Call $m \in S$ the least element of S. We know $1 \notin S$ by hypothesis. So $m > 1$, and $m-1$ is a natural number as well. Since m is the least element of S, we know that $P(m-1)$ is true. But the induction step says that $P(m-1+1) = P(m)$ is true, contradicting the statement that $m \in S$. Therefore, S is empty and $P(n)$ is true for all $n \in \mathbb{N}$. □

Sometimes it is convenient to start at a different number than 1, all that changes is the labeling. The assumption that $P(n)$ is true in "if $P(n)$ is true, then $P(n+1)$ is true" is usually called the *induction hypothesis*.

Example 0.3.7: Let us prove that for all $n \in \mathbb{N}$,

$$2^{n-1} \leq n! \qquad (\text{recall } n! = 1 \cdot 2 \cdot 3 \cdots n).$$

We let $P(n)$ be the statement that $2^{n-1} \leq n!$ is true. By plugging in $n = 1$, we see that $P(1)$ is true.

Suppose $P(n)$ is true. That is, suppose $2^{n-1} \leq n!$ holds. Multiply both sides by 2 to obtain

$$2^n \leq 2(n!).$$

As $2 \leq (n+1)$ when $n \in \mathbb{N}$, we have $2(n!) \leq (n+1)(n!) = (n+1)!$. That is,

$$2^n \leq 2(n!) \leq (n+1)!,$$

and hence $P(n+1)$ is true. By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. In other words, $2^{n-1} \leq n!$ is true for all $n \in \mathbb{N}$.

*To be completely rigorous, this equivalence is only true if we also assume as an axiom that $n-1$ exists for all natural numbers bigger than 1, which we do. In this book, we are assuming all the usual arithmetic holds.

Example 0.3.8: We claim that for all $c \neq 1$,

$$1 + c + c^2 + \cdots + c^n = \frac{1 - c^{n+1}}{1 - c}.$$

Proof: It is easy to check that the equation holds with $n = 1$. Suppose it is true for n. Then

$$\begin{aligned} 1 + c + c^2 + \cdots + c^n + c^{n+1} &= (1 + c + c^2 + \cdots + c^n) + c^{n+1} \\ &= \frac{1 - c^{n+1}}{1 - c} + c^{n+1} \\ &= \frac{1 - c^{n+1} + (1 - c)c^{n+1}}{1 - c} \\ &= \frac{1 - c^{n+2}}{1 - c}. \end{aligned}$$

Sometimes, it is easier to use in the inductive step that $P(k)$ is true for all $k = 1, 2, \ldots, n$, not just for $k = n$. This principle is called *strong induction* and is equivalent to the normal induction above. The proof of that equivalence is left as an exercise.

Theorem 0.3.9 (Principle of strong induction)**.** *Let $P(n)$ be a statement depending on a natural number n. Suppose that*

(i) (basis statement) $P(1)$ *is true.*

(ii) (induction step) *If $P(k)$ is true for all $k = 1, 2, \ldots, n$, then $P(n+1)$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

0.3.3 Functions

Informally, a *set-theoretic function* f taking a set A to a set B is a mapping that to each $x \in A$ assigns a unique $y \in B$. We write $f\colon A \to B$. An example function $f\colon S \to T$ taking $S := \{0, 1, 2\}$ to $T := \{0, 2\}$ can be defined by assigning $f(0) := 2$, $f(1) := 2$, and $f(2) := 0$. That is, a function $f\colon A \to B$ is a black box, into which we stick an element of A and the function spits out an element of B. Sometimes f is called a *mapping* or a *map*, and we say *f maps A to B*.

Often, functions are defined by some sort of formula; however, you should really think of a function as just a very big table of values. The subtle issue here is that a single function can have several formulas, all giving the same function. Also, for many functions, there is no formula that expresses its values.

To define a function rigorously, first let us define the Cartesian product.

Definition 0.3.10. Let A and B be sets. The *Cartesian product* is the set of tuples defined as

$$A \times B := \bigl\{ (x, y) : x \in A, y \in B \bigr\}.$$

For instance, $\{a, b\} \times \{c, d\} = \bigl\{ (a, c), (a, d), (b, c), (b, d) \bigr\}$. A more complicated example is the set $[0, 1] \times [0, 1]$: a subset of the plane bounded by a square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. When A and B are the same set we sometimes use a superscript 2 to denote such a product. For example, $[0, 1]^2 = [0, 1] \times [0, 1]$ or $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (the Cartesian plane).

Definition 0.3.11. A *function* $f : A \to B$ is a subset f of $A \times B$ such that for each $x \in A$, there exists a unique $y \in B$ for which $(x, y) \in f$. We write $f(x) = y$. Sometimes the set f is called the *graph* of the function rather than the function itself.

The set A is called the *domain* of f (and sometimes confusingly denoted $D(f)$). The set

$$R(f) := \{y \in B : \text{there exists an } x \in A \text{ such that } f(x) = y\}$$

is called the *range* of f. The set B is called the *codomain* of f.

It is possible that the range $R(f)$ is a proper subset of the codomain B, while the domain of f is always equal to A. We generally assume that the domain of f is nonempty.

Example 0.3.12: From calculus, you are most familiar with functions taking real numbers to real numbers. However, you saw some other types of functions as well. The derivative is a function mapping the set of differentiable functions to the set of all functions. Another example is the Laplace transform, which also takes functions to functions. Yet another example is the function that takes a continuous function g defined on the interval $[0, 1]$ and returns the number $\int_0^1 g(x)\,dx$.

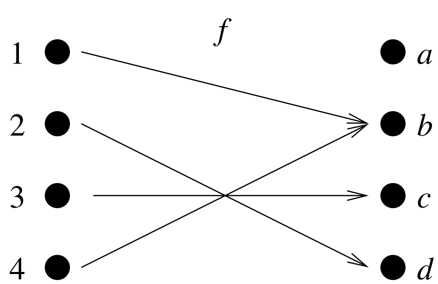
Definition 0.3.13. Consider a function $f : A \to B$. Define the *image* (or *direct image*) of a subset $C \subset A$ as

$$f(C) := \big\{f(x) \in B : x \in C\big\}.$$

Define the *inverse image* of a subset $D \subset B$ as

$$f^{-1}(D) := \big\{x \in A : f(x) \in D\big\}.$$

In particular, $R(f) = f(A)$, the range is the direct image of the domain A.



$$\begin{aligned}
&f(\{1,2,3,4\}) = \{b,c,d\} \\
&f(\{1,2,4\}) = \{b,d\} \\
&f(\{1\}) = \{b\} \\
&f^{-1}(\{a,b,c\}) = \{1,3,4\} \\
&f^{-1}(\{a\}) = \emptyset \\
&f^{-1}(\{b\}) = \{1,4\}
\end{aligned}$$

Figure 3: Example of direct and inverse images for the function $f : \{1,2,3,4\} \to \{a,b,c,d\}$ defined by $f(1) := b$, $f(2) := d$, $f(3) := c$, $f(4) := b$.

Example 0.3.14: Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \sin(\pi x)$. Then $f\big([0, 1/2]\big) = [0, 1]$, $f^{-1}\big(\{0\}\big) = \mathbb{Z}$, etc.

Proposition 0.3.15. *Consider $f : A \to B$. Let C, D be subsets of B. Then*

$$\begin{aligned}
f^{-1}(C \cup D) &= f^{-1}(C) \cup f^{-1}(D), \\
f^{-1}(C \cap D) &= f^{-1}(C) \cap f^{-1}(D), \\
f^{-1}(C^c) &= \big(f^{-1}(C)\big)^c.
\end{aligned}$$

Read the last line of the proposition as $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$.

Proof. We start with the union. If $x \in f^{-1}(C \cup D)$, then x is taken to C or D, that is, $f(x) \in C$ or $f(x) \in D$. Thus $f^{-1}(C \cup D) \subset f^{-1}(C) \cup f^{-1}(D)$. Conversely if $x \in f^{-1}(C)$, then $x \in f^{-1}(C \cup D)$. Similarly for $x \in f^{-1}(D)$. Hence $f^{-1}(C \cup D) \supset f^{-1}(C) \cup f^{-1}(D)$, and we have equality.

The rest of the proof is left as an exercise. □

The proposition does not hold for direct images. We do have the following weaker result.

Proposition 0.3.16. *Consider $f \colon A \to B$. Let C, D be subsets of A. Then*

$$f(C \cup D) = f(C) \cup f(D),$$
$$f(C \cap D) \subset f(C) \cap f(D).$$

The proof is left as an exercise.

Definition 0.3.17. Let $f \colon A \to B$ be a function. The function f is said to be *injective* or *one-to-one* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. In other words, f is injective if for all $y \in B$, the set $f^{-1}(\{y\})$ is empty or consists of a single element. We call such an f an *injection*.

If $f(A) = B$, then we say f is *surjective* or *onto*. In other words, f is surjective if the range and the codomain of f are equal. We call such an f a *surjection*.

If f is both an surjective and injective, then we say f is *bijective* or that f is a *bijection*.

When $f \colon A \to B$ is a bijection, then the inverse image of a single element, $f^{-1}(\{y\})$, is always a unique element of A. We then consider f^{-1} as a function $f^{-1} \colon B \to A$ and we write simply $f^{-1}(y)$. In this case, we call f^{-1} the *inverse function* of f. For instance, for the bijection $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^3$, we have $f^{-1}(x) = \sqrt[3]{x}$.

Definition 0.3.18. Consider $f \colon A \to B$ and $g \colon B \to C$. The *composition* of the functions f and g is the function $g \circ f \colon A \to C$ defined as

$$(g \circ f)(x) := g\bigl(f(x)\bigr).$$

For example, if $f \colon \mathbb{R} \to \mathbb{R}$ is $f(x) := x^3$ and $g \colon \mathbb{R} \to \mathbb{R}$ is $g(y) = \sin(y)$, then $(g \circ f)(x) = \sin(x^3)$.

0.3.4 Relations and equivalence classes

We often compare two objects in some way. We say $1 < 2$ for natural numbers, or $1/2 = 2/4$ for rational numbers, or $\{a, c\} \subset \{a, b, c\}$ for sets. The '$<$', '$=$', and '$\subset$' are examples of relations.

Definition 0.3.19. Given a set A, a *binary relation* on A is a subset $\mathscr{R} \subset A \times A$, which are those pairs where the relation is said to hold. Instead of $(a, b) \in \mathscr{R}$, we write $a \mathscr{R} b$.

Example 0.3.20: Take $A := \{1, 2, 3\}$.

Consider the relation '$<$'. The corresponding set of pairs is $\bigl\{(1,2), (1,3), (2,3)\bigr\}$. So $1 < 2$ holds as $(1,2)$ is in the corresponding set of pairs, but $3 < 1$ does not hold as $(3,1)$ is not in the set.

Similarly, the relation '$=$' is defined by the set of pairs $\bigl\{(1,1), (2,2), (3,3)\bigr\}$.

Any subset of $A \times A$ is a relation. Let us define the relation $\dagger$ via $\bigl\{(1,2), (2,1), (2,3), (3,1)\bigr\}$, then $1 \dagger 2$ and $3 \dagger 1$ are true, but $1 \dagger 3$ is not.

Definition 0.3.21. Let $\mathscr{R}$ be a relation on a set A. Then $\mathscr{R}$ is said to be

(i) *Reflexive* if $a \mathscr{R} a$ for all $a \in A$.

(ii) *Symmetric* if $a \mathscr{R} b$ implies $b \mathscr{R} a$.

(iii) *Transitive* if $a \mathscr{R} b$ and $b \mathscr{R} c$ implies $a \mathscr{R} c$.

If $\mathscr{R}$ is reflexive, symmetric, and transitive, then it is said to be an *equivalence relation*.

Example 0.3.22: Let $A := \{1,2,3\}$ as above. The relation '$<$' is transitive, but neither reflexive nor symmetric. The relation '$\leq$' defined by $\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ is reflexive and transitive, but not symmetric. Finally, a relation '$\star$' defined by $\{(1,1),(1,2),(2,1),(2,2),(3,3)\}$ is an equivalence relation.

Equivalence relations are useful in that they divide a set into sets of "equivalent" elements.

Definition 0.3.23. Let A be a set and $\mathscr{R}$ an equivalence relation. An *equivalence class* of $a \in A$, often denoted by $[a]$, is the set $\{x \in A : a \mathscr{R} x\}$.

For example, given the relation '$\star$' above, there are two equivalence classes, $[1] = [2] = \{1,2\}$ and $[3] = \{3\}$.

Reflexivity guarantees that $a \in [a]$. Symmetry guarantees that if $b \in [a]$, then $a \in [b]$. Finally, transitivity guarantees that if $a \in [b]$ and $b \in [c]$, then $a \in [c]$. In particular, we have the following proposition, whose proof is an exercise.

Proposition 0.3.24. *If $\mathscr{R}$ is an equivalence relation on a set A, then every $a \in A$ is in exactly one equivalence class. In particular, $a \mathscr{R} b$ if and only $[a] = [b]$.*

Example 0.3.25: The set of rational numbers can be defined as equivalence classes of a pair of an integer and a natural number, that is elements of $\mathbb{Z} \times \mathbb{N}$. The relation is defined by $(a,b) \sim (c,d)$ whenever $ad = bc$. It is left as an exercise to prove that '$\sim$' is an equivalence relation. Usually the equivalence class $[(a,b)]$ is written as a/b.

0.3.5 Cardinality

A subtle issue in set theory and one generating a considerable amount of confusion among students is that of cardinality, or "size" of sets. The concept of cardinality is important in modern mathematics in general and in analysis in particular. In this section, we will see the first really unexpected theorem.

Definition 0.3.26. Let A and B be sets. We say A and B have the same *cardinality* when there exists a bijection $f : A \to B$. We denote by $|A|$ the equivalence class of all sets with the same cardinality as A and we simply call $|A|$ the cardinality of A.

For example, $\{1,2,3\}$ has the same cardinality as $\{a,b,c\}$ by defining a bijection $f(1) := a$, $f(2) := b$, $f(3) := c$. Clearly the bijection is not unique.

The existence of a bijection really is an equivalence relation. The identity, $f(x) := x$, is a bijection showing reflexivity. If f is a bijection, then so is f^{-1} showing symmetricity. If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f$ is a bijection of A and C showing transitivity. A set A has the same cardinality as the empty set if and only if A itself is the empty set: If B is nonempty, then no function $f : B \to \emptyset$ can exist. In particular, there is no bijection of B and $\emptyset$.

Definition 0.3.27. Suppose A has the same cardinality as $\{1,2,3,\ldots,n\}$ for some $n \in \mathbb{N}$. We then write $|A| := n$. If A is empty, we write $|A| := 0$. In either case, we say that A is *finite*.

We say A is *infinite* or "of infinite cardinality" if A is not finite.

That the notation $|A| = n$ is justified we leave as an exercise. That is, for each nonempty finite set A, there exists a unique natural number n such that there exists a bijection from A to $\{1,2,3,\ldots,n\}$.

We can order sets by size.

Definition 0.3.28. We write

$$|A| \leq |B|$$

if there exists an injection from A to B. We write $|A| = |B|$ if A and B have the same cardinality. We write $|A| < |B|$ if $|A| \leq |B|$, but A and B do not have the same cardinality.

We state without proof that A and B have the same cardinality if and only if $|A| \leq |B|$ and $|B| \leq |A|$. This is the so-called Cantor–Bernstein–Schröder theorem. Furthermore, if A and B are any two sets, we can always write $|A| \leq |B|$ or $|B| \leq |A|$. The issues surrounding this last statement are very subtle. As we do not require either of these two statements, we omit proofs.

The truly interesting cases of cardinality are infinite sets. We will distinguish two types of infinite cardinality.

Definition 0.3.29. If $|A| = |\mathbb{N}|$, then A is said to be *countably infinite*. If A is finite or countably infinite, then we say A is *countable*. If A is not countable, then A is said to be *uncountable*.

The cardinality of $\mathbb{N}$ is usually denoted as $\aleph_0$ (read as aleph-naught)*.

Example 0.3.30: The set of even natural numbers has the same cardinality as $\mathbb{N}$. Proof: Let $E \subset \mathbb{N}$ be the set of even natural numbers. Given $k \in E$, write $k = 2n$ for some $n \in \mathbb{N}$. Then $f(n) := 2n$ defines a bijection $f : \mathbb{N} \to E$.

In fact, let us mention without proof the following characterization of infinite sets: *A set is infinite if and only if it is in one-to-one correspondence with a proper subset of itself.*

Example 0.3.31: $\mathbb{N} \times \mathbb{N}$ is a countably infinite set. Proof: Arrange the elements of $\mathbb{N} \times \mathbb{N}$ as follows $(1,1)$, $(1,2)$, $(2,1)$, $(1,3)$, $(2,2)$, $(3,1)$, $\ldots$. That is, always write down first all the elements whose two entries sum to k, then write down all the elements whose entries sum to $k+1$ and so on. Define a bijection with $\mathbb{N}$ by letting 1 go to $(1,1)$, 2 go to $(1,2)$, and so on. See Figure 4.

Example 0.3.32: The set of rational numbers is countable. Proof: (informal) For positive rational numbers follow the same procedure as in the previous example, writing $1/1$, $1/2$, $2/1$, etc. However, leave out fractions (such as $2/2$) that have already appeared. The list would continue: $1/3$, $3/1$, $1/4$, $2/3$, etc. For all rational numbers, include 0 and the negative numbers: 0, $1/1$, $-1/1$, $1/2$, $-1/2$, etc.

For completeness, we mention the following statements from the exercises. *If $A \subset B$ and B is countable, then A is countable.* The contrapositive of the statement is that if A is uncountable, then B is uncountable. As a consequence, if $|A| < |\mathbb{N}|$, then A is finite. Similarly, if B is finite and $A \subset B$, then A is finite.

*For the fans of the TV show *Futurama*, there is a movie theater in one episode called an $\aleph_0$-plex.

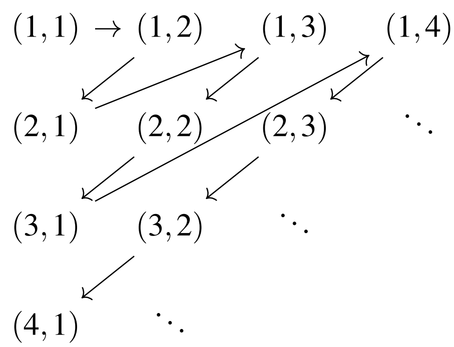


Figure 4: Showing $\mathbb{N} \times \mathbb{N}$ is countable.

We give the first truly striking result. First, we need a notation for the set of all subsets of a set.

Definition 0.3.33. The *power set* of a set A, denoted by $\mathscr{P}(A)$, is the set of all subsets of A.

For example, if $A := \{1,2\}$, then $\mathscr{P}(A) = \big\{\emptyset, \{1\}, \{2\}, \{1,2\}\big\}$. In particular, $|A| = 2$ and $|\mathscr{P}(A)| = 4 = 2^2$. In general, for a finite set A of cardinality n, the cardinality of $\mathscr{P}(A)$ is 2^n. This fact is left as an exercise. Hence, for a finite set A, the cardinality of $\mathscr{P}(A)$ is strictly larger than the cardinality of A. What is an unexpected and striking fact is that this statement is still true for infinite sets.

Theorem 0.3.34 (Cantor*)**.** $|A| < |\mathscr{P}(A)|$. *In particular, there exists no surjection from A onto $\mathscr{P}(A)$.*

Proof. There exists an injection $f \colon A \to \mathscr{P}(A)$. For $x \in A$, define $f(x) := \{x\}$. Thus, $|A| \leq |\mathscr{P}(A)|$.

To finish the proof, we must show that no function $g \colon A \to \mathscr{P}(A)$ is a surjection. Suppose $g \colon A \to \mathscr{P}(A)$ is a function. So for $x \in A$, $g(x)$ is a subset of A. Define the set

$$B := \big\{x \in A : x \notin g(x)\big\}.$$

We claim that B is not in the range of g and hence g is not a surjection. Suppose for contradiction that there exists an x_0 such that $g(x_0) = B$. Either $x_0 \in B$ or $x_0 \notin B$. If $x_0 \in B$, then $x_0 \notin g(x_0) = B$, which is a contradiction. If $x_0 \notin B$, then $x_0 \in g(x_0) = B$, which is again a contradiction. Thus such an x_0 does not exist. Therefore, B is not in the range of g, and g is not a surjection. As g was an arbitrary function, no surjection exists. $\square$

One particular consequence of this theorem is that there do exist uncountable sets, as $\mathscr{P}(\mathbb{N})$ must be uncountable. A related fact is that the set of real numbers (which we study in the next chapter) is uncountable. The existence of uncountable sets may seem unintuitive, and the theorem caused quite a controversy at the time it was announced. The theorem not only says that uncountable sets exist, but that there in fact exist progressively larger and larger infinite sets $\mathbb{N}$, $\mathscr{P}(\mathbb{N})$, $\mathscr{P}(\mathscr{P}(\mathbb{N}))$, $\mathscr{P}(\mathscr{P}(\mathscr{P}(\mathbb{N})))$, etc.

*Named after the German mathematician Georg Ferdinand Ludwig Philipp Cantor (1845–1918).

0.3.6 Exercises

Exercise* 0.3.1*: *Show* $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Exercise* 0.3.2*: *Prove that the principle of strong induction is equivalent to the standard induction.*

Exercise* 0.3.3*: *Finish the proof of Proposition 0.3.15.*

Exercise* 0.3.4*:

a) *Prove Proposition 0.3.16.*
b) *Find an example for which equality of sets in* $f(C \cap D) \subset f(C) \cap f(D)$ *fails. That is, find an* f, A, B, C, *and* D *such that* $f(C \cap D)$ *is a proper subset of* $f(C) \cap f(D)$.

Exercise* 0.3.5** (Tricky):*** *Prove that if* A *is nonempty and finite, then there exists a unique* $n \in \mathbb{N}$ *such that there exists a bijection between* A *and* $\{1,2,3,\ldots,n\}$. *In other words, the notation* $|A| := n$ *is justified. Hint: Show that if* $n > m$, *then there is no injection from* $\{1,2,3,\ldots,n\}$ *to* $\{1,2,3,\ldots,m\}$.

Exercise* 0.3.6*: *Prove:*

a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Exercise* 0.3.7*: *Let* $A \Delta B$ *denote the* symmetric difference*, that is, the set of all elements that belong to either* A *or* B*, but not to both* A *and* B.

a) *Draw a Venn diagram for* $A \Delta B$.
b) *Show* $A \Delta B = (A \setminus B) \cup (B \setminus A)$.
c) *Show* $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Exercise* 0.3.8*: *For each* $n \in \mathbb{N}$, *let* $A_n := \{(n+1)k : k \in \mathbb{N}\}$.

a) *Find* $A_1 \cap A_2$.
b) *Find* $\bigcup_{n=1}^{\infty} A_n$.
c) *Find* $\bigcap_{n=1}^{\infty} A_n$.

Exercise* 0.3.9*: *Determine* $\mathscr{P}(S)$ *(the power set) for each of the following:*

a) $S = \emptyset$,
b) $S = \{1\}$,
c) $S = \{1,2\}$,
d) $S = \{1,2,3,4\}$.

Exercise* 0.3.10*: *Let* $f : A \to B$ *and* $g : B \to C$ *be functions.*

a) *Prove that if* $g \circ f$ *is injective, then* f *is injective.*
b) *Prove that if* $g \circ f$ *is surjective, then* g *is surjective.*
c) *Find an explicit example where* $g \circ f$ *is bijective, but neither* f *nor* g *is bijective.*

Exercise* 0.3.11*: *Prove by induction that* $n < 2^n$ *for all* $n \in \mathbb{N}$.

Exercise* 0.3.12*: *Show that for a finite set* A *of cardinality* n, *the cardinality of* $\mathscr{P}(A)$ *is* 2^n.

Exercise* 0.3.13*: *Prove* $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ *for all* $n \in \mathbb{N}$.

Exercise* 0.3.14*: *Prove* $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ *for all* $n \in \mathbb{N}$.

Exercise* 0.3.15*: *Prove that* $n^3 + 5n$ *is divisible by* 6 *for all* $n \in \mathbb{N}$.

Exercise* 0.3.16*: *Find the smallest* $n \in \mathbb{N}$ *such that* $2(n+5)^2 < n^3$ *and call it* n_0. *Show that* $2(n+5)^2 < n^3$ *for all* $n \geq n_0$.

Exercise* 0.3.17*: *Find all* $n \in \mathbb{N}$ *such that* $n^2 < 2^n$.

Exercise* 0.3.18*: *Prove the well ordering property of* $\mathbb{N}$ *using the principle of induction.*

Exercise* 0.3.19*: *Give an example of a countably infinite collection of finite sets* $A_1, A_2, \ldots$, *whose union is not a finite set.*

Exercise* 0.3.20*: *Give an example of a countably infinite collection of infinite sets* $A_1, A_2, \ldots$, *with* $A_j \cap A_k$ *being infinite for all* j *and* k, *such that* $\bigcap_{j=1}^{\infty} A_j$ *is nonempty and finite.*

Exercise* 0.3.21*: *Suppose* $A \subset B$ *and* B *is finite. Prove that* A *is finite. That is, if* A *is nonempty, construct a bijection of* A *to* $\{1, 2, \ldots, n\}$.

Exercise* 0.3.22*: *Prove Proposition 0.3.24. That is, prove that if* $\mathscr{R}$ *is an equivalence relation on a set* A, *then every* $a \in A$ *is in exactly one equivalence class. Then prove that* $a \mathscr{R} b$ *if and only if* $[a] = [b]$.

Exercise* 0.3.23*: *Prove that the relation '*$\sim$*' in Example 0.3.25 is an equivalence relation.*

Exercise* 0.3.24*:

a) Suppose $A \subset B$ *and* B *is countably infinite. By constructing a bijection, show that* A *is countable (that is,* A *is empty, finite, or countably infinite).*

b) Use part a) to show that if $|A| < |\mathbb{N}|$, *then* A *is finite.*

Exercise* 0.3.25** (Challenging):*** *Suppose* $|\mathbb{N}| \leq |S|$, *or in other words,* S *contains a countably infinite subset. Show that there exists a countably infinite subset* $A \subset S$ *and a bijection between* $S \setminus A$ *and* S.

Chapter 1

Real Numbers

1.1 Basic properties

Note: 1.5 lectures

The main object we work with in analysis is the set of real numbers. As this set is so fundamental, often much time is spent on formally constructing the set of real numbers. However, we take an easier approach here and just assume that a set with the correct properties exists. We start with the definitions of those properties.

Definition 1.1.1. An *ordered set* is a set S together with a relation $<$ such that

(i) *(trichotomy)* For all $x, y \in S$, exactly one of $x < y$, $x = y$, or $y < x$ holds.

(ii) *(transitivity)* If $x, y, z \in S$ are such that $x < y$ and $y < z$, then $x < z$.

We write $x \leq y$ if $x < y$ or $x = y$. We define $>$ and $\geq$ in the obvious way.

The set of rational numbers $\mathbb{Q}$ is an ordered set by letting $x < y$ if and only if $y - x$ is a positive rational number, that is if $y - x = p/q$ where $p, q \in \mathbb{N}$. Similarly, $\mathbb{N}$ and $\mathbb{Z}$ are also ordered sets.

There are other ordered sets than sets of numbers. For example, the set of countries can be ordered by landmass, so India $>$ Lichtenstein. A typical ordered set that you have used since primary school is the dictionary. It is the ordered set of words where the order is the so-called lexicographic ordering. Such ordered sets often appear, for example, in computer science. In this book we will mostly be interested in ordered sets of numbers.

Definition 1.1.2. Let $E \subset S$, where S is an ordered set.

(i) If there exists a $b \in S$ such that $x \leq b$ for all $x \in E$, then we say E is *bounded above* and b is an *upper bound* of E.

(ii) If there exists a $b \in S$ such that $x \geq b$ for all $x \in E$, then we say E is *bounded below* and b is a *lower bound* of E.

(iii) If there exists an upper bound b_0 of E such that whenever b is an upper bound for E we have $b_0 \leq b$, then b_0 is called the *least upper bound* or the *supremum* of E. See Figure 1.1. We write

$$\sup E := b_0.$$

(iv) Similarly, if there exists a lower bound b_0 of E such that whenever b is a lower bound for E we have $b_0 \geq b$, then b_0 is called the *greatest lower bound* or the *infimum* of E. We write

$$\inf E := b_0.$$

When a set E is both bounded above and bounded below, we say simply that E is *bounded*.

The notation $\sup E$ and $\inf E$ is justified as the supremum (or infimum) is unique (if it exists): If b and b' are suprema of E, then $b \leq b'$ and $b' \leq b$, because both b and b' are the least upper bounds, so $b = b'$.

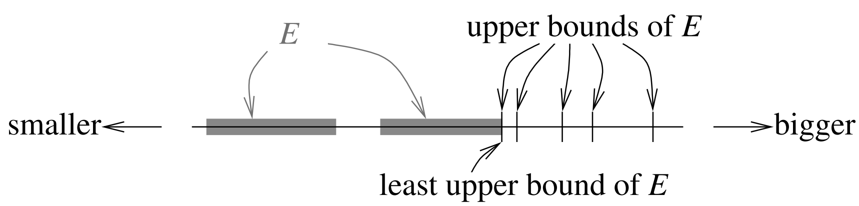


Figure 1.1: A set E bounded above and the least upper bound of E.

A simple example: Let $S := \{a,b,c,d,e\}$ be ordered as $a < b < c < d < e$, and let $E := \{a,c\}$. Then c, d, and e are upper bounds of E, and c is the least upper bound or supremum of E.

A supremum or infimum for E (even if it exists) need not be in E. The set $E := \{x \in \mathbb{Q} : x < 1\}$ has a least upper bound of 1, but 1 is not in the set E itself. The set $G := \{x \in \mathbb{Q} : x \leq 1\}$ also has an upper bound of 1, and in this case $1 \in G$. The set $P := \{x \in \mathbb{Q} : x \geq 0\}$ has no upper bound (why?) and therefore it cannot have a least upper bound. The set P does have a greatest lower bound: 0.

Definition 1.1.3. An ordered set S has the *least-upper-bound property* if every nonempty subset $E \subset S$ that is bounded above has a least upper bound, that is $\sup E$ exists in S.

The *least-upper-bound property* is sometimes called the *completeness property* or the *Dedekind completeness property*[*]. As we will note in the next section, the real numbers have this property.

Example 1.1.4: The set $\mathbb{Q}$ of rational numbers does not have the least-upper-bound property. The subset $\{x \in \mathbb{Q} : x^2 < 2\}$ does not have a supremum in $\mathbb{Q}$. We will see later (Example 1.2.3) that the supremum is $\sqrt{2}$, which is not rational[†]. Suppose $x \in \mathbb{Q}$ such that $x^2 = 2$. Write $x = {}^m/_n$ in lowest terms. So $({}^m/_n)^2 = 2$ or $m^2 = 2n^2$. Hence, m^2 is divisible by 2, and so m is divisible by 2. Write $m = 2k$ and so $(2k)^2 = 2n^2$. Divide by 2 and note that $2k^2 = n^2$, and hence n is divisible by 2. But that is a contradiction as ${}^m/_n$ is in lowest terms.

That $\mathbb{Q}$ does not have the least-upper-bound property is one of the most important reasons why we work with $\mathbb{R}$ in analysis. The set $\mathbb{Q}$ is just fine for algebraists. But us analysts require the least-upper-bound property to do any work. We also require our real numbers to have many algebraic properties. In particular, we require that they are a field.

[*]Named after the German mathematician Julius Wilhelm Richard Dedekind (1831–1916).

[†]This is true for all other roots of 2, and interestingly, the fact that $\sqrt[k]{2}$ is never rational for $k > 1$ implies no piano can ever be perfectly tuned in all keys. See for example: `https://youtu.be/1Hqm0dYKUx4`.

Definition 1.1.5. A set F is called a *field* if it has two operations defined on it, addition $x+y$ and multiplication xy, and if it satisfies the following axioms:

(A1) If $x \in F$ and $y \in F$, then $x+y \in F$.

(A2) *(commutativity of addition)* $x+y = y+x$ for all $x,y \in F$.

(A3) *(associativity of addition)* $(x+y)+z = x+(y+z)$ for all $x,y,z \in F$.

(A4) There exists an element $0 \in F$ such that $0+x = x$ for all $x \in F$.

(A5) For every element $x \in F$, there exists an element $-x \in F$ such that $x+(-x) = 0$.

(M1) If $x \in F$ and $y \in F$, then $xy \in F$.

(M2) *(commutativity of multiplication)* $xy = yx$ for all $x,y \in F$.

(M3) *(associativity of multiplication)* $(xy)z = x(yz)$ for all $x,y,z \in F$.

(M4) There exists an element $1 \in F$ (and $1 \neq 0$) such that $1x = x$ for all $x \in F$.

(M5) For every $x \in F$ such that $x \neq 0$ there exists an element $1/x \in F$ such that $x(1/x) = 1$.

(D) *(distributive law)* $x(y+z) = xy+xz$ for all $x,y,z \in F$.

Example 1.1.6: The set $\mathbb{Q}$ of rational numbers is a field. On the other hand $\mathbb{Z}$ is not a field, as it does not contain multiplicative inverses. For example, there is no $x \in \mathbb{Z}$ such that $2x = 1$, so (M5) is not satisfied. You can check that (M5) is the only property that fails*.

We will assume the basic facts about fields that are easily proved from the axioms. For example, $0x = 0$ is easily proved by noting that $xx = (0+x)x = 0x+xx$, using (A4), (D), and (M2). Then using (A5) on xx, along with (A2), (A3), and (A4), we obtain $0 = 0x$.

Definition 1.1.7. A field F is said to be an *ordered field* if F is also an ordered set such that

(i) For $x,y,z \in F$, $x<y$ implies $x+z<y+z$.

(ii) For $x,y \in F$, $x>0$ and $y>0$ implies $xy>0$.

If $x>0$, we say x is *positive*. If $x<0$, we say x is *negative*. We also say x is *nonnegative* if $x \geq 0$, and x is *nonpositive* if $x \leq 0$.

It can be checked that the rational numbers $\mathbb{Q}$ with the standard ordering is an ordered field.

Proposition 1.1.8. *Let F be an ordered field and $x,y,z,w \in F$. Then*

(i) If $x>0$, then $-x<0$ (and vice versa).

(ii) If $x>0$ and $y<z$, then $xy<xz$.

(iii) If $x<0$ and $y<z$, then $xy>xz$.

(iv) If $x \neq 0$, then $x^2>0$.

(v) If $0<x<y$, then $0<1/y<1/x$.

(vi) If $0<x<y$, then $x^2<y^2$.

(vii) If $x \leq y$ and $z \leq w$, then $x+z \leq y+w$.

*An algebraist would say that $\mathbb{Z}$ is an ordered ring, or perhaps more precisely a commutative ordered ring.

Note that (iv) implies in particular that $1 > 0$.

Proof. Let us prove (i). The inequality $x > 0$ implies by item (i) of definition of ordered fields that $x + (-x) > 0 + (-x)$. Apply the algebraic properties of fields to obtain $0 > -x$. The "vice versa" follows by similar calculation.

For (ii), notice that $y < z$ implies $0 < z - y$ by item (i) of the definition of ordered fields. Apply item (ii) of the definition of ordered fields to obtain $0 < x(z-y)$. By algebraic properties, $0 < xz - xy$. Again by item (i) of the definition, $xy < xz$.

Part (iii) is left as an exercise.

To prove part (iv) first suppose $x > 0$. By item (ii) of the definition of ordered fields, $x^2 > 0$ (use $y = x$). If $x < 0$, we use part (iii) of this proposition, where we plug in $y = x$ and $z = 0$.

To prove part (v), notice that $1/x$ cannot be equal to zero (why?). Suppose $1/x < 0$, then $-1/x > 0$ by (i). Apply part (ii) (as $x > 0$) to obtain $x(-1/x) > 0x$ or $-1 > 0$, which contradicts $1 > 0$ by using part (i) again. Hence $1/x > 0$. Similarly, $1/y > 0$. Thus $(1/x)(1/y) > 0$ by definition of ordered field and by part (ii)

$$(1/x)(1/y)x < (1/x)(1/y)y.$$

By algebraic properties we get $1/y < 1/x$.

Parts (vi) and (vii) are left as exercises. $\square$

The product of two positive numbers (elements of an ordered field) is positive. However, it is not true that if the product is positive, then each of the two factors must be positive. For instance, $(-1)(-1) = 1 > 0$.

Proposition 1.1.9. *Let $x, y \in F$, where F is an ordered field. If $xy > 0$, then either both x and y are positive, or both are negative.*

Proof. We show the contrapositive: If either one of x or y is zero, or if x and y have opposite signs, then xy is not positive. If either x or y is zero, then xy is zero and hence not positive. Hence assume that x and y are nonzero and have opposite signs. Without loss of generality suppose $x > 0$ and $y < 0$. Multiply $y < 0$ by x to get $xy < 0x = 0$. $\square$

Example 1.1.10: The reader may also know about the *complex numbers*, usually denoted by $\mathbb{C}$. That is, $\mathbb{C}$ is the set of numbers of the form $x + iy$, where x and y are real numbers, and i is the imaginary number, a number such that $i^2 = -1$. The reader may remember from algebra that $\mathbb{C}$ is also a field; however, it is not an ordered field. While one can make $\mathbb{C}$ into an ordered set in some way, it is not possible to put an order on $\mathbb{C}$ that would make it an ordered field: In every ordered field, $-1 < 0$ and $x^2 > 0$ for all nonzero x, but in $\mathbb{C}$, $i^2 = -1$.

Finally, an ordered field that has the least-upper-bound property has the corresponding property for greatest lower bounds.

Proposition 1.1.11. *Let F be an ordered field with the least-upper-bound property. Let $A \subset F$ be a nonempty set that is bounded below. Then* $\inf A$ *exists.*

Proof. Let $B := \{-x : x \in A\}$. Let $b \in F$ be a lower bound for A: If $x \in A$, then $x \geq b$. In other words, $-x \leq -b$. So $-b$ is an upper bound for B. Since F has the least-upper-bound property, $c := \sup B$ exists, and $c \leq -b$. As $y \leq c$ for all $y \in B$, then $-c \leq x$ for all $x \in A$. So $-c$ is a lower bound for A. As $-c \geq b$, $-c$ is the greatest lower bound of A. $\square$

1.1.1 Exercises

Exercise* 1.1.1*: *Prove part (iii) of Proposition 1.1.8. That is, let F be an ordered field and $x, y, z \in F$. Prove If $x < 0$ and $y < z$, then $xy > xz$.*

Exercise* 1.1.2*: *Let S be an ordered set. Let $A \subset S$ be a nonempty finite subset. Then A is bounded. Furthermore, $\inf A$ exists and is in A and $\sup A$ exists and is in A. Hint: Use induction.*

Exercise* 1.1.3*: *Prove part (vi) of Proposition 1.1.8. That is, let $x, y \in F$, where F is an ordered field, such that $0 < x < y$. Show that $x^2 < y^2$.*

Exercise* 1.1.4*: *Let S be an ordered set. Let $B \subset S$ be bounded (above and below). Let $A \subset B$ be a nonempty subset. Suppose all the* infs *and* sups *exist. Show that*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Exercise* 1.1.5*: *Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A. Suppose $b \in A$. Show that $b = \sup A$.*

Exercise* 1.1.6*: *Let S be an ordered set. Let $A \subset S$ be nonempty and bounded above. Suppose $\sup A$ exists and $\sup A \notin A$. Show that A contains a countably infinite subset.*

Exercise* 1.1.7*: *Find a (nonstandard) ordering of the set of natural numbers $\mathbb{N}$ such that there exists a nonempty proper subset $A \subsetneq \mathbb{N}$ and such that $\sup A$ exists in $\mathbb{N}$, but $\sup A \notin A$. To keep things straight it might be a good idea to use a different notation for the nonstandard ordering such as $n \prec m$.*

Exercise* 1.1.8*: *Let $F := \{0, 1, 2\}$.*

a) *Prove that there is exactly one way to define addition and multiplication so that F is a field if 0 and 1 have their usual meaning of (A4) and (M4).*
b) *Show that F cannot be an ordered field.*

Exercise* 1.1.9*: *Let S be an ordered set and A is a nonempty subset such that $\sup A$ exists. Suppose there is a $B \subset A$ such that whenever $x \in A$ there is a $y \in B$ such that $x \leq y$. Show that $\sup B$ exists and $\sup B = \sup A$.*

Exercise* 1.1.10*: *Let D be the ordered set of all possible words (not just English words, all strings of letters of arbitrary length) using the Latin alphabet using only lower case letters. The order is the lexicographic order as in a dictionary (e.g.* aa $<$ aaa $<$ dog $<$ door*). Let A be the subset of D containing the words whose first letter is 'a' (e.g.* a $\in A$, abcd $\in A$*). Show that A has a supremum and find what it is.*

Exercise* 1.1.11*: *Let F be an ordered field and $x, y, z, w \in F$.*

a) *Prove part (vii) of Proposition 1.1.8. That is, if $x \leq y$ and $z \leq w$, then $x + z \leq y + w$.*
b) *Prove that if $x < y$ and $z \leq w$, then $x + z < y + w$.*

Exercise* 1.1.12*: *Prove that any ordered field must contain a countably infinite set.*

Exercise* 1.1.13*: *Let $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, where elements of $\mathbb{N}$ are ordered in the usual way amongst themselves, and $k < \infty$ for every $k \in \mathbb{N}$. Show $\mathbb{N}_\infty$ is an ordered set and that every subset $E \subset \mathbb{N}_\infty$ has a supremum in $\mathbb{N}_\infty$ (make sure to also handle the case of an empty set).*

Exercise* 1.1.14*: *Let $S := \{a_k : k \in \mathbb{N}\} \cup \{b_k : k \in \mathbb{N}\}$, ordered such that $a_k < b_j$ for every k and j, $a_k < a_m$ whenever $k < m$, and $b_k > b_m$ whenever $k < m$.*

a) *Show that S is an ordered set.*
b) *Show that every subset of S is bounded (both above and below).*
c) *Find a bounded subset of S that has no least upper bound.*

1.2 The set of real numbers

Note: 2 lectures, the extended real numbers are optional

1.2.1 The set of real numbers

We finally get to the real number system. To simplify matters, instead of constructing the real number set from the rational numbers, we simply state their existence as a theorem without proof. Notice that $\mathbb{Q}$ is an ordered field.

Theorem 1.2.1. *There exists a unique* ordered field $\mathbb{R}$ with the least-upper-bound property such that $\mathbb{Q} \subset \mathbb{R}$.*

Note that also $\mathbb{N} \subset \mathbb{Q}$. We saw that $1 > 0$. By induction (exercise) we can prove that $n > 0$ for all $n \in \mathbb{N}$. Similarly, we verify simple statements about rational numbers. For example, we proved that if $n > 0$, then $1/n > 0$. Then $m < k$ implies $m/n < k/n$.

Let us prove one of the most basic but useful results about the real numbers. The following proposition is essentially how an analyst proves an inequality.

Proposition 1.2.2. *If $x \in \mathbb{R}$ is such that $x \leq \varepsilon$ for all $\varepsilon \in \mathbb{R}$ where $\varepsilon > 0$, then $x \leq 0$.*

Proof. If $x > 0$, then $0 < x/2 < x$ (why?). Taking $\varepsilon = x/2$ obtains a contradiction. Thus $x \leq 0$. $\square$

Another useful version of this idea is the following equivalent statement for nonnegative numbers: *If $x \geq 0$ is such that $x \leq \varepsilon$ for all $\varepsilon > 0$, then $x = 0$.* And to prove that $x \geq 0$ in the first place, an analyst might prove that all $x \geq -\varepsilon$ for all $\varepsilon > 0$. From now on, when we say $x \geq 0$ or $\varepsilon > 0$, we automatically mean that $x \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$.

A related simple fact is that any time we have two real numbers $a < b$, then there is another real number c such that $a < c < b$. Take, for example, $c = \frac{a+b}{2}$ (why?). In fact, there are infinitely many real numbers between a and b. We will use this fact in the next example.

The most useful property of $\mathbb{R}$ for analysts is not just that it is an ordered field, but that it has the least-upper-bound property. Essentially, we want $\mathbb{Q}$, but we also want to take suprema (and infima) willy-nilly. So what we do is take $\mathbb{Q}$ and throw in enough numbers to obtain $\mathbb{R}$.

We mentioned already that $\mathbb{R}$ contains elements that are not in $\mathbb{Q}$ because of the least-upper-bound property. Let us prove it. We saw there is no rational square root of two. The set $\{x \in \mathbb{Q} : x^2 < 2\}$ implies the existence of the real number $\sqrt{2}$, although this fact requires a bit of work. See also Exercise 1.2.14.

Example 1.2.3: Claim: *There exists a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$. We denote r by $\sqrt{2}$.*

Proof. Take the set $A := \{x \in \mathbb{R} : x^2 < 2\}$. We first show that A is bounded above and nonempty. The equation $x \geq 2$ implies $x^2 \geq 4$ (see Exercise 1.1.3), so if $x^2 < 2$, then $x < 2$, and A is bounded above. As $1 \in A$, the set A is nonempty. We can therefore find the supremum.

Let $r := \sup A$. We will show that $r^2 = 2$ by showing that $r^2 \geq 2$ and $r^2 \leq 2$. This is the way analysts show equality, by showing two inequalities. We already know that $r \geq 1 > 0$.

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*Uniqueness is up to isomorphism, but we wish to avoid excessive use of algebra. For us, it is simply enough to assume that a set of real numbers exists. See Rudin [R2] for the construction and more details.

In the following, it may seem we are pulling certain expressions out of a hat. When writing a proof such as this we would, of course, come up with the expressions only after playing around with what we wish to prove. The order in which we write the proof is not necessarily the order in which we come up with the proof.

Let us first show that $r^2 \geq 2$. Take a positive number s such that $s^2 < 2$. We wish to find an $h > 0$ such that $(s+h)^2 < 2$. As $2 - s^2 > 0$, we have $\frac{2-s^2}{2s+1} > 0$. Choose an $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$. Furthermore, assume $h < 1$. Estimate,

$$\begin{aligned} (s+h)^2 - s^2 &= h(2s+h) \\ &< h(2s+1) && \bigl(\text{since } h < 1\bigr) \\ &< 2 - s^2 && \bigl(\text{since } h < \tfrac{2-s^2}{2s+1}\bigr). \end{aligned}$$

Therefore, $(s+h)^2 < 2$. Hence $s+h \in A$, but as $h > 0$, we have $s+h > s$. So $s < r = \sup A$. As s was an arbitrary positive number such that $s^2 < 2$, it follows that $r^2 \geq 2$.

Now take a positive number s such that $s^2 > 2$. We wish to find an $h > 0$ such that $(s-h)^2 > 2$ and $s-h$ is still positive. As $s^2 - 2 > 0$, we have $\frac{s^2-2}{2s} > 0$. Let $h := \frac{s^2-2}{2s}$, and check $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$. Estimate,

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh && \bigl(\text{since } h^2 > 0 \text{ as } h \neq 0\bigr) \\ &= s^2 - 2 && \bigl(\text{since } h = \tfrac{s^2-2}{2s}\bigr). \end{aligned}$$

By subtracting s^2 from both sides and multiplying by -1, we find $(s-h)^2 > 2$. Therefore, $s - h \notin A$.

Moreover, if $x \geq s-h$, then $x^2 \geq (s-h)^2 > 2$ (as $x > 0$ and $s-h > 0$) and so $x \notin A$. Thus, $s-h$ is an upper bound for A. However, $s - h < s$, or in other words, $s > r = \sup A$. Hence, $r^2 \leq 2$.

Together, $r^2 \geq 2$ and $r^2 \leq 2$ imply $r^2 = 2$. The existence part is finished. We still need to handle uniqueness. Suppose $s \in \mathbb{R}$ such that $s^2 = 2$ and $s > 0$. Thus $s^2 = r^2$. However, if $0 < s < r$, then $s^2 < r^2$. Similarly, $0 < r < s$ implies $r^2 < s^2$. Hence $s = r$. □

The number $\sqrt{2} \notin \mathbb{Q}$. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational* numbers. We just saw that $\mathbb{R} \setminus \mathbb{Q}$ is nonempty. Not only is it nonempty, we will see later that it is very large indeed.

Using the same technique as above, we can show that a positive real number $x^{1/n}$ exists for all $n \in \mathbb{N}$ and all $x > 0$. That is, for each $x > 0$, there exists a unique positive real number r such that $r^n = x$. The proof is left as an exercise.

1.2.2 Archimedean property

As we have seen, there are plenty of real numbers in any interval. But there are also infinitely many rational numbers in any interval. The following is one of the fundamental facts about the real numbers. The two parts of the next theorem are actually equivalent, even though it may not seem like that at first sight.

Theorem 1.2.4.

(i) (Archimedean property)* *If* $x, y \in \mathbb{R}$ *and* $x > 0$*, then there exists an* $n \in \mathbb{N}$ *such that*

$$nx > y.$$

(ii) ($\mathbb{Q}$ is dense in $\mathbb{R}$) *If* $x, y \in \mathbb{R}$ *and* $x < y$*, then there exists an* $r \in \mathbb{Q}$ *such that* $x < r < y$*.*

Proof. Let us prove (i). Divide through by x. Then (i) says that for every real number $t := y/x$, we can find $n \in \mathbb{N}$ such that $n > t$. In other words, (i) says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above. Suppose for contradiction that $\mathbb{N}$ is bounded above. Let $b := \sup \mathbb{N}$. The number $b - 1$ cannot possibly be an upper bound for $\mathbb{N}$ as it is strictly less than b (the least upper bound). Thus there exists an $m \in \mathbb{N}$ such that $m > b - 1$. Add one to obtain $m + 1 > b$, contradicting b being an upper bound.

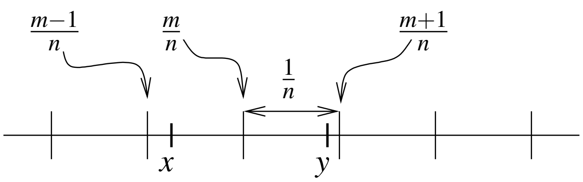

Figure 1.2: Idea of the proof of the density of $\mathbb{Q}$: Find n such that $y - x > 1/n$, then take the least m such that $m/n > x$.

Let us tackle (ii). See Figure 1.2 for a picture of the idea behind the proof. First assume $x \geq 0$. Note that $y - x > 0$. By (i), there exists an $n \in \mathbb{N}$ such that

$$n(y - x) > 1 \qquad \text{or} \qquad y - x > 1/n.$$

Again by (i) the set $A := \{k \in \mathbb{N} : k > nx\}$ is nonempty. By the well ordering property of $\mathbb{N}$, A has a least element m, and as $m \in A$, then $m > nx$. Divide through by n to get $x < m/n$. As m is the least element of A, $m - 1 \notin A$. If $m > 1$, then $m - 1 \in \mathbb{N}$, but $m - 1 \notin A$ and so $m - 1 \leq nx$. If $m = 1$, then $m - 1 = 0$, and $m - 1 \leq nx$ still holds as $x \geq 0$. In other words,

$$m - 1 \leq nx \qquad \text{or} \qquad m \leq nx + 1.$$

On the other hand, from $n(y - x) > 1$ we obtain $ny > 1 + nx$. Hence $ny > 1 + nx \geq m$, and therefore $y > m/n$. Putting everything together we obtain $x < m/n < y$. So take $r = m/n$.

Now assume $x < 0$. If $y > 0$, then just take $r = 0$. If $y \leq 0$, then $0 \leq -y < -x$, and we find a rational q such that $-y < q < -x$. Then take $r = -q$. □

Let us state and prove a simple but useful corollary of the Archimedean property.

Corollary 1.2.5. $\inf\{1/n : n \in \mathbb{N}\} = 0$.

Proof. Let $A := \{1/n : n \in \mathbb{N}\}$. Obviously A is not empty. Furthermore, $1/n > 0$ for all $n \in \mathbb{N}$, and so 0 is a lower bound, and $b := \inf A$ exists. As 0 is a lower bound, then $b \geq 0$. Take an arbitrary $a > 0$. By the Archimedean property there exists an n such that $na > 1$, or in other words $a > 1/n \in A$. Therefore, a cannot be a lower bound for A. Hence $b = 0$. □

*Named after the Ancient Greek mathematician Archimedes of Syracuse (c. 287 BC – c. 212 BC). This property is Axiom V from Archimedes' "On the Sphere and Cylinder" 225 BC.

1.2.3 Using supremum and infimum

Suprema and infima are compatible with algebraic operations. For a set $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ define

$$x + A := \{x + y \in \mathbb{R} : y \in A\},$$
$$xA := \{xy \in \mathbb{R} : y \in A\}.$$

For example, if $A = \{1,2,3\}$, then $5 + A = \{6,7,8\}$ and $3A = \{3,6,9\}$.

Proposition 1.2.6. *Let $A \subset \mathbb{R}$ be nonempty.*

(i) If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$.

(ii) If $x \in \mathbb{R}$ and A is bounded below, then $\inf(x + A) = x + \inf A$.

(iii) If $x > 0$ and A is bounded above, then $\sup(xA) = x(\sup A)$.

(iv) If $x > 0$ and A is bounded below, then $\inf(xA) = x(\inf A)$.

(v) If $x < 0$ and A is bounded below, then $\sup(xA) = x(\inf A)$.

(vi) If $x < 0$ and A is bounded above, then $\inf(xA) = x(\sup A)$.

Do note that multiplying a set by a negative number switches supremum for an infimum and vice versa. Also, as the proposition implies that supremum (resp. infimum) of $x + A$ or xA exists, it also implies that $x + A$ or xA is nonempty and bounded above (resp. below).

Proof. Let us only prove the first statement. The rest are left as exercises.

Suppose b is an upper bound for A. That is, $y \leq b$ for all $y \in A$. Then $x + y \leq x + b$ for all $y \in A$, and so $x + b$ is an upper bound for $x + A$. In particular, if $b = \sup A$, then

$$\sup(x + A) \leq x + b = x + \sup A.$$

The opposite inequality is similar. If b is an upper bound for $x + A$, then $x + y \leq b$ for all $y \in A$ and so $y \leq b - x$ for all $y \in A$. So $b - x$ is an upper bound for A. If $b = \sup(x + A)$, then

$$\sup A \leq b - x = \sup(x + A) - x.$$

The result follows. □

Sometimes we need to apply supremum or infimum twice. Here is an example.

Proposition 1.2.7. *Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and* $\sup A \leq \inf B$.

Proof. Any $x \in A$ is a lower bound for B. Therefore $x \leq \inf B$ for all $x \in A$, so $\inf B$ is an upper bound for A. Hence, $\sup A \leq \inf B$. □

We must be careful about strict inequalities and taking suprema and infima. Note that $x < y$ whenever $x \in A$ and $y \in B$ still only implies $\sup A \leq \inf B$, and not a strict inequality. This is an important subtle point that comes up often. For example, take $A := \{0\}$ and take $B := \{1/n : n \in \mathbb{N}\}$. Then $0 < 1/n$ for all $n \in \mathbb{N}$. However, $\sup A = 0$ and $\inf B = 0$.

The proof of the following often used elementary fact is left to the reader. A similar statement holds for infima.

Proposition 1.2.8. *If $S \subset \mathbb{R}$ is nonempty and bounded above, then for every $\varepsilon > 0$ there exists an $x \in S$ such that $(\sup S) - \varepsilon < x \leq \sup S$.*

To make using suprema and infima even easier, we may want to write $\sup A$ and $\inf A$ without worrying about A being bounded and nonempty. We make the following natural definitions.

Definition 1.2.9. Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\sup A := -\infty$.

(ii) If A is not bounded above, then $\sup A := \infty$.

(iii) If A is empty, then $\inf A := \infty$.

(iv) If A is not bounded below, then $\inf A := -\infty$.

For convenience, ∞ and $-\infty$ are sometimes treated as if they were numbers, except we do not allow arbitrary arithmetic with them. We make $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ into an ordered set by letting

$$-\infty < \infty \quad \text{and} \quad -\infty < x \quad \text{and} \quad x < \infty \quad \text{for all } x \in \mathbb{R}.$$

The set $\mathbb{R}^*$ is called the set of *extended real numbers*. It is possible to define some arithmetic on $\mathbb{R}^*$. Most operations are extended in an obvious way, but we must leave $\infty - \infty$, $0 \cdot (\pm\infty)$, and $\frac{\pm\infty}{\pm\infty}$ undefined. We refrain from using this arithmetic, it leads to easy mistakes as $\mathbb{R}^*$ is not a field. Now we can take suprema and infima without fear of emptiness or unboundedness. In this book, we mostly avoid using $\mathbb{R}^*$ outside of exercises, and leave such generalizations to the interested reader.

1.2.4 Maxima and minima

By Exercise 1.1.2, a finite set of numbers always has a supremum or an infimum that is contained in the set itself. In this case we usually do not use the words supremum or infimum.

When a set A of real numbers is bounded above, such that $\sup A \in A$, then we can use the word *maximum* and the notation $\max A$ to denote the supremum. Similarly for infimum: When a set A is bounded below and $\inf A \in A$, then we can use the word *minimum* and the notation $\min A$. For example,

$$\max\{1, 2.4, \pi, 100\} = 100,$$
$$\min\{1, 2.4, \pi, 100\} = 1.$$

While writing sup and inf may be technically correct in this situation, max and min are generally used to emphasize that the supremum or infimum is in the set itself.

1.2.5 Exercises

***Exercise* 1.2.1:** *Prove that if $t > 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $\dfrac{1}{n^2} < t$.*

***Exercise* 1.2.2:** *Prove that if $t \geq 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.*

***Exercise* 1.2.3:** *Finish the proof of Proposition 1.2.6.*

***Exercise* 1.2.4:** *Let $x, y \in \mathbb{R}$. Suppose $x^2 + y^2 = 0$. Prove that $x = 0$ and $y = 0$.*

***Exercise* 1.2.5:** *Show that $\sqrt{3}$ is irrational.*

***Exercise* 1.2.6:** *Let $n \in \mathbb{N}$. Show that either $\sqrt{n}$ is either an integer or it is irrational.*

***Exercise* 1.2.7:** *Prove the* arithmetic-geometric mean inequality. *That is, for two positive real numbers x, y, we have*

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Furthermore, equality occurs if and only if $x = y$.

***Exercise* 1.2.8:** *Show that for every pair of real numbers x and y such that $x < y$, there exists an irrational number s such that $x < s < y$. Hint: Apply the density of $\mathbb{Q}$ to $\dfrac{x}{\sqrt{2}}$ and $\dfrac{y}{\sqrt{2}}$.*

***Exercise* 1.2.9:** *Let A and B be two nonempty bounded sets of real numbers. Let $C := \{a + b : a \in A, b \in B\}$. Show that C is a bounded set and that*

$$\sup C = \sup A + \sup B \qquad \text{and} \qquad \inf C = \inf A + \inf B.$$

***Exercise* 1.2.10:** *Let A and B be two nonempty bounded sets of nonnegative real numbers. Define the set $C := \{ab : a \in A, b \in B\}$. Show that C is a bounded set and that*

$$\sup C = (\sup A)(\sup B) \qquad \text{and} \qquad \inf C = (\inf A)(\inf B).$$

***Exercise* 1.2.11** (Hard)**:** *Given $x > 0$ and $n \in \mathbb{N}$, show that there exists a unique positive real number r such that $x = r^n$. Usually r is denoted by $x^{1/n}$.*

***Exercise* 1.2.12** (Easy)**:** *Prove Proposition 1.2.8.*

***Exercise* 1.2.13:** *Prove the so-called* Bernoulli's inequality*: *If $1 + x > 0$, then for all $n \in \mathbb{N}$, we have $(1 + x)^n \geq 1 + nx$.*

***Exercise* 1.2.14:** *Prove* $\sup\{x \in \mathbb{Q} : x^2 < 2\} = \sup\{x \in \mathbb{R} : x^2 < 2\}$.

***Exercise* 1.2.15:**

a) *Prove that given $y \in \mathbb{R}$, we have* $\sup\{x \in \mathbb{Q} : x < y\} = y$.
b) *Let $A \subset \mathbb{Q}$ be a set that is bounded above such that whenever $x \in A$ and $t \in \mathbb{Q}$ with $t < x$, then $t \in A$. Further suppose $\sup A \notin A$. Show that there exists a $y \in \mathbb{R}$ such that $A = \{x \in \mathbb{Q} : x < y\}$. A set such as A is called a* Dedekind cut.
c) *Show that there is a bijection between $\mathbb{R}$ and Dedekind cuts.*

Note: Dedekind used sets as in part b) in his construction of the real numbers.

***Exercise* 1.2.16:** *Prove that if $A \subset \mathbb{Z}$ is a nonempty subset bounded below, then there exists a least element in A. Now describe why this statement would simplify the proof of Theorem 1.2.4* part *(ii) so that you do not have to assume $x \geq 0$.*

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*Named after the Swiss mathematician Jacob Bernoulli (1655–1705).

***Exercise* 1.2.17:** *Let us suppose we know $x^{1/n}$ exists for every $x > 0$ and every $n \in \mathbb{N}$ (see Exercise 1.2.11 above). For integers p and $q > 0$ where ${}^p/_q$ is in lowest terms, define $x^{p/q} := (x^{1/q})^p$.*

a) *Show that the power is well-defined even if the fraction is not in lowest terms: If ${}^p/_q = {}^m/_k$ where m and $k > 0$ are integers, then $(x^{1/q})^p = (x^{1/m})^k$.*

b) *Let x and y be two positive numbers and r a rational number. Assuming $r > 0$, show $x < y$ if and only if $x^r < y^r$. Then suppose $r < 0$ and show: $x < y$ if and only if $x^r > y^r$.*

c) *Suppose $x > 1$ and r, s are rational where $r < s$. Show $x^r < x^s$. If $0 < x < 1$ and $r < s$, show that $x^r > x^s$. Hint: Write r and s with the same denominator.*

d) *(Challenging)* For an irrational $z \in \mathbb{R} \setminus \mathbb{Q}$ and $x > 1$ define $x^z := \sup\{x^r : r \leq z, r \in \mathbb{Q}\}$, for $x = 1$ define $1^z = 1$, and for $0 < x < 1$ define $x^z := \inf\{x^r : r \leq z, r \in \mathbb{Q}\}$. Prove the two assertions of part b) for all real z.*

---

*In §5.4 we will define exponential and the logarithm and define $x^z := \exp(z \ln x)$. We will then have sufficient machinery to make proofs of these assertions far easier. At this point, however, we do not yet have these tools.

1.3 Absolute value and bounded functions

Note: 0.5–1 lecture

A concept we will encounter over and over is the concept of *absolute value*. You want to think of the absolute value as the "size" of a real number. Let us give a formal definition.

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Let us give the main features of the absolute value as a proposition.

Proposition 1.3.1.

(i) $|x| \geq 0$, *moreover,* $|x| = 0$ *if and only if* $x = 0$.

(ii) $|-x| = |x|$ *for all* $x \in \mathbb{R}$.

(iii) $|xy| = |x|\,|y|$ *for all* $x, y \in \mathbb{R}$.

(iv) $|x|^2 = x^2$ *for all* $x \in \mathbb{R}$.

(v) $|x| \leq y$ *if and only if* $-y \leq x \leq y$.

(vi) $-|x| \leq x \leq |x|$ *for all* $x \in \mathbb{R}$.

Proof. (i): First suppose $x \geq 0$. Then $|x| = x \geq 0$. Also, $|x| = x = 0$ if and only if $x = 0$. On the other hand, if $x < 0$, then $|x| = -x > 0$, and $|x|$ is never zero.

(ii): If $x > 0$, then $-x < 0$ and so $|-x| = -(-x) = x = |x|$. Similarly when $x < 0$, or $x = 0$.

(iii): If x or y is zero, then the result is immediate. When x and y are both positive, then $|x|\,|y| = xy$. xy is also positive and hence $xy = |xy|$. If x and y are both negative, then xy is still positive and $xy = |xy|$, and $|x|\,|y| = (-x)(-y) = xy$. Next assume $x > 0$ and $y < 0$. Then $|x|\,|y| = x(-y) = -(xy)$. Now xy is negative and hence $|xy| = -(xy)$. Similarly if $x < 0$ and $y > 0$.

(iv): Immediate if $x \geq 0$. If $x < 0$, then $|x|^2 = (-x)^2 = x^2$.

(v): Suppose $|x| \leq y$. If $x \geq 0$, then $x \leq y$. It follows that $y \geq 0$, leading to $-y \leq 0 \leq x$. So $-y \leq x \leq y$ holds. If $x < 0$, then $|x| \leq y$ means $-x \leq y$. Negating both sides we get $x \geq -y$. Again $y \geq 0$ and so $y \geq 0 > x$. Hence, $-y \leq x \leq y$.

On the other hand, suppose $-y \leq x \leq y$ is true. If $x \geq 0$, then $x \leq y$ is equivalent to $|x| \leq y$. If $x < 0$, then $-y \leq x$ implies $(-x) \leq y$, which is equivalent to $|x| \leq y$.

(vi): Apply (v) with $y = |x|$. □

A property used frequently enough to give it a name is the so-called *triangle inequality*.

Proposition 1.3.2 (Triangle Inequality)**.** $|x+y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

Proof. Proposition 1.3.1 gives $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Add these two inequalities to obtain

$$-\big(|x| + |y|\big) \leq x + y \leq |x| + |y|.$$

Apply Proposition 1.3.1 again to find $|x+y| \leq |x| + |y|$. □

There are other often applied versions of the triangle inequality.

Corollary 1.3.3. *Let $x, y \in \mathbb{R}$.*

(i) (reverse triangle inequality) $\big|(|x| - |y|)\big| \leq |x - y|$.

(ii) $|x - y| \leq |x| + |y|$.

Proof. Let us plug in $x = a - b$ and $y = b$ into the standard triangle inequality to obtain

$$|a| = |a - b + b| \leq |a - b| + |b|,$$

or $|a| - |b| \leq |a - b|$. Switching the roles of a and b we find $|b| - |a| \leq |b - a| = |a - b|$. Applying Proposition 1.3.1, we obtain the reverse triangle inequality.

The second version of the triangle inequality is obtained from the standard one by just replacing y with $-y$, and noting $|-y| = |y|$. □

Corollary 1.3.4. *Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then*

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Proof. We proceed by induction. The conclusion holds trivially for $n = 1$, and for $n = 2$ it is the standard triangle inequality. Suppose the corollary holds for n. Take $n + 1$ numbers $x_1, x_2, \ldots, x_{n+1}$ and first use the standard triangle inequality, then the induction hypothesis

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n + x_{n+1}| &\leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}| \\ &\leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|. \end{aligned}$$ □

Let us see an example of the use of the triangle inequality.

Example 1.3.5: Find a number M such that $|x^2 - 9x + 1| \leq M$ for all $-1 \leq x \leq 5$.

Using the triangle inequality, write

$$|x^2 - 9x + 1| \leq |x^2| + |9x| + |1| = |x|^2 + 9|x| + 1.$$

The expression $|x|^2 + 9|x| + 1$ is largest when $|x|$ is largest (why?). In the interval provided, $|x|$ is largest when $x = 5$ and so $|x| = 5$. One possibility for M is

$$M = 5^2 + 9(5) + 1 = 71.$$

There are, of course, other M that work. The bound of 71 is much higher than it need be, but we didn't ask for the best possible M, just one that works.

The last example leads us to the concept of bounded functions.

Definition 1.3.6. Suppose $f \colon D \to \mathbb{R}$ is a function. We say f is *bounded* if there exists a number M such that $|f(x)| \leq M$ for all $x \in D$.

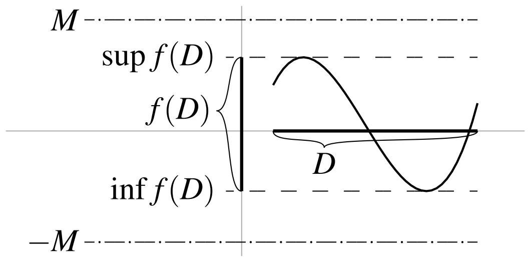

Figure 1.3: Example of a bounded function, a bound M, and its supremum and infimum.

In the example, we proved $x^2 - 9x + 1$ is bounded when considered as a function on $D = \{x : -1 \leq x \leq 5\}$. On the other hand, if we consider the same polynomial as a function on the whole real line $\mathbb{R}$, then it is not bounded.

For a function $f : D \to \mathbb{R}$, we write (see Figure 1.3 for an example)

$$\sup_{x \in D} f(x) := \sup f(D),$$
$$\inf_{x \in D} f(x) := \inf f(D).$$

We also sometimes replace the "$x \in D$" with an expression. For example if, as before, $f(x) = x^2 - 9x + 1$, for $-1 \leq x \leq 5$, a little bit of calculus shows

$$\sup_{x \in D} f(x) = \sup_{-1 \leq x \leq 5} (x^2 - 9x + 1) = 11, \qquad \inf_{x \in D} f(x) = \inf_{-1 \leq x \leq 5} (x^2 - 9x + 1) = {}^{-77}\!/_{4}.$$

Proposition 1.3.7. *If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ (D nonempty) are bounded* functions and*

$$f(x) \leq g(x) \qquad \textit{for all } x \in D,$$

then

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \qquad \textit{and} \qquad \inf_{x \in D} f(x) \leq \inf_{x \in D} g(x). \tag{1.1}$$

Be careful with the variables. The x on the left side of the inequality in (1.1) is different from the x on the right. You should really think of, say, the first inequality as

$$\sup_{x \in D} f(x) \leq \sup_{y \in D} g(y).$$

Let us prove this inequality. If b is an upper bound for $g(D)$, then $f(x) \leq g(x) \leq b$ for all $x \in D$, and hence b is also an upper bound for $f(D)$, or $f(x) \leq b$ for all $x \in D$. Take the least upper bound of $g(D)$ to get that for all $x \in D$

$$f(x) \leq \sup_{y \in D} g(y).$$

*The boundedness hypothesis is for simplicity, it can be dropped if we allow for the extended real numbers.

Therefore, $\sup_{y \in D} g(y)$ is an upper bound for $f(D)$ and thus greater than or equal to the least upper bound of $f(D)$.

$$\sup_{x \in D} f(x) \leq \sup_{y \in D} g(y).$$

The second inequality (the statement about the inf) is left as an exercise (Exercise 1.3.4).

A common mistake is to conclude

$$\sup_{x \in D} f(x) \leq \inf_{y \in D} g(y). \tag{1.2}$$

The inequality (1.2) is not true given the hypothesis of the proposition above. For this stronger inequality we need the stronger hypothesis

$$f(x) \leq g(y) \qquad \text{for all } x \in D \text{ and } y \in D.$$

The proof as well as a counterexample is left as an exercise (Exercise 1.3.5).

1.3.1 Exercises

***Exercise* 1.3.1:** *Show that $|x-y| < \varepsilon$ if and only if $x-\varepsilon < y < x+\varepsilon$.*

***Exercise* 1.3.2:** *Show:* *a)* $\max\{x,y\} = \frac{x+y+|x-y|}{2}$ *b)* $\min\{x,y\} = \frac{x+y-|x-y|}{2}$

***Exercise* 1.3.3:** *Find a number M such that $|x^3 - x^2 + 8x| \leq M$ for all $-2 \leq x \leq 10$.*

***Exercise* 1.3.4:** *Finish the proof of Proposition 1.3.7. That is, prove that given a set D, and two bounded functions $f \colon D \to \mathbb{R}$ and $g \colon D \to \mathbb{R}$ such that $f(x) \leq g(x)$ for all $x \in D$, then*

$$\inf_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

***Exercise* 1.3.5:** *Let $f \colon D \to \mathbb{R}$ and $g \colon D \to \mathbb{R}$ be functions (D nonempty).*

a) Suppose $f(x) \leq g(y)$ for all $x \in D$ and $y \in D$. Show that

$$\sup_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

b) Find a specific D, f, and g, such that $f(x) \leq g(x)$ for all $x \in D$, but

$$\sup_{x \in D} f(x) > \inf_{x \in D} g(x).$$

***Exercise* 1.3.6:** *Prove Proposition 1.3.7 without the assumption that the functions are bounded. Hint: You need to use the extended real numbers.*

***Exercise* 1.3.7:** *Let D be a nonempty set. Suppose $f \colon D \to \mathbb{R}$ and $g \colon D \to \mathbb{R}$ are bounded functions.*

a) Show

$$\sup_{x \in D} \bigl(f(x)+g(x)\bigr) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x) \qquad \text{and} \qquad \inf_{x \in D} \bigl(f(x)+g(x)\bigr) \geq \inf_{x \in D} f(x) + \inf_{x \in D} g(x).$$

b) Find examples where we obtain strict inequalities.

Exercise* 1.3.8*: *Suppose $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are bounded functions and $\alpha \in \mathbb{R}$.*

a) *Show that $\alpha f\colon D \to \mathbb{R}$ defined by $(\alpha f)(x) := \alpha f(x)$ is a bounded function.*

b) *Show that $f+g\colon D \to \mathbb{R}$ defined by $(f+g)(x) := f(x)+g(x)$ is a bounded function.*

Exercise* 1.3.9*: *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be functions, $\alpha \in \mathbb{R}$, and recall what $f+g$ and αf means from the previous exercise.*

a) *Prove that if $f+g$ and g are bounded, then f is bounded.*

b) *Find an example where f and g are both unbounded, but $f+g$ is bounded.*

c) *Prove that if f is bounded but g is unbounded, then $f+g$ is unbounded.*

d) *Find an example where f is unbounded but αf is bounded.*

1.4 Intervals and the size of $\mathbb{R}$

Note: 0.5–1 lecture (proof of uncountability of $\mathbb{R}$ can be optional)

You surely saw the notation for intervals before, but let us give a formal definition here. For $a, b \in \mathbb{R}$ such that $a < b$ we define

$$\begin{aligned}
[a,b] &:= \{x \in \mathbb{R} : a \leq x \leq b\}, \\
(a,b) &:= \{x \in \mathbb{R} : a < x < b\}, \\
(a,b] &:= \{x \in \mathbb{R} : a < x \leq b\}, \\
[a,b) &:= \{x \in \mathbb{R} : a \leq x < b\}.
\end{aligned}$$

The interval $[a,b]$ is called a *closed interval* and (a,b) is called an *open interval*. The intervals of the form $(a,b]$ and $[a,b)$ are called *half-open intervals*.

The intervals above were all *bounded intervals*, since both a and b were real numbers. We define *unbounded intervals*,

$$\begin{aligned}
[a,\infty) &:= \{x \in \mathbb{R} : a \leq x\}, \\
(a,\infty) &:= \{x \in \mathbb{R} : a < x\}, \\
(-\infty,b] &:= \{x \in \mathbb{R} : x \leq b\}, \\
(-\infty,b) &:= \{x \in \mathbb{R} : x < b\}.
\end{aligned}$$

For completeness, we define $(-\infty,\infty) := \mathbb{R}$. The intervals $[a,\infty)$, $(-\infty,b]$, and $\mathbb{R}$ are sometimes called *unbounded closed intervals*, and (a,∞), $(-\infty,b)$, and $\mathbb{R}$ are sometimes called *unbounded open intervals*.

The proof of the following proposition is left as an exercise. In short, an interval is a set with at least two points that contains all points between any two points.*

Proposition 1.4.1. *A set $I \subset \mathbb{R}$ is an interval if and only if I contains at least 2 points and for all $a, c \in I$ and $b \in \mathbb{R}$ such that $a < b < c$, we have $b \in I$.*

We have already seen that every open interval (a,b) (where $a < b$ of course) must be nonempty. For example, it contains the number $\frac{a+b}{2}$. An unexpected fact is that from a set-theoretic perspective, all intervals have the same "size," that is, they all have the same cardinality. For example the map $f(x) := 2x$ takes the interval $[0,1]$ bijectively to the interval $[0,2]$.

Maybe more interestingly, the function $f(x) := \tan(x)$ is a bijective map from $(-\pi/2, \pi/2)$ to $\mathbb{R}$. Hence the bounded interval $(-\pi/2, \pi/2)$ has the same cardinality as $\mathbb{R}$. It is not completely straightforward to construct a bijective map from $[0,1]$ to $(0,1)$, but it is possible.

And do not worry, there does exist a way to measure the "size" of subsets of real numbers that "sees" the difference between $[0,1]$ and $[0,2]$. However, its proper definition requires much more machinery than we have right now.

Let us say more about the cardinality of intervals and hence about the cardinality of $\mathbb{R}$. We have seen that there exist irrational numbers, that is $\mathbb{R} \setminus \mathbb{Q}$ is nonempty. The question is: How

*Sometimes single point sets and the empty set are also called intervals, but in this book, intervals have at least 2 points. That is, we only defined the bounded intervals if $a < b$.

many irrational numbers are there? It turns out there are a lot more irrational numbers than rational numbers. We have seen that $\mathbb{Q}$ is countable, and we will show that $\mathbb{R}$ is uncountable. In fact, the cardinality of $\mathbb{R}$ is the same as the cardinality of $\mathscr{P}(\mathbb{N})$, although we will not prove this claim here.

Theorem 1.4.2 (Cantor)**.** *$\mathbb{R}$ is uncountable.*

We give a version of Cantor's original proof from 1874 as this proof requires the least setup. Normally this proof is stated as a contradiction, but a proof by contrapositive is easier to understand.

Proof. Let $X \subset \mathbb{R}$ be a countably infinite subset such that for every pair of real numbers $a < b$, there is an $x \in X$ such that $a < x < b$. Were $\mathbb{R}$ countable, we could take $X = \mathbb{R}$. We will show that X is necessarily a proper subset, and so X cannot equal $\mathbb{R}$, and $\mathbb{R}$ must be uncountable.

As X is countably infinite, there is a bijection from $\mathbb{N}$ to X. We write X as a sequence of real numbers $x_1, x_2, x_3, \ldots$, such that each number in X is given by x_n for some $n \in \mathbb{N}$.

We inductively construct two sequences of real numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$. Let $a_1 := x_1$ and $b_1 := x_1 + 1$. Note that $a_1 < b_1$ and $x_1 \notin (a_1, b_1)$. For some $k > 1$, suppose a_j and b_j have been defined for $j = 1, 2, \ldots, k-1$, suppose the open interval (a_j, b_j) does not contain x_ℓ for $\ell = 1, 2, \ldots, j$, and suppose $a_1 < a_2 < \cdots < a_{k-1} < b_{k-1} < \cdots < b_2 < b_1$.

(i) Define $a_k := x_n$, where n is the smallest $n \in \mathbb{N}$ such that $x_n \in (a_{k-1}, b_{k-1})$. Such an x_n exists by our assumption on X, and $n \geq k$ by the assumption on (a_{k-1}, b_{k-1}).

(ii) Next, define b_k to be some real number in (a_k, b_{k-1}).

Notice that $a_{k-1} < a_k < b_k < b_{k-1}$. Also notice that (a_k, b_k) does not contain x_k and hence does not contain x_j for $j = 1, 2, \ldots, k$. The two sequences are now defined.

Claim: $a_n < b_m$ *for all n and m in* $\mathbb{N}$. Proof: Let us first assume $n < m$. Then $a_n < a_{n+1} < \cdots < a_{m-1} < a_m < b_m$. Similarly for $n > m$. The claim follows.

Let $A := \{a_n : n \in \mathbb{N}\}$ and $B := \{b_n : n \in \mathbb{N}\}$. By Proposition 1.2.7 and the claim above,

$$\sup A \leq \inf B.$$

Define $y := \sup A$. The number y cannot be a member of A: If $y = a_n$ for some n, then $y < a_{n+1}$, which is impossible. Similarly, y cannot be a member of B. Therefore, $a_n < y$ for all $n \in \mathbb{N}$ and $y < b_n$ for all $n \in \mathbb{N}$. In other words, for every $n \in \mathbb{N}$, we have $y \in (a_n, b_n)$. By the construction of the sequence, $x_n \notin (a_n, b_n)$, and so $y \neq x_n$. As this was true for all $n \in \mathbb{N}$, we have that $y \notin X$.

We have constructed a real number y that is not in X, and thus X is a proper subset of $\mathbb{R}$. The sequence $x_1, x_2, \ldots$ cannot contain all elements of $\mathbb{R}$ and thus $\mathbb{R}$ is uncountable. $\square$

1.4.1 Exercises

Exercise* 1.4.1*: *For $a < b$, construct an explicit bijection from $(a, b]$ to $(0, 1]$.*

Exercise* 1.4.2*: *Suppose $f\colon [0, 1] \to (0, 1)$ is a bijection. Using f, construct a bijection from $[-1, 1]$ to $\mathbb{R}$.*

Exercise* 1.4.3*: *Prove Proposition 1.4.1. That is, suppose $I \subset \mathbb{R}$ is a subset with at least 2 elements such that if $a < b < c$ and $a, c \in I$, then $b \in I$. Prove that I is one of the nine types of intervals explicitly given in this section. Furthermore, prove that the intervals given in this section all satisfy this property.*

Exercise* 1.4.4** (Hard):*** *Construct an explicit bijection from* $(0,1]$ *to* $(0,1)$. *Hint: One approach is as follows: First map* $(1/2,1]$ *to* $(0,1/2]$, *then map* $(1/4,1/2]$ *to* $(1/2,3/4]$, *etc. Write down the map explicitly, that is, write down an algorithm that tells you exactly what number goes where. Then prove that the map is a bijection.*

Exercise* 1.4.5** (Hard):*** *Construct an explicit bijection from* $[0,1]$ *to* $(0,1)$.

Exercise* 1.4.6*:

a) *Show that every closed interval* $[a,b]$ *is the intersection of countably many open intervals.*
b) *Show that every open interval* (a,b) *is a countable union of closed intervals.*
c) *Show that an intersection of a possibly infinite family of bounded closed intervals,* $\bigcap_{\lambda \in I} [a_\lambda, b_\lambda]$, *is either empty, a single point, or a bounded closed interval.*

Exercise* 1.4.7*: *Suppose* S *is a set of disjoint open intervals in* $\mathbb{R}$. *That is, if* $(a,b) \in S$ *and* $(c,d) \in S$, *then either* $(a,b) = (c,d)$ *or* $(a,b) \cap (c,d) = \emptyset$. *Prove* S *is a countable set.*

Exercise* 1.4.8*: *Prove that the cardinality of* $[0,1]$ *is the same as the cardinality of* $(0,1)$ *by showing that* $|[0,1]| \leq |(0,1)|$ *and* $|(0,1)| \leq |[0,1]|$. *See Definition 0.3.28. This proof requires the Cantor–Bernstein–Schröder theorem we stated without proof. Note that this proof does not give you an explicit bijection.*

Exercise* 1.4.9** (Challenging):*** *A number* x *is* algebraic *if* x *is a root of a polynomial with integer coefficients, in other words,* $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ *where all* $a_n \in \mathbb{Z}$.

a) *Show that there are only countably many algebraic numbers.*
b) *Show that there exist non-algebraic (transcendental) numbers (follow in the footsteps of Cantor, use the uncountability of* $\mathbb{R}$*).*

Hint: Feel free to use the fact that a polynomial of degree n *has at most* n *real roots.*

Exercise* 1.4.10** (Challenging):*** *Let* F *be the set of all functions* $f : \mathbb{R} \to \mathbb{R}$. *Prove* $|\mathbb{R}| < |F|$ *using Cantor's Theorem 0.3.34.*[*]

[*]Interestingly, if C is the set of continuous functions, then $|\mathbb{R}| = |C|$.

1.5 Decimal representation of the reals

Note: 1 lecture (optional)

We often think of real numbers as their *decimal representation*. For a positive integer n, we find the digits $d_K, d_{K-1}, \ldots, d_2, d_1, d_0$ for some K, where each d_j is an integer between 0 and 9, then

$$n = d_K 10^K + d_{K-1} 10^{K-1} + \cdots + d_2 10^2 + d_1 10 + d_0.$$

We often assume $d_K \neq 0$. To represent n we write the sequence of digits: $n = d_K d_{K-1} \cdots d_2 d_1 d_0$. By a (decimal) *digit*, we mean an integer between 0 and 9.

Similarly, we represent some rational numbers. That is, for certain numbers x, we can find negative integer $-M$, a positive integer K, and digits $d_K, d_{K-1}, \ldots, d_1, d_0, d_{-1}, \ldots, d_{-M}$, such that

$$x = d_K 10^K + d_{K-1} 10^{K-1} + \cdots + d_2 10^2 + d_1 10 + d_0 + d_{-1} 10^{-1} + d_{-2} 10^{-2} + \cdots + d_{-M} 10^{-M}.$$

We write $x = d_K d_{K-1} \cdots d_1 d_0 \,.\, d_{-1} d_{-2} \cdots d_{-M}$.

Not every real number has such a representation, even the simple rational number $1/3$ does not. The irrational number $\sqrt{2}$ does not have such a representation either. To get a representation for all real numbers, we must allow infinitely many digits.

Let us consider only real numbers in the interval $(0,1]$. If we find a representation for these, adding integers to them obtains a representation for all real numbers. Take an infinite sequence of decimal digits:

$$0.d_1 d_2 d_3 \ldots.$$

That is, we have a digit d_j for every $j \in \mathbb{N}$. We renumbered the digits to avoid the negative signs. We call the number

$$D_n := \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n}.$$

the truncation of x to n decimal digits. We say this sequence of digits represents a real number x if

$$x = \sup_{n \in \mathbb{N}} \left(\frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} \right) = \sup_{n \in \mathbb{N}} D_n.$$

Proposition 1.5.1.

(i) Every infinite sequence of digits $0.d_1 d_2 d_3 \ldots$ represents a unique real number $x \in [0,1]$, and

$$D_n \leq x \leq D_n + \frac{1}{10^n} \qquad \textit{for all } n \in \mathbb{N}.$$

(ii) For every $x \in (0,1]$ there exists an infinite sequence of digits $0.d_1 d_2 d_3 \ldots$ that represents x. There exists a unique representation such that

$$D_n < x \leq D_n + \frac{1}{10^n} \qquad \textit{for all } n \in \mathbb{N}.$$

Proof. We start with the first item. Take an arbitrary infinite sequence of digits $0.d_1d_2d_3\ldots$. Use the geometric sum formula to write

$$\begin{aligned} D_n = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots + \frac{d_n}{10^n} &\leq \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots + \frac{9}{10^n} \\ &= \frac{9}{10}\left(1 + {}^1\!/\!{}_{10} + ({}^1\!/\!{}_{10})^2 + \cdots + ({}^1\!/\!{}_{10})^{n-1}\right) \\ &= \frac{9}{10}\left(\frac{1-({}^1\!/\!{}_{10})^n}{1-{}^1\!/\!{}_{10}}\right) = 1 - ({}^1\!/\!{}_{10})^n < 1. \end{aligned}$$

In particular, $D_n < 1$ for all n. A sum of nonnegative numbers is nonnegative so $D_n \geq 0$, and hence

$$0 \leq \sup_{n\in\mathbb{N}} D_n \leq 1.$$

Therefore, $0.d_1d_2d_3\ldots$ represents a unique number $x := \sup_{n\in\mathbb{N}} D_n \in [0,1]$. As x is a supremum, then $D_n \leq x$. Take $m \in \mathbb{N}$. If $m < n$, then $D_m - D_n \leq 0$. If $m > n$, then computing as above

$$D_m - D_n = \frac{d_{n+1}}{10^{n+1}} + \frac{d_{n+2}}{10^{n+2}} + \frac{d_{n+3}}{10^{n+3}} + \cdots + \frac{d_m}{10^m} \leq \frac{1}{10^n}\left(1-({}^1\!/\!{}_{10})^{m-n}\right) < \frac{1}{10^n}.$$

Take the supremum over m to find

$$x - D_n \leq \frac{1}{10^n}.$$

We move on to the second item. Take any $x \in (0,1]$. First let us tackle the existence. For convenience, let $D_0 := 0$. Then, $D_0 < x \leq D_0 + 10^{-0}$. Suppose we defined the digits $d_1, d_2, \ldots, d_n$, and that $D_k < x \leq D_k + 10^{-k}$, for $k = 0,1,2,\ldots,n$. We need to define d_{n+1}.

By the Archimedean property of the real numbers, find an integer j such that $x - D_n \leq j10^{-(n+1)}$. Take the least such j and obtain

$$(j-1)10^{-(n+1)} < x - D_n \leq j10^{-(n+1)}. \tag{1.3}$$

Let $d_{n+1} := j-1$. As $D_n < x$, then $d_{n+1} = j-1 \geq 0$. On the other hand, since $x - D_n \leq 10^{-n}$, we have that j is at most 10, and therefore $d_{n+1} \leq 9$. So d_{n+1} is a decimal digit. Since $D_{n+1} = D_n + d_{n+1}10^{-(n+1)}$ add D_n to the inequality (1.3) above:

$$\begin{aligned} D_{n+1} = D_n + (j-1)10^{-(n+1)} < x &\leq D_n + j10^{-(n+1)} \\ &= D_n + (j-1)10^{-(n+1)} + 10^{-(n+1)} = D_{n+1} + 10^{-(n+1)}. \end{aligned}$$

And so $D_{n+1} < x \leq D_{n+1} + 10^{-(n+1)}$ holds. We inductively defined an infinite sequence of digits $0.d_1d_2d_3\ldots$.

Consider $D_n < x \leq D_n + 10^{-n}$. As $D_n < x$ for all n, then $\sup\{D_n : n \in \mathbb{N}\} \leq x$. The second inequality for D_n implies

$$x - \sup\{D_m : m \in \mathbb{N}\} \leq x - D_n \leq 10^{-n}.$$

As the inequality holds for all n and 10^{-n} can be made arbitrarily small (see Exercise 1.5.8), we have $x \leq \sup\{D_m : m \in \mathbb{N}\}$. Therefore, $\sup\{D_m : m \in \mathbb{N}\} = x$.

What is left to show is the uniqueness. Suppose $0.e_1e_2e_3\ldots$ is another representation of x. Let E_n be the n-digit truncation of $0.e_1e_2e_3\ldots$, and suppose $E_n < x \leq E_n + 10^{-n}$ for all $n \in \mathbb{N}$. Suppose for some $K \in \mathbb{N}$, $e_n = d_n$ for all $n < K$, so $D_{K-1} = E_{K-1}$. Then

$$E_K = D_{K-1} + e_K 10^{-K} < x \leq E_K + 10^{-K} = D_{K-1} + e_K 10^{-K} + 10^{-K}.$$

Subtracting D_{K-1} and multiplying by 10^K we get

$$e_K < (x - D_{K-1})10^K \leq e_K + 1.$$

Similarly,

$$d_K < (x - D_{K-1})10^K \leq d_K + 1.$$

Hence, both e_K and d_K are the largest integer j such that $j < (x - D_{K-1})10^K$, and therefore $e_K = d_K$. That is, the representation is unique. □

The representation is not unique if we do not require $D_n < x$ for all n. For example, for the number $1/2$, the method in the proof obtains the representation

$$0.49999\ldots.$$

However, $1/2$ also has the representation $0.50000\ldots$.

The only numbers that have nonunique representations are ones that end either in an infinite sequence of 0s or 9s, because the only representation for which $D_n = x$ is one where all digits past the nth digit are zero. In this case there are exactly two representations of x (see the exercises).

Let us give another proof of the uncountability of the reals using decimal representations. This is Cantor's second proof, and is probably better known. This proof may seem shorter, but it is because we already did the hard part above and we are left with a slick trick to prove that $\mathbb{R}$ is uncountable. This trick is called *Cantor diagonalization* and finds use in other proofs as well.

Theorem 1.5.2 (Cantor)**.** *The set $(0,1]$ is uncountable.*

Proof. Let $X := \{x_1, x_2, x_3, \ldots\}$ be any countable subset of real numbers in $(0,1]$. We will construct a real number not in X. Let

$$x_n = 0.d_1^n d_2^n d_3^n \ldots$$

be the unique representation from the proposition, that is, d_j^n is the jth digit of the nth number. Let

$$e_n := \begin{cases} 1 & \text{if } d_n^n \neq 1, \\ 2 & \text{if } d_n^n = 1. \end{cases}$$

Let E_n be the n-digit truncation of $y = 0.e_1e_2e_3\ldots$. Because all the digits are nonzero we get $E_n < E_{n+1} \leq y$. Therefore

$$E_n < y \leq E_n + 10^{-n}$$

for all n, and the representation is the unique one for y from the proposition. For every n, the nth digit of y is different from the nth digit of x_n, so $y \neq x_n$. Therefore $y \notin X$, and as X was an arbitrary countable subset, $(0,1]$ must be uncountable. See Figure 1.4 for an example. □

$$
\begin{array}{rccccccc}
x_1 = & 0. & \boxed{1} & 3 & 2 & 1 & 0 & \cdots \\
x_2 = & 0. & 7 & \boxed{9} & 4 & 1 & 3 & \cdots \\
x_3 = & 0. & 3 & 0 & \boxed{1} & 3 & 4 & \cdots \\
x_4 = & 0. & 8 & 9 & 2 & \boxed{5} & 6 & \cdots \\
x_5 = & 0. & 1 & 6 & 0 & 2 & \boxed{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Number not in the list:
$y = 0.21211\ldots$

Figure 1.4: Example of Cantor diagonalization, the diagonal digits d_n^n marked.

Using decimal digits we can also find lots of numbers that are not rational. The following proposition is true for every rational number, but we give it only for $x \in (0,1]$ for simplicity.

Proposition 1.5.3. *If $x \in (0,1]$ is a rational number and $x = 0.d_1 d_2 d_3 \ldots$, then the decimal digits eventually start repeating. That is, there are positive integers N and P, such that for all $n \geq N$, $d_n = d_{n+P}$.*

Proof. Suppose $x = p/q$ for positive integers p and q. Suppose also that x is a number with a unique representation, as otherwise we have seen above that both its representations are repeating, see also Exercise 1.5.3. This also means that $x \neq 1$ so $p < q$.

To compute the first digit we take $10p$ and divide by q. Let d_1 be the quotient, and the remainder r_1 is some integer between 0 and $q-1$. That is, d_1 is the largest integer such that $d_1 q \leq 10p$ and then $r_1 = 10p - d_1 q$. As $p < q$, then $d_1 < 10$, so d_1 is a digit. Furthermore,

$$\frac{d_1}{10} \leq \frac{p}{q} = \frac{d_1}{10} + \frac{r_1}{10q} \leq \frac{d_1}{10} + \frac{1}{10}.$$

The first inequality must be strict since x has a unique representation. That is, d_1 really is the first digit. What is left is $r_1/(10q)$. This is the same as computing the first digit of r_1/q. To compute d_2 divide $10r_1$ by q, and so on. After computing $n-1$ digits, we have $p/q = D_{n-1} + r_{n-1}/(10^n q)$. To get the nth digit, divide $10r_{n-1}$ by q to get quotient d_n, remainder r_n, and the inequalities

$$\frac{d_n}{10} \leq \frac{r_{n-1}}{q} = \frac{d_n}{10} + \frac{r_n}{10q} \leq \frac{d_n}{10} + \frac{1}{10}.$$

Dividing by 10^{n-1} and adding D_{n-1} we find

$$D_n \leq D_{n-1} + \frac{r_{n-1}}{10^n q} = \frac{p}{q} \leq D_n + \frac{1}{10^n}.$$

By uniqueness we really have the nth digit d_n from the construction.

The new digit depends only the remainder from the previous step. There are at most q possible remainders and hence at some step the process must start repeating itself, and P is at most q. $\square$

The converse of the proposition is also true and is left as an exercise.

Example 1.5.4: The number

$$x = 0.101001000100001000001\ldots,$$

is irrational. That is, the digits are n zeros, then a one, then $n+1$ zeros, then a one, and so on and so forth. The fact that x is irrational follows from the proposition; the digits never start repeating. For every P, if we go far enough, we find a 1 followed by at least $P+1$ zeros.

1.5.1 Exercises

Exercise* 1.5.1** (Easy):*** *What is the decimal representation of* 1 *guaranteed by Proposition 1.5.1? Make sure to show that it does satisfy the condition.*

Exercise* 1.5.2*: *Prove the converse of Proposition 1.5.3, that is, if the digits in the decimal representation of x are eventually repeating, then x must be rational.*

Exercise* 1.5.3*: *Show that real numbers $x \in (0,1)$ with nonunique decimal representation are exactly the rational numbers that can be written as $\frac{m}{10^n}$ for some integers m and n. In this case show that there exist exactly two representations of x.*

Exercise* 1.5.4*: *Let $b \geq 2$ be an integer. Define a representation of a real number in $[0,1]$ in terms of base b rather than base 10 and prove Proposition 1.5.1 for base b.*

Exercise* 1.5.5*: *Using the previous exercise with $b = 2$ (binary), show that cardinality of $\mathbb{R}$ is the same as the cardinality of $\mathscr{P}(\mathbb{N})$, obtaining yet another (though related) proof that $\mathbb{R}$ is uncountable. Hint: Construct two injections, one from $[0,1]$ to $\mathscr{P}(\mathbb{N})$ and one from $\mathscr{P}(\mathbb{N})$ to $[0,1]$. Hint 2: Given a set $A \subset \mathbb{N}$, let the nth binary digit of x be 1 if $n \in A$.*

Exercise* 1.5.6** (Challenging):*** *Explicitly construct an injection from $[0,1] \times [0,1]$ to $[0,1]$ (think about why this is so surprising**). Then describe the set of numbers in $[0,1]$ not in the image of your injection (unless, of course, you managed to construct a bijection). Hint: Consider even and odd digits of the decimal expansion.*

Exercise* 1.5.7*: *Prove that if $x = p/q \in (0,1]$ is a rational number, $q > 1$, then the period P of repeating digits in the decimal representation of x is in fact less than or equal to $q-1$.*

Exercise* 1.5.8*: *Prove that if $b \in \mathbb{N}$ and $b \geq 2$, then for every $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $b^{-n} < \varepsilon$. Hint: One possibility is to first prove that $b^n > n$ for all $n \in \mathbb{N}$ by induction.*

Exercise* 1.5.9*: *Explicitly construct an injection $f \colon \mathbb{R} \to \mathbb{R} \setminus \mathbb{Q}$ using Proposition 1.5.3.*

*With quite a bit more work (or by applying the Cantor–Bernstein–Schröder theorem) one can prove that there is a bijection. When he proved this result, Cantor apparently wrote "I see it but I don't believe it."

Chapter 2

Sequences and Series

2.1 Sequences and limits

Note: 2.5 lectures

Analysis is essentially about taking limits. The most basic type of a limit is a limit of a sequence of real numbers. We have already seen sequences used informally. Let us give the formal definition.

Definition 2.1.1. A *sequence* (of real numbers) is a function $x\colon \mathbb{N} \to \mathbb{R}$. Instead of $x(n)$, we usually denote the nth element in the sequence by x_n. We use the notation $\{x_n\}$, or more precisely

$$\{x_n\}_{n=1}^{\infty},$$

to denote a sequence.

A sequence $\{x_n\}$ is *bounded* if there exists a $B \in \mathbb{R}$ such that

$$|x_n| \leq B \qquad \text{for all } n \in \mathbb{N}.$$

In other words, the sequence $\{x_n\}$ is bounded whenever the set $\{x_n : n \in \mathbb{N}\}$ is bounded, or equivalently when it is bounded as a function.

When we need to give a concrete sequence we often give each term as a formula in terms of n. For example, $\{1/n\}_{n=1}^{\infty}$, or simply $\{1/n\}$, stands for the sequence $1, 1/2, 1/3, 1/4, 1/5, \ldots$. The sequence $\{1/n\}$ is a bounded sequence ($B = 1$ suffices). On the other hand the sequence $\{n\}$ stands for $1, 2, 3, 4, \ldots$, and this sequence is not bounded (why?).

While the notation for a sequence is similar* to that of a set, the notions are distinct. For example, the sequence $\{(-1)^n\}$ is the sequence $-1, 1, -1, 1, -1, 1, \ldots$, whereas the set of values, the *range of the sequence*, is just the set $\{-1, 1\}$. We can write this set as $\{(-1)^n : n \in \mathbb{N}\}$. When ambiguity can arise, we use the words *sequence* or *set* to distinguish the two concepts.

Another example of a sequence is the so-called *constant sequence*. That is a sequence $\{c\} = c, c, c, c, \ldots$ consisting of a single constant $c \in \mathbb{R}$ repeating indefinitely.

We now get to the idea of a *limit of a sequence*. We will see in Proposition 2.1.6 that the notation below is well-defined. That is, if a limit exists, then it is unique. So it makes sense to talk about *the* limit of a sequence.

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*[BS] use the notation (x_n) to denote a sequence instead of $\{x_n\}$, which is what [R2] uses. Both are common.

Definition 2.1.2. A sequence $\{x_n\}$ is said to *converge* to a number $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq M$. The number x is said to be the *limit* of $\{x_n\}$. We write

$$\lim_{n\to\infty} x_n := x.$$

A sequence that converges is said to be *convergent*. Otherwise, we say the sequence *diverges* or that it is *divergent*.

It is good to know intuitively what a limit means. It means that eventually every number in the sequence is close to the number x. More precisely, we can get arbitrarily close to the limit, provided we go far enough in the sequence. It does not mean we ever reach the limit. It is possible, and quite common, that there is no x_n in the sequence that equals the limit x. We illustrate the concept in Figure 2.1. In the figure we first think of the sequence as a graph, as it is a function of $\mathbb{N}$. Secondly we also plot it as a sequence of labeled points on the real line.

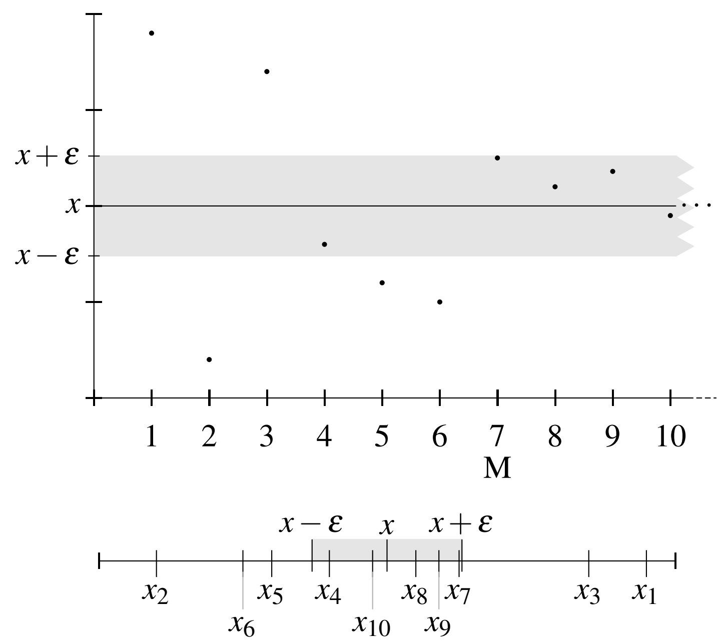

Figure 2.1: Illustration of convergence. On top, we show the first ten points of the sequence as a graph with M and the interval around the limit x marked. On bottom, the points of the same sequence are marked on the number line.

When we write $\lim x_n = x$ for some real number x, we are saying two things: first, that $\{x_n\}$ is convergent, and second, that the limit is x.

The definition above is one of the most important definitions in analysis, and it is necessary to understand it perfectly. The key point in the definition is that given *any* $\varepsilon > 0$, we can find an M. The M can depend on ε, so we only pick an M once we know ε. Let us illustrate convergence on a few examples.

Example 2.1.3: The constant sequence $1, 1, 1, 1, \ldots$ is convergent and the limit is 1. For every $\varepsilon > 0$, we pick $M = 1$.

Example 2.1.4: Claim: *The sequence* $\{1/n\}$ *is convergent and*

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Proof: Given an $\varepsilon > 0$, we find an $M \in \mathbb{N}$ such that $0 < 1/M < \varepsilon$ (Archimedean property at work). Then for all $n \geq M$,

$$|x_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{M} < \varepsilon.$$

Example 2.1.5: The sequence $\{(-1)^n\}$ is divergent. Proof: If there were a limit x, then for $\varepsilon = \frac{1}{2}$ we expect an M that satisfies the definition. Suppose such an M exists. Then for an even $n \geq M$ we compute

$$1/2 > |x_n - x| = |1 - x| \qquad \text{and} \qquad 1/2 > |x_{n+1} - x| = |-1 - x|.$$

But

$$2 = |1 - x - (-1 - x)| \leq |1 - x| + |-1 - x| < 1/2 + 1/2 = 1,$$

and that is a contradiction.

Proposition 2.1.6. *A convergent sequence has a unique limit.*

The proof of this proposition exhibits a useful technique in analysis. Many proofs follow the same general scheme. We want to show a certain quantity is zero. We write the quantity using the triangle inequality as two quantities, and we estimate each one by arbitrarily small numbers.

Proof. Suppose the sequence $\{x_n\}$ has limits x and y. Take an arbitrary $\varepsilon > 0$. From the definition find an M_1 such that for all $n \geq M_1$, $|x_n - x| < \varepsilon/2$. Similarly, find an M_2 such that for all $n \geq M_2$, we have $|x_n - y| < \varepsilon/2$. Now take an n such that $n \geq M_1$ and also $n \geq M_2$, and estimate

$$\begin{aligned} |y - x| &= |x_n - x - (x_n - y)| \\ &\leq |x_n - x| + |x_n - y| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As $|y - x| < \varepsilon$ for all $\varepsilon > 0$, then $|y - x| = 0$ and $y = x$. Hence the limit (if it exists) is unique. □

Proposition 2.1.7. *A convergent sequence* $\{x_n\}$ *is bounded.*

Proof. Suppose $\{x_n\}$ converges to x. Thus there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $|x_n - x| < 1$. Let $B_1 := |x| + 1$ and note that for $n \geq M$,

$$\begin{aligned} |x_n| &= |x_n - x + x| \\ &\leq |x_n - x| + |x| \\ &< 1 + |x| = B_1. \end{aligned}$$

The set $\{|x_1|, |x_2|, \ldots, |x_{M-1}|\}$ is a finite set and hence let

$$B_2 := \max\{|x_1|, |x_2|, \ldots, |x_{M-1}|\}.$$

Let $B := \max\{B_1, B_2\}$. Then for all $n \in \mathbb{N}$,

$$|x_n| \leq B.$$

□

The sequence $\{(-1)^n\}$ shows that the converse does not hold. A bounded sequence is not necessarily convergent.

Example 2.1.8: Let us show the sequence $\left\{\frac{n^2+1}{n^2+n}\right\}$ converges and

$$\lim_{n\to\infty} \frac{n^2+1}{n^2+n} = 1.$$

Given $\varepsilon > 0$, find $M \in \mathbb{N}$ such that $\frac{1}{M} < \varepsilon$. Then for all $n \geq M$,

$$\begin{aligned}\left|\frac{n^2+1}{n^2+n} - 1\right| = \left|\frac{n^2+1-(n^2+n)}{n^2+n}\right| &= \left|\frac{1-n}{n^2+n}\right| \\ &= \frac{n-1}{n^2+n} \\ &\leq \frac{n}{n^2+n} = \frac{1}{n+1} \\ &\leq \frac{1}{n} \leq \frac{1}{M} < \varepsilon.\end{aligned}$$

Therefore, $\lim \frac{n^2+1}{n^2+n} = 1$. This example shows that sometimes to get what you want, you must throw away some information to get a simpler estimate.

2.1.1 Monotone sequences

The simplest type of a sequence is a monotone sequence. Checking that a monotone sequence converges is as easy as checking that it is bounded. It is also easy to find the limit for a convergent monotone sequence, provided we can find the supremum or infimum of a countable set of numbers.

Definition 2.1.9. A sequence $\{x_n\}$ is *monotone increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{x_n\}$ is *monotone decreasing* if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. If a sequence is either monotone increasing or monotone decreasing, we can simply say the sequence is *monotone*.*

For example, $\{n\}$ is monotone increasing, $\{1/n\}$ is monotone decreasing, the constant sequence $\{1\}$ is both monotone increasing and monotone decreasing, and $\{(-1)^n\}$ is not monotone. First few terms of a sample monotone increasing sequence are shown in Figure 2.2.

Proposition 2.1.10. *A monotone sequence $\{x_n\}$ is bounded if and only if it is convergent.*

Furthermore, if $\{x_n\}$ is monotone increasing and bounded, then

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

If $\{x_n\}$ is monotone decreasing and bounded, then

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

*Some authors use the word *monotonic*.

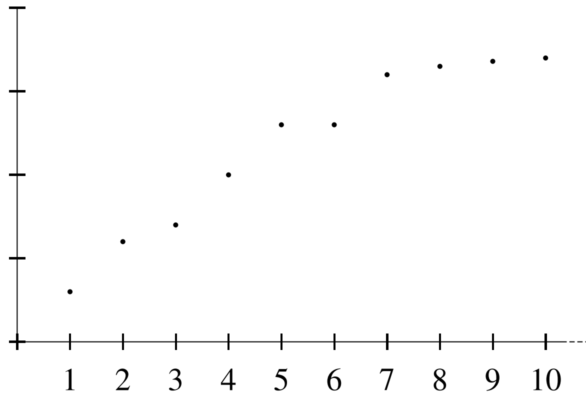

Figure 2.2: First few terms of a monotone increasing sequence as a graph.

Proof. Consider a monotone increasing sequence $\{x_n\}$. Suppose the sequence is bounded, that is, the set $\{x_n : n \in \mathbb{N}\}$ is bounded. Let

$$x := \sup\{x_n : n \in \mathbb{N}\}.$$

Let $\varepsilon > 0$ be arbitrary. As x is the supremum, then there must be at least one $M \in \mathbb{N}$ such that $x_M > x - \varepsilon$ (because x is the supremum). As $\{x_n\}$ is monotone increasing, then it is easy to see (by induction) that $x_n \geq x_M$ for all $n \geq M$. Hence for all $n \geq M$,

$$|x_n - x| = x - x_n \leq x - x_M < \varepsilon.$$

Therefore, the sequence converges to x, so a bounded monotone increasing sequence converges. For the other direction, we have already proved that a convergent sequence is bounded.

The proof for monotone decreasing sequences is left as an exercise. $\square$

Example 2.1.11: Take the sequence $\left\{\frac{1}{\sqrt{n}}\right\}$.

The sequence is bounded below as $\frac{1}{\sqrt{n}} > 0$ for all $n \in \mathbb{N}$. Let us show that it is monotone decreasing. We start with $\sqrt{n+1} \geq \sqrt{n}$ (why is that true?). From this inequality we obtain

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}.$$

So the sequence is monotone decreasing and bounded below (hence bounded). Proposition 2.1.10 says that that the sequence is convergent and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \inf\left\{\frac{1}{\sqrt{n}} : n \in \mathbb{N}\right\}.$$

We already know that the infimum is greater than or equal to 0, as 0 is a lower bound. Take a number $b \geq 0$ such that $b \leq \frac{1}{\sqrt{n}}$ for all n. We square both sides to obtain

$$b^2 \leq \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}.$$

We have seen before that this implies that $b^2 \leq 0$ (a consequence of the Archimedean property). As $b^2 \geq 0$ as well, we have $b^2 = 0$ and so $b = 0$. Hence, $b = 0$ is the greatest lower bound, and $\lim \frac{1}{\sqrt{n}} = 0$.

Example 2.1.12: A word of caution: We must show that a monotone sequence is bounded in order to use Proposition 2.1.10 to conclude a sequence converges. The sequence $\{1 + 1/2 + \cdots + 1/n\}$ is a monotone increasing sequence that grows very slowly. We will see, once we get to series, that this sequence has no upper bound and so does not converge. It is not at all obvious that this sequence has no upper bound.

A common example of where monotone sequences arise is the following proposition. The proof is left as an exercise.

Proposition 2.1.13. *Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n, y_n \in S$ and*

$$\sup S = \lim_{n\to\infty} x_n \qquad \text{and} \qquad \inf S = \lim_{n\to\infty} y_n.$$

2.1.2 Tail of a sequence

Definition 2.1.14. For a sequence $\{x_n\}$, the *K-tail* (where $K \in \mathbb{N}$), or just the *tail*, of $\{x_n\}$ is the sequence starting at $K+1$, usually written as

$$\{x_{n+K}\}_{n=1}^{\infty} \qquad \text{or} \qquad \{x_n\}_{n=K+1}^{\infty}.$$

For example, the 4-tail of $\{1/n\}$ is $1/5, 1/6, 1/7, 1/8, \ldots$. The 0-tail of a sequence is the sequence itself. The convergence and the limit of a sequence only depends on its tail.

Proposition 2.1.15. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then the following statements are equivalent:*

(i) The sequence $\{x_n\}_{n=1}^{\infty}$ converges.

(ii) The K-tail $\{x_{n+K}\}_{n=1}^{\infty}$ converges for all $K \in \mathbb{N}$.

(iii) The K-tail $\{x_{n+K}\}_{n=1}^{\infty}$ converges for some $K \in \mathbb{N}$.

Furthermore, if any (and hence all) of the limits exist, then for all $K \in \mathbb{N}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}.$$

Proof. It is clear that (ii) implies (iii). We will therefore show first that (i) implies (ii), and then we will show that (iii) implies (i). That is,

$$\begin{tikzcd}
& \text{(ii)} \arrow[dr, Rightarrow] & \\
\text{(i)} \arrow[ur, Rightarrow, "\text{to prove}"] & & \text{(iii)} \arrow[ll, Rightarrow, "\text{to prove}"]
\end{tikzcd}$$

In the process we will also show that the limits are equal.

We start with (i) implies (ii). Suppose $\{x_n\}$ converges to some $x \in \mathbb{R}$. Let $K \in \mathbb{N}$ be arbitrary, and define $y_n := x_{n+K}$. We wish to show that $\{y_n\}$ converges to x. Given an $\varepsilon > 0$, there exists an

$M \in \mathbb{N}$ such that $|x - x_n| < \varepsilon$ for all $n \geq M$. Note that $n \geq M$ implies $n + K \geq M$. Therefore, for all $n \geq M$, we have

$$|x - y_n| = |x - x_{n+K}| < \varepsilon.$$

Consequently, $\{y_n\}$ converges to x.

Let us move to (iii) implies (i). Let $K \in \mathbb{N}$ be given, define $y_n := x_{n+K}$, and suppose that $\{y_n\}$ converges to $x \in \mathbb{R}$. That is, given an $\varepsilon > 0$, there exists an $M' \in \mathbb{N}$ such that $|x - y_n| < \varepsilon$ for all $n \geq M'$. Let $M := M' + K$. Then $n \geq M$ implies $n - K \geq M'$. Thus, whenever $n \geq M$, we have

$$|x - x_n| = |x - y_{n-K}| < \varepsilon.$$

Therefore, $\{x_n\}$ converges to x. □

At the end of the day, the limit does not care about how the sequence begins, it only cares about the tail of the sequence. The beginning of the sequence may be arbitrary.

For example, the sequence defined by $x_n := \frac{n}{n^2+16}$ is decreasing if we start at $n = 4$ (it is increasing before). That is: $\{x_n\} = 1/17, 1/10, 3/25, 1/8, 5/41, 3/26, 7/65, 1/10, 9/97, 5/58, \ldots$, and

$$1/17 < 1/10 < 3/25 < 1/8 > 5/41 > 3/26 > 7/65 > 1/10 > 9/97 > 5/58 > \ldots.$$

If we throw away the first 3 terms and look at the 3-tail, it is decreasing. The proof is left as an exercise. Since the 3-tail is monotone and bounded below by zero, it is convergent, and therefore the sequence is convergent.

2.1.3 Subsequences

It is useful to sometimes consider only some terms of a sequence. A subsequence of $\{x_n\}$ is a sequence that contains only some of the numbers from $\{x_n\}$ in the same order.

Definition 2.1.16. Let $\{x_n\}$ be a sequence. Let $\{n_i\}$ be a strictly increasing sequence of natural numbers, that is, $n_i < n_{i+1}$ for all i (in other words $n_1 < n_2 < n_3 < \cdots$). The sequence

$$\{x_{n_i}\}_{i=1}^{\infty}$$

is called a *subsequence* of $\{x_n\}$.

So the subsequence is the sequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$. Consider the sequence $\{1/n\}$. The sequence $\{1/3n\}$ is a subsequence. To see how these two sequences fit in the definition, take $n_i := 3i$. The numbers in the subsequence must come from the original sequence. So $1, 0, 1/3, 0, 1/5, \ldots$ is not a subsequence of $\{1/n\}$. Similarly, order must be preserved. So the sequence $1, 1/3, 1/2, 1/5, \ldots$ is not a subsequence of $\{1/n\}$.

A tail of a sequence is one special type of a subsequence. For an arbitrary subsequence, we have the following proposition about convergence.

Proposition 2.1.17. *If $\{x_n\}$ is a convergent sequence, then every subsequence $\{x_{n_i}\}$ is also convergent, and*

$$\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_i}.$$

Proof. Suppose $\lim_{n\to\infty} x_n = x$. So for every $\varepsilon > 0$ there is an $M \in \mathbb{N}$ such that for all $n \geq M$,

$$|x_n - x| < \varepsilon.$$

It is not hard to prove (do it!) by induction that $n_i \geq i$. Hence $i \geq M$ implies $n_i \geq M$. Thus, for all $i \geq M$,

$$|x_{n_i} - x| < \varepsilon,$$

and we are done. □

Example 2.1.18: Existence of a convergent subsequence does not imply convergence of the sequence itself. Take the sequence $0, 1, 0, 1, 0, 1, \ldots$. That is, $x_n = 0$ if n is odd, and $x_n = 1$ if n is even. The sequence $\{x_n\}$ is divergent; however, the subsequence $\{x_{2n}\}$ converges to 1 and the subsequence $\{x_{2n+1}\}$ converges to 0. Compare Proposition 2.3.7.

2.1.4 Exercises

In the following exercises, feel free to use what you know from calculus to find the limit, if it exists. But you must *prove* that you found the correct limit, or prove that the series is divergent.

***Exercise* 2.1.1:** *Is the sequence $\{3n\}$ bounded? Prove or disprove.*

***Exercise* 2.1.2:** *Is the sequence $\{n\}$ convergent? If so, what is the limit?*

***Exercise* 2.1.3:** *Is the sequence $\left\{\frac{(-1)^n}{2n}\right\}$ convergent? If so, what is the limit?*

***Exercise* 2.1.4:** *Is the sequence $\{2^{-n}\}$ convergent? If so, what is the limit?*

***Exercise* 2.1.5:** *Is the sequence $\left\{\frac{n}{n+1}\right\}$ convergent? If so, what is the limit?*

***Exercise* 2.1.6:** *Is the sequence $\left\{\frac{n}{n^2+1}\right\}$ convergent? If so, what is the limit?*

***Exercise* 2.1.7:** *Let $\{x_n\}$ be a sequence.*

a) *Show that $\lim x_n = 0$ (that is, the limit exists and is zero) if and only if $\lim |x_n| = 0$.*
b) *Find an example such that $\{|x_n|\}$ converges and $\{x_n\}$ diverges.*

***Exercise* 2.1.8:** *Is the sequence $\left\{\frac{2^n}{n!}\right\}$ convergent? If so, what is the limit?*

***Exercise* 2.1.9:** *Show that the sequence $\left\{\frac{1}{\sqrt[3]{n}}\right\}$ is monotone and bounded. Then use Proposition 2.1.10 to find the limit.*

***Exercise* 2.1.10:** *Show that the sequence $\left\{\frac{n+1}{n}\right\}$ is monotone and bounded. Then use Proposition 2.1.10 to find the limit.*

***Exercise* 2.1.11:** *Finish the proof of Proposition 2.1.10 for monotone decreasing sequences.*

Exercise* 2.1.12*: *Prove Proposition 2.1.13.*

Exercise* 2.1.13*: *Let $\{x_n\}$ be a convergent monotone sequence. Suppose there exists a $k \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} x_n = x_k.$$

Show that $x_n = x_k$ for all $n \geq k$.

Exercise* 2.1.14*: *Find a convergent subsequence of the sequence $\{(-1)^n\}$.*

Exercise* 2.1.15*: *Let $\{x_n\}$ be a sequence defined by*

$$x_n := \begin{cases} n & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}$$

a) Is the sequence bounded? (prove or disprove)

b) Is there a convergent subsequence? If so, find it.

Exercise* 2.1.16*: *Let $\{x_n\}$ be a sequence. Suppose there are two convergent subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$. Suppose*

$$\lim_{i\to\infty} x_{n_i} = a \qquad \text{and} \qquad \lim_{i\to\infty} x_{m_i} = b,$$

where $a \neq b$. Prove that $\{x_n\}$ is not convergent, without using Proposition 2.1.17.

Exercise* 2.1.17** (Tricky):*** *Find a sequence $\{x_n\}$ such that for every $y \in \mathbb{R}$, there exists a subsequence $\{x_{n_i}\}$ converging to y.*

Exercise* 2.1.18** (Easy):*** *Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$. Suppose for every $\varepsilon > 0$, there is an M such that $|x_n - x| \leq \varepsilon$ for all $n \geq M$. Show that $\lim x_n = x$.*

Exercise* 2.1.19** (Easy):*** *Let $\{x_n\}$ be a sequence and $x \in \mathbb{R}$ such that there exists a $k \in \mathbb{N}$ such that for all $n \geq k$, $x_n = x$. Prove that $\{x_n\}$ converges to x.*

Exercise* 2.1.20*: *Let $\{x_n\}$ be a sequence and define a sequence $\{y_n\}$ by $y_{2k} := x_{k^2}$ and $y_{2k-1} := x_k$ for all $k \in \mathbb{N}$. Prove that $\{x_n\}$ converges if and only if $\{y_n\}$ converges. Furthermore, prove that if they converge, then $\lim x_n = \lim y_n$.*

Exercise* 2.1.21*: *Show that the 3-tail of the sequence defined by $x_n := \frac{n}{n^2+16}$ is monotone decreasing. Hint: Suppose $n \geq m \geq 4$ and consider the numerator of the expression $x_n - x_m$.*

Exercise* 2.1.22*: *Suppose that $\{x_n\}$ is a sequence such that the subsequences $\{x_{2n}\}$, $\{x_{2n-1}\}$, and $\{x_{3n}\}$ all converge. Show that $\{x_n\}$ is convergent.*

Exercise* 2.1.23*: *Suppose that $\{x_n\}$ is a monotone increasing sequence that has a convergent subsequence. Show that $\{x_n\}$ is convergent. Note: So Proposition 2.1.17 is an "if and only if" for monotone sequences.*

2.2 Facts about limits of sequences

Note: 2–2.5 lectures, recursively defined sequences can safely be skipped

In this section we go over some basic results about the limits of sequences. We start by looking at how sequences interact with inequalities.

2.2.1 Limits and inequalities

A basic lemma about limits and inequalities is the so-called squeeze lemma. It allows us to show convergence of sequences in difficult cases if we find two other simpler convergent sequences that "squeeze" the original sequence.

Lemma 2.2.1 (Squeeze lemma)**.** *Let $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ be sequences such that*

$$a_n \leq x_n \leq b_n \quad \textit{for all } n \in \mathbb{N}.$$

Suppose $\{a_n\}$ and $\{b_n\}$ converge and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Then $\{x_n\}$ converges and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Proof. Let $x := \lim a_n = \lim b_n$. Let $\varepsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$, we have that $|a_n - x| < \varepsilon$, and an M_2 such that for all $n \geq M_2$, we have $|b_n - x| < \varepsilon$. Set $M := \max\{M_1, M_2\}$. Suppose $n \geq M$. In particular, $x - a_n < \varepsilon$, or $x - \varepsilon < a_n$. Similarly, $b_n < x + \varepsilon$. Putting everything together, we find

$$x - \varepsilon < a_n \leq x_n \leq b_n < x + \varepsilon.$$

In other words, $-\varepsilon < x_n - x < \varepsilon$ or $|x_n - x| < \varepsilon$. So $\{x_n\}$ converges to x. See Figure 2.3. □

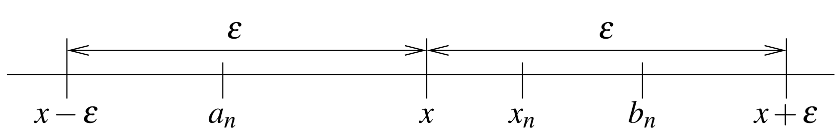

Figure 2.3: Squeeze lemma proof in picture.

Example 2.2.2: One application of the squeeze lemma is to compute limits of sequences using limits that we already know. For example, consider the sequence $\{\frac{1}{n\sqrt{n}}\}$. Since $\sqrt{n} \geq 1$ for all $n \in \mathbb{N}$, we have

$$0 \leq \frac{1}{n\sqrt{n}} \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. We already know $\lim {1}/{n} = 0$. Hence, using the constant sequence $\{0\}$ and the sequence $\{{1}/{n}\}$ in the squeeze lemma, we conclude

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0.$$

Limits, when they exist, preserve non-strict inequalities.

Lemma 2.2.3. *Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences and*

$$x_n \leq y_n,$$

for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Proof. Let $x := \lim x_n$ and $y := \lim y_n$. Let $\varepsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$, we have $|x_n - x| < \varepsilon/2$. Find an M_2 such that for all $n \geq M_2$, we have $|y_n - y| < \varepsilon/2$. In particular, for some $n \geq \max\{M_1, M_2\}$, we have $x - x_n < \varepsilon/2$ and $y_n - y < \varepsilon/2$. We add these inequalities to obtain

$$y_n - x_n + x - y < \varepsilon, \qquad \text{or} \qquad y_n - x_n < y - x + \varepsilon.$$

Since $x_n \leq y_n$, we have $0 \leq y_n - x_n$ and hence $0 < y - x + \varepsilon$. In other words,

$$x - y < \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we obtain $x - y \leq 0$. Therefore, $x \leq y$. □

The next corollary follows by using constant sequences in Lemma 2.2.3. The proof is left as an exercise.

Corollary 2.2.4.

(i) If $\{x_n\}$ is a convergent sequence such that $x_n \geq 0$, then

$$\lim_{n\to\infty} x_n \geq 0.$$

(ii) Let $a, b \in \mathbb{R}$ and let $\{x_n\}$ be a convergent sequence such that

$$a \leq x_n \leq b,$$

for all $n \in \mathbb{N}$. Then

$$a \leq \lim_{n\to\infty} x_n \leq b.$$

In Lemma 2.2.3 and Corollary 2.2.4 we cannot simply replace all the non-strict inequalities with strict inequalities. For example, let $x_n := -1/n$ and $y_n := 1/n$. Then $x_n < y_n$, $x_n < 0$, and $y_n > 0$ for all n. However, these inequalities are not preserved by the limit operation as $\lim x_n = \lim y_n = 0$. The moral of this example is that strict inequalities may become non-strict inequalities when limits are applied; if we know $x_n < y_n$ for all n, we may only conclude

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

This issue is a common source of errors.

2.2.2 Continuity of algebraic operations

Limits interact nicely with algebraic operations.

Proposition 2.2.5. *Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences.*

(i) The sequence $\{z_n\}$, where $z_n := x_n + y_n$, converges and

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n.$$

(ii) The sequence $\{z_n\}$, where $z_n := x_n - y_n$, converges and

$$\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n.$$

(iii) The sequence $\{z_n\}$, where $z_n := x_n y_n$, converges and

$$\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty} z_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right).$$

(iv) If $\lim y_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then the sequence $\{z_n\}$, where $z_n := \dfrac{x_n}{y_n}$, converges and

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty} z_n = \frac{\lim x_n}{\lim y_n}.$$

Proof. We start with (i). Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences and write $z_n := x_n + y_n$. Let $x := \lim x_n$, $y := \lim y_n$, and $z := x + y$.

Let $\varepsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$, we have $|x_n - x| < \varepsilon/2$. Find an M_2 such that for all $n \geq M_2$, we have $|y_n - y| < \varepsilon/2$. Take $M := \max\{M_1, M_2\}$. For all $n \geq M$, we have

$$\begin{aligned}
|z_n - z| &= |(x_n + y_n) - (x + y)| \\
&= |x_n - x + y_n - y| \\
&\leq |x_n - x| + |y_n - y| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore (i) is proved. Proof of (ii) is almost identical and is left as an exercise.

Let us tackle (iii). Suppose again that $\{x_n\}$ and $\{y_n\}$ are convergent sequences and write $z_n := x_n y_n$. Let $x := \lim x_n$, $y := \lim y_n$, and $z := xy$.

Let $\varepsilon > 0$ be given. Let $K := \max\{|x|, |y|, \varepsilon/3, 1\}$. Find an M_1 such that for all $n \geq M_1$, we have $|x_n - x| < \frac{\varepsilon}{3K}$. Find an M_2 such that for all $n \geq M_2$, we have $|y_n - y| < \frac{\varepsilon}{3K}$. Take $M := \max\{M_1, M_2\}$. For all $n \geq M$, we have

$$\begin{aligned}
|z_n - z| &= |(x_n y_n) - (xy)| \\
&= |(x_n - x + x)(y_n - y + y) - xy| \\
&= |(x_n - x)y + x(y_n - y) + (x_n - x)(y_n - y)| \\
&\leq |(x_n - x)y| + |x(y_n - y)| + |(x_n - x)(y_n - y)| \\
&= |x_n - x|\,|y| + |x|\,|y_n - y| + |x_n - x|\,|y_n - y| \\
&< \frac{\varepsilon}{3K}K + K\frac{\varepsilon}{3K} + \frac{\varepsilon}{3K}\frac{\varepsilon}{3K} \qquad \text{(now notice that } \tfrac{\varepsilon}{3K} \leq 1 \text{ and } K \geq 1\text{)} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Finally, we examine (iv). Instead of proving (iv) directly, we prove the following simpler claim:

Claim: *If* $\{y_n\}$ *is a convergent sequence such that* $\lim y_n \neq 0$ *and* $y_n \neq 0$ *for all* $n \in \mathbb{N}$*, then* $\{1/y_n\}$ *converges and*

$$\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{\lim y_n}.$$

Once the claim is proved, we take the sequence $\{1/y_n\}$, multiply it by the sequence $\{x_n\}$ and apply item (iii).

Proof of claim: Let $\varepsilon > 0$ be given. Let $y := \lim y_n$. As $|y| \neq 0$, then $\min\left\{|y|^2 \frac{\varepsilon}{2}, \frac{|y|}{2}\right\} > 0$. Find an M such that for all $n \geq M$, we have

$$|y_n - y| < \min\left\{|y|^2 \frac{\varepsilon}{2}, \frac{|y|}{2}\right\}.$$

For all $n \geq M$, we have $|y - y_n| < |y|/2$, and so

$$|y| = |y - y_n + y_n| \leq |y - y_n| + |y_n| < \frac{|y|}{2} + |y_n|.$$

Subtracting $|y|/2$ from both sides we obtain $|y|/2 < |y_n|$, or in other words,

$$\frac{1}{|y_n|} < \frac{2}{|y|}.$$

We finish the proof of the claim:

$$\begin{aligned}
\left|\frac{1}{y_n} - \frac{1}{y}\right| &= \left|\frac{y - y_n}{y y_n}\right| \\
&= \frac{|y - y_n|}{|y|\,|y_n|} \\
&\leq \frac{|y - y_n|}{|y|} \frac{2}{|y|} \\
&< \frac{|y|^2 \frac{\varepsilon}{2}}{|y|} \frac{2}{|y|} = \varepsilon.
\end{aligned}$$

And we are done. $\square$

By plugging in constant sequences, we get several easy corollaries. If $c \in \mathbb{R}$ and $\{x_n\}$ is a convergent sequence, then for example

$$\lim_{n\to\infty} c x_n = c \left(\lim_{n\to\infty} x_n\right) \qquad \text{and} \qquad \lim_{n\to\infty} (c + x_n) = c + \lim_{n\to\infty} x_n.$$

Similarly, we find such equalities for constant subtraction and division.

As we can take limits past multiplication we can show (exercise) that $\lim x_n^k = (\lim x_n)^k$ for all $k \in \mathbb{N}$. That is, we can take limits past powers. Let us see if we can do the same with roots.

Proposition 2.2.6. *Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$. Then*

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}.$$

Of course, to even make this statement, we need to apply Corollary 2.2.4 to show that $\lim x_n \geq 0$, so that we can take the square root without worry.

Proof. Let $\{x_n\}$ be a convergent sequence and let $x := \lim x_n$. As we just mentioned, $x \geq 0$.

First suppose $x = 0$. Let $\varepsilon > 0$ be given. Then there is an M such that for all $n \geq M$, we have $x_n = |x_n| < \varepsilon^2$, or in other words, $\sqrt{x_n} < \varepsilon$. Hence,

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \sqrt{x_n} < \varepsilon.$$

Now suppose $x > 0$ (and hence $\sqrt{x} > 0$).

$$\begin{aligned}\left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right| \\ &= \frac{1}{\sqrt{x_n} + \sqrt{x}} \left|x_n - x\right| \\ &\leq \frac{1}{\sqrt{x}} \left|x_n - x\right|.\end{aligned}$$

We leave the rest of the proof to the reader. □

A similar proof works for the kth root. That is, we also obtain $\lim x_n^{1/k} = (\lim x_n)^{1/k}$. We leave this to the reader as a challenging exercise.

We may also want to take the limit past the absolute value sign. The converse of this proposition is not true, see Exercise 2.1.7 part b).

Proposition 2.2.7. *If $\{x_n\}$ is a convergent sequence, then $\{|x_n|\}$ is convergent and*

$$\lim_{n\to\infty} |x_n| = \left|\lim_{n\to\infty} x_n\right|.$$

Proof. We simply note the reverse triangle inequality

$$\big|\, |x_n| - |x| \,\big| \leq |x_n - x|.$$

Hence if $|x_n - x|$ can be made arbitrarily small, so can $\big|\, |x_n| - |x| \,\big|$. Details are left to the reader. □

Let us see an example putting the propositions above together. Since $\lim {}^{1}\!/_{n} = 0$, then

$$\lim_{n\to\infty} \left|\sqrt{1 + {}^{1}\!/_{n}} - {}^{100}\!/_{n^2}\right| = \left|\sqrt{1 + (\lim {}^{1}\!/_{n})} - 100(\lim {}^{1}\!/_{n})(\lim {}^{1}\!/_{n})\right| = 1.$$

That is, the limit on the left-hand side exists because the right-hand side exists. You really should read the equality above from right to left.

On the other hand you must apply the propositions carefully. For example, by rewriting the expression with common denominator first we find

$$\lim_{n\to\infty} \left(\frac{n^2}{n+1} - n\right) = -1.$$

However, $\left\{\frac{n^2}{n+1}\right\}$ and $\{n\}$ are not convergent, so $\left(\lim \frac{n^2}{n+1}\right) - \left(\lim n\right)$ is nonsense.

2.2.3 Recursively defined sequences

Now that we know we can interchange limits and algebraic operations, we can compute the limits of many sequences. One such class are recursively defined sequences, that is, sequences where the next number in the sequence is computed using a formula from a fixed number of preceding elements in the sequence.

Example 2.2.8: Let $\{x_n\}$ be defined by $x_1 := 2$ and

$$x_{n+1} := x_n - \frac{x_n^2 - 2}{2x_n}.$$

We must first find out if this sequence is well-defined; we must show we never divide by zero. Then we must find out if the sequence converges. Only then can we attempt to find the limit.

So let us prove x_n exists and $x_n > 0$ for all n (so the sequence is well-defined and bounded below). Let us show this by induction. We know that $x_1 = 2 > 0$. For the induction step, suppose $x_n > 0$. Then

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{2x_n^2 - x_n^2 + 2}{2x_n} = \frac{x_n^2 + 2}{2x_n}.$$

It is always true that $x_n^2 + 2 > 0$, and as $x_n > 0$, then $\frac{x_n^2+2}{2x_n} > 0$ and hence $x_{n+1} > 0$.

Next let us show that the sequence is monotone decreasing. If we show that $x_n^2 - 2 \geq 0$ for all n, then $x_{n+1} \leq x_n$ for all n. Obviously $x_1^2 - 2 = 4 - 2 = 2 > 0$. For an arbitrary n, we have

$$x_{n+1}^2 - 2 = \left(\frac{x_n^2 + 2}{2x_n}\right)^2 - 2 = \frac{x_n^4 + 4x_n^2 + 4 - 8x_n^2}{4x_n^2} = \frac{x_n^4 - 4x_n^2 + 4}{4x_n^2} = \frac{\left(x_n^2 - 2\right)^2}{4x_n^2}.$$

Since squares are nonnegative, $x_{n+1}^2 - 2 \geq 0$ for all n. Therefore, $\{x_n\}$ is monotone decreasing and bounded ($x_n > 0$ for all n), and so the limit exists. It remains to find the limit.

Write

$$2x_n x_{n+1} = x_n^2 + 2.$$

Since $\{x_{n+1}\}$ is the 1-tail of $\{x_n\}$, it converges to the same limit. Let us define $x := \lim x_n$. Take the limit of both sides to obtain

$$2x^2 = x^2 + 2,$$

or $x^2 = 2$. As $x_n > 0$ for all n we get $x \geq 0$, and therefore $x = \sqrt{2}$.

You may have seen the sequence above before. It is *Newton's method** for finding the square root of 2. This method comes up often in practice and converges very rapidly. We used the fact that $x_1^2 - 2 > 0$, although it was not strictly needed to show convergence by considering a tail of the sequence. The sequence converges as long as $x_1 \neq 0$, although with a negative x_1 we would arrive at $x = -\sqrt{2}$. By replacing the 2 in the numerator we obtain the square root of any positive number. These statements are left as an exercise.

You should, however, be careful. Before taking any limits, you must make sure the sequence converges. Let us see an example.

*Named after the English physicist and mathematician Isaac Newton (1642–1726/7).

Example 2.2.9: Suppose $x_1 := 1$ and $x_{n+1} := x_n^2 + x_n$. If we blindly assumed that the limit exists (call it x), then we would get the equation $x = x^2 + x$, from which we might conclude $x = 0$. However, it is not hard to show that $\{x_n\}$ is unbounded and therefore does not converge.

The thing to notice in this example is that the method still works, but it depends on the initial value x_1. If we set $x_1 := 0$, then the sequence converges and the limit really is 0. An entire branch of mathematics, called dynamics, deals precisely with these issues. See Exercise 2.2.14.

2.2.4 Some convergence tests

It is not always necessary to go back to the definition of convergence to prove that a sequence is convergent. We first give a simple convergence test. The main idea is that $\{x_n\}$ converges to x if and only if $\{|x_n - x|\}$ converges to zero.

Proposition 2.2.10. *Let $\{x_n\}$ be a sequence. Suppose there is an $x \in \mathbb{R}$ and a convergent sequence $\{a_n\}$ such that*

$$\lim_{n\to\infty} a_n = 0$$

and

$$|x_n - x| \leq a_n \qquad \text{for all } n \in \mathbb{N}.$$

Then $\{x_n\}$ converges and $\lim x_n = x$.

Proof. Let $\varepsilon > 0$ be given. Note that $a_n \geq 0$ for all n. Find an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $a_n = |a_n - 0| < \varepsilon$. Then, for all $n \geq M$, we have

$$|x_n - x| \leq a_n < \varepsilon.$$

□

As the proposition shows, to study when a sequence has a limit is the same as studying when another sequence goes to zero. In general, it may be hard to decide if a sequence converges, but for certain sequences there exist easy to apply tests that tell us if the sequence converges or not. Let us see one such test. First, let us compute the limit of a certain specific sequence.

Proposition 2.2.11. *Let $c > 0$.*

(i) If $c < 1$, then

$$\lim_{n\to\infty} c^n = 0.$$

(ii) If $c > 1$, then $\{c^n\}$ is unbounded.

Proof. First consider $c < 1$. As $c > 0$, then $c^n > 0$ for all $n \in \mathbb{N}$ by induction. As $c < 1$, then $c^{n+1} < c^n$ for all n. So $\{c^n\}$ is a decreasing sequence that is bounded below. Hence, it is convergent. Let $L := \lim c^n$. The 1-tail $\{c^{n+1}\}$ also converges to L. Taking the limit of both sides of $c^{n+1} = c \cdot c^n$, we obtain $L = cL$, or $(1-c)L = 0$. It follows that $L = 0$ as $c \neq 1$.

Now consider $c > 1$. Let $B > 0$ be arbitrary. As $1/c < 1$, then $\left\{(1/c)^n\right\}$ converges to 0. Hence for some large enough n, we get

$$\frac{1}{c^n} = \left(\frac{1}{c}\right)^n < \frac{1}{B}.$$

In other words, $c^n > B$, and B is not an upper bound for $\{c^n\}$. As B was arbitrary, $\{c^n\}$ is unbounded. □

In the proposition above, the ratio of the $(n+1)$th term and the nth term is c. We generalize this simple result to a larger class of sequences. The following lemma will come up again once we get to series.

Lemma 2.2.12 (Ratio test for sequences)**.** *Let $\{x_n\}$ be a sequence such that $x_n \neq 0$ for all n and such that the limit*

$$L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \qquad \textit{exists.}$$

(i) If $L < 1$, then $\{x_n\}$ converges and $\lim x_n = 0$.

(ii) If $L > 1$, then $\{x_n\}$ is unbounded (hence diverges).

If L exists, but $L = 1$, the lemma says nothing. We cannot make any conclusion based on that information alone. For example, the sequence $\{1/n\}$ converges to zero, but $L = 1$. The constant sequence $\{1\}$ converges to 1, not zero, and $L = 1$. The sequence $\{(-1)^n\}$ does not converge at all, and $L = 1$ as well. Finally, the sequence $\{n\}$ is unbounded, yet again $L = 1$. The statement of the lemma may be strengthened somewhat, see exercises 2.2.13 and 2.3.15.

Proof. Suppose $L < 1$. As $\frac{|x_{n+1}|}{|x_n|} \geq 0$ for all n, then $L \geq 0$. Pick r such that $L < r < 1$. We wish to compare the sequence $\{x_n\}$ to the sequence $\{r^n\}$. The idea is that while the ratio $\frac{|x_{n+1}|}{|x_n|}$ is not going to be less than L eventually, it will eventually be less than r, which is still less than 1. The intuitive idea of the proof is illustrated in Figure 2.4.

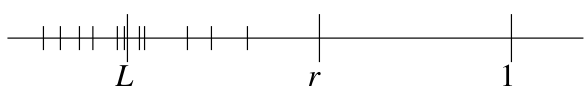

Figure 2.4: Proof of ratio test in picture. The short lines represent the ratios $\frac{|x_{n+1}|}{|x_n|}$ approaching L.

As $r - L > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L.$$

Therefore, for $n \geq M$,

$$\frac{|x_{n+1}|}{|x_n|} - L < r - L \qquad \text{or} \qquad \frac{|x_{n+1}|}{|x_n|} < r.$$

For $n > M$ (that is for $n \geq M+1$) write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} < |x_M| rr\cdots r = |x_M| r^{n-M} = (|x_M| r^{-M}) r^n.$$

The sequence $\{r^n\}$ converges to zero and hence $|x_M| r^{-M} r^n$ converges to zero. By Proposition 2.2.10, the M-tail of $\{x_n\}$ converges to zero and therefore $\{x_n\}$ converges to zero.

Now suppose $L > 1$. Pick r such that $1 < r < L$. As $L - r > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < L - r.$$

Therefore,

$$\frac{|x_{n+1}|}{|x_n|} > r.$$

Again for $n > M$, write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} > |x_M| rr \cdots r = |x_M| r^{n-M} = (|x_M| r^{-M}) r^n.$$

The sequence $\{r^n\}$ is unbounded (since $r > 1$), and so $\{x_n\}$ cannot be bounded (if $|x_n| \leq B$ for all n, then $r^n < \frac{B}{|x_M|} r^M$ for all $n > M$, which is impossible). Consequently, $\{x_n\}$ cannot converge. $\square$

Example 2.2.13: A simple application of the lemma above is to prove

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

Proof: Compute

$$\frac{2^{n+1}/(n+1)!}{2^n/n!} = \frac{2^{n+1}}{2^n} \frac{n!}{(n+1)!} = \frac{2}{n+1}.$$

It is not hard to see that $\left\{\frac{2}{n+1}\right\}$ converges to zero. The conclusion follows by the lemma.

Example 2.2.14: A more complicated (and useful) application of the ratio test is to prove

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Proof: Let $\varepsilon > 0$ be given. Consider the sequence $\left\{\frac{n}{(1+\varepsilon)^n}\right\}$. Compute

$$\frac{(n+1)/(1+\varepsilon)^{n+1}}{n/(1+\varepsilon)^n} = \frac{n+1}{n} \frac{1}{1+\varepsilon}.$$

The limit of $\frac{n+1}{n} = 1 + \frac{1}{n}$ as $n \to \infty$ is 1, and so

$$\lim_{n\to\infty} \frac{(n+1)/(1+\varepsilon)^{n+1}}{n/(1+\varepsilon)^n} = \frac{1}{1+\varepsilon} < 1.$$

Therefore, $\left\{\frac{n}{(1+\varepsilon)^n}\right\}$ converges to 0. In particular, there exists an M such that for $n \geq M$, we have $\frac{n}{(1+\varepsilon)^n} < 1$, or $n < (1+\varepsilon)^n$, or $n^{1/n} < 1+\varepsilon$. As $n \geq 1$, then $n^{1/n} \geq 1$, and so $0 \leq n^{1/n} - 1 < \varepsilon$. Consequently, $\lim n^{1/n} = 1$.

2.2.5 Exercises

***Exercise* 2.2.1:** *Prove Corollary 2.2.4. Hint: Use constant sequences and Lemma 2.2.3.*

***Exercise* 2.2.2:** *Prove part (ii) of Proposition 2.2.5.*

Exercise* 2.2.3*: *Prove that if $\{x_n\}$ is a convergent sequence, $k \in \mathbb{N}$, then*

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

Hint: Use induction.

Exercise* 2.2.4*: *Suppose $x_1 := \frac{1}{2}$ and $x_{n+1} := x_n^2$. Show that $\{x_n\}$ converges and find $\lim x_n$. Hint: You cannot divide by zero!*

Exercise* 2.2.5*: *Let $x_n := \frac{n-\cos(n)}{n}$. Use the squeeze lemma to show that $\{x_n\}$ converges and find the limit.*

Exercise* 2.2.6*: *Let $x_n := \frac{1}{n^2}$ and $y_n := \frac{1}{n}$. Define $z_n := \frac{x_n}{y_n}$ and $w_n := \frac{y_n}{x_n}$. Do $\{z_n\}$ and $\{w_n\}$ converge? What are the limits? Can you apply Proposition 2.2.5? Why or why not?*

Exercise* 2.2.7*: *True or false, prove or find a counterexample. If $\{x_n\}$ is a sequence such that $\{x_n^2\}$ converges, then $\{x_n\}$ converges.*

Exercise* 2.2.8*: *Show that*

$$\lim_{n\to\infty} \frac{n^2}{2^n} = 0.$$

Exercise* 2.2.9*: *Suppose $\{x_n\}$ is a sequence and suppose for some $x \in \mathbb{R}$, the limit*

$$L := \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|}$$

exists and $L < 1$. Show that $\{x_n\}$ converges to x.

Exercise* 2.2.10** (Challenging):*** *Let $\{x_n\}$ be a convergent sequence such that $x_n \geq 0$ and $k \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} x_n^{1/k} = \left(\lim_{n\to\infty} x_n\right)^{1/k}.$$

Hint: Find an expression q such that $\frac{x_n^{1/k} - x^{1/k}}{x_n - x} = \frac{1}{q}$.

Exercise* 2.2.11*: *Let $r > 0$. Show that starting with an arbitrary $x_1 \neq 0$, the sequence defined by*

$$x_{n+1} := x_n - \frac{x_n^2 - r}{2x_n}$$

converges to $\sqrt{r}$ if $x_1 > 0$ and $-\sqrt{r}$ if $x_1 < 0$.

Exercise* 2.2.12*:

a) Suppose $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a sequence converging to 0. Show that $\{a_n b_n\}$ converges to 0.

b) Find an example where $\{a_n\}$ is unbounded, $\{b_n\}$ converges to 0, and $\{a_n b_n\}$ is not convergent.

c) Find an example where $\{a_n\}$ is bounded, $\{b_n\}$ converges to some $x \neq 0$, and $\{a_n b_n\}$ is not convergent.

Exercise* 2.2.13** (Easy):*** *Prove the following stronger version of Lemma 2.2.12, the ratio test. Suppose $\{x_n\}$ is a sequence such that $x_n \neq 0$ for all n.*

a) Prove that if there exists an $r < 1$ and $M \in \mathbb{N}$ such that for all $n \geq M$, we have

$$\frac{|x_{n+1}|}{|x_n|} \leq r,$$

then $\{x_n\}$ converges to 0.

b) Prove that if there exists an $r > 1$ and $M \in \mathbb{N}$ such that for all $n \geq M$, we have

$$\frac{|x_{n+1}|}{|x_n|} \geq r,$$

then $\{x_n\}$ is unbounded.

Exercise* 2.2.14*: *Suppose $x_1 := c$ and $x_{n+1} := x_n^2 + x_n$. Show that $\{x_n\}$ converges if and only if $-1 \leq c \leq 0$, in which case it converges to 0.*

Exercise* 2.2.15*: *Prove $\lim\limits_{n\to\infty} (n^2+1)^{1/n} = 1$.*

Exercise* 2.2.16*: *Prove that $\left\{(n!)^{1/n}\right\}$ is unbounded. Hint: Show that $\left\{\frac{C^n}{n!}\right\}$ converges to zero for all $C > 0$.*

2.3 Limit superior, limit inferior, and Bolzano–Weierstrass

Note: 1–2 lectures, alternative proof of BW optional

In this section we study bounded sequences and their subsequences. In particular, we define the so-called limit superior and limit inferior of a bounded sequence and talk about limits of subsequences. Furthermore, we prove the Bolzano–Weierstrass theorem*, an indispensable tool in analysis, showing the existence of convergent subsequences.

We have seen that every convergent sequence is bounded, although there exist many bounded divergent sequences. For example, the sequence $\{(-1)^n\}$ is bounded, but divergent. All is not lost, however, and we can still compute certain limits with a bounded divergent sequence.

2.3.1 Upper and lower limits

There are ways of creating monotone sequences out of any sequence, and in this fashion we get the so-called *limit superior* and *limit inferior*. These limits always exist for bounded sequences.

If a sequence $\{x_n\}$ is bounded, then the set $\{x_k : k \in \mathbb{N}\}$ is bounded. For every n, the set $\{x_k : k \geq n\}$ is also bounded (as it is a subset), so we take its supremum and infimum.

Definition 2.3.1. Let $\{x_n\}$ be a bounded sequence. Define the sequences $\{a_n\}$ and $\{b_n\}$ by $a_n := \sup\{x_k : k \geq n\}$ and $b_n := \inf\{x_k : k \geq n\}$. Define, if the limits exist,

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} a_n,$$
$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} b_n.$$

For a bounded sequence, liminf and limsup always exist (see below). It is possible to define liminf and limsup for unbounded sequences if we allow ∞ and $-\infty$, and we do so later in this section. It is not hard to generalize the following results to include unbounded sequences; however, we first restrict our attention to bounded ones.

Proposition 2.3.2. *Let $\{x_n\}$ be a bounded sequence. Let a_n and b_n be as in the definition above.*

(i) The sequence $\{a_n\}$ is bounded monotone decreasing and $\{b_n\}$ is bounded monotone increasing. In particular, $\liminf x_n$ *and* $\limsup x_n$ *exist.*

(ii) $\limsup\limits_{n\to\infty} x_n = \inf\{a_n : n \in \mathbb{N}\}$ *and* $\liminf\limits_{n\to\infty} x_n = \sup\{b_n : n \in \mathbb{N}\}$.

(iii) $\liminf\limits_{n\to\infty} x_n \leq \limsup\limits_{n\to\infty} x_n$.

Proof. Let us see why $\{a_n\}$ is a decreasing sequence. As a_n is the least upper bound for $\{x_k : k \geq n\}$, it is also an upper bound for the subset $\{x_k : k \geq (n+1)\}$. Therefore a_{n+1}, the least upper bound for $\{x_k : k \geq (n+1)\}$, has to be less than or equal to a_n, the least upper bound for $\{x_k : k \geq n)\}$. That is, $a_n \geq a_{n+1}$ for all n. Similarly (an exercise), $\{b_n\}$ is an increasing sequence. It is left as an exercise to show that if $\{x_n\}$ is bounded, then $\{a_n\}$ and $\{b_n\}$ must be bounded.

The second item follows as the sequences $\{a_n\}$ and $\{b_n\}$ are monotone and bounded.

*Named after the Czech mathematician Bernhard Placidus Johann Nepomuk Bolzano (1781–1848), and the German mathematician Karl Theodor Wilhelm Weierstrass (1815–1897).

For the third item, note that $b_n \leq a_n$, as the inf of a nonempty set is less than or equal to its sup. The sequences $\{a_n\}$ and $\{b_n\}$ converge to the limsup and the liminf respectively. Apply Lemma 2.2.3 to obtain

$$\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n. \qquad \square$$

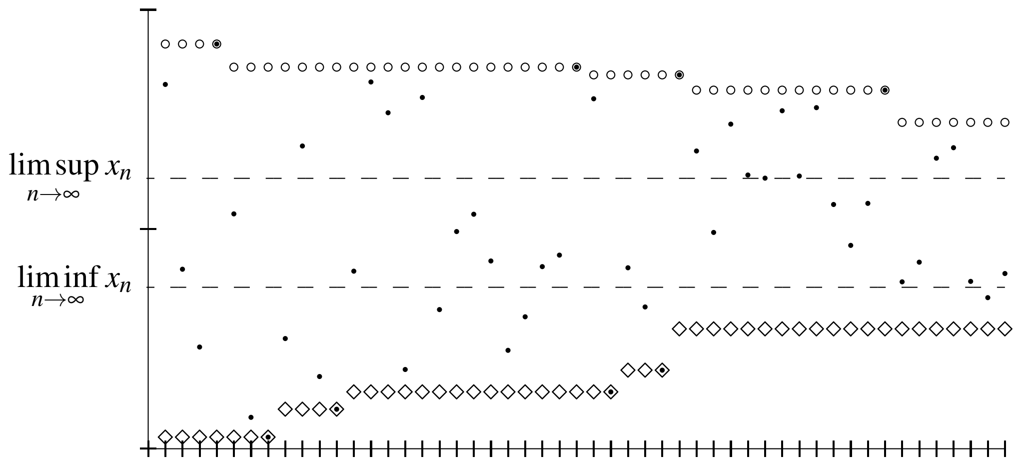

Figure 2.5: First 50 terms of an example sequence. Terms x_n of the sequence are marked with dots ($\bullet$), a_n are marked with circles ($\circ$), and b_n are marked with diamonds ($\diamond$).

Example 2.3.3: Let $\{x_n\}$ be defined by

$$x_n := \begin{cases} \frac{n+1}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Let us compute the lim inf and lim sup of this sequence. See also Figure 2.6. First the limit inferior:

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \bigl(\inf\{x_k : k \geq n\}\bigr) = \lim_{n\to\infty} 0 = 0.$$

For the limit superior, we write

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \bigl(\sup\{x_k : k \geq n\}\bigr).$$

It is not hard to see that

$$\sup\{x_k : k \geq n\} = \begin{cases} \frac{n+1}{n} & \text{if } n \text{ is odd,} \\ \frac{n+2}{n+1} & \text{if } n \text{ is even.} \end{cases}$$

We leave it to the reader to show that the limit is 1. That is,

$$\limsup_{n\to\infty} x_n = 1.$$

Do note that the sequence $\{x_n\}$ is not a convergent sequence.

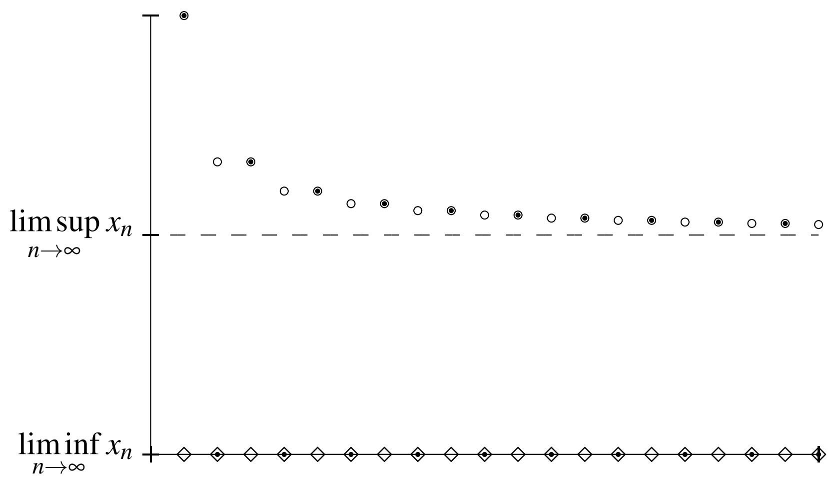

Figure 2.6: First 20 terms of the sequence in Example 2.3.3. The marking is the same as in Figure 2.5.

We associate certain subsequences with lim sup and lim inf. It is important to notice that $\{a_n\}$ and $\{b_n\}$ are not necessarily subsequences of $\{x_n\}$, nor do they have to even consist of the same numbers. For example, for the sequence $\{1/n\}$, $b_n = 0$ for all $n \in N$.

Theorem 2.3.4. *If $\{x_n\}$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}$ such that*

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n.$$

Similarly, there exists a (perhaps different) subsequence $\{x_{m_k}\}$ such that

$$\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n.$$

Proof. Define $a_n := \sup\{x_k : k \geq n\}$. Write $x := \limsup x_n = \lim a_n$. We define the subsequence inductively. Let $n_1 := 1$ and suppose we have defined the subsequence until n_k for some k. Pick some $m > n_k$ such that

$$a_{(n_k+1)} - x_m < \frac{1}{k+1}.$$

We can do this as $a_{(n_k+1)}$ is a supremum of the set $\{x_n : n \geq n_k + 1\}$ and hence there are elements of the sequence arbitrarily close (or even possibly equal) to the supremum. Set $n_{k+1} := m$. The subsequence $\{x_{n_k}\}$ is defined. Next we need to prove that it converges and has the right limit.

For all $k \geq 2$, we have $a_{(n_{k-1}+1)} \geq a_{n_k}$ (why?) and $a_{n_k} \geq x_{n_k}$. Therefore, for every $k \geq 2$,

$$\begin{aligned} |a_{n_k} - x_{n_k}| &= a_{n_k} - x_{n_k} \\ &\leq a_{(n_{k-1}+1)} - x_{n_k} \\ &< \frac{1}{k}. \end{aligned}$$

Let us show that $\{x_{n_k}\}$ converges to x. Note that the subsequence need not be monotone. Let $\varepsilon > 0$ be given. As $\{a_n\}$ converges to x, the subsequence $\{a_{n_k}\}$ converges to x. Thus there exists an

$M_1 \in \mathbb{N}$ such that for all $k \geq M_1$, we have

$$|a_{n_k} - x| < \frac{\varepsilon}{2}.$$

Find an $M_2 \in \mathbb{N}$ such that

$$\frac{1}{M_2} \leq \frac{\varepsilon}{2}.$$

Take $M := \max\{M_1, M_2, 2\}$ and compute. For all $k \geq M$, we have

$$\begin{aligned}
|x - x_{n_k}| &= |a_{n_k} - x_{n_k} + x - a_{n_k}| \\
&\leq |a_{n_k} - x_{n_k}| + |x - a_{n_k}| \\
&< \frac{1}{k} + \frac{\varepsilon}{2} \\
&\leq \frac{1}{M_2} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

We leave the statement for $\liminf$ as an exercise. □

2.3.2 Using limit inferior and limit superior

The advantage of $\liminf$ and $\limsup$ is that we can always write them down for any (bounded) sequence. If we could somehow compute them, we could also compute the limit of the sequence if it exists, or show that the sequence diverges. Working with $\liminf$ and $\limsup$ is a little bit like working with limits, although there are subtle differences.

Proposition 2.3.5. *Let $\{x_n\}$ be a bounded sequence. Then $\{x_n\}$ converges if and only if*

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Furthermore, if $\{x_n\}$ converges, then

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Proof. Let a_n and b_n be as in Definition 2.3.1. In particular, for all $n \in \mathbb{N}$,

$$b_n \leq x_n \leq a_n.$$

If $\liminf x_n = \limsup x_n$, then we know that $\{a_n\}$ and $\{b_n\}$ both converge to the same limit. By the squeeze lemma (Lemma 2.2.1), $\{x_n\}$ converges and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n.$$

Now suppose $\{x_n\}$ converges to x. By Theorem 2.3.4, there exists a subsequence $\{x_{n_k}\}$ that converges to $\limsup x_n$. As $\{x_n\}$ converges to x, every subsequence converges to x and therefore $\limsup x_n = \lim x_{n_k} = x$. Similarly, $\liminf x_n = x$. □

Limit superior and limit inferior behave nicely with subsequences.

Proposition 2.3.6. *Suppose $\{x_n\}$ is a bounded sequence and $\{x_{n_k}\}$ is a subsequence. Then*

$$\liminf_{n\to\infty} x_n \leq \liminf_{k\to\infty} x_{n_k} \leq \limsup_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n.$$

Proof. The middle inequality has been proved already. We will prove the third inequality, and leave the first inequality as an exercise.

We want to prove that $\limsup x_{n_k} \leq \limsup x_n$. Define $a_j := \sup\{x_k : k \geq j\}$ as usual. Also define $c_j := \sup\{x_{n_k} : k \geq j\}$. It is not true that $\{c_j\}$ is necessarily a subsequence of $\{a_j\}$. However, as $n_k \geq k$ for all k, we have that $\{x_{n_k} : k \geq j\} \subset \{x_k : k \geq j\}$. A supremum of a subset is less than or equal to the supremum of the set and therefore

$$c_j \leq a_j \qquad \text{for all } j.$$

Lemma 2.2.3 gives

$$\lim_{j\to\infty} c_j \leq \lim_{j\to\infty} a_j,$$

which is the desired conclusion. □

Limit superior and limit inferior are the largest and smallest subsequential limits. If the subsequence $\{x_{n_k}\}$ in the previous proposition is convergent, then $\liminf x_{n_k} = \lim x_{n_k} = \limsup x_{n_k}$. Therefore,

$$\liminf_{n\to\infty} x_n \leq \lim_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n.$$

Similarly, we get the following useful test for convergence of a bounded sequence. We leave the proof as an exercise.

Proposition 2.3.7. *A bounded sequence $\{x_n\}$ is convergent and converges to x if and only if every convergent subsequence $\{x_{n_k}\}$ converges to x.*

2.3.3 Bolzano–Weierstrass theorem

While it is not true that a bounded sequence is convergent, the Bolzano–Weierstrass theorem tells us that we can at least find a convergent subsequence. The version of Bolzano–Weierstrass that we present in this section is the Bolzano–Weierstrass for sequences of real numbers.

Theorem 2.3.8 (Bolzano–Weierstrass)**.** *Suppose a sequence $\{x_n\}$ of real numbers is bounded. Then there exists a convergent subsequence $\{x_{n_i}\}$.*

Proof. We use Theorem 2.3.4. It says that there exists a subsequence whose limit is $\limsup x_n$. □

The reader might complain right now that Theorem 2.3.4 is strictly stronger than the Bolzano–Weierstrass theorem as presented above. That is true. However, Theorem 2.3.4 only applies to the real line, but Bolzano–Weierstrass applies in more general contexts (that is, in $\mathbb{R}^n$) with pretty much the exact same statement.

As the theorem is so important to analysis, we present an explicit proof. The idea of the following proof also generalizes to different contexts.

Alternate proof of Bolzano–Weierstrass. As the sequence is bounded, then there exist two numbers $a_1 < b_1$ such that $a_1 \leq x_n \leq b_1$ for all $n \in \mathbb{N}$. We will define a subsequence $\{x_{n_i}\}$ and two sequences $\{a_i\}$ and $\{b_i\}$, such that $\{a_i\}$ is monotone increasing, $\{b_i\}$ is monotone decreasing, $a_i \leq x_{n_i} \leq b_i$ and such that $\lim a_i = \lim b_i$. That x_{n_i} converges then follows by the squeeze lemma.

We define the sequences inductively. We will always have that $a_i < b_i$, and that $x_n \in [a_i, b_i]$ for infinitely many $n \in \mathbb{N}$. We have already defined a_1 and b_1. We take $n_1 := 1$, that is $x_{n_1} = x_1$. Suppose that up to some $k \in \mathbb{N}$, we have defined the subsequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$, and the sequences $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$. Let $y := \frac{a_k+b_k}{2}$. Clearly $a_k < y < b_k$. If there exist infinitely many $j \in \mathbb{N}$ such that $x_j \in [a_k, y]$, then set $a_{k+1} := a_k$, $b_{k+1} := y$, and pick $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in [a_k, y]$. If there are not infinitely many j such that $x_j \in [a_k, y]$, then it must be true that there are infinitely many $j \in \mathbb{N}$ such that $x_j \in [y, b_k]$. In this case pick $a_{k+1} := y$, $b_{k+1} := b_k$, and pick $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in [y, b_k]$.

We now have the sequences defined. What is left to prove is that $\lim a_i = \lim b_i$. The limits exist as the sequences are monotone. In the construction, $b_i - a_i$ is cut in half in each step. Therefore, $b_{i+1} - a_{i+1} = \frac{b_i - a_i}{2}$. By induction,

$$b_i - a_i = \frac{b_1 - a_1}{2^{i-1}}.$$

Let $x := \lim a_i$. As $\{a_i\}$ is monotone,

$$x = \sup\{a_i : i \in \mathbb{N}\}.$$

Let $y := \lim b_i = \inf\{b_i : i \in \mathbb{N}\}$. Since $a_i < b_i$ for all i, then $x \leq y$. As the sequences are monotone, then for all i, we have (why?)

$$y - x \leq b_i - a_i = \frac{b_1 - a_1}{2^{i-1}}.$$

Because $\frac{b_1-a_1}{2^{i-1}}$ is arbitrarily small and $y - x \geq 0$, we have $y - x = 0$. Finish by the squeeze lemma. □

Yet another proof of the Bolzano–Weierstrass theorem is to show the following claim, which is left as a challenging exercise. *Claim: Every sequence has a monotone subsequence.*

2.3.4 Infinite limits

Just as for infima and suprema, it is possible to allow certain limits to be infinite. That is, we write $\lim x_n = \infty$ or $\lim x_n = -\infty$ for certain divergent sequences.

Definition 2.3.9. We say $\{x_n\}$ *diverges to infinity** if for every $K \in \mathbb{R}$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n > K$. In this case we write

$$\lim_{n\to\infty} x_n := \infty.$$

Similarly, if for every $K \in \mathbb{R}$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n < K$, we say $\{x_n\}$ *diverges to minus infinity* and we write

$$\lim_{n\to\infty} x_n := -\infty.$$

*Sometimes it is said that $\{x_n\}$ *converges to infinity*.

With this definition and allowing ∞ and $-\infty$, we can write $\lim x_n$ for any monotone sequence.

Proposition 2.3.10. *Suppose $\{x_n\}$ is a monotone unbounded sequence. Then*

$$\lim_{n\to\infty} x_n = \begin{cases} \infty & \text{if } \{x_n\} \text{ is increasing,} \\ -\infty & \text{if } \{x_n\} \text{ is decreasing.} \end{cases}$$

Proof. The case of monotone increasing follows from Exercise 2.3.14 part c) below. Let us do monotone decreasing. Suppose $\{x_n\}$ is decreasing and unbounded, that is, for every $K \in \mathbb{R}$, there is an $M \in \mathbb{N}$ such that $x_M < K$. By monotonicity $x_n \leq x_M < K$ for all $n \geq M$. Therefore, $\lim x_n = -\infty$. $\square$

Example 2.3.11:

$$\lim_{n\to\infty} n = \infty, \qquad \lim_{n\to\infty} n^2 = \infty, \qquad \lim_{n\to\infty} -n = -\infty.$$

We leave verification to the reader.

We may also allow $\liminf$ and $\limsup$ to take on the values ∞ and $-\infty$, so that we can apply $\liminf$ and $\limsup$ to absolutely any sequence, not just a bounded one. Unfortunately, the sequences $\{a_n\}$ and $\{b_n\}$ are not sequences of real numbers but of extended real numbers. In particular, a_n can equal ∞ for some n, and b_n can equal $-\infty$. So we have no definition for the limits. But since the extended real numbers are still an ordered set, we can take suprema and infima.

Definition 2.3.12. Let $\{x_n\}$ be an unbounded sequence of real numbers. Define sequences of extended real numbers by $a_n := \sup\{x_k : k \geq n\}$ and $b_n := \inf\{x_k : k \geq n\}$. Define

$$\limsup_{n\to\infty} x_n := \inf\{a_n : n \in \mathbb{N}\}, \qquad \text{and} \qquad \liminf_{n\to\infty} x_n := \sup\{b_n : n \in \mathbb{N}\}.$$

This definition agrees with the definition for bounded sequences whenever $\lim a_n$ or $\lim b_n$ makes sense including possibly ∞ and $-\infty$.

Proposition 2.3.13. *Let $\{x_n\}$ be an unbounded sequence. Define $\{a_n\}$ and $\{b_n\}$ as above. Then $\{a_n\}$ is decreasing, and $\{b_n\}$ is increasing. If a_n is a real number for every n, then $\limsup x_n = \lim a_n$. If b_n is a real number for every n, then $\liminf x_n = \lim b_n$.*

Proof. As before, $a_n = \sup\{x_k : k \geq n\} \geq \sup\{x_k : k \geq n+1\} = a_{n+1}$. So $\{a_n\}$ is decreasing. Similarly, $\{b_n\}$ is increasing.

If the sequence $\{a_n\}$ is a sequence of real numbers, then $\lim a_n = \inf\{a_n : n \in \mathbb{N}\}$. This follows from Proposition 2.1.10 if $\{a_n\}$ is bounded and Proposition 2.3.10 if $\{a_n\}$ is unbounded. We proceed similarly with $\{b_n\}$. $\square$

The definition behaves as expected with $\limsup$ and $\liminf$, see exercises 2.3.13 and 2.3.14.

Example 2.3.14: Suppose $x_n := 0$ for odd n and $x_n := n$ for even n. Then $a_n = \infty$ for all n, since for every M, there exists an even k such that $x_k = k \geq M$. On the other hand, $b_n = 0$ for all n, as for every n, the set $\{b_k : k \geq n\}$ consists of 0 and nonnegative numbers. So,

$$\lim_{n\to\infty} x_n \quad \text{does not exist}, \qquad \limsup_{n\to\infty} x_n = \infty, \qquad \liminf_{n\to\infty} x_n = 0.$$

2.3.5 Exercises

Exercise* 2.3.1*: *Suppose $\{x_n\}$ is a bounded sequence. Define a_n and b_n as in Definition 2.3.1. Show that $\{a_n\}$ and $\{b_n\}$ are bounded.*

Exercise* 2.3.2*: *Suppose $\{x_n\}$ is a bounded sequence. Define b_n as in Definition 2.3.1. Show that $\{b_n\}$ is an increasing sequence.*

Exercise* 2.3.3*: *Finish the proof of Proposition 2.3.6. That is, suppose $\{x_n\}$ is a bounded sequence and $\{x_{n_k}\}$ is a subsequence. Prove $\liminf\limits_{n\to\infty} x_n \leq \liminf\limits_{k\to\infty} x_{n_k}$.*

Exercise* 2.3.4*: *Prove Proposition 2.3.7.*

Exercise* 2.3.5*:

a) *Let $x_n := \dfrac{(-1)^n}{n}$. Find $\limsup x_n$ and $\liminf x_n$.*

b) *Let $x_n := \dfrac{(n-1)(-1)^n}{n}$. Find $\limsup x_n$ and $\liminf x_n$.*

Exercise* 2.3.6*: *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences such that $x_n \leq y_n$ for all n. Then show that*

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$$

and

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n.$$

Exercise* 2.3.7*: *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences.*

a) *Show that $\{x_n + y_n\}$ is bounded.*

b) *Show that*

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) \leq \liminf_{n\to\infty} (x_n + y_n).$$

Hint: Find a subsequence $\{x_{n_i} + y_{n_i}\}$ of $\{x_n + y_n\}$ that converges. Then find a subsequence $\{x_{n_{m_i}}\}$ of $\{x_{n_i}\}$ that converges. Then apply what you know about limits.

c) *Find an explicit $\{x_n\}$ and $\{y_n\}$ such that*

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) < \liminf_{n\to\infty} (x_n + y_n).$$

Hint: Look for examples that do not have a limit.

Exercise* 2.3.8*: *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences (from the previous exercise we know that $\{x_n + y_n\}$ is bounded).*

a) *Show that*

$$(\limsup_{n\to\infty} x_n) + (\limsup_{n\to\infty} y_n) \geq \limsup_{n\to\infty} (x_n + y_n).$$

Hint: See previous exercise.

b) *Find an explicit $\{x_n\}$ and $\{y_n\}$ such that*

$$(\limsup_{n\to\infty} x_n) + (\limsup_{n\to\infty} y_n) > \limsup_{n\to\infty} (x_n + y_n).$$

Hint: See previous exercise.

Exercise* 2.3.9*: *If $S \subset \mathbb{R}$ is a set, then $x \in \mathbb{R}$ is a* cluster point *if for every $\varepsilon > 0$, the set $(x-\varepsilon, x+\varepsilon) \cap S \setminus \{x\}$ is not empty. That is, if there are points of S arbitrarily close to x. For example, $S := \{1/n : n \in \mathbb{N}\}$ has a unique (only one) cluster point 0, but $0 \notin S$. Prove the following version of the Bolzano–Weierstrass theorem:*

Theorem. Let $S \subset \mathbb{R}$ be a bounded infinite set, then there exists at least one cluster point of S.

Hint: If S is infinite, then S contains a countably infinite subset. That is, there is a sequence $\{x_n\}$ of distinct numbers in S.

Exercise* 2.3.10** (Challenging):***

a) *Prove that every sequence contains a monotone subsequence. Hint: Call $n \in \mathbb{N}$ a* peak *if $a_m \leq a_n$ for all $m \geq n$. There are two possibilities: Either the sequence has at most finitely many peaks, or it has infinitely many peaks.*
b) *Conclude the Bolzano–Weierstrass theorem.*

Exercise* 2.3.11*: *Prove a stronger version of Proposition 2.3.7. Suppose $\{x_n\}$ is a sequence such that every subsequence $\{x_{n_i}\}$ has a subsequence $\{x_{n_{m_i}}\}$ that converges to x.*

a) *First show that $\{x_n\}$ is bounded.*
b) *Now show that $\{x_n\}$ converges to x.*

Exercise* 2.3.12*: *Let $\{x_n\}$ be a bounded sequence.*

a) *Prove that there exists an s such that for every $r > s$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n < r$.*
b) *If s is a number as in a), then prove $\limsup x_n \leq s$.*
c) *Show that if S is the set of all s as in a), then $\limsup x_n = \inf S$.*

Exercise* 2.3.13** (Easy):*** *Suppose $\{x_n\}$ is such that $\liminf x_n = -\infty$, $\limsup x_n = \infty$.*

a) *Show that $\{x_n\}$ is not convergent, and also that neither $\lim x_n = \infty$ nor $\lim x_n = -\infty$ is true.*
b) *Find an example of such a sequence.*

Exercise* 2.3.14*: *Let $\{x_n\}$ be a sequence.*

a) *Show that $\lim x_n = \infty$ if and only if $\liminf x_n = \infty$.*
b) *Then show that $\lim x_n = -\infty$ if and only if $\limsup x_n = -\infty$.*
c) *If $\{x_n\}$ is monotone increasing, show that either $\lim x_n$ exists and is finite or $\lim x_n = \infty$. In either case, $\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$.*

Exercise* 2.3.15*: *Prove the following stronger version of Lemma 2.2.12, the ratio test. Suppose $\{x_n\}$ is a sequence such that $x_n \neq 0$ for all n.*

a) *Prove that if*
$$\limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} < 1,$$
then $\{x_n\}$ converges to 0.
b) *Prove that if*
$$\liminf_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} > 1,$$
then $\{x_n\}$ is unbounded.

Exercise* 2.3.16*: *Suppose* $\{x_n\}$ *is a bounded sequence,* $a_n := \sup\{x_k : k \geq n\}$ *as before. Suppose that for some* $\ell \in \mathbb{N}$, $a_\ell \notin \{x_k : k \geq \ell\}$. *Then show that* $a_j = a_\ell$ *for all* $j \geq \ell$, *and hence* $\limsup x_n = a_\ell$.

Exercise* 2.3.17*: *Suppose* $\{x_n\}$ *is a sequence, and* $a_n := \sup\{x_k : k \geq n\}$ *and* $b_n := \sup\{x_k : k \geq n\}$ *as before.*

a) Prove that if $a_\ell = \infty$ *for some* $\ell \in \mathbb{N}$, *then* $\limsup x_n = \infty$.

b) Prove that if $b_\ell = -\infty$ *for some* $\ell \in \mathbb{N}$, *then* $\liminf x_n = -\infty$.

Exercise* 2.3.18*: *Suppose* $\{x_n\}$ *is a sequence such that both* $\liminf x_n$ *and* $\limsup x_n$ *are finite. Prove that* $\{x_n\}$ *is bounded.*

Exercise* 2.3.19*: *Suppose* $\{x_n\}$ *is a bounded sequence, and* $\varepsilon > 0$ *is given. Prove that there exists an* M *such that for all* $k \geq M$,

$$x_k - \left(\limsup_{n\to\infty} x_n\right) < \varepsilon \qquad \text{and} \qquad \left(\liminf_{n\to\infty} x_n\right) - x_k < \varepsilon.$$

Exercise* 2.3.20*: *Extend Theorem 2.3.4 to unbounded sequences: Suppose that* $\{x_n\}$ *is a sequence. If* $\limsup x_n = \infty$, *then prove that there exists a subsequence* $\{x_{n_i}\}$ *converging to* ∞. *Then prove the same result for* $-\infty$, *and then prove both statements for* $\liminf$.

2.4 Cauchy sequences

Note: 0.5–1 lecture

Often we wish to describe a certain number by a sequence that converges to it. In this case, it is impossible to use the number itself in the proof that the sequence converges. It would be nice if we could check for convergence without knowing the limit.

Definition 2.4.1. A sequence $\{x_n\}$ is a *Cauchy sequence** if for every $\varepsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $k \geq M$, we have

$$\lvert x_n - x_k \rvert < \varepsilon.$$

Informally, being Cauchy means that the terms of the sequence are eventually all arbitrarily close to each other. We might expect such a sequence to be convergent, and we would be correct due to $\mathbb{R}$ having the least-upper-bound property. Before we prove this fact, we look at some examples.

Example 2.4.2: The sequence $\{1/n\}$ is a Cauchy sequence.

Proof: Given $\varepsilon > 0$, find M such that $M > 2/\varepsilon$. Then for $n,k \geq M$, we have $1/n < \varepsilon/2$ and $1/k < \varepsilon/2$. Therefore, for $n,k \geq M$, we have

$$\left\lvert \frac{1}{n} - \frac{1}{k} \right\rvert \leq \left\lvert \frac{1}{n} \right\rvert + \left\lvert \frac{1}{k} \right\rvert < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Example 2.4.3: The sequence $\{\frac{n+1}{n}\}$ is a Cauchy sequence.

Proof: Given $\varepsilon > 0$, find M such that $M > 2/\varepsilon$. Then for $n,k \geq M$, we have $1/n < \varepsilon/2$ and $1/k < \varepsilon/2$. Therefore, for $n,k \geq M$, we have

$$\begin{aligned}
\left\lvert \frac{n+1}{n} - \frac{k+1}{k} \right\rvert &= \left\lvert \frac{k(n+1) - n(k+1)}{nk} \right\rvert \\
&= \left\lvert \frac{kn + k - nk - n}{nk} \right\rvert \\
&= \left\lvert \frac{k-n}{nk} \right\rvert \\
&\leq \left\lvert \frac{k}{nk} \right\rvert + \left\lvert \frac{-n}{nk} \right\rvert \\
&= \frac{1}{n} + \frac{1}{k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Proposition 2.4.4. *A Cauchy sequence is bounded.*

Proof. Suppose $\{x_n\}$ is Cauchy. Pick an M such that for all $n,k \geq M$, we have $\lvert x_n - x_k \rvert < 1$. In particular, for all $n \geq M$,

$$\lvert x_n - x_M \rvert < 1.$$

*Named after the French mathematician Augustin-Louis Cauchy (1789–1857).

By the reverse triangle inequality, $|x_n| - |x_M| \leq |x_n - x_M| < 1$. Hence for $n \geq M$,

$$|x_n| < 1 + |x_M|.$$

Let

$$B := \max\{|x_1|, |x_2|, \ldots, |x_{M-1}|, 1 + |x_M|\}.$$

Then $|x_n| \leq B$ for all $n \in \mathbb{N}$. □

Theorem 2.4.5. *A sequence of real numbers is Cauchy if and only if it converges.*

Proof. Let $\varepsilon > 0$ be given and suppose $\{x_n\}$ converges to x. Then there exists an M such that for $n \geq M$,

$$|x_n - x| < \frac{\varepsilon}{2}.$$

Hence for $n \geq M$ and $k \geq M$,

$$|x_n - x_k| = |x_n - x + x - x_k| \leq |x_n - x| + |x - x_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Alright, that direction was easy. Now suppose $\{x_n\}$ is Cauchy. We have shown that $\{x_n\}$ is bounded. For a bounded sequence, liminf and limsup exist, and this is where we use the least-upper-bound property. If we show that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n,$$

then $\{x_n\}$ must be convergent by Proposition 2.3.5.

Define $a := \limsup x_n$ and $b := \liminf x_n$. By Theorem 2.3.4, there exist subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$, such that

$$\lim_{i\to\infty} x_{n_i} = a \qquad \text{and} \qquad \lim_{i\to\infty} x_{m_i} = b.$$

Given an $\varepsilon > 0$, there exists an M_1 such that $|x_{n_i} - a| < \varepsilon/3$ for all $i \geq M_1$ and an M_2 such that $|x_{m_i} - b| < \varepsilon/3$ for all $i \geq M_2$. There also exists an M_3 such that $|x_n - x_k| < \varepsilon/3$ for all $n, k \geq M_3$. Let $M := \max\{M_1, M_2, M_3\}$. If $i \geq M$, then $n_i \geq M$ and $m_i \geq M$. Hence,

$$\begin{aligned} |a - b| &= |a - x_{n_i} + x_{n_i} - x_{m_i} + x_{m_i} - b| \\ &\leq |a - x_{n_i}| + |x_{n_i} - x_{m_i}| + |x_{m_i} - b| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

As $|a - b| < \varepsilon$ for all $\varepsilon > 0$, then $a = b$ and the sequence converges. □

Remark 2.4.6. The statement of this proposition is sometimes used to define the completeness property of the real numbers. We say a set is *Cauchy-complete* (or sometimes just *complete*) if every Cauchy sequence converges. Above, we proved that as $\mathbb{R}$ has the least-upper-bound property, then $\mathbb{R}$ is Cauchy-complete. One can construct $\mathbb{R}$ via "completing" $\mathbb{Q}$ by "throwing in" just enough points to make all Cauchy sequences converge (we omit the details). The resulting field has the least-upper-bound property. The advantage of using Cauchy sequences to define completeness is that this idea generalizes to more abstract settings such as metric spaces, see chapter 7.

The Cauchy criterion is stronger than $|x_{n+1} - x_n|$ (or $\left|x_{n+j} - x_n\right|$ for a fixed j) going to zero as n goes to infinity. When we get to the partial sums of the harmonic series (see Example 2.5.11 in the next section), we will have a sequence such that $x_{n+1} - x_n = {}^1/_n$, yet $\{x_n\}$ is divergent. In fact, for that sequence, $\lim_{n\to\infty} \left|x_{n+j} - x_n\right| = 0$ for every $j \in \mathbb{N}$ (confer Exercise 2.5.12). The key point in the definition of Cauchy is that n and k vary independently and can be arbitrarily far apart.

2.4.1 Exercises

Exercise* 2.4.1*: *Prove that* $\{\frac{n^2-1}{n^2}\}$ *is Cauchy using directly the definition of Cauchy sequences.*

Exercise* 2.4.2*: *Let* $\{x_n\}$ *be a sequence such that there exists a positive* $C < 1$ *and for all* n*,*

$$|x_{n+1} - x_n| \leq C\,|x_n - x_{n-1}|\,.$$

Prove that $\{x_n\}$ *is Cauchy. Hint: You can freely use the formula (for* $C \neq 1$*)*

$$1 + C + C^2 + \cdots + C^n = \frac{1 - C^{n+1}}{1 - C}.$$

Exercise* 2.4.3** (Challenging):*** *Suppose* F *is an ordered field that contains the rational numbers* $\mathbb{Q}$*, such that* $\mathbb{Q}$ *is dense, that is: Whenever* $x, y \in F$ *are such that* $x < y$*, then there exists a* $q \in \mathbb{Q}$ *such that* $x < q < y$*. Say a sequence* $\{x_n\}_{n=1}^{\infty}$ *of rational numbers is Cauchy if given every* $\varepsilon \in \mathbb{Q}$ *with* $\varepsilon > 0$*, there exists an* M *such that for all* $n, k \geq M$*, we have* $|x_n - x_k| < \varepsilon$*. Suppose every Cauchy sequence of rational numbers has a limit in* F*. Prove that* F *has the least-upper-bound property.*

Exercise* 2.4.4*: *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences such that* $\lim y_n = 0$*. Suppose that for all* $k \in \mathbb{N}$ *and for all* $m \geq k$*, we have*

$$|x_m - x_k| \leq y_k.$$

Show that $\{x_n\}$ *is Cauchy.*

Exercise* 2.4.5*: *Suppose a Cauchy sequence* $\{x_n\}$ *is such that for every* $M \in \mathbb{N}$*, there exists a* $k \geq M$ *and an* $n \geq M$ *such that* $x_k < 0$ *and* $x_n > 0$*. Using simply the definition of a Cauchy sequence and of a convergent sequence, show that the sequence converges to* 0*.*

Exercise* 2.4.6*: *Suppose* $|x_n - x_k| \leq {}^n/_{k^2}$ *for all* n *and* k*. Show that* $\{x_n\}$ *is Cauchy.*

Exercise* 2.4.7*: *Suppose* $\{x_n\}$ *is a Cauchy sequence such that for infinitely many* n*,* $x_n = c$*. Using only the definition of Cauchy sequence prove that* $\lim x_n = c$*.*

Exercise* 2.4.8*: *True or false, prove or find a counterexample: If* $\{x_n\}$ *is a Cauchy sequence, then there exists an* M *such that for all* $n \geq M$*, we have* $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$*.*

2.5 Series

Note: 2 lectures

A fundamental object in mathematics is that of a series. In fact, when the foundations of analysis were being developed, the motivation was to understand series. Understanding series is important in applications of analysis. For example, solutions to differential equations are often given as series, and differential equations are the basis for understanding almost all of modern science.

2.5.1 Definition

Definition 2.5.1. Given a sequence $\{x_n\}$, we write the formal object

$$\sum_{n=1}^{\infty} x_n \qquad \text{or sometimes just} \qquad \sum x_n$$

and call it a *series*. A series *converges* if the sequence $\{s_k\}$ defined by

$$s_k := \sum_{n=1}^{k} x_n = x_1 + x_2 + \cdots + x_k,$$

converges. The numbers s_k are called *partial sums*. If $x := \lim s_k$, we write

$$\sum_{n=1}^{\infty} x_n = x.$$

In this case, we cheat a little and treat $\sum_{n=1}^{\infty} x_n$ as a number.

If the sequence $\{s_k\}$ diverges, we say the series is *divergent*. In this case, $\sum x_n$ is simply a formal object and not a number.

In other words, for a convergent series, we have

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} \sum_{n=1}^{k} x_n.$$

We only have this equality if the limit on the right actually exists. If the series does not converge, the right-hand side does not make sense (the limit does not exist). Therefore, be careful as $\sum x_n$ means two different things (a notation for the series itself or the limit of the partial sums), and you must use context to distinguish.

Remark 2.5.2. It is sometimes convenient to start the series at an index different from 1. For instance, we can write

$$\sum_{n=0}^{\infty} r^n = \sum_{n=1}^{\infty} r^{n-1}.$$

The left-hand side is more convenient to write.

Remark 2.5.3. It is common to write the series $\sum x_n$ as

$$x_1 + x_2 + x_3 + \cdots$$

with the understanding that the ellipsis indicates a series and not a simple sum. We do not use this notation as it is the sort of informal notation that leads to mistakes in proofs.

Example 2.5.4: The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges and the limit is 1. That is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2^n} = 1.$$

Proof: First we prove the following equality

$$\left(\sum_{n=1}^{k} \frac{1}{2^n} \right) + \frac{1}{2^k} = 1.$$

The equality is immediate when $k = 1$. The proof for general k follows by induction, which we leave to the reader. See Figure 2.7 for an illustration.

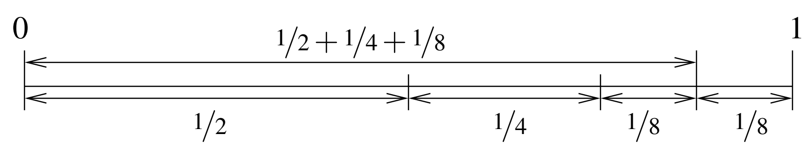

Figure 2.7: The equality $\left(\sum_{n=1}^{k} \frac{1}{2^n}\right) + \frac{1}{2^k} = 1$ illustrated for $k = 3$.

Let s_k be the partial sum. We write

$$|1 - s_k| = \left| 1 - \sum_{n=1}^{k} \frac{1}{2^n} \right| = \left| \frac{1}{2^k} \right| = \frac{1}{2^k}.$$

The sequence $\left\{ \frac{1}{2^k} \right\}$, and therefore $\left\{ |1 - s_k| \right\}$, converges to zero. So, $\{s_k\}$ converges to 1.

Proposition 2.5.5. *Suppose* $-1 < r < 1$. *Then the* geometric series $\sum_{n=0}^{\infty} r^n$ *converges, and*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Details of the proof are left as an exercise. The proof consists of showing

$$\sum_{n=0}^{k-1} r^n = \frac{1 - r^k}{1 - r},$$

and then taking the limit as k goes to ∞. Geometric series is one of the most important series, and in fact it is one of the few series for which we can so explicitly find the limit.

As for sequences we can talk about a *tail of a series*.

Proposition 2.5.6. *Let $\sum x_n$ be a series. Let $M \in \mathbb{N}$. Then*

$$\sum_{n=1}^{\infty} x_n \quad \textit{converges if and only if} \quad \sum_{n=M}^{\infty} x_n \quad \textit{converges.}$$

Proof. We look at partial sums of the two series (for $k \geq M$)

$$\sum_{n=1}^{k} x_n = \left(\sum_{n=1}^{M-1} x_n \right) + \sum_{n=M}^{k} x_n.$$

Note that $\sum_{n=1}^{M-1} x_n$ is a fixed number. Use Proposition 2.2.5 to finish the proof. □

2.5.2 Cauchy series

Definition 2.5.7. A series $\sum x_n$ is said to be *Cauchy* or a *Cauchy series* if the sequence of partial sums $\{s_n\}$ is a Cauchy sequence.

A sequence of real numbers converges if and only if it is Cauchy. Therefore, a series is convergent if and only if it is Cauchy. The series $\sum x_n$ is Cauchy if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$, such that for every $n \geq M$ and $k \geq M$, we have

$$\left| \left(\sum_{j=1}^{k} x_j \right) - \left(\sum_{j=1}^{n} x_j \right) \right| < \varepsilon.$$

Without loss of generality we assume $n < k$. Then we write

$$\left| \left(\sum_{j=1}^{k} x_j \right) - \left(\sum_{j=1}^{n} x_j \right) \right| = \left| \sum_{j=n+1}^{k} x_j \right| < \varepsilon.$$

We have proved the following simple proposition.

Proposition 2.5.8. *The series $\sum x_n$ is Cauchy if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for every $n \geq M$ and every $k > n$, we have*

$$\left| \sum_{j=n+1}^{k} x_j \right| < \varepsilon.$$

2.5.3 Basic properties

Proposition 2.5.9. *Let $\sum x_n$ be a convergent series. Then the sequence $\{x_n\}$ is convergent and*

$$\lim_{n\to\infty} x_n = 0.$$

Proof. Let $\varepsilon > 0$ be given. As $\sum x_n$ is convergent, it is Cauchy. Thus we find an M such that for every $n \geq M$, we have

$$\varepsilon > \left| \sum_{j=n+1}^{n+1} x_j \right| = \left| x_{n+1} \right|.$$

Hence for every $n \geq M+1$, we have $|x_n| < \varepsilon$. □

Example 2.5.10: If $r \geq 1$ or $r \leq -1$, then the geometric series $\sum_{n=0}^{\infty} r^n$ diverges.

Proof: $|r^n| = |r|^n \geq 1^n = 1$. So the terms do not go to zero and the series cannot converge.

So if a series converges, the terms of the series go to zero. The implication, however, goes only one way. Let us give an example.

Example 2.5.11: The series $\sum \frac{1}{n}$ diverges (despite the fact that $\lim \frac{1}{n} = 0$). This is the famous *harmonic series**.

Proof: We will show that the sequence of partial sums is unbounded, and hence cannot converge. Write the partial sums s_n for $n = 2^k$ as:

$$\begin{aligned}
s_1 &= 1, \\
s_2 &= (1) + \left(\frac{1}{2}\right), \\
s_4 &= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right), \\
s_8 &= (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right), \\
&\vdots \\
s_{2^k} &= 1 + \sum_{j=1}^{k} \left(\sum_{m=2^{j-1}+1}^{2^j} \frac{1}{m} \right).
\end{aligned}$$

Notice $1/3 + 1/4 \geq 1/4 + 1/4 = 1/2$ and $1/5 + 1/6 + 1/7 + 1/8 \geq 1/8 + 1/8 + 1/8 + 1/8 = 1/2$. More generally

$$\sum_{m=2^{k-1}+1}^{2^k} \frac{1}{m} \geq \sum_{m=2^{k-1}+1}^{2^k} \frac{1}{2^k} = (2^{k-1}) \frac{1}{2^k} = \frac{1}{2}.$$

Therefore,

$$s_{2^k} = 1 + \sum_{j=1}^{k} \left(\sum_{m=2^{j-1}+1}^{2^j} \frac{1}{m} \right) \geq 1 + \sum_{j=1}^{k} \frac{1}{2} = 1 + \frac{k}{2}.$$

As $\{\frac{k}{2}\}$ is unbounded by the Archimedean property, that means that $\{s_{2^k}\}$ is unbounded, and therefore $\{s_n\}$ is unbounded. Hence $\{s_n\}$ diverges, and consequently $\sum \frac{1}{n}$ diverges.

Convergent series are linear. That is, we can multiply them by constants and add them and these operations are done term by term.

Proposition 2.5.12 (Linearity of series)**.** *Let $\alpha \in \mathbb{R}$ and $\sum x_n$ and $\sum y_n$ be convergent series. Then*

(i) $\sum \alpha x_n$ is a convergent series and

$$\sum_{n=1}^{\infty} \alpha x_n = \alpha \sum_{n=1}^{\infty} x_n.$$

*The divergence of the harmonic series was known long before the theory of series was made rigorous. The proof we give is the earliest proof and was given by Nicole Oresme (1323?–1382).

(ii) $\sum (x_n + y_n)$ *is a convergent series and*

$$\sum_{n=1}^{\infty} (x_n + y_n) = \left(\sum_{n=1}^{\infty} x_n \right) + \left(\sum_{n=1}^{\infty} y_n \right).$$

Proof. For the first item, we simply write the kth partial sum

$$\sum_{n=1}^{k} \alpha x_n = \alpha \left(\sum_{n=1}^{k} x_n \right).$$

We look at the right-hand side and note that the constant multiple of a convergent sequence is convergent. Hence, we take the limit of both sides to obtain the result.

For the second item we also look at the kth partial sum

$$\sum_{n=1}^{k} (x_n + y_n) = \left(\sum_{n=1}^{k} x_n \right) + \left(\sum_{n=1}^{k} y_n \right).$$

We look at the right-hand side and note that the sum of convergent sequences is convergent. Hence, we take the limit of both sides to obtain the proposition. □

An example of a useful application of the first item is the following formula. If $|r| < 1$ and $j \in \mathbb{N}$, then

$$\sum_{n=j}^{\infty} r^n = \frac{r^j}{1-r}.$$

The formula follows by using the geometric series and multiplying by r^j:

$$r^j \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty} r^{n+j} = \sum_{n=j}^{\infty} r^n.$$

Multiplying series is not as simple as adding, see the next section. It is not true, of course, that we multiply term by term. That strategy does not work even for finite sums: $(a+b)(c+d) \neq ac + bd$.

2.5.4 Absolute convergence

As monotone sequences are easier to work with than arbitrary sequences, it is usually easier to work with series $\sum x_n$, where $x_n \geq 0$ for all n. The sequence of partial sums is then monotone increasing and converges if it is bounded above. Let us formalize this statement as a proposition.

Proposition 2.5.13. *If $x_n \geq 0$ for all n, then $\sum x_n$ converges if and only if the sequence of partial sums is bounded above.*

As the limit of a monotone increasing sequence is the supremum, then when $x_n \geq 0$ for all n, we have the inequality

$$\sum_{n=1}^{k} x_n \leq \sum_{n=1}^{\infty} x_n.$$

If we allow infinite limits, the inequality still holds even when the series diverges to infinity, although in that case it is not terribly useful.

We will see that the following common criterion for convergence of series has big implications for how the series can be manipulated.

Definition 2.5.14. A series $\sum x_n$ *converges absolutely* if the series $\sum |x_n|$ converges. If a series converges, but does not converge absolutely, we say it is *conditionally convergent*.

Proposition 2.5.15. *If the series $\sum x_n$ converges absolutely, then it converges.*

Proof. A series is convergent if and only if it is Cauchy. Hence suppose $\sum |x_n|$ is Cauchy. That is, for every $\varepsilon > 0$, there exists an M such that for all $k \geq M$ and all $n > k$, we have

$$\sum_{j=k+1}^{n} |x_j| = \left| \sum_{j=k+1}^{n} |x_j| \right| < \varepsilon.$$

We apply the triangle inequality for a finite sum to obtain

$$\left| \sum_{j=k+1}^{n} x_j \right| \leq \sum_{j=k+1}^{n} |x_j| < \varepsilon.$$

Hence $\sum x_n$ is Cauchy, and therefore it converges. □

If $\sum x_n$ converges absolutely, the limits of $\sum x_n$ and $\sum |x_n|$ are generally different. Computing one does not help us compute the other. However, the computation above leads to a useful inequality for absolutely convergent series, a series version of the triangle inequality, a proof of which we leave as an exercise:

$$\left| \sum_{j=1}^{\infty} x_j \right| \leq \sum_{j=1}^{\infty} |x_j| .$$

Absolutely convergent series have many wonderful properties. For example, absolutely convergent series can be rearranged arbitrarily, or we can multiply such series together easily. Conditionally convergent series on the other hand often do not behave as one would expect. See the next section.

We leave as an exercise to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges, although the reader should finish this section before trying. On the other hand, we proved

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. Therefore, $\sum \frac{(-1)^n}{n}$ is a conditionally convergent series.

2.5.5 Comparison test and the p-series

We noted above that for a series to converge the terms not only have to go to zero, but they have to go to zero "fast enough." If we know about convergence of a certain series, we can use the following comparison test to see if the terms of another series go to zero "fast enough."

Proposition 2.5.16 (Comparison test)**.** *Let $\sum x_n$ and $\sum y_n$ be series such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$.*

(i) If $\sum y_n$ converges, then so does $\sum x_n$.

(ii) If $\sum x_n$ diverges, then so does $\sum y_n$.

Proof. As the terms of the series are all nonnegative, the sequences of partial sums are both monotone increasing. Since $x_n \leq y_n$ for all n, the partial sums satisfy for all k

$$\sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k} y_n. \tag{2.1}$$

If the series $\sum y_n$ converges, the partial sums for the series are bounded. Therefore, the right-hand side of (2.1) is bounded for all k; there exists some $B \in \mathbb{R}$ such that $\sum_{n=1}^{k} y_n \leq B$ for all k, and so

$$\sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k} y_n \leq B.$$

Hence the partial sums for $\sum x_n$ are also bounded. Since the partial sums are a monotone increasing sequence they are convergent. The first item is thus proved.

On the other hand if $\sum x_n$ diverges, the sequence of partial sums must be unbounded since it is monotone increasing. That is, the partial sums for $\sum x_n$ are eventually bigger than any real number. Putting this together with (2.1) we see that for every $B \in \mathbb{R}$, there is a k such that

$$B \leq \sum_{n=1}^{k} x_n \leq \sum_{n=1}^{k} y_n.$$

Hence the partial sums for $\sum y_n$ are also unbounded, and $\sum y_n$ also diverges. □

A useful series to use with the comparison test is the p-series*.

Proposition 2.5.17 (p-series or the p-test)**.** *For $p \in \mathbb{R}$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

*We have not yet defined x^p for $x > 0$ and an arbitrary $p \in \mathbb{R}$. The definition is $x^p := \exp(p \ln x)$. We will define the logarithm and the exponential in §5.4. For now you can just think of rational p where $x^{k/m} = (x^{1/m})^k$. See also Exercise 1.2.17.

Proof. First suppose $p \leq 1$. As $n \geq 1$, we have $\frac{1}{n^p} \geq \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^p}$ must diverge for all $p \leq 1$ by the comparison test.

Now suppose $p > 1$. We proceed as we did for the harmonic series, but instead of showing that the sequence of partial sums is unbounded, we show that it is bounded. The terms of the series are positive, so the sequence of partial sums is monotone increasing and converges if it is bounded above. Let s_n denote the nth partial sum.

$$\begin{aligned}
s_1 &= 1, \\
s_3 &= (1) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right), \\
s_7 &= (1) + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right), \\
&\ \vdots \\
s_{2^k-1} &= 1 + \sum_{j=1}^{k-1} \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{m^p} \right).
\end{aligned}$$

Instead of estimating from below, we estimate from above. As p is positive, then $2^p < 3^p$, and hence $\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p}$. Similarly, $\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$. Therefore, for all $k \geq 2$,

$$\begin{aligned}
s_{2^k-1} &= 1 + \sum_{j=1}^{k-1} \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{m^p} \right) \\
&< 1 + \sum_{j=1}^{k-1} \left(\sum_{m=2^j}^{2^{j+1}-1} \frac{1}{(2^j)^p} \right) \\
&= 1 + \sum_{j=1}^{k-1} \left(\frac{2^j}{(2^j)^p} \right) \\
&= 1 + \sum_{j=1}^{k-1} \left(\frac{1}{2^{p-1}} \right)^j.
\end{aligned}$$

As $p > 1$, then $\frac{1}{2^{p-1}} < 1$. Proposition 2.5.5 says that

$$\sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j$$

converges. Thus,

$$s_{2^k-1} < 1 + \sum_{j=1}^{k-1} \left(\frac{1}{2^{p-1}} \right)^j \leq 1 + \sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j.$$

For every n there is a $k \geq 2$ such that $n \leq 2^k - 1$, and as $\{s_n\}$ is a monotone sequence, $s_n \leq s_{2^k-1}$. So for all n,

$$s_n < 1 + \sum_{j=1}^{\infty} \left(\frac{1}{2^{p-1}} \right)^j$$

Thus the sequence of partial sums is bounded, and the series converges. □

Neither the p-series test nor the comparison test tell us what the sum converges to. They only tell us that a limit of the partial sums exists. For instance, while we know that $\sum 1/n^2$ converges, it is far harder to find* that the limit is $\pi^2/6$. If we treat $\sum 1/n^p$ as a function of p, we get the so-called Riemann ζ function. Understanding the behavior of this function contains one of the most famous unsolved problems in mathematics today and has applications in seemingly unrelated areas such as modern cryptography.

Example 2.5.18: The series $\sum \frac{1}{n^2+1}$ converges.

Proof: First, $\frac{1}{n^2+1} < \frac{1}{n^2}$ for all $n \in \mathbb{N}$. The series $\sum \frac{1}{n^2}$ converges by the p-series test. Therefore, by the comparison test, $\sum \frac{1}{n^2+1}$ converges.

2.5.6 Ratio test

Suppose $r > 0$. The ratio of two subsequent terms in the geometric series $\sum r^n$ is $\frac{r^{n+1}}{r^n} = r$, and the series converges whenever $r < 1$. Just as for sequences, this fact can be generalized to more arbitrary series as long as we have such a ratio "in the limit." We then compare the tail of a series to the geometric series.

Proposition 2.5.19 (Ratio test)**.** *Let $\sum x_n$ be a series, $x_n \neq 0$ for all n, and such that*

$$L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \qquad \textit{exists.}$$

(i) If $L < 1$, then $\sum x_n$ converges absolutely.

(ii) If $L > 1$, then $\sum x_n$ diverges.

Although the test as stated is often sufficient, it can be strengthened a bit, see Exercise 2.5.6.

Proof. If $L > 1$, then Lemma 2.2.12 says that the sequence $\{x_n\}$ diverges. Since it is a necessary condition for the convergence of series that the terms go to zero, we know that $\sum x_n$ must diverge.

Thus suppose $L < 1$. We will argue that $\sum |x_n|$ must converge. The proof is similar to that of Lemma 2.2.12. Of course $L \geq 0$. Pick r such that $L < r < 1$. As $r - L > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$,

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L.$$

Therefore,

$$\frac{|x_{n+1}|}{|x_n|} < r.$$

For $n > M$ (that is for $n \geq M+1$), write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} < |x_M| r r \cdots r = |x_M| r^{n-M} = (|x_M| r^{-M}) r^n.$$

*Demonstration of this fact is what made the Swiss mathematician Leonhard Paul Euler (1707–1783) famous.

For $k > M$, write the partial sum as

$$\begin{aligned}\sum_{n=1}^{k} |x_n| &= \left(\sum_{n=1}^{M} |x_n|\right) + \left(\sum_{n=M+1}^{k} |x_n|\right)\\ &< \left(\sum_{n=1}^{M} |x_n|\right) + \left(\sum_{n=M+1}^{k} (|x_M|\, r^{-M}) r^n\right)\\ &= \left(\sum_{n=1}^{M} |x_n|\right) + (|x_M|\, r^{-M}) \left(\sum_{n=M+1}^{k} r^n\right).\end{aligned}$$

As $0 < r < 1$, the geometric series $\sum_{n=0}^{\infty} r^n$ converges, so $\sum_{n=M+1}^{\infty} r^n$ converges as well. We take the limit as k goes to infinity on the right-hand side above to obtain

$$\begin{aligned}\sum_{n=1}^{k} |x_n| &< \left(\sum_{n=1}^{M} |x_n|\right) + (|x_M|\, r^{-M}) \left(\sum_{n=M+1}^{k} r^n\right)\\ &\leq \left(\sum_{n=1}^{M} |x_n|\right) + (|x_M|\, r^{-M}) \left(\sum_{n=M+1}^{\infty} r^n\right).\end{aligned}$$

The right-hand side is a number that does not depend on k. Hence the sequence of partial sums of $\sum |x_n|$ is bounded and $\sum |x_n|$ is convergent. Thus $\sum x_n$ is absolutely convergent. $\square$

Example 2.5.20: The series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

converges absolutely.

Proof: We write

$$\lim_{n\to\infty} \frac{2^{(n+1)}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

Therefore, the series converges absolutely by the ratio test.

2.5.7 Exercises

Exercise 2.5.1: *Suppose the kth partial sum of* $\sum\limits_{n=1}^{\infty} x_n$ *is* $s_k = \frac{k}{k+1}$*. Find the series, that is find* x_n*, prove that the series converges, and then find the limit.*

Exercise 2.5.2: *Prove Proposition 2.5.5, that is for* $-1 < r < 1$ *prove*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Hint: See Example 0.3.8.

Exercise 2.5.3: *Decide the convergence or divergence of the following series.*

a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{9n+1}$ *b)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ *c)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ *d)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *e)* $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Exercise* 2.5.4*:

a) *Prove that if* $\sum_{n=1}^{\infty} x_n$ *converges, then* $\sum_{n=1}^{\infty} (x_{2n} + x_{2n+1})$ *also converges.*

b) *Find an explicit example where the converse does not hold.*

Exercise* 2.5.5*: *For* $j = 1, 2, \ldots, n$*, let* $\{x_{j,k}\}_{k=1}^{\infty}$ *denote n sequences. Suppose that for each* $j \in \mathbb{N}$*,*

$$\sum_{k=1}^{\infty} x_{j,k}$$

is convergent. Prove

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{\infty} x_{j,k} \right) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{n} x_{j,k} \right).$$

Exercise* 2.5.6*: *Prove the following stronger version of the ratio test: Let* $\sum x_n$ *be a series.*

a) *If there is an N and a* $\rho < 1$ *such that* $\frac{|x_{n+1}|}{|x_n|} < \rho$ *for all* $n \geq N$*, then the series converges absolutely. (Remark: Equivalently the condition can be stated as* $\limsup_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} < 1$*.)*

b) *If there is an N such that* $\frac{|x_{n+1}|}{|x_n|} \geq 1$ *for all* $n \geq N$*, then the series diverges.*

Exercise* 2.5.7** (Challenging):*** *Suppose* $\{x_n\}$ *is a decreasing sequence and* $\sum x_n$ *converges. Prove* $\lim_{n\to\infty} n x_n = 0$.

Exercise* 2.5.8*: *Show that* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges. Hint: Consider the sum of two subsequent entries.*

Exercise* 2.5.9*:

a) *Prove that if* $\sum x_n$ *and* $\sum y_n$ *converge absolutely, then* $\sum x_n y_n$ *converges absolutely.*

b) *Find an explicit example where the converse does not hold.*

c) *Find an explicit example where all three series are absolutely convergent, are not just finite sums, and* $(\sum x_n)(\sum y_n) \neq \sum x_n y_n$*. That is, show that series are not multiplied term-by-term.*

Exercise* 2.5.10*: *Prove the triangle inequality for series: If* $\sum x_n$ *converges absolutely, then*

$$\left| \sum_{n=1}^{\infty} x_n \right| \leq \sum_{n=1}^{\infty} |x_n|.$$

Exercise* 2.5.11*: *Prove the* limit comparison test*. That is, prove that if* $a_n > 0$ *and* $b_n > 0$ *for all n, and*

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty,$$

then either $\sum a_n$ *and* $\sum b_n$ *both converge or both diverge.*

Exercise* 2.5.12*: *Let* $x_n := \sum_{j=1}^{n} 1/j$*. Show that for every k, we get* $\lim_{n\to\infty} |x_{n+k} - x_n| = 0$*, yet* $\{x_n\}$ *is not Cauchy.*

Exercise* 2.5.13*: *Let* s_k *be the kth partial sum of* $\sum x_n$*.*

a) *Suppose that there exists an* $m \in \mathbb{N}$ *such that* $\lim_{k\to\infty} s_{mk}$ *exists and* $\lim x_n = 0$*. Show that* $\sum x_n$ *converges.*

b) *Find an example where* $\lim_{k\to\infty} s_{2k}$ *exists and* $\lim x_n \neq 0$ *(and therefore* $\sum x_n$ *diverges).*

c) *(Challenging) Find an example where* $\lim x_n = 0$*, and there exists a subsequence* $\{s_{k_j}\}$ *such that* $\lim_{j\to\infty} s_{k_j}$ *exists, but* $\sum x_n$ *still diverges.*

Exercise* 2.5.14*: *Suppose* $\sum x_n$ *converges and* $x_n \geq 0$ *for all* n*. Prove that* $\sum x_n^2$ *converges.*

Exercise* 2.5.15** (Challenging):*** *Suppose* $\{x_n\}$ *is a decreasing sequence of positive numbers. The proof of convergence/divergence for the* p*-series generalizes. Prove the so-called* Cauchy condensation principle*:*

$$\sum_{n=1}^{\infty} x_n \qquad \textit{converges if and only if} \qquad \sum_{n=1}^{\infty} 2^n x_{2^n} \qquad \textit{converges}.$$

Exercise* 2.5.16*: *Use the Cauchy condensation principle (see Exercise 2.5.15) to decide the convergence of*

a) $\sum \frac{\ln n}{n^2}$ *b)* $\sum \frac{1}{n \ln n}$ *c)* $\sum \frac{1}{n(\ln n)^2}$ *d)* $\sum \frac{1}{n(\ln n)(\ln \ln n)^2}$

For the series to be well-defined you need to start some of the series at $n = 2$*. Note that only the tails of some of these series satisfy the hypotheses of the principle; you should argue why that is sufficient.*
Hint: Feel free to use the identity $\ln(2^n) = n \ln 2$*.*

Exercise* 2.5.17** (Challenging):*** *Prove* Abel's theorem*:*

Theorem. Suppose $\sum x_n$ is a series whose partial sums are a bounded sequence, $\{\lambda_n\}$ is a sequence with $\lim \lambda_n = 0$, and $\sum |\lambda_{n+1} - \lambda_n|$ is convergent. Then $\sum \lambda_n x_n$ is convergent.

2.6 More on series

Note: up to 2–3 lectures (optional, can safely be skipped or covered partially)

2.6.1 Root test

A test similar to the ratio test is the so-called *root test*. In fact, the proof of this test is similar and somewhat easier. Again, the idea is to generalize what happens for the geometric series.

Proposition 2.6.1 (Root test)**.** *Let $\sum x_n$ be a series and let*

$$L := \limsup_{n\to\infty} |x_n|^{1/n}.$$

(i) If $L < 1$, then $\sum x_n$ converges absolutely.

(ii) If $L > 1$, then $\sum x_n$ diverges.

Proof. If $L > 1$, then there exists* a subsequence $\{x_{n_k}\}$ such that $L = \lim_{k\to\infty} |x_{n_k}|^{1/n_k}$. Let r be such that $L > r > 1$. There exists an M such that for all $k \geq M$, we have $|x_{n_k}|^{1/n_k} > r > 1$, or in other words $|x_{n_k}| > r^{n_k} > 1$. The subsequence $\{|x_{n_k}|\}$, and therefore also $\{|x_n|\}$, cannot possibly converge to zero, and so the series diverges.

Now suppose $L < 1$. Pick r such that $L < r < 1$. By definition of limit supremum, there is an M such that for all $n \geq M$,

$$\sup\left\{|x_k|^{1/k} : k \geq n\right\} < r.$$

Therefore, for all $n \geq M$,

$$|x_n|^{1/n} < r, \qquad \text{or in other words} \qquad |x_n| < r^n.$$

Let $k > M$, and estimate the kth partial sum:

$$\sum_{n=1}^{k} |x_n| = \left(\sum_{n=1}^{M} |x_n|\right) + \left(\sum_{n=M+1}^{k} |x_n|\right) \leq \left(\sum_{n=1}^{M} |x_n|\right) + \left(\sum_{n=M+1}^{k} r^n\right).$$

As $0 < r < 1$, the geometric series $\sum_{n=M+1}^{\infty} r^n$ converges to $\frac{r^{M+1}}{1-r}$. As everything is positive,

$$\sum_{n=1}^{k} |x_n| \leq \left(\sum_{n=1}^{M} |x_n|\right) + \frac{r^{M+1}}{1-r}.$$

Thus the sequence of partial sums of $\sum |x_n|$ is bounded, and the series converges. Therefore, $\sum x_n$ converges absolutely. □

*In case $L = \infty$, see Exercise 2.3.20. Alternatively, note that if $L = \infty$, then $\left\{|x_n|^{1/n}\right\}$ and thus $\{x_n\}$ is unbounded.

2.6.2 Alternating series test

The tests we have seen so far only addressed absolute convergence. The following test gives a large supply of conditionally convergent series.

Proposition 2.6.2 (Alternating series)**.** *Let $\{x_n\}$ be a monotone decreasing sequence of positive real numbers such that $\lim x_n = 0$. Then*

$$\sum_{n=1}^{\infty} (-1)^n x_n$$

converges.

Proof. Let $s_m := \sum_{k=1}^{m} (-1)^k x_k$ be the mth partial sum. Then write

$$s_{2n} = \sum_{k=1}^{2n} (-1)^k x_k = (-x_1 + x_2) + \cdots + (-x_{2n-1} + x_{2n}) = \sum_{k=1}^{n} (-x_{2k-1} + x_{2k}).$$

The sequence $\{x_k\}$ is decreasing and so $(-x_{2k-1} + x_{2k}) \leq 0$ for all k. Therefore, the subsequence $\{s_{2n}\}$ of partial sums is a decreasing sequence. Similarly, $(x_{2k} - x_{2k+1}) \geq 0$, and so

$$s_{2n} = -x_1 + (x_2 - x_3) + \cdots + (x_{2n-2} - x_{2n-1}) + x_{2n} \geq -x_1.$$

The sequence $\{s_{2n}\}$ is decreasing and bounded below, so it converges. Let $a := \lim s_{2n}$.

We wish to show that $\lim s_m = a$ (and not just for the subsequence). Notice

$$s_{2n+1} = s_{2n} + x_{2n+1}.$$

Given $\varepsilon > 0$, pick M such that $|s_{2n} - a| < \varepsilon/2$ whenever $2n \geq M$. Since $\lim x_n = 0$, we also make M possibly larger to obtain $x_{2n+1} < \varepsilon/2$ whenever $2n \geq M$. If $2n \geq M$, we have $|s_{2n} - a| < \varepsilon/2 < \varepsilon$, so we just need to check the situation for s_{2n+1}:

$$|s_{2n+1} - a| = |s_{2n} - a + x_{2n+1}| \leq |s_{2n} - a| + x_{2n+1} < \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \square$$

Notably, there exist conditionally convergent series where the absolute values of the terms go to zero arbitrarily slowly. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$$

converges for arbitrarily small $p > 0$, but it does not converge absolutely when $p \leq 1$.

2.6.3 Rearrangements

Absolutely convergent series behave as we imagine they should. For example, absolutely convergent series can be summed in any order whatsoever. Nothing of the sort holds for conditionally convergent series (see Example 2.6.4 and Exercise 2.6.3).

Consider a series

$$\sum_{n=1}^{\infty} x_n.$$

Given a bijective function $\sigma \colon \mathbb{N} \to \mathbb{N}$, the corresponding rearrangement is the following series:

$$\sum_{k=1}^{\infty} x_{\sigma(k)}.$$

We simply sum the series in a different order.

Proposition 2.6.3. *Let $\sum x_n$ be an absolutely convergent series converging to a number x. Let $\sigma \colon \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum x_{\sigma(n)}$ is absolutely convergent and converges to x.*

In other words, a rearrangement of an absolutely convergent series converges (absolutely) to the same number.

Proof. Let $\varepsilon > 0$ be given. As $\sum x_n$ is absolutely convergent, take M such that

$$\left| \left(\sum_{n=1}^{M} x_n \right) - x \right| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \sum_{n=M+1}^{\infty} \lvert x_n \rvert < \frac{\varepsilon}{2}.$$

As σ is a bijection, there exists a number K such that for each $n \leq M$, there exists $k \leq K$ such that $\sigma(k) = n$. In other words $\{1,2,\ldots,M\} \subset \sigma\bigl(\{1,2,\ldots,K\}\bigr)$.

For $N \geq K$, let $Q := \max \sigma\bigl(\{1,2,\ldots,N\}\bigr)$. Compute

$$\begin{aligned}
\left| \left(\sum_{n=1}^{N} x_{\sigma(n)} \right) - x \right| &= \left| \left(\sum_{n=1}^{M} x_n + \sum_{\substack{n=1 \\ \sigma(n) > M}}^{N} x_{\sigma(n)} \right) - x \right| \\
&\leq \left| \left(\sum_{n=1}^{M} x_n \right) - x \right| + \sum_{\substack{n=1 \\ \sigma(n) > M}}^{N} \lvert x_{\sigma(n)} \rvert \\
&\leq \left| \left(\sum_{n=1}^{M} x_n \right) - x \right| + \sum_{n=M+1}^{Q} \lvert x_n \rvert \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

So $\sum x_{\sigma(n)}$ converges to x. To see that the convergence is absolute, we apply the argument above to $\sum \lvert x_n \rvert$ to show that $\sum \lvert x_{\sigma(n)} \rvert$ converges. $\square$

Example 2.6.4: Let us show that the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$, which does not converge absolutely, can be rearranged to converge to anything. The odd terms and the even terms diverge to plus infinity and minus infinity respectively (prove this!):

$$\sum_{m=1}^{\infty} \frac{1}{2m-1} = \infty, \qquad \text{and} \qquad \sum_{m=1}^{\infty} \frac{-1}{2m} = -\infty.$$

Let $a_n := \frac{(-1)^{n+1}}{n}$ for simplicity, let an arbitrary number $L \in \mathbb{R}$ be given, and set $\sigma(1) := 1$. Suppose we have defined $\sigma(n)$ for all $n \leq N$. If

$$\sum_{n=1}^{N} a_{\sigma(n)} \leq L,$$

then let $\sigma(N+1) := k$ be the smallest odd $k \in \mathbb{N}$ that we have not used yet, that is, $\sigma(n) \neq k$ for all $n \leq N$. Otherwise, let $\sigma(N+1) := k$ be the smallest even k that we have not yet used.

By construction $\sigma\colon \mathbb{N} \to \mathbb{N}$ is one-to-one. It is also onto, because if we keep adding either odd (resp. even) terms, eventually we pass L and switch to the evens (resp. odds). So we switch infinitely many times.

Finally, let N be the N where we just pass L and switch. For example, suppose we have just switched from odd to even (so we start subtracting), and let $N' > N$ be where we first switch back from even to odd. Then

$$L + \frac{1}{\sigma(N)} \geq \sum_{n=1}^{N-1} a_{\sigma(n)} > \sum_{n=1}^{N'-1} a_{\sigma(n)} > L - \frac{1}{\sigma(N')}.$$

And similarly for switching in the other direction. Therefore, the sum up to $N'-1$ is within $\frac{1}{\min\{\sigma(N),\sigma(N')\}}$ of L. As we switch infinitely many times we obtain that $\sigma(N) \to \infty$ and $\sigma(N') \to \infty$, and hence

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} \frac{(-1)^{\sigma(n)+1}}{\sigma(n)} = L.$$

Here is an example to illustrate the proof. Suppose $L = 1.2$, then the order is

$$1 + 1/3 - 1/2 + 1/5 + 1/7 + 1/9 - 1/4 + 1/11 + 1/13 - 1/6 + 1/15 + 1/17 + 1/19 - 1/8 + \cdots .$$

At this point we are no more than $1/8$ from the limit.

2.6.4 Multiplication of series

As we have already mentioned, multiplication of series is somewhat harder than addition. If at least one of the series converges absolutely, then we can use the following theorem. For this result, it is convenient to start the series at 0, rather than at 1.

Theorem 2.6.5 (Mertens' theorem*). *Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two convergent series, converging to A and B respectively. If at least one of the series converges absolutely, then the series $\sum_{n=0}^{\infty} c_n$ where*

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{j=0}^{n} a_j b_{n-j},$$

converges to AB.

The series $\sum c_n$ is called the *Cauchy product* of $\sum a_n$ and $\sum b_n$.

Proof. Suppose $\sum a_n$ converges absolutely, and let $\varepsilon > 0$ be given. In this proof instead of picking complicated estimates just to make the final estimate come out as less than ε, let us simply obtain an estimate that depends on ε and can be made arbitrarily small.

Write

$$A_m := \sum_{n=0}^{m} a_n, \qquad B_m := \sum_{n=0}^{m} b_n.$$

*Proved by the German mathematician Franz Mertens (1840–1927).

We rearrange the mth partial sum of $\sum c_n$:

$$\begin{aligned}
\left| \left(\sum_{n=0}^{m} c_n \right) - AB \right| &= \left| \left(\sum_{n=0}^{m} \sum_{j=0}^{n} a_j b_{n-j} \right) - AB \right| \\
&= \left| \left(\sum_{n=0}^{m} B_n a_{m-n} \right) - AB \right| \\
&= \left| \left(\sum_{n=0}^{m} (B_n - B) a_{m-n} \right) + BA_m - AB \right| \\
&\leq \left(\sum_{n=0}^{m} |B_n - B| \, |a_{m-n}| \right) + |B| \, |A_m - A|
\end{aligned}$$

We can surely make the second term on the right-hand side go to zero. The trick is to handle the first term. Pick K such that for all $m \geq K$, we have $|A_m - A| < \varepsilon$ and also $|B_m - B| < \varepsilon$. Finally, as $\sum a_n$ converges absolutely, make sure that K is large enough such that for all $m \geq K$,

$$\sum_{n=K}^{m} |a_n| < \varepsilon.$$

As $\sum b_n$ converges, then we have that $B_{\max} := \sup\{|B_n - B| : n = 0, 1, 2, \ldots\}$ is finite. Take $m \geq 2K$, then in particular $m - K + 1 > K$. So

$$\begin{aligned}
\sum_{n=0}^{m} |B_n - B| \, |a_{m-n}| &= \left(\sum_{n=0}^{m-K} |B_n - B| \, |a_{m-n}| \right) + \left(\sum_{n=m-K+1}^{m} |B_n - B| \, |a_{m-n}| \right) \\
&\leq \left(\sum_{n=K}^{m} |a_n| \right) B_{\max} + \left(\sum_{n=0}^{K-1} \varepsilon \, |a_n| \right) \\
&\leq \varepsilon B_{\max} + \varepsilon \left(\sum_{n=0}^{\infty} |a_n| \right).
\end{aligned}$$

Therefore, for $m \geq 2K$, we have

$$\begin{aligned}
\left| \left(\sum_{n=0}^{m} c_n \right) - AB \right| &\leq \left(\sum_{n=0}^{m} |B_n - B| \, |a_{m-n}| \right) + |B| \, |A_m - A| \\
&\leq \varepsilon B_{\max} + \varepsilon \left(\sum_{n=0}^{\infty} |a_n| \right) + |B| \, \varepsilon = \varepsilon \left(B_{\max} + \left(\sum_{n=0}^{\infty} |a_n| \right) + |B| \right).
\end{aligned}$$

The expression in the parenthesis on the right-hand side is a fixed number. Hence, we can make the right-hand side arbitrarily small by picking a small enough $\varepsilon > 0$. So $\sum_{n=0}^{\infty} c_n$ converges to AB. $\square$

Example 2.6.6: If both series are only conditionally convergent, the Cauchy product series need not even converge. Suppose we take $a_n = b_n = (-1)^n \frac{1}{\sqrt{n+1}}$. The series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$ converges by the alternating series test; however, it does not converge absolutely as can be seen from the p-test. Let us look at the Cauchy product.

$$c_n = (-1)^n \left(\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{2n}} + \frac{1}{\sqrt{3(n-1)}} + \cdots + \frac{1}{\sqrt{n+1}} \right) = (-1)^n \sum_{j=0}^{n} \frac{1}{\sqrt{(j+1)(n-j+1)}}.$$

Therefore,

$$|c_n| = \sum_{j=0}^{n} \frac{1}{\sqrt{(j+1)(n-j+1)}} \geq \sum_{j=0}^{n} \frac{1}{\sqrt{(n+1)(n+1)}} = 1.$$

The terms do not go to zero and hence $\sum c_n$ cannot converge.

2.6.5 Power series

Fix $x_0 \in \mathbb{R}$. A *power series* about x_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

A power series is really a function of x, and many important functions in analysis can be written as a power series. We use the convention that $0^0 = 1$ (if $x = x_0$ and $n = 0$).

We say that a power series is *convergent* if there is at least one $x \neq x_0$ that makes the series converge. If $x = x_0$, then the series always converges since all terms except the first are zero. If the series does not converge for any point $x \neq x_0$, we say that the series is *divergent*.

Example 2.6.7: The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

is absolutely convergent for all $x \in \mathbb{R}$ using the ratio test: For any $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{\big(1/(n+1)!\big)\, x^{n+1}}{(1/n!)\, x^n} = \lim_{n\to\infty} \frac{x}{n+1} = 0.$$

Recall from calculus that this series converges to e^x.

Example 2.6.8: The series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

converges absolutely for all $x \in (-1,1)$ via the ratio test:

$$\lim_{n\to\infty} \left| \frac{\big(1/(n+1)\big)\, x^{n+1}}{(1/n)\, x^n} \right| = \lim_{n\to\infty} |x| \frac{n}{n+1} = |x| < 1.$$

The series converges at $x = -1$, as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test. But the power series does not converge absolutely at $x = -1$, because $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. The series diverges at $x = 1$. When $|x| > 1$, then the series diverges via the ratio test.

Example 2.6.9: The series

$$\sum_{n=1}^{\infty} n^n x^n$$

diverges for all $x \neq 0$. Let us apply the root test

$$\limsup_{n\to\infty} \left\lvert n^n x^n \right\rvert^{1/n} = \limsup_{n\to\infty} n\,\lvert x\rvert = \infty.$$

Therefore, the series diverges for all $x \neq 0$.

Convergence of power series in general works analogously to one of the three examples above.

Proposition 2.6.10. *Let $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ be a power series. If the series is convergent, then either it converges at all $x \in \mathbb{R}$, or there exists a number ρ, such that the series converges absolutely on the interval $(x_0-\rho, x_0+\rho)$ and diverges when $x < x_0-\rho$ or $x > x_0+\rho$.*

The number ρ is called the *radius of convergence* of the power series. We write $\rho = \infty$ if the series converges for all x, and we write $\rho = 0$ if the series is divergent. At the endpoints, that is if $x = x_0+\rho$ or $x = x_0-\rho$, the proposition says nothing, and the series might or might not converge. See Figure 2.8. In Example 2.6.8 the radius of convergence is $\rho = 1$. In Example 2.6.7 the radius of convergence is $\rho = \infty$, and in Example 2.6.9 the radius of convergence is $\rho = 0$.

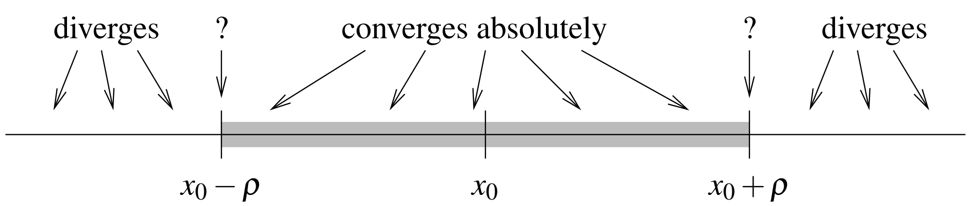

Figure 2.8: Convergence of a power series.

Proof. Write

$$R := \limsup_{n\to\infty} \lvert a_n\rvert^{1/n}.$$

We use the root test to prove the proposition:

$$L = \limsup_{n\to\infty} \left\lvert a_n (x-x_0)^n \right\rvert^{1/n} = \lvert x-x_0\rvert \limsup_{n\to\infty} \lvert a_n\rvert^{1/n} = \lvert x-x_0\rvert R.$$

In particular, if $R = \infty$, then $L = \infty$ for every $x \neq x_0$, and the series diverges by the root test. On the other hand, if $R = 0$, then $L = 0$ for every x, and the series converges absolutely for all x.

Suppose $0 < R < \infty$. The series converges absolutely if $1 > L = R\,\lvert x-x_0\rvert$, or in other words when

$$\lvert x-x_0\rvert < {}^{1}/_{R}.$$

The series diverges when $1 < L = R\,\lvert x-x_0\rvert$, or

$$\lvert x-x_0\rvert > {}^{1}/_{R}.$$

Letting $\rho := {}^{1}/_{R}$ completes the proof. $\square$

It may be useful to restate what we have learned in the proof as a separate proposition.

Proposition 2.6.11. *Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series, and let*

$$R := \limsup_{n\to\infty} \lvert a_n \rvert^{1/n}.$$

If $R = \infty$, the power series is divergent. If $R = 0$, then the power series converges everywhere. Otherwise, the radius of convergence $\rho = 1/R$.

Often, radius of convergence is written as $\rho = 1/R$ in all three cases, with the understanding of what ρ should be if $R = 0$ or $R = \infty$.

Convergent power series can be added and multiplied together, and multiplied by constants. The proposition has a straight forward proof using what we know about series in general, and power series in particular. We leave the proof to the reader.

Proposition 2.6.12. *Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ be two convergent power series with radius of convergence at least $\rho > 0$ and $\alpha \in \mathbb{R}$. Then for all x such that $\lvert x-x_0 \rvert < \rho$, we have*

$$\left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right) + \left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} (a_n+b_n)(x-x_0)^n,$$

$$\alpha\left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} \alpha a_n(x-x_0)^n,$$

and

$$\left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$.

That is, after performing the algebraic operations, the radius of convergence of the resulting series is at least ρ. For all x with $\lvert x-x_0 \rvert < \rho$, we have two convergent series so their term by term addition and multiplication by constants follows by what we learned in the last section. For multiplication of two power series, the series are absolutely convergent inside the radius of convergence and that is why for those x we can apply Mertens' theorem. Note that after applying an algebraic operation the radius of convergence could increase. See the exercises.

Let us look at some examples of power series. Polynomials are simply finite power series. That is, a polynomial is a power series where the a_n are zero for all n large enough. We expand a polynomial as a power series about any point x_0 by writing the polynomial as a polynomial in $(x-x_0)$. For example, $2x^2-3x+4$ as a power series around $x_0 = 1$ is

$$2x^2-3x+4 = 3+(x-1)+2(x-1)^2.$$

We can also expand *rational functions* (that is, ratios of polynomials) as power series, although we will not completely prove this fact here. Notice that a series for a rational function only defines the function on an interval even if the function is defined elsewhere. For example, for the *geometric series*, we have that for $x \in (-1,1)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

The series diverges when $|x| > 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions as power series around x_0, as long as the denominator is not zero at x_0. We state without proof that this is always possible, and we give an example of such a computation using the geometric series.

Example 2.6.13: Let us expand $\frac{x}{1+2x+x^2}$ as a power series around the origin ($x_0 = 0$) and find the radius of convergence.

Write $1+2x+x^2 = (1+x)^2 = \bigl(1-(-x)\bigr)^2$, and suppose $|x| < 1$. Compute

$$\begin{aligned}
\frac{x}{1+2x+x^2} &= x\left(\frac{1}{1-(-x)}\right)^2 \\
&= x\left(\sum_{n=0}^{\infty} (-1)^n x^n\right)^2 \\
&= x\left(\sum_{n=0}^{\infty} c_n x^n\right) \\
&= \sum_{n=0}^{\infty} c_n x^{n+1}.
\end{aligned}$$

Using the formula for the product of series, we obtain $c_0 = 1$, $c_1 = -1-1 = -2$, $c_2 = 1+1+1 = 3$, etc. Hence, for $|x| < 1$,

$$\frac{x}{1+2x+x^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^n.$$

The radius of convergence is at least 1. We leave it to the reader to verify that the radius of convergence is exactly equal to 1.

You can use the method of partial fractions you know from calculus. For example, to find the power series for $\frac{x^3+x}{x^2-1}$ at 0, write

$$\frac{x^3+x}{x^2-1} = x + \frac{1}{1+x} - \frac{1}{1-x} = x + \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} x^n.$$

2.6.6 Exercises

Exercise* 2.6.1*: *Decide the convergence or divergence of the following series.*

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{2n+1}}$ *b)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n-1)}{n}$ *c)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/10}}$ *d)* $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{2n}}$

Exercise* 2.6.2*: *Suppose both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely. Show that the product series, $\sum_{n=0}^{\infty} c_n$ where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$, also converges absolutely.*

Exercise* 2.6.3** (Challenging):*** *Let $\sum a_n$ be conditionally convergent. Show that given an arbitrary $x \in \mathbb{R}$ there exists a rearrangement of $\sum a_n$ such that the rearranged series converges to x. Hint: See Example 2.6.4.*

Exercise* 2.6.4*:

a) *Show that the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ has a rearrangement such that whenever $x < y$, there exists a partial sum s_n of the rearranged series such that $x < s_n < y$.*

b) *Show that the rearrangement you found does not converge. See Example 2.6.4.*

c) *Show that for every $x \in \mathbb{R}$, there exists a subsequence of partial sums $\{s_{n_k}\}$ of your rearrangement such that $\lim s_{n_k} = x$.*

Exercise* 2.6.5*: *For the following power series, find if they are convergent or not, and if so find their radius of convergence.*

a) $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$ *b)* $\displaystyle\sum_{n=0}^{\infty} n x^n$ *c)* $\displaystyle\sum_{n=0}^{\infty} n!\, x^n$ *d)* $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n)!}(x-10)^n$ *e)* $\displaystyle\sum_{n=0}^{\infty} x^{2n}$ *f)* $\displaystyle\sum_{n=0}^{\infty} n!\, x^{n!}$

Exercise* 2.6.6*: *Suppose $\sum a_n x^n$ converges for $x = 1$.*

a) *What can you say about the radius of convergence?*

b) *If you further know that at $x = 1$ the convergence is not absolute, what can you say?*

Exercise* 2.6.7*: *Expand $\dfrac{x}{4-x^2}$ as a power series around $x_0 = 0$ and compute its radius of convergence.*

Exercise* 2.6.8*:

a) *Find an example where the radii of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ are both 1, but the radius of convergence of the sum of the two series is infinite.*

b) *(Trickier) Find an example where the radii of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ are both 1, but the radius of convergence of the product of the two series is infinite.*

Exercise* 2.6.9*: *Figure out how to compute the radius of convergence using the ratio test. That is, suppose $\sum a_n x^n$ is a power series and $R := \lim \frac{\lvert a_{n+1}\rvert}{\lvert a_n\rvert}$ exists or is ∞. Find the radius of convergence and prove your claim.*

Exercise* 2.6.10*:

a) *Prove that $\lim n^{1/n} = 1$ using the following procedure: Write $n^{1/n} = 1 + b_n$ and note $b_n > 0$. Then show that $(1+b_n)^n \geq \frac{n(n-1)}{2} b_n^2$ and use this to show that $\lim b_n = 0$.*

b) *Use the result of part a) to show that if $\sum a_n x^n$ is a convergent power series with radius of convergence R, then $\sum n a_n x^n$ is also convergent with the same radius of convergence.*

There are different notions of summability (convergence) of a series than just the one we have seen. A common one is *Cesàro summability**. Let $\sum a_n$ be a series and let s_n be the nth partial sum. The series is said to be Cesàro summable to a if

$$a = \lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

*Named for the Italian mathematician Ernesto Cesàro (1859–1906).

***Exercise* 2.6.11** (Challenging)*:*

a) If $\sum a_n$ is convergent to a (in the usual sense), show that $\sum a_n$ is Cesàro summable (see above) to a.

b) Show that in the sense of Cesàro $\sum (-1)^n$ is summable to $1/2$.

c) Let $a_n := k$ when $n = k^3$ for some $k \in \mathbb{N}$, $a_n := -k$ when $n = k^3 + 1$ for some $k \in \mathbb{N}$, otherwise let $a_n := 0$. Show that $\sum a_n$ diverges in the usual sense, (partial sums are unbounded), but it is Cesàro summable to 0 (seems a little paradoxical at first sight).

***Exercise* 2.6.12** (Challenging)*: Show that the monotonicity in the alternating series test is necessary. That is, find a sequence of positive real numbers $\{x_n\}$ with $\lim x_n = 0$ but such that $\sum (-1)^n x_n$ diverges.*

Exercise* 2.6.13: Find a series such that $\sum x_n$ converges but $\sum x_n^2$ diverges. Hint: Compare Exercise 2.5.14.*

Exercise* 2.6.14: Suppose $\{c_n\}$ is a sequence. Prove that for every $r \in (0,1)$, there exists a strictly increasing sequence $\{n_k\}$ of natural numbers ($n_{k+1} > n_k$) such that*

$$\sum_{k=1}^{\infty} c_k x^{n_k}$$

converges absolutely for all $x \in [-r,r]$.

Chapter 3

Continuous Functions

3.1 Limits of functions

Note: 2–3 lectures

Before we define continuity of functions, we must visit a somewhat more general notion of a limit. Given a function $f: S \to \mathbb{R}$, we want to see how $f(x)$ behaves as x tends to a certain point.

3.1.1 Cluster points

First, we return to a concept we have previously seen in an exercise. When moving within the set S we can only approach points that have elements of S arbitrarily near.

Definition 3.1.1. Let $S \subset \mathbb{R}$ be a set. A number $x \in \mathbb{R}$ is called a *cluster point* of S if for every $\varepsilon > 0$, the set $(x-\varepsilon, x+\varepsilon) \cap S \setminus \{x\}$ is not empty.

That is, x is a cluster point of S if there are points of S arbitrarily close to x. Another way of phrasing the definition is to say that x is a cluster point of S if for every $\varepsilon > 0$, there exists a $y \in S$ such that $y \neq x$ and $|x-y| < \varepsilon$. Note that a cluster point of S need not lie in S.

Let us see some examples.

(i) The set $\{1/n : n \in \mathbb{N}\}$ has a unique cluster point zero.

(ii) The cluster points of the open interval $(0,1)$ are all points in the closed interval $[0,1]$.

(iii) The set of cluster points of $\mathbb{Q}$ is the whole real line $\mathbb{R}$.

(iv) The set of cluster points of $[0,1) \cup \{2\}$ is the interval $[0,1]$.

(v) The set $\mathbb{N}$ has no cluster points in $\mathbb{R}$.

Proposition 3.1.2. *Let $S \subset \mathbb{R}$. Then $x \in \mathbb{R}$ is a cluster point of S if and only if there exists a convergent sequence of numbers $\{x_n\}$ such that $x_n \neq x$ and $x_n \in S$ for all n, and* $\lim x_n = x$.

Proof. First suppose x is a cluster point of S. For every $n \in \mathbb{N}$, pick x_n to be an arbitrary point of $(x-1/n, x+1/n) \cap S \setminus \{x\}$, which is nonempty because x is a cluster point of S. Then x_n is within $1/n$ of x, that is,

$$|x-x_n| < 1/n.$$

As $\{1/n\}$ converges to zero, $\{x_n\}$ converges to x.

On the other hand, if we start with a sequence of numbers $\{x_n\}$ in S converging to x such that $x_n \neq x$ for all n, then for every $\varepsilon > 0$ there is an M such that, in particular, $|x_M - x| < \varepsilon$. That is, $x_M \in (x-\varepsilon, x+\varepsilon) \cap S \setminus \{x\}$. $\square$

3.1.2 Limits of functions

If a function f is defined on a set S and c is a cluster point of S, then we define the limit of $f(x)$ as x gets close to c. It is irrelevant for the definition whether f is defined at c or not. Even if the function is defined at c, the limit of the function as x goes to c can very well be different from $f(c)$.

Definition 3.1.3. Let $f \colon S \to \mathbb{R}$ be a function and c a cluster point of $S \subset \mathbb{R}$. Suppose there exists an $L \in \mathbb{R}$ and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in S \setminus \{c\}$ and $|x-c| < \delta$, we have

$$|f(x) - L| < \varepsilon.$$

We then say $f(x)$ *converges* to L as x goes to c. We say L is the *limit* of $f(x)$ as x goes to c. We write

$$\lim_{x \to c} f(x) := L,$$

or

$$f(x) \to L \quad \text{as} \quad x \to c.$$

If no such L exists, then we say that the limit does not exist or that f *diverges* at c.

Again the notation and language we are using above assumes the limit is unique even though we have not yet proved uniqueness. Let us do that now.

Proposition 3.1.4. *Let c be a cluster point of $S \subset \mathbb{R}$ and let $f \colon S \to \mathbb{R}$ be a function such that $f(x)$ converges as x goes to c. Then the limit of $f(x)$ as x goes to c is unique.*

Proof. Let L_1 and L_2 be two numbers that both satisfy the definition. Take an $\varepsilon > 0$ and find a $\delta_1 > 0$ such that $|f(x) - L_1| < \varepsilon/2$ for all $x \in S \setminus \{c\}$ with $|x-c| < \delta_1$. Also find $\delta_2 > 0$ such that $|f(x) - L_2| < \varepsilon/2$ for all $x \in S \setminus \{c\}$ with $|x-c| < \delta_2$. Put $\delta := \min\{\delta_1, \delta_2\}$. Suppose $x \in S$, $|x-c| < \delta$, and $x \neq c$. As $\delta > 0$ and c is a cluster point, such an x exists. Then

$$|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As $|L_1 - L_2| < \varepsilon$ for arbitrary $\varepsilon > 0$, then $L_1 = L_2$. $\square$

Example 3.1.5: Consider $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$. Then for any $c \in \mathbb{R}$,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x^2 = c^2.$$

Proof: Let $c \in \mathbb{R}$ be fixed, and suppose $\varepsilon > 0$ is given. Write

$$\delta := \min\left\{1, \frac{\varepsilon}{2|c|+1}\right\}.$$

Take $x \neq c$ such that $|x-c| < \delta$. In particular, $|x-c| < 1$. By reverse triangle inequality, we get

$$|x| - |c| \leq |x-c| < 1.$$

Adding $2|c|$ to both sides, we obtain $|x|+|c| < 2|c|+1$. We compute

$$\begin{aligned} \left|f(x)-c^2\right| &= \left|x^2-c^2\right| \\ &= |(x+c)(x-c)| \\ &= |x+c|\,|x-c| \\ &\leq (|x|+|c|)\,|x-c| \\ &< (2|c|+1)\,|x-c| \\ &< (2|c|+1)\frac{\varepsilon}{2|c|+1} = \varepsilon. \end{aligned}$$

Example 3.1.6: Define $f \colon [0,1) \to \mathbb{R}$ by

$$f(x) := \begin{cases} x & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then

$$\lim_{x \to 0} f(x) = 0,$$

even though $f(0) = 1$.

Proof: Let $\varepsilon > 0$ be given. Let $\delta := \varepsilon$. For $x \in [0,1)$, $x \neq 0$, and $|x-0| < \delta$, we get

$$|f(x)-0| = |x| < \delta = \varepsilon.$$

3.1.3 Sequential limits

Let us connect the limit as defined above with limits of sequences.

Lemma 3.1.7. *Let $S \subset \mathbb{R}$, let c be a cluster point of S, let $f \colon S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$.*

Then $f(x) \to L$ as $x \to c$ if and only if for every sequence $\{x_n\}$ of numbers such that $x_n \in S \setminus \{c\}$ for all n, and such that $\lim x_n = c$, we have that the sequence $\{f(x_n)\}$ converges to L.

Proof. Suppose $f(x) \to L$ as $x \to c$, and $\{x_n\}$ is a sequence such that $x_n \in S \setminus \{c\}$ and $\lim x_n = c$. We wish to show that $\{f(x_n)\}$ converges to L. Let $\varepsilon > 0$ be given. Find a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x-c| < \delta$, then $|f(x)-L| < \varepsilon$. As $\{x_n\}$ converges to c, find an M such that for $n \geq M$, we have that $|x_n - c| < \delta$. Therefore, for $n \geq M$,

$$|f(x_n) - L| < \varepsilon.$$

Thus $\{f(x_n)\}$ converges to L.

For the other direction, we use proof by contrapositive. Suppose it is not true that $f(x) \to L$ as $x \to c$. The negation of the definition is that there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $x \in S \setminus \{c\}$, where $|x-c| < \delta$ and $|f(x)-L| \geq \varepsilon$.

Let us use $1/n$ for δ in the statement above to construct a sequence $\{x_n\}$. We have that there exists an $\varepsilon > 0$ such that for every n, there exists a point $x_n \in S \setminus \{c\}$, where $|x_n - c| < 1/n$ and $|f(x_n) - L| \geq \varepsilon$. The sequence $\{x_n\}$ just constructed converges to c, but the sequence $\{f(x_n)\}$ does not converge to L. And we are done. $\square$

It is possible to strengthen the reverse direction of the lemma by simply stating that $\{f(x_n)\}$ converges without requiring a specific limit. See Exercise 3.1.11.

Example 3.1.8: $\lim\limits_{x\to 0} \sin(1/x)$ does not exist, but $\lim\limits_{x\to 0} x\sin(1/x) = 0$. See Figure 3.1.

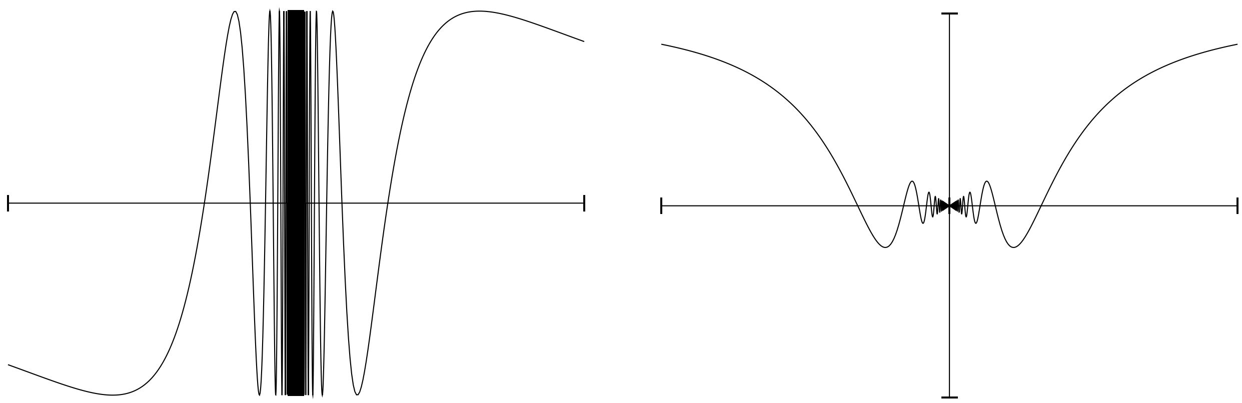

Figure 3.1: Graphs of $\sin(1/x)$ and $x\sin(1/x)$. Note that the computer cannot properly graph $\sin(1/x)$ near zero as it oscillates too fast.

Proof: We start with $\sin(1/x)$. Define a sequence by $x_n := \frac{1}{\pi n + \pi/2}$. It is not hard to see that $\lim x_n = 0$. Furthermore,

$$\sin(1/x_n) = \sin(\pi n + \pi/2) = (-1)^n.$$

Therefore, $\{\sin(1/x_n)\}$ does not converge. By Lemma 3.1.7, $\lim_{x\to 0} \sin(1/x)$ does not exist.

Now consider $x\sin(1/x)$. Let $\{x_n\}$ be a sequence such that $x_n \neq 0$ for all n, and such that $\lim x_n = 0$. Notice that $|\sin(t)| \leq 1$ for all $t \in \mathbb{R}$. Therefore,

$$|x_n \sin(1/x_n) - 0| = |x_n| \, |\sin(1/x_n)| \leq |x_n| \, .$$

As x_n goes to 0, then $|x_n|$ goes to zero, and hence $\{x_n \sin(1/x_n)\}$ converges to zero. By Lemma 3.1.7, $\lim\limits_{x\to 0} x\sin(1/x) = 0$.

Keep in mind the phrase "for every sequence" in the lemma. For example, take $\sin(1/x)$ and the sequence given by $x_n := 1/\pi n$. Then $\{\sin(1/x_n)\}$ is the constant zero sequence, and therefore converges to zero, but the limit of $\sin(1/x)$ as $x \to 0$ does not exist.

Using Lemma 3.1.7, we can start applying everything we know about sequential limits to limits of functions. Let us give a few important examples.

Corollary 3.1.9. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist, and*

$$f(x) \leq g(x) \qquad \text{for all } x \in S.$$

Then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

Proof. Take $\{x_n\}$ be a sequence of numbers in $S \setminus \{c\}$ that converges to c. Let

$$L_1 := \lim_{x\to c} f(x), \qquad \text{and} \qquad L_2 := \lim_{x\to c} g(x).$$

Lemma 3.1.7 says that $\{f(x_n)\}$ converges to L_1 and $\{g(x_n)\}$ converges to L_2. We also have $f(x_n) \leq g(x_n)$. We obtain $L_1 \leq L_2$ using Lemma 2.2.3. $\square$

By applying constant functions, we get the following corollary. The proof is left as an exercise.

Corollary 3.1.10. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Suppose there are two real numbers a and b such that*

$$a \leq f(x) \leq b \qquad \textit{for all } x \in S.$$

Then

$$a \leq \lim_{x\to c} f(x) \leq b.$$

Using Lemma 3.1.7 in the same way as above, we also get the following corollaries, whose proofs are again left as exercises.

Corollary 3.1.11. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $h : S \to \mathbb{R}$ are functions such that*

$$f(x) \leq g(x) \leq h(x) \qquad \textit{for all } x \in S.$$

Suppose the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

Then the limit of $g(x)$ as x goes to c exists and

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

Corollary 3.1.12. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist. Then*

(i) $\displaystyle\lim_{x\to c}\big(f(x)+g(x)\big) = \left(\lim_{x\to c} f(x)\right) + \left(\lim_{x\to c} g(x)\right).$

(ii) $\displaystyle\lim_{x\to c}\big(f(x)-g(x)\big) = \left(\lim_{x\to c} f(x)\right) - \left(\lim_{x\to c} g(x)\right).$

(iii) $\displaystyle\lim_{x\to c}\big(f(x)g(x)\big) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right).$

(iv) *If* $\displaystyle\lim_{x\to c} g(x) \neq 0$, *and* $g(x) \neq 0$ *for all* $x \in S \setminus \{c\}$, *then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Corollary 3.1.13. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Then*

$$\lim_{x\to c} \lvert f(x)\rvert = \left\lvert \lim_{x\to c} f(x)\right\rvert.$$

3.1.4 Limits of restrictions and one-sided limits

Sometimes we work with the function defined on a subset.

Definition 3.1.14. Let $f\colon S \to \mathbb{R}$ be a function and $A \subset S$. Define the function $f|_A\colon A \to \mathbb{R}$ by

$$f|_A(x) := f(x) \qquad \text{for } x \in A.$$

The function $f|_A$ is called the *restriction* of f to A.

The function $f|_A$ is simply the function f taken on a smaller domain. The following proposition is the analogue of taking a tail of a sequence.

Proposition 3.1.15. *Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, and let $f\colon S \to \mathbb{R}$ be a function. Suppose $A \subset S$ is such that there is some $\alpha > 0$ such that $(A \setminus \{c\}) \cap (c-\alpha, c+\alpha) = (S \setminus \{c\}) \cap (c-\alpha, c+\alpha)$.*

(i) The point c is a cluster point of A if and only if c is a cluster point of S.

(ii) Supposing c is a cluster point of S, then $f(x) \to L$ as $x \to c$ if and only if $f|_A(x) \to L$ as $x \to c$.

Proof. First, let c be a cluster point of A. Since $A \subset S$, then if $(A \setminus \{c\}) \cap (c-\varepsilon, c+\varepsilon)$ is nonempty for every $\varepsilon > 0$, then $(S \setminus \{c\}) \cap (c-\varepsilon, c+\varepsilon)$ is nonempty for every $\varepsilon > 0$. Thus c is a cluster point of S. Second, suppose c is a cluster point of S. Then for $\varepsilon > 0$ such that $\varepsilon < \alpha$ we get that $(A \setminus \{c\}) \cap (c-\varepsilon, c+\varepsilon) = (S \setminus \{c\}) \cap (c-\varepsilon, c+\varepsilon)$, which is nonempty. This is true for all $\varepsilon < \alpha$ and hence $(A \setminus \{c\}) \cap (c-\varepsilon, c+\varepsilon)$ must be nonempty for all $\varepsilon > 0$. Thus c is a cluster point of A.

Now suppose c is a cluster point of S and $f(x) \to L$ as $x \to c$. That is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x-c| < \delta$, then $|f(x) - L| < \varepsilon$. Because $A \subset S$, if x is in $A \setminus \{c\}$, then x is in $S \setminus \{c\}$, and hence $f|_A(x) \to L$ as $x \to c$.

Finally suppose $f|_A(x) \to L$ as $x \to c$. For every $\varepsilon > 0$ there is a $\delta' > 0$ such that if $x \in A \setminus \{c\}$ and $|x-c| < \delta'$, then $\left|f|_A(x) - L\right| < \varepsilon$. Take $\delta := \min\{\delta', \alpha\}$. Now suppose $x \in S \setminus \{c\}$ and $|x-c| < \delta$. As $|x-c| < \alpha$, then $x \in A \setminus \{c\}$, and as $|x-c| < \delta'$, we have $|f(x) - L| = \left|f|_A(x) - L\right| < \varepsilon$. $\square$

The hypothesis of the proposition is necessary. For an arbitrary restriction we generally only get implication in only one direction, see Exercise 3.1.6.

The usual notation for the limit is

$$\lim_{\substack{x \to c \\ x \in A}} f(x) := \lim_{x \to c} f|_A(x).$$

The most common use of restriction with respect to limits are the *one-sided limits**.

Definition 3.1.16. Let $f\colon S \to \mathbb{R}$ be function and let c be a cluster point of $S \cap (c, \infty)$. Then if the limit of the restriction of f to $S \cap (c, \infty)$ as $x \to c$ exists, define

$$\lim_{x \to c^+} f(x) := \lim_{x \to c} f|_{S \cap (c,\infty)}(x).$$

Similarly, if c is a cluster point of $S \cap (-\infty, c)$ and the limit of the restriction as $x \to c$ exists, define

$$\lim_{x \to c^-} f(x) := \lim_{x \to c} f|_{S \cap (-\infty,c)}(x).$$

*There are a plethora of notations for one-sided limits. E.g. for $\lim\limits_{x \to c^-} f(x)$ one sees $\lim\limits_{\substack{x \to c \\ x < c}} f(x)$, $\lim\limits_{x \uparrow c} f(x)$, or $\lim\limits_{x \nearrow c} f(x)$.

The proposition above does not apply to one-sided limits. It is possible to have one-sided limits, but no limit at a point. For example, define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) := 1$ for $x < 0$ and $f(x) := 0$ for $x \geq 0$. We leave it to the reader to verify that

$$\lim_{x \to 0^-} f(x) = 1, \qquad \lim_{x \to 0^+} f(x) = 0, \qquad \lim_{x \to 0} f(x) \quad \text{does not exist.}$$

We have the following replacement.

Proposition 3.1.17. *Let $S \subset \mathbb{R}$ be such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, let $f\colon S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then c is a cluster point of S and*

$$\lim_{x \to c} f(x) = L \qquad \text{if and only if} \qquad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

That is, a limit exists if both one-sided limits exist and are equal, and vice versa. The proof is a straightforward application of the definition of limit and is left as an exercise. The key point is that $\bigl(S \cap (-\infty, c)\bigr) \cup \bigl(S \cap (c, \infty)\bigr) = S \setminus \{c\}$.

3.1.5 Exercises

Exercise* 3.1.1*: *Find the limit (and prove it of course) or prove that the limit does not exist*

a) $\displaystyle\lim_{x \to c} \sqrt{x}$, *for* $c \geq 0$ *b)* $\displaystyle\lim_{x \to c} x^2 + x + 1$, *for* $c \in \mathbb{R}$ *c)* $\displaystyle\lim_{x \to 0} x^2 \cos(1/x)$
d) $\displaystyle\lim_{x \to 0} \sin(1/x) \cos(1/x)$ *e)* $\displaystyle\lim_{x \to 0} \sin(x) \cos(1/x)$

Exercise* 3.1.2*: *Prove Corollary 3.1.10.*

Exercise* 3.1.3*: *Prove Corollary 3.1.11.*

Exercise* 3.1.4*: *Prove Corollary 3.1.12.*

Exercise* 3.1.5*: *Let $A \subset S$. Show that if c is a cluster point of A, then c is a cluster point of S. Note the difference from Proposition 3.1.15.*

Exercise* 3.1.6*: *Let $A \subset S$. Suppose c is a cluster point of A and it is also a cluster point of S. Let $f\colon S \to \mathbb{R}$ be a function. Show that if $f(x) \to L$ as $x \to c$, then $f|_A(x) \to L$ as $x \to c$. Note the difference from Proposition 3.1.15.*

Exercise* 3.1.7*: *Find an example of a function $f\colon [-1,1] \to \mathbb{R}$, where for $A := [0,1]$, we have $f|_A(x) \to 0$ as $x \to 0$, but the limit of $f(x)$ as $x \to 0$ does not exist. Note why you cannot apply Proposition 3.1.15.*

Exercise* 3.1.8*: *Find example functions f and g such that the limit of neither $f(x)$ nor $g(x)$ exists as $x \to 0$, but such that the limit of $f(x) + g(x)$ exists as $x \to 0$.*

Exercise* 3.1.9*: *Let c_1 be a cluster point of $A \subset \mathbb{R}$ and c_2 be a cluster point of $B \subset \mathbb{R}$. Suppose $f\colon A \to B$ and $g\colon B \to \mathbb{R}$ are functions such that $f(x) \to c_2$ as $x \to c_1$ and $g(y) \to L$ as $y \to c_2$. If $c_2 \in B$, also suppose that $g(c_2) = L$. Let $h(x) := g\bigl(f(x)\bigr)$ and show $h(x) \to L$ as $x \to c_1$. Hint: Note that $f(x)$ could equal c_2 for many $x \in A$, see also Exercise 3.1.14.*

Exercise* 3.1.10** (note*):*** *Let c be a cluster point of $A \subset \mathbb{R}$, and $f\colon A \to \mathbb{R}$ be a function. Suppose for every sequence $\{x_n\}$ in A, such that $\lim x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy. Prove that $\lim_{x \to c} f(x)$ exists.*

*This exercise is almost identical to the next one. It will be replaced in the next major edition.

Exercise* 3.1.11*: *Prove the following stronger version of one direction of Lemma 3.1.7: Let $S \subset \mathbb{R}$, c be a cluster point of S, and $f\colon S \to \mathbb{R}$ be a function. Suppose that for every sequence $\{x_n\}$ in $S \setminus \{c\}$ such that* $\lim x_n = c$ *the sequence $\{f(x_n)\}$ is convergent. Then show that the limit of $f(x)$ as $x \to c$ exists.*

Exercise* 3.1.12*: *Prove Proposition 3.1.17.*

Exercise* 3.1.13*: *Suppose $S \subset \mathbb{R}$ and c is a cluster point of S. Suppose $f\colon S \to \mathbb{R}$ is bounded. Show that there exists a sequence $\{x_n\}$ with $x_n \in S \setminus \{c\}$ and $\lim x_n = c$ such that $\{f(x_n)\}$ converges.*

Exercise* 3.1.14** (Challenging):*** *Show that the hypothesis that $g(c_2) = L$ in Exercise 3.1.9 is necessary. That is, find f and g such that $f(x) \to c_2$ as $x \to c_1$ and $g(y) \to L$ as $y \to c_2$, but $g\bigl(f(x)\bigr)$ does not go to L as $x \to c_1$.*

Exercise* 3.1.15*: *Show that the condition of being a cluster point is necessary to have a reasonable definition of a limit. That is, suppose c is not a cluster point of $S \subset \mathbb{R}$, and $f\colon S \to \mathbb{R}$ is a function. Show that every L would satisfy the definition of limit at c without the condition on c being a cluster point.*

Exercise* 3.1.16*:

a) Prove Corollary 3.1.13.

b) Find an example showing that the converse of the corollary does not hold.

3.2 Continuous functions

Note: 2–2.5 lectures

You undoubtedly heard of continuous functions in your schooling. A high-school criterion for this concept is that a function is continuous if we can draw its graph without lifting the pen from the paper. While that intuitive concept may be useful in simple situations, we require rigor. The following definition took three great mathematicians (Bolzano, Cauchy, and finally Weierstrass) to get correctly and its final form dates only to the late 1800s.

3.2.1 Definition and basic properties

Definition 3.2.1. Let $S \subset \mathbb{R}$, $c \in S$, and let $f \colon S \to \mathbb{R}$ be a function. We say that f is *continuous at c* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S$ and $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

When $f \colon S \to \mathbb{R}$ is continuous at all $c \in S$, then we simply say f is a *continuous function*.

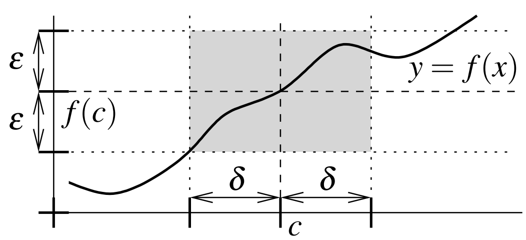

Figure 3.2: For $|x-c| < \delta$, the graph of $f(x)$ should be within the gray region.

If f is continuous for all $c \in A$, we say f is *continuous on $A \subset S$*. A straightforward exercise (Exercise 3.2.7) shows that this implies that $f|_A$ is continuous, although the converse does not hold.

Continuity may be the most important definition to understand in analysis, and it is not an easy one. See Figure 3.2. Note that δ not only depends on ε, but also on c; we need not pick one δ for all $c \in S$. It is no accident that the definition of continuity is similar to the definition of a limit of a function. The main feature of continuous functions is that these are precisely the functions that behave nicely with limits.

Proposition 3.2.2. *Consider a function $f \colon S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}$ and let $c \in S$. Then:*

(i) If c is not a cluster point of S, then f is continuous at c.

(ii) If c is a cluster point of S, then f is continuous at c if and only if the limit of $f(x)$ as $x \to c$ exists and

$$\lim_{x \to c} f(x) = f(c).$$

(iii) The function f is continuous at c if and only if for every sequence $\{x_n\}$ where $x_n \in S$ and $\lim x_n = c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

Proof. We start with the first item. Suppose c is not a cluster point of S. Then there exists a $\delta > 0$ such that $S \cap (c-\delta, c+\delta) = \{c\}$. For any $\varepsilon > 0$, simply pick this given δ. The only $x \in S$ such that $|x-c| < \delta$ is $x = c$. Then $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon$.

Let us move to the second item. Suppose c is a cluster point of S. Let us first suppose that $\lim_{x \to c} f(x) = f(c)$. Then for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Also $|f(c) - f(c)| = 0 < \varepsilon$, so the definition of continuity at c is satisfied. On the other hand, suppose f is continuous at c. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $x \in S$ where $|x-c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$. Then the statement is, of course, still true if $x \in S \setminus \{c\} \subset S$. Therefore, $\lim_{x \to c} f(x) = f(c)$.

For the third item, first suppose f is continuous at c. Let $\{x_n\}$ be a sequence such that $x_n \in S$ and $\lim x_n = c$. Let $\varepsilon > 0$ be given. Find a $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ for all $x \in S$ where $|x-c| < \delta$. Find an $M \in \mathbb{N}$ such that for $n \geq M$, we have $|x_n - c| < \delta$. Then for $n \geq M$, we have that $|f(x_n) - f(c)| < \varepsilon$, so $\{f(x_n)\}$ converges to $f(c)$.

We prove the other direction of the third item by contrapositive. Suppose f is not continuous at c. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in S$ such that $|x-c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon$. Let us define a sequence $\{x_n\}$ as follows. Let $x_n \in S$ be such that $|x_n - c| < 1/n$ and $|f(x_n) - f(c)| \geq \varepsilon$. Now $\{x_n\}$ is a sequence of numbers in S such that $\lim x_n = c$ and such that $|f(x_n) - f(c)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Thus $\{f(x_n)\}$ does not converge to $f(c)$. It may or may not converge, but it definitely does not converge to $f(c)$. $\square$

The last item in the proposition is particularly powerful. It allows us to quickly apply what we know about limits of sequences to continuous functions and even to prove that certain functions are continuous. It can also be strengthened, see Exercise 3.2.13.

Example 3.2.3: The function $f \colon (0,\infty) \to \mathbb{R}$ defined by $f(x) := 1/x$ is continuous.

Proof: Fix $c \in (0,\infty)$. Let $\{x_n\}$ be a sequence in $(0,\infty)$ such that $\lim x_n = c$. Then we know that

$$f(c) = \frac{1}{c} = \frac{1}{\lim x_n} = \lim_{n\to\infty} \frac{1}{x_n} = \lim_{n\to\infty} f(x_n).$$

Thus f is continuous at c. As f is continuous at all $c \in (0,\infty)$, f is continuous.

We have previously shown $\lim_{x\to c} x^2 = c^2$ directly. Therefore the function x^2 is continuous. We can use the continuity of algebraic operations with respect to limits of sequences, which we proved in the previous chapter, to prove a much more general result.

Proposition 3.2.4. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a* polynomial. *That is*

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0,$$

for some constants $a_0, a_1, \ldots, a_d$. Then f is continuous.

Proof. Fix $c \in \mathbb{R}$. Let $\{x_n\}$ be a sequence such that $\lim x_n = c$. Then

$$\begin{aligned} f(c) &= a_d c^d + a_{d-1} c^{d-1} + \cdots + a_1 c + a_0 \\ &= a_d (\lim x_n)^d + a_{d-1} (\lim x_n)^{d-1} + \cdots + a_1 (\lim x_n) + a_0 \\ &= \lim_{n\to\infty} \left(a_d x_n^d + a_{d-1} x_n^{d-1} + \cdots + a_1 x_n + a_0 \right) = \lim_{n\to\infty} f(x_n). \end{aligned}$$

Thus f is continuous at c. As f is continuous at all $c \in \mathbb{R}$, f is continuous. $\square$

By similar reasoning, or by appealing to Corollary 3.1.12, we can prove the following proposition. The proof is left as an exercise.

Proposition 3.2.5. *Let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ be functions continuous at $c \in S$.*

(i) The function $h\colon S \to \mathbb{R}$ defined by $h(x) := f(x) + g(x)$ is continuous at c.

(ii) The function $h\colon S \to \mathbb{R}$ defined by $h(x) := f(x) - g(x)$ is continuous at c.

(iii) The function $h\colon S \to \mathbb{R}$ defined by $h(x) := f(x)g(x)$ is continuous at c.

(iv) If $g(x) \neq 0$ for all $x \in S$, the function $h\colon S \to \mathbb{R}$ defined by $h(x) := \frac{f(x)}{g(x)}$ is continuous at c.

Example 3.2.6: The functions $\sin(x)$ and $\cos(x)$ are continuous. In the following computations we use the sum-to-product trigonometric identities. We also use the simple facts that $\lvert\sin(x)\rvert \leq \lvert x\rvert$, $\lvert\cos(x)\rvert \leq 1$, and $\lvert\sin(x)\rvert \leq 1$.

$$\begin{aligned}
\lvert\sin(x) - \sin(c)\rvert &= \left\lvert 2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)\right\rvert \\
&= 2\left\lvert\sin\left(\frac{x-c}{2}\right)\right\rvert\left\lvert\cos\left(\frac{x+c}{2}\right)\right\rvert \\
&\leq 2\left\lvert\sin\left(\frac{x-c}{2}\right)\right\rvert \\
&\leq 2\left\lvert\frac{x-c}{2}\right\rvert = \lvert x-c\rvert
\end{aligned}$$

$$\begin{aligned}
\lvert\cos(x) - \cos(c)\rvert &= \left\lvert -2\sin\left(\frac{x-c}{2}\right)\sin\left(\frac{x+c}{2}\right)\right\rvert \\
&= 2\left\lvert\sin\left(\frac{x-c}{2}\right)\right\rvert\left\lvert\sin\left(\frac{x+c}{2}\right)\right\rvert \\
&\leq 2\left\lvert\sin\left(\frac{x-c}{2}\right)\right\rvert \\
&\leq 2\left\lvert\frac{x-c}{2}\right\rvert = \lvert x-c\rvert
\end{aligned}$$

The claim that sin and cos are continuous follows by taking an arbitrary sequence $\{x_n\}$ converging to c, or by applying the definition of continuity directly. Details are left to the reader.

3.2.2 Composition of continuous functions

You probably already realized that one of the basic tools in constructing complicated functions out of simple ones is composition. Recall that for two functions f and g, the composition $f \circ g$ is defined by $(f \circ g)(x) := f\bigl(g(x)\bigr)$. A composition of continuous functions is again continuous.

Proposition 3.2.7. *Let $A, B \subset \mathbb{R}$ and $f\colon B \to \mathbb{R}$ and $g\colon A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g\colon A \to \mathbb{R}$ is continuous at c.*

Proof. Let $\{x_n\}$ be a sequence in A such that $\lim x_n = c$. As g is continuous at c, we have $\{g(x_n)\}$ converges to $g(c)$. As f is continuous at $g(c)$, we have $\{f(g(x_n))\}$ converges to $f(g(c))$. Thus $f \circ g$ is continuous at c. $\square$

Example 3.2.8: Claim: $\bigl(\sin(1/x)\bigr)^2$ *is a continuous function on* $(0,\infty)$.

Proof: The function $1/x$ is continuous on $(0,\infty)$ and $\sin(x)$ is continuous on $(0,\infty)$ (actually on $\mathbb{R}$, but $(0,\infty)$ is the range for $1/x$). Hence the composition $\sin(1/x)$ is continuous. Also, x^2 is continuous on the interval $(-1,1)$ (the range of sin). Thus the composition $\bigl(\sin(1/x)\bigr)^2$ is continuous on $(0,\infty)$.

3.2.3 Discontinuous functions

When f is not continuous at c, we say f is *discontinuous* at c, or that it has a *discontinuity* at c. The following proposition is a useful test and follows immediately from third item of Proposition 3.2.2.

Proposition 3.2.9. *Let $f \colon S \to \mathbb{R}$ be a function and $c \in S$. Suppose there exists a sequence $\{x_n\}$, $x_n \in S$, and $\lim x_n = c$ such that $\{f(x_n)\}$ does not converge to $f(c)$. Then f is discontinuous at c.*

Again, saying that $\{f(x_n)\}$ does not converge to $f(c)$ means that it either does not converge at all, or it converges to something other than $f(c)$.

Example 3.2.10: The function $f \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

is not continuous at 0.

Proof: Take the sequence $\{-1/n\}$, which converges to 0. Then $f(-1/n) = -1$ for every n, and so $\lim f(-1/n) = -1$, but $f(0) = 1$. Thus the function is not continuous at 0. See Figure 3.3.

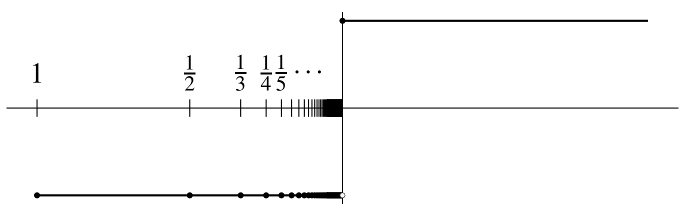

Figure 3.3: Graph of the jump discontinuity. The values of $f(-1/n)$ and $f(0)$ are marked as black dots.

Notice that $f(1/n) = 1$ for all $n \in \mathbb{N}$. Hence, $\lim f(1/n) = f(0) = 1$. So $\{f(x_n)\}$ may converge to $f(0)$ for some specific sequence $\{x_n\}$ going to 0, despite the function being discontinuous at 0.

Finally, consider $f\left(\frac{(-1)^n}{n}\right) = (-1)^n$. This sequence diverges.

Example 3.2.11: For an extreme example, take the so-called *Dirichlet function**.

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

The function f is discontinuous at all $c \in \mathbb{R}$.

*Named after the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805–1859).

Proof: Suppose c is rational. Take a sequence $\{x_n\}$ of irrational numbers such that $\lim x_n = c$ (why can we?). Then $f(x_n) = 0$ and so $\lim f(x_n) = 0$, but $f(c) = 1$. If c is irrational, take a sequence of rational numbers $\{x_n\}$ that converges to c (why can we?). Then $\lim f(x_n) = 1$, but $f(c) = 0$.

Let us test the limits of our intuition. Can there exist a function continuous at all irrational numbers, but discontinuous at all rational numbers? There are rational numbers arbitrarily close to any irrational number. Perhaps strangely, the answer is yes, such a function exists. The following example is called the *Thomae function*[*] or the *popcorn function*.

Example 3.2.12: Define $f\colon (0,1) \to \mathbb{R}$ as

$$f(x) := \begin{cases} 1/k & \text{if } x = m/k, \text{ where } m,k \in \mathbb{N} \text{ and } m \text{ and } k \text{ have no common divisors,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

See the graph of f in Figure 3.4. We claim that f is continuous at all irrational c and discontinuous at all rational c.

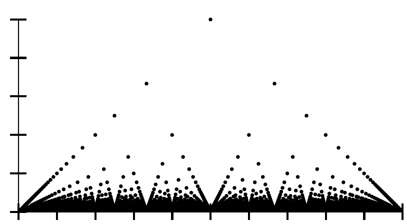

Figure 3.4: Graph of the "popcorn function."

Proof: Suppose $c = m/k$ is rational. Take a sequence of irrational numbers $\{x_n\}$ such that $\lim x_n = c$. Then $\lim f(x_n) = \lim 0 = 0$, but $f(c) = 1/k \neq 0$. So f is discontinuous at c.

Now let c be irrational, so $f(c) = 0$. Take a sequence $\{x_n\}$ in $(0,1)$ such that $\lim x_n = c$. Given $\varepsilon > 0$, find $K \in \mathbb{N}$ such that $1/K < \varepsilon$ by the Archimedean property. If $m/k \in (0,1)$ is in lowest terms (no common divisors), then $0 < m < k$. So there are only finitely many rational numbers in $(0,1)$ whose denominator k in lowest terms is less than K. As $\lim x_n = c$, every number not equal to c can appear at most finitely many times in $\{x_n\}$. Hence, there is an M such that for $n \geq M$, all the numbers x_n that are rational have a denominator larger than or equal to K. Thus for $n \geq M$,

$$|f(x_n) - 0| = f(x_n) \leq 1/K < \varepsilon.$$

Therefore, f is continuous at irrational c.

Let us end on an easier example.

Example 3.2.13: Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) := 0$ if $x \neq 0$ and $g(0) := 1$. Then g is not continuous at zero, but continuous everywhere else (why?). The point $x = 0$ is called a *removable discontinuity*. That is because if we would change the definition of g, by insisting that $g(0)$ be 0, we would obtain

[*]Named after the German mathematician Carl Johannes Thomae (1840–1921).

a continuous function. On the other hand, let f be the function of Example 3.2.10. Then f does not have a removable discontinuity at 0. No matter how we would define $f(0)$ the function would still fail to be continuous. The difference is that $\lim_{x\to 0} g(x)$ exists while $\lim_{x\to 0} f(x)$ does not.

We stay with this example to show another phenomenon. Let $A := \{0\}$, then $g|_A$ is continuous (why?), while g is not continuous on A. Similarly, if $B := \mathbb{R} \setminus \{0\}$, then $g|_B$ is also continuous, and g is in fact continuous on B.

3.2.4 Exercises

Exercise* 3.2.1*: *Using the definition of continuity directly prove that* $f\colon \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) := x^2$ *is continuous.*

Exercise* 3.2.2*: *Using the definition of continuity directly prove that* $f\colon (0,\infty) \to \mathbb{R}$ *defined by* $f(x) := 1/x$ *is continuous.*

Exercise* 3.2.3*: *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the definition of continuity directly prove that f *is continuous at* 1 *and discontinuous at* 2*.*

Exercise* 3.2.4*: *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f *continuous? Prove your assertion.*

Exercise* 3.2.5*: *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) := \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f *continuous? Prove your assertion.*

Exercise* 3.2.6*: *Prove Proposition 3.2.5.*

Exercise* 3.2.7*: *Prove the following statement. Let* $S \subset \mathbb{R}$ *and* $A \subset S$*. Let* $f\colon S \to \mathbb{R}$ *be a continuous function. Then the restriction* $f|_A$ *is continuous.*

Exercise* 3.2.8*: *Suppose* $S \subset \mathbb{R}$*, such that* $(c-\alpha, c+\alpha) \subset S$ *for some* $c \in \mathbb{R}$ *and* $\alpha > 0$*. Let* $f\colon S \to \mathbb{R}$ *be a function and* $A := (c-\alpha, c+\alpha)$*. Prove that if* $f|_A$ *is continuous at* c*, then* f *is continuous at* c*.*

Exercise* 3.2.9*: *Give an example of functions* $f\colon \mathbb{R} \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *such that the function* h *defined by* $h(x) := f(x) + g(x)$ *is continuous, but* f *and* g *are not continuous. Can you find* f *and* g *that are nowhere continuous, but* h *is a continuous function?*

Exercise* 3.2.10*: *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *be continuous functions. Suppose that for all rational numbers* r*,* $f(r) = g(r)$*. Show that* $f(x) = g(x)$ *for all* x*.*

Exercise* 3.2.11*: *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c-\alpha, c+\alpha)$, we have $f(x) > 0$.*

Exercise* 3.2.12*: *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function. Show that f is continuous.*

Exercise* 3.2.13*: *Let $f : S \to \mathbb{R}$ be a function and $c \in S$, such that for every sequence $\{x_n\}$ in S with $\lim x_n = c$, the sequence $\{f(x_n)\}$ converges. Show that f is continuous at c.*

Exercise* 3.2.14*: *Suppose $f : [-1,0] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are continuous and $f(0) = g(0)$. Define $h : [-1,1] \to \mathbb{R}$ by $h(x) := f(x)$ if $x \leq 0$ and $h(x) := g(x)$ if $x > 0$. Show that h is continuous.*

Exercise* 3.2.15*: *Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$, and suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $|f(x) - f(y)| \leq g(x-y)$ for all x and y. Show that f is continuous.*

Exercise* 3.2.16** (Challenging):*** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and such that $f(x+y) = f(x) + f(y)$ for every x and y. Show that $f(x) = ax$ for some $a \in \mathbb{R}$. Hint: Show that $f(nx) = nf(x)$, then show f is continuous on $\mathbb{R}$. Then show that ${}^{f(x)}/_{x} = f(1)$ for all rational x.*

Exercise* 3.2.17*: *Suppose $S \subset \mathbb{R}$ and let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be continuous functions. Define $p : S \to \mathbb{R}$ by $p(x) := \max\{f(x), g(x)\}$ and $q : S \to \mathbb{R}$ by $q(x) := \min\{f(x), g(x)\}$. Prove that p and q are continuous.*

Exercise* 3.2.18*: *Suppose $f : [-1,1] \to \mathbb{R}$ is a function continuous at all $x \in [-1,1] \setminus \{0\}$. Show that for every ε such that $0 < \varepsilon < 1$, there exists a function $g : [-1,1] \to \mathbb{R}$ continuous on all of $[-1,1]$, such that $f(x) = g(x)$ for all $x \in [-1,-\varepsilon] \cup [\varepsilon,1]$, and $|g(x)| \leq |f(x)|$ for all $x \in [-1,1]$.*

Exercise* 3.2.19** (Challenging):*** *A function $f : I \to \mathbb{R}$ is* convex *if whenever $a \leq x \leq b$ for a, x, b in I, we have $f(x) \leq f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a}$. In other words, if the line drawn between $\bigl(a, f(a)\bigr)$ and $\bigl(b, f(b)\bigr)$ is above the graph of f.*

a) Prove that if $I = (\alpha, \beta)$ an open interval and $f : I \to \mathbb{R}$ is convex, then f is continuous.

b) Find an example of a convex $f : [0,1] \to \mathbb{R}$ which is not continuous.

3.3 Min-max and intermediate value theorems

Note: 1.5 lectures

Continuous functions on closed and bounded intervals are quite well behaved.

3.3.1 Min-max or extreme value theorem

Recall a function $f: [a,b] \to \mathbb{R}$ is *bounded* if there exists a $B \in \mathbb{R}$ such that $|f(x)| \leq B$ for all $x \in [a,b]$. We have the following lemma.

Lemma 3.3.1. *A continuous function $f: [a,b] \to \mathbb{R}$ is bounded.*

Proof. Let us prove this claim by contrapositive. Suppose f is not bounded. Then for each $n \in \mathbb{N}$, there is an $x_n \in [a,b]$, such that

$$|f(x_n)| \geq n.$$

The sequence $\{x_n\}$ is bounded as $a \leq x_n \leq b$. By the Bolzano–Weierstrass theorem, there is a convergent subsequence $\{x_{n_i}\}$. Let $x := \lim x_{n_i}$. Since $a \leq x_{n_i} \leq b$ for all i, then $a \leq x \leq b$. The sequence $\{f(x_{n_i})\}$ is not bounded as $|f(x_{n_i})| \geq n_i \geq i$. Thus f is not continuous at x as

$$f(x) = f\left(\lim_{i\to\infty} x_{n_i}\right), \qquad \text{but} \qquad \lim_{i\to\infty} f(x_{n_i}) \ \text{does not exist.} \qquad \square$$

Notice a key point of the proof. Boundedness of $[a,b]$ allows us to use Bolzano–Weierstrass, while the fact that it is closed gives us that the limit is back in $[a,b]$. The technique is a common one: Find a sequence with a certain property, then use Bolzano–Weierstrass to make such a sequence that also converges.

Recall from calculus that $f: S \to \mathbb{R}$ achieves an *absolute minimum* at $c \in S$ if

$$f(x) \geq f(c) \qquad \text{for all } x \in S.$$

On the other hand, f achieves an *absolute maximum* at $c \in S$ if

$$f(x) \leq f(c) \qquad \text{for all } x \in S.$$

If such a $c \in S$ exists, then f *achieves an absolute minimum (resp. absolute maximum) on S*.

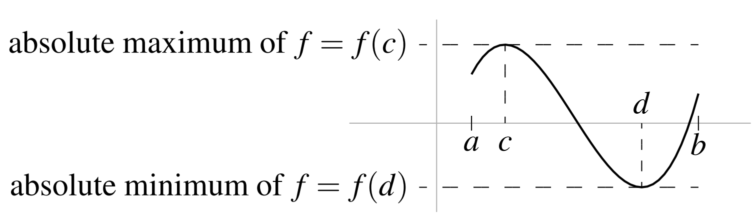

Figure 3.5: $f: [a,b] \to \mathbb{R}$ achieves an absolute maximum $f(c)$ at c, and an absolute minimum $f(d)$ at d.

If S is a closed and bounded interval, then a continuous f must achieve an absolute minimum and an absolute maximum on S.

Theorem 3.3.2 (Minimum-maximum theorem / Extreme value theorem)**.** *A continuous function $f\colon [a,b] \to \mathbb{R}$ on a closed and bounded interval $[a,b]$ achieves both an absolute minimum and an absolute maximum on $[a,b]$.*

Proof. The lemma says that f is bounded, so the set $f\big([a,b]\big) = \big\{ f(x) : x \in [a,b] \big\}$ has a supremum and an infimum. There exist sequences in the set $f\big([a,b]\big)$ that approach its supremum and its infimum. That is, there are sequences $\big\{ f(x_n) \big\}$ and $\big\{ f(y_n) \big\}$, where x_n and y_n are in $[a,b]$, such that

$$\lim_{n\to\infty} f(x_n) = \inf f\big([a,b]\big) \qquad \text{and} \qquad \lim_{n\to\infty} f(y_n) = \sup f\big([a,b]\big).$$

We are not done yet; we need to find where the minima and the maxima are. The problem is that the sequences $\{x_n\}$ and $\{y_n\}$ need not converge. We know $\{x_n\}$ and $\{y_n\}$ are bounded (their elements belong to a bounded interval $[a,b]$). Apply the Bolzano–Weierstrass theorem, to find convergent subsequences $\{x_{n_i}\}$ and $\{y_{m_i}\}$. Let

$$x := \lim_{i\to\infty} x_{n_i} \qquad \text{and} \qquad y := \lim_{i\to\infty} y_{m_i}.$$

As $a \leq x_{n_i} \leq b$ for all i, we have $a \leq x \leq b$. Similarly, $a \leq y \leq b$. So x and y are in $[a,b]$. A limit of a subsequence is the same as the limit of the sequence, and we can take a limit past the continuous function f:

$$\inf f([a,b]) = \lim_{n\to\infty} f(x_n) = \lim_{i\to\infty} f(x_{n_i}) = f\Big(\lim_{i\to\infty} x_{n_i}\Big) = f(x).$$

Similarly,

$$\sup f([a,b]) = \lim_{n\to\infty} f(y_n) = \lim_{i\to\infty} f(y_{m_i}) = f\Big(\lim_{i\to\infty} y_{m_i}\Big) = f(y).$$

Therefore, f achieves an absolute minimum at x and f achieves an absolute maximum at y. □

Example 3.3.3: The function $f(x) := x^2+1$ defined on the interval $[-1,2]$ achieves a minimum at $x = 0$ when $f(0) = 1$. It achieves a maximum at $x = 2$ where $f(2) = 5$. Do note that the domain of definition matters. If we instead took the domain to be $[-10,10]$, then $x = 2$ would no longer be a maximum of f. Instead the maximum would be achieved at either $x = 10$ or $x = -10$.

We show by examples that the different hypotheses of the theorem are truly needed.

Example 3.3.4: The function $f(x) := x$, defined on the whole real line, achieves neither a minimum, nor a maximum. So it is important that we are looking at a bounded interval.

Example 3.3.5: The function $f(x) := 1/x$, defined on $(0,1)$ achieves neither a minimum, nor a maximum. The values of the function are unbounded as we approach 0. Also as we approach $x = 1$, the values of the function approach 1, but $f(x) > 1$ for all $x \in (0,1)$. There is no $x \in (0,1)$ such that $f(x) = 1$. So it is important that we are looking at a closed interval.

Example 3.3.6: Continuity is important. Define $f\colon [0,1] \to \mathbb{R}$ by $f(x) := 1/x$ for $x > 0$ and let $f(0) := 0$. The function does not achieve a maximum. The problem is that the function is not continuous at 0.

3.3.2 Bolzano's intermediate value theorem

Bolzano's intermediate value theorem is one of the cornerstones of analysis. It is sometimes only called the intermediate value theorem, or just Bolzano's theorem. To prove Bolzano's theorem we prove the following simpler lemma.

Lemma 3.3.7. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) < 0$ and $f(b) > 0$. Then there exists a number $c \in (a,b)$ such that $f(c) = 0$.*

Proof. We define two sequences $\{a_n\}$ and $\{b_n\}$ inductively:

(i) Let $a_1 := a$ and $b_1 := b$.

(ii) If $f\left(\frac{a_n+b_n}{2}\right) \geq 0$, let $a_{n+1} := a_n$ and $b_{n+1} := \frac{a_n+b_n}{2}$.

(iii) If $f\left(\frac{a_n+b_n}{2}\right) < 0$, let $a_{n+1} := \frac{a_n+b_n}{2}$ and $b_{n+1} := b_n$.

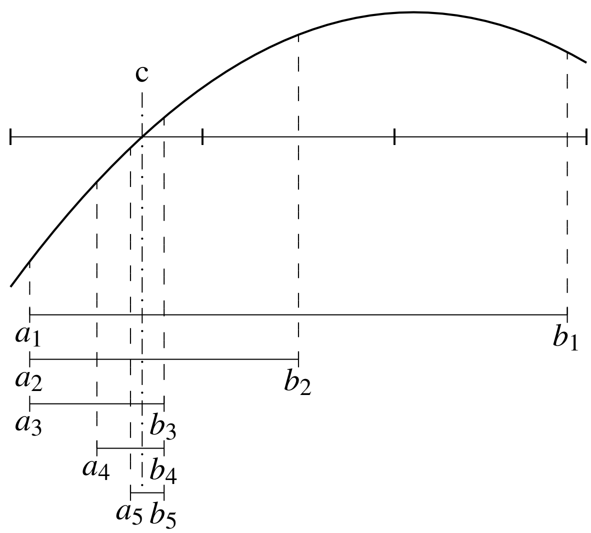

Figure 3.6: Finding roots (bisection method).

See Figure 3.6 for an example defining the first five steps. If $a_n < b_n$, then $a_n < \frac{a_n+b_n}{2} < b_n$. So $a_{n+1} < b_{n+1}$. Thus by induction $a_n < b_n$ for all n. Furthermore, $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$ for all n, that is, the sequences are monotone. As $a_n < b_n \leq b_1 = b$ and $b_n > a_n \geq a_1 = a$ for all n, the sequences are also bounded. Therefore, the sequences converge. Let $c := \lim a_n$ and $d := \lim b_n$, where also $a \leq c \leq d \leq b$. We need to show that $c = d$. Notice

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}.$$

By induction,

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} = 2^{1-n}(b-a).$$

As $2^{1-n}(b-a)$ converges to zero, we take the limit as n goes to infinity to get

$$d-c = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} 2^{1-n}(b-a) = 0.$$

In other words, $d = c$.

By construction, for all n, we have

$$f(a_n) < 0 \qquad \text{and} \qquad f(b_n) \geq 0.$$

Since $\lim a_n = \lim b_n = c$ and as f is continuous, we may take limits in those inequalities:

$$f(c) = \lim f(a_n) \leq 0 \qquad \text{and} \qquad f(c) = \lim f(b_n) \geq 0.$$

As $f(c) \geq 0$ and $f(c) \leq 0$, we conclude $f(c) = 0$. Thus also $c \neq a$ and $c \neq b$, so $a < c < b$. □

Theorem 3.3.8 (Bolzano's intermediate value theorem)**.** *Let $f \colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $y \in \mathbb{R}$ is such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then there exists a $c \in (a,b)$ such that $f(c) = y$.*

The theorem says that a continuous function on a closed interval achieves all the values between the values at the endpoints.

Proof. If $f(a) < y < f(b)$, then define $g(x) := f(x) - y$. Then $g(a) < 0$ and $g(b) > 0$, and we apply Lemma 3.3.7 to g to find c. If $g(c) = 0$, then $f(c) = y$.

Similarly, if $f(a) > y > f(b)$, then define $g(x) := y - f(x)$. Again, $g(a) < 0$ and $g(b) > 0$, and we apply Lemma 3.3.7 to find c. As before, if $g(c) = 0$, then $f(c) = y$. □

If a function is continuous, then the restriction to a subset is continuous; if $f \colon S \to \mathbb{R}$ is continuous and $[a,b] \subset S$, then $f|_{[a,b]}$ is also continuous. We generally apply the theorem to a function continuous on some large set S, but we restrict our attention to an interval.

The proof of the lemma tells us how to find the root c. The proof is not only useful for us pure mathematicians, it is a useful idea in applied mathematics, where it is called the *bisection method*.

Example 3.3.9 (Bisection method)**:** The polynomial $f(x) := x^3 - 2x^2 + x - 1$ has a real root in $(1,2)$. We simply notice that $f(1) = -1$ and $f(2) = 1$. Hence there must exist a point $c \in (1,2)$ such that $f(c) = 0$. To find a better approximation of the root we follow the proof of Lemma 3.3.7. We look at 1.5 and find that $f(1.5) = -0.625$. Therefore, there is a root of the polynomial in $(1.5,2)$. Next we look at 1.75 and note that $f(1.75) \approx -0.016$. Hence there is a root of f in $(1.75,2)$. Next we look at 1.875 and find that $f(1.875) \approx 0.44$, thus there is a root in $(1.75,1.875)$. We follow this procedure until we gain sufficient precision. In fact, the root is at $c \approx 1.7549$.

The technique above is the simplest method of finding roots of polynomials, which is perhaps the most common problem in applied mathematics. In general, finding roots is hard to do quickly, precisely, and automatically. There are other, faster methods of finding roots of polynomials, such as Newton's method. One advantage of the method above is its simplicity. The moment we find an initial interval where the intermediate value theorem applies, we are guaranteed to find a root up to a desired precision in finitely many steps. Furthermore, the bisection method finds roots of any continuous function, not just a polynomial.

The theorem guarantees at least one c such that $f(c) = y$, but there may be many different roots of the equation $f(c) = y$. If we follow the procedure of the proof, we are guaranteed to find approximations to one such root. We need to work harder to find any other roots.

Polynomials of even degree may not have any real roots. There is no real number x such that $x^2 + 1 = 0$. Odd polynomials, on the other hand, always have at least one real root.

Proposition 3.3.10. *Let $f(x)$ be a polynomial of odd degree. Then f has a real root.*

Proof. Suppose f is a polynomial of odd degree d. We write

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0,$$

where $a_d \neq 0$. We divide by a_d to obtain a *monic polynomial*[*]

$$g(x) := x^d + b_{d-1} x^{d-1} + \cdots + b_1 x + b_0,$$

where $b_k = \nicefrac{a_k}{a_d}$. Let us show that $g(n)$ is positive for some large $n \in \mathbb{N}$. We first compare the highest order term with the rest:

$$\begin{aligned}
\left\lvert \frac{b_{d-1} n^{d-1} + \cdots + b_1 n + b_0}{n^d} \right\rvert &= \frac{\lvert b_{d-1} n^{d-1} + \cdots + b_1 n + b_0 \rvert}{n^d} \\
&\leq \frac{\lvert b_{d-1} \rvert n^{d-1} + \cdots + \lvert b_1 \rvert n + \lvert b_0 \rvert}{n^d} \\
&\leq \frac{\lvert b_{d-1} \rvert n^{d-1} + \cdots + \lvert b_1 \rvert n^{d-1} + \lvert b_0 \rvert n^{d-1}}{n^d} \\
&= \frac{n^{d-1} \bigl(\lvert b_{d-1} \rvert + \cdots + \lvert b_1 \rvert + \lvert b_0 \rvert \bigr)}{n^d} \\
&= \frac{1}{n} \bigl(\lvert b_{d-1} \rvert + \cdots + \lvert b_1 \rvert + \lvert b_0 \rvert \bigr).
\end{aligned}$$

Therefore,

$$\lim_{n \to \infty} \frac{b_{d-1} n^{d-1} + \cdots + b_1 n + b_0}{n^d} = 0.$$

Thus there exists an $M \in \mathbb{N}$ such that

$$\left\lvert \frac{b_{d-1} M^{d-1} + \cdots + b_1 M + b_0}{M^d} \right\rvert < 1,$$

which implies

$$-(b_{d-1} M^{d-1} + \cdots + b_1 M + b_0) < M^d.$$

Therefore, $g(M) > 0$.

Next, consider $g(-n)$ for $n \in \mathbb{N}$. By a similar argument, there exists a $K \in \mathbb{N}$ such that $b_{d-1}(-K)^{d-1} + \cdots + b_1(-K) + b_0 < K^d$ and therefore $g(-K) < 0$ (see Exercise 3.3.5). In the proof, make sure you use the fact that d is odd. In particular, if d is odd, then $(-n)^d = -(n^d)$.

We appeal to the intermediate value theorem to find a $c \in [-K, M]$, such that $g(c) = 0$. As $g(x) = \frac{f(x)}{a_d}$, then $f(c) = 0$, and the proof is done. $\square$

[*]The word *monic* means that the coefficient of x^d is 1.

Example 3.3.11: Interestingly, there exist discontinuous functions with the intermediate value property. The function

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0; however, f has the intermediate value property: Whenever $a < b$ and y is such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, there exists a c such that $f(y) = c$. Proof is left as Exercise 3.3.4.

The intermediate value theorem says that if $f: [a,b] \to \mathbb{R}$ is continuous, then $f\big([a,b]\big)$ contains all the values between $f(a)$ and $f(b)$. In fact, more is true. Combining all the results of this section one can prove the following useful corollary whose proof is left as an exercise.

Corollary 3.3.12. *If $f: [a,b] \to \mathbb{R}$ is continuous, then the direct image $f([a,b])$ is a closed and bounded interval or a single number.*

3.3.3 Exercises

***Exercise* 3.3.1:** *Find an example of a discontinuous function $f: [0,1] \to \mathbb{R}$ where the conclusion of the intermediate value theorem fails.*

***Exercise* 3.3.2:** *Find an example of a* bounded *discontinuous function $f: [0,1] \to \mathbb{R}$ that has neither an absolute minimum nor an absolute maximum.*

***Exercise* 3.3.3:** *Let $f: (0,1) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0,1)$ (but perhaps not both).*

***Exercise* 3.3.4:** *Let*

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f has the intermediate value property. That is, whenever $a < b$, if there exists a y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then there exists a $c \in (a,b)$ such that $f(c) = y$.

***Exercise* 3.3.5:** *Suppose $g(x)$ is a monic polynomial of odd degree d, that is,*

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0,$$

for some real numbers $b_0, b_1, \ldots, b_{d-1}$. Show that there exists a $K \in \mathbb{N}$ such that $g(-K) < 0$. Hint: Make sure to use the fact that d is odd. You will have to use that $(-n)^d = -(n^d)$.

***Exercise* 3.3.6:** *Suppose $g(x)$ is a monic polynomial of positive even degree d, that is,*

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0,$$

for some real numbers $b_0, b_1, \ldots, b_{d-1}$. Suppose $g(0) < 0$. Show that g has at least two distinct real roots.

***Exercise* 3.3.7:** *Prove Corollary 3.3.12: Suppose $f: [a,b] \to \mathbb{R}$ is a continuous function. Prove that the direct image $f([a,b])$ is a closed and bounded interval or a single number.*

Exercise* 3.3.8*: *Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period $P > 0$. That is, $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. Show that f achieves an absolute minimum and an absolute maximum.*

Exercise* 3.3.9** (Challenging):*** *Suppose $f(x)$ is a bounded polynomial, in other words, there is an M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that f must be a constant.*

Exercise* 3.3.10*: *Suppose $f: [0,1] \to [0,1]$ is continuous. Show that f has a fixed point, in other words, show that there exists an $x \in [0,1]$ such that $f(x) = x$.*

Exercise* 3.3.11*: *Find an example of a continuous bounded function $f: \mathbb{R} \to \mathbb{R}$ that does not achieve an absolute minimum nor an absolute maximum on $\mathbb{R}$.*

Exercise* 3.3.12*: *Suppose $f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $x \leq f(x) \leq x+1$ for all $x \in \mathbb{R}$. Find $f(\mathbb{R})$.*

Exercise* 3.3.13*: *True/False, prove or find a counterexample. If $f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f|_{\mathbb{Z}}$ is bounded, then f is bounded.*

Exercise* 3.3.14*: *Suppose $f: [0,1] \to (0,1)$ is a bijection. Prove that f is not continuous.*

Exercise* 3.3.15*: *Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous.*

a) Prove that if there is a c such that $f(c)f(-c) < 0$, then there is a $d \in \mathbb{R}$ such that $f(d) = 0$.

b) Find a continuous function f such that $f(\mathbb{R}) = \mathbb{R}$, but $f(x)f(-x) \geq 0$ for all $x \in \mathbb{R}$.

Exercise* 3.3.16*: *Suppose $g(x)$ is a monic polynomial of even degree d, that is,*

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0,$$

for some real numbers $b_0, b_1, \ldots, b_{d-1}$. Show that g achieves an absolute minimum on $\mathbb{R}$.

Exercise* 3.3.17*: *Suppose $f(x)$ is a polynomial of degree d and $f(\mathbb{R}) = \mathbb{R}$. Show that d is odd.*

3.4 Uniform continuity

Note: 1.5–2 lectures (continuous extension can be optional)

3.4.1 Uniform continuity

We made a fuss of saying that the δ in the definition of continuity depended on the point c. There are situations when it is advantageous to have a δ independent of any point, and so we give a name to this concept.

Definition 3.4.1. Let $S \subset \mathbb{R}$, and let $f \colon S \to \mathbb{R}$ be a function. Suppose for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, c \in S$ and $\lvert x - c \rvert < \delta$, then $\lvert f(x) - f(c) \rvert < \varepsilon$. Then we say f is *uniformly continuous*.

A uniformly continuous function must be continuous. The only difference in the definitions is that in uniform continuity, for a given $\varepsilon > 0$ we pick a $\delta > 0$ that works for all $c \in S$. That is, δ can no longer depend on c, it only depends on ε. The domain of definition of the function makes a difference now. A function that is not uniformly continuous on a larger set, may be uniformly continuous when restricted to a smaller set. We will say *uniformly continuous on* X to mean that f restricted to X is uniformly continuous, or perhaps to just emphasize the domain. Note that x and c are not treated any differently in this definition.

Example 3.4.2: $f \colon [0,1] \to \mathbb{R}$, defined by $f(x) := x^2$ is uniformly continuous.

Proof: Note that $0 \leq x, c \leq 1$. Then

$$\lvert x^2 - c^2 \rvert = \lvert x + c \rvert \lvert x - c \rvert \leq (\lvert x \rvert + \lvert c \rvert) \lvert x - c \rvert \leq (1+1) \lvert x - c \rvert .$$

Therefore, given $\varepsilon > 0$, let $\delta := \varepsilon/2$. If $\lvert x - c \rvert < \delta$, then $\lvert x^2 - c^2 \rvert < \varepsilon$.

On the other hand, $g \colon \mathbb{R} \to \mathbb{R}$, defined by $g(x) := x^2$ is not uniformly continuous.

Proof: Suppose it is uniformly continuous, then for every $\varepsilon > 0$, there would exist a $\delta > 0$ such that if $\lvert x - c \rvert < \delta$, then $\lvert x^2 - c^2 \rvert < \varepsilon$. Take $x > 0$ and let $c := x + \delta/2$. Write

$$\varepsilon > \lvert x^2 - c^2 \rvert = \lvert x + c \rvert \lvert x - c \rvert = (2x + \delta/2)\delta/2 \geq \delta x.$$

Therefore, $x < \varepsilon/\delta$ for all $x > 0$, which is a contradiction.

Example 3.4.3: The function $f \colon (0,1) \to \mathbb{R}$, defined by $f(x) := 1/x$ is not uniformly continuous.

Proof: Given $\varepsilon > 0$, then $\varepsilon > \lvert 1/x - 1/y \rvert$ holds if and only if

$$\varepsilon > \lvert 1/x - 1/y \rvert = \frac{\lvert y - x \rvert}{\lvert xy \rvert} = \frac{\lvert y - x \rvert}{xy},$$

or

$$\lvert x - y \rvert < xy\varepsilon.$$

Suppose $\varepsilon < 1$, and we wish to see if a small $\delta > 0$ would work. If $x \in (0,1)$ and $y = x + \delta/2 \in (0,1)$, then $\lvert x - y \rvert = \delta/2 < \delta$. We plug y into the inequality above to get $\delta/2 < x\bigl(x + \delta/2\bigr)\varepsilon < x$. If the definition of uniform continuity is satisfied, then the inequality $\delta/2 < x$ holds for all $x > 0$. But then $\delta \leq 0$. Therefore, there is no single $\delta > 0$ that works for all points.

The examples show that if f is defined on an interval that is either not closed or not bounded, then f can be continuous, but not uniformly continuous. For a closed and bounded interval $[a,b]$, we can, however, make the following statement.

Theorem 3.4.4. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Proof. We prove the statement by contrapositive. Suppose f is not uniformly continuous. We will prove that there is some $c \in [a,b]$ where f is not continuous. Let us negate the definition of uniformly continuous. There exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exist points x, y in $[a,b]$ with $\lvert x-y \rvert < \delta$ and $\lvert f(x) - f(y) \rvert \geq \varepsilon$.

So for the $\varepsilon > 0$ above, we find sequences $\{x_n\}$ and $\{y_n\}$ such that $\lvert x_n - y_n \rvert < 1/n$ and such that $\lvert f(x_n) - f(y_n) \rvert \geq \varepsilon$. By Bolzano–Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}$. Let $c := \lim x_{n_k}$. As $a \leq x_{n_k} \leq b$ for all k, we have $a \leq c \leq b$. Estimate

$$\lvert y_{n_k} - c \rvert = \lvert y_{n_k} - x_{n_k} + x_{n_k} - c \rvert \leq \lvert y_{n_k} - x_{n_k} \rvert + \lvert x_{n_k} - c \rvert < 1/n_k + \lvert x_{n_k} - c \rvert .$$

As $1/n_k$ and $\lvert x_{n_k} - c \rvert$ both go to zero when k goes to infinity, $\{y_{n_k}\}$ converges and the limit is c. We now show that f is not continuous at c. Estimate

$$\begin{aligned} \lvert f(x_{n_k}) - f(c) \rvert &= \lvert f(x_{n_k}) - f(y_{n_k}) + f(y_{n_k}) - f(c) \rvert \\ &\geq \lvert f(x_{n_k}) - f(y_{n_k}) \rvert - \lvert f(y_{n_k}) - f(c) \rvert \\ &\geq \varepsilon - \lvert f(y_{n_k}) - f(c) \rvert . \end{aligned}$$

Or in other words,

$$\lvert f(x_{n_k}) - f(c) \rvert + \lvert f(y_{n_k}) - f(c) \rvert \geq \varepsilon .$$

At least one of the sequences $\{f(x_{n_k})\}$ or $\{f(y_{n_k})\}$ cannot converge to $f(c)$, otherwise the left-hand side of the inequality would go to zero while the right-hand side is positive. Thus f cannot be continuous at c. ☐

As before, note what is key in the proof: We can apply Bolzano–Weierstrass because the interval $[a,b]$ is bounded, and the limit of the subsequence is back in $[a,b]$ because the interval is closed.

3.4.2 Continuous extension

Before we get to continuous extension, we show the following useful lemma. It says that uniformly continuous functions behave nicely with respect to Cauchy sequences. The new issue here is that for a Cauchy sequence we no longer know where the limit ends up; it may not end up in the domain of the function.

Lemma 3.4.5. *Let $f\colon S \to \mathbb{R}$ be a uniformly continuous function. Let $\{x_n\}$ be a Cauchy sequence in S. Then $\{f(x_n)\}$ is Cauchy.*

Proof. Let $\varepsilon > 0$ be given. There is a $\delta > 0$ such that $\lvert f(x) - f(y) \rvert < \varepsilon$ whenever $x, y \in S$ and $\lvert x - y \rvert < \delta$. Find an $M \in \mathbb{N}$ such that for all $n, k \geq M$, we have $\lvert x_n - x_k \rvert < \delta$. Then for all $n, k \geq M$, we have $\lvert f(x_n) - f(x_k) \rvert < \varepsilon$. ☐

An application of the lemma above is the following extension result. It says that a function on an open interval is uniformly continuous if and only if it can be extended to a continuous function on the closed interval.

Proposition 3.4.6. *A function $f\colon (a,b) \to \mathbb{R}$ is uniformly continuous if and only if the limits*

$$L_a := \lim_{x\to a} f(x) \qquad \textit{and} \qquad L_b := \lim_{x\to b} f(x)$$

exist and the function $\widetilde{f}\colon [a,b] \to \mathbb{R}$ defined by

$$\widetilde{f}(x) := \begin{cases} f(x) & \textit{if } x \in (a,b), \\ L_a & \textit{if } x = a, \\ L_b & \textit{if } x = b, \end{cases}$$

is continuous.

Proof. One direction is not difficult. If $\widetilde{f}$ is continuous, then it is uniformly continuous by Theorem 3.4.4. As f is the restriction of $\widetilde{f}$ to (a,b), then f is also uniformly continuous (easy exercise).

Now suppose f is uniformly continuous. We must first show that the limits L_a and L_b exist. Let us concentrate on L_a. Take a sequence $\{x_n\}$ in (a,b) such that $\lim x_n = a$. The sequence $\{x_n\}$ is Cauchy, so by Lemma 3.4.5 the sequence $\{f(x_n)\}$ is Cauchy and thus convergent. We have some number $L_1 := \lim f(x_n)$. Take another sequence $\{y_n\}$ in (a,b) such that $\lim y_n = a$. By the same reasoning we get $L_2 := \lim f(y_n)$. If we show that $L_1 = L_2$, then the limit $L_a = \lim_{x\to a} f(x)$ exists. Let $\varepsilon > 0$ be given. Find $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \nicefrac{\varepsilon}{3}$. Find $M \in \mathbb{N}$ such that for $n \geq M$, we have $|a-x_n| < \nicefrac{\delta}{2}$, $|a-y_n| < \nicefrac{\delta}{2}$, $|f(x_n)-L_1| < \nicefrac{\varepsilon}{3}$, and $|f(y_n)-L_2| < \nicefrac{\varepsilon}{3}$. Then for $n \geq M$,

$$|x_n - y_n| = |x_n - a + a - y_n| \leq |x_n - a| + |a - y_n| < \nicefrac{\delta}{2} + \nicefrac{\delta}{2} = \delta.$$

So

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x_n) + f(x_n) - f(y_n) + f(y_n) - L_2| \\ &\leq |L_1 - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - L_2| \\ &\leq \nicefrac{\varepsilon}{3} + \nicefrac{\varepsilon}{3} + \nicefrac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Therefore, $L_1 = L_2$. Thus L_a exists. To show that L_b exists is left as an exercise.

Now that we know that the limits L_a and L_b exist, we are done. If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a} \widetilde{f}(x)$ exists (see Proposition 3.1.15). Similarly with L_b. Hence $\widetilde{f}$ is continuous at a and b. And since f is continuous at $c \in (a,b)$, then $\widetilde{f}$ is continuous at $c \in (a,b)$. $\square$

A common application of this proposition (together with Proposition 3.1.17) is the following. Suppose $f\colon (-1,0) \cup (0,1) \to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to 0} f(x)$ exists and the function has what is called an *removable singularity*, that is, we can extend the function to a continuous function on $(-1,1)$.

3.4.3 Lipschitz continuous functions

Definition 3.4.7. A function $f : S \to \mathbb{R}$ is *Lipschitz continuous*[*], if there exists a $K \in \mathbb{R}$, such that

$$\lvert f(x) - f(y) \rvert \leq K \lvert x - y \rvert \qquad \text{for all } x \text{ and } y \text{ in } S.$$

A large class of functions is Lipschitz continuous. Be careful, just as for uniformly continuous functions, the domain of definition of the function is important. See the examples below and the exercises. First, we justify the use of the word *continuous*.

Proposition 3.4.8. *A Lipschitz continuous function is uniformly continuous.*

Proof. Let $f : S \to \mathbb{R}$ be a function and let K be a constant such that $\lvert f(x) - f(y) \rvert \leq K \lvert x - y \rvert$ for all x, y in S. Let $\varepsilon > 0$ be given. Take $\delta := \varepsilon/K$. For all x and y in S such that $\lvert x - y \rvert < \delta$, we have

$$\lvert f(x) - f(y) \rvert \leq K \lvert x - y \rvert < K\delta = K \frac{\varepsilon}{K} = \varepsilon.$$

Therefore, f is uniformly continuous. □

We interpret Lipschitz continuity geometrically. Let f be a Lipschitz continuous function with some constant K. We rewrite the inequality to say that for $x \neq y$, we have

$$\left\lvert \frac{f(x) - f(y)}{x - y} \right\rvert \leq K.$$

The quantity $\frac{f(x)-f(y)}{x-y}$ is the slope of the line between the points $\bigl(x, f(x)\bigr)$ and $\bigl(y, f(y)\bigr)$, that is, a *secant line*. Therefore, f is Lipschitz continuous if and only if every line that intersects the graph of f in at least two distinct points has slope less than or equal to K. See Figure 3.7.

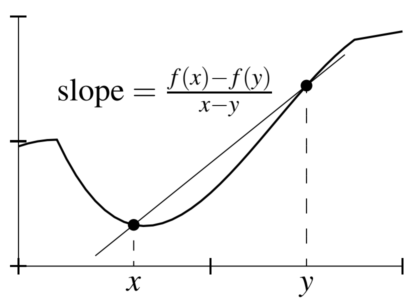

Figure 3.7: The slope of a secant line. A function is Lipschitz if $\lvert \text{slope} \rvert = \left\lvert \frac{f(x)-f(y)}{x-y} \right\rvert \leq K$ for all x and y.

Example 3.4.9: The functions $\sin(x)$ and $\cos(x)$ are Lipschitz continuous. In Example 3.2.6 we have seen the following two inequalities.

$$\lvert \sin(x) - \sin(y) \rvert \leq \lvert x - y \rvert \qquad \text{and} \qquad \lvert \cos(x) - \cos(y) \rvert \leq \lvert x - y \rvert .$$

Hence sine and cosine are Lipschitz continuous with $K = 1$.

[*]Named after the German mathematician Rudolf Otto Sigismund Lipschitz (1832–1903).

Example 3.4.10: The function $f\colon [1,\infty) \to \mathbb{R}$ defined by $f(x) := \sqrt{x}$ is Lipschitz continuous. Proof:

$$\left\lvert\sqrt{x}-\sqrt{y}\right\rvert = \left\lvert \frac{x-y}{\sqrt{x}+\sqrt{y}} \right\rvert = \frac{\lvert x-y\rvert}{\sqrt{x}+\sqrt{y}}.$$

As $x \geq 1$ and $y \geq 1$, we see that $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}$. Therefore,

$$\left\lvert\sqrt{x}-\sqrt{y}\right\rvert = \left\lvert \frac{x-y}{\sqrt{x}+\sqrt{y}} \right\rvert \leq \frac{1}{2}\lvert x-y\rvert .$$

On the other hand, $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) := \sqrt{x}$ is not Lipschitz continuous. Let us see why: Suppose

$$\left\lvert\sqrt{x}-\sqrt{y}\right\rvert \leq K\lvert x-y\rvert ,$$

for some K. Set $y = 0$ to obtain $\sqrt{x} \leq Kx$. If $K > 0$, then for $x > 0$ we then get $1/K \leq \sqrt{x}$. This cannot possibly be true for all $x > 0$. Thus no such $K > 0$ exists and f is not Lipschitz continuous.

The last example is a function that is uniformly continuous but not Lipschitz continuous. To see that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$, note that it is uniformly continuous on $[0,1]$ by Theorem 3.4.4. It is also Lipschitz (and therefore uniformly continuous) on $[1,\infty)$. It is not hard (exercise) to show that this means that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

3.4.4 Exercises

***Exercise* 3.4.1:** *Let $f\colon S \to \mathbb{R}$ be uniformly continuous. Let $A \subset S$. Then the restriction $f|_A$ is uniformly continuous.*

***Exercise* 3.4.2:** *Let $f\colon (a,b) \to \mathbb{R}$ be a uniformly continuous function. Finish the proof of Proposition 3.4.6 by showing that the limit $\lim\limits_{x\to b} f(x)$ exists.*

***Exercise* 3.4.3:** *Show that $f\colon (c,\infty) \to \mathbb{R}$ for some $c > 0$ and defined by $f(x) := 1/x$ is Lipschitz continuous.*

***Exercise* 3.4.4:** *Show that $f\colon (0,\infty) \to \mathbb{R}$ defined by $f(x) := 1/x$ is not Lipschitz continuous.*

***Exercise* 3.4.5:** *Let A, B be intervals. Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ be uniformly continuous functions such that $f(x) = g(x)$ for $x \in A \cap B$. Define the function $h\colon A \cup B \to \mathbb{R}$ by $h(x) := f(x)$ if $x \in A$ and $h(x) := g(x)$ if $x \in B \setminus A$.*

a) Prove that if $A \cap B \neq \emptyset$, then h is uniformly continuous.

b) Find an example where $A \cap B = \emptyset$ and h is not even continuous.

***Exercise* 3.4.6** (Challenging)**:** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a polynomial of degree $d \geq 2$. Show that f is not Lipschitz continuous.*

***Exercise* 3.4.7:** *Let $f\colon (0,1) \to \mathbb{R}$ be a bounded continuous function. Show that the function $g(x) := x(1-x)f(x)$ is uniformly continuous.*

***Exercise* 3.4.8:** *Show that $f\colon (0,\infty) \to \mathbb{R}$ defined by $f(x) := \sin(1/x)$ is not uniformly continuous.*

***Exercise* 3.4.9** (Challenging)**:** *Let $f\colon \mathbb{Q} \to \mathbb{R}$ be a uniformly continuous function. Show that there exists a uniformly continuous function $\widetilde{f}\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = \widetilde{f}(x)$ for all $x \in \mathbb{Q}$.*

Exercise* 3.4.10*:

a) Find a continuous $f\colon (0,1) \to \mathbb{R}$ and a sequence $\{x_n\}$ in $(0,1)$ that is Cauchy, but such that $\{f(x_n)\}$ is not Cauchy.

b) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

Exercise* 3.4.11*: *Prove:*

a) If $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ are uniformly continuous, then $h\colon S \to \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is uniformly continuous.

b) If $f\colon S \to \mathbb{R}$ is uniformly continuous and $a \in \mathbb{R}$, then $h\colon S \to \mathbb{R}$ given by $h(x) := af(x)$ is uniformly continuous.

Exercise* 3.4.12*: *Prove:*

a) If $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ are Lipschitz, then $h\colon S \to \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is Lipschitz.

b) If $f\colon S \to \mathbb{R}$ is Lipschitz and $a \in \mathbb{R}$, then $h\colon S \to \mathbb{R}$ given by $h(x) := af(x)$ is Lipschitz.

Exercise* 3.4.13*:

a) If $f\colon [0,1] \to \mathbb{R}$ is given by $f(x) := x^m$ for an integer $m \geq 0$, show f is Lipschitz and find the best (the smallest) Lipschitz constant K (depending on m of course). Hint: $(x-y)(x^{m-1} + x^{m-2}y + x^{m-3}y^2 + \cdots + xy^{m-2} + y^{m-1}) = x^m - y^m$.

b) Using the previous exercise, show that if $f\colon [0,1] \to \mathbb{R}$ is a polynomial, that is, $f(x) := a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$, then f is Lipschitz.

Exercise* 3.4.14*: *Suppose for $f\colon [0,1] \to \mathbb{R}$, we have $|f(x) - f(y)| \leq K\,|x-y|$ for all x,y in $[0,1]$, and $f(0) = f(1) = 0$. Prove that $|f(x)| \leq K/2$ for all $x \in [0,1]$. Further show by example that $K/2$ is the best possible, that is, there exists such a continuous function for which $|f(x)| = K/2$ for some $x \in [0,1]$.*

Exercise* 3.4.15*: *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period $P > 0$. That is, $f(x+P) = f(x)$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous.*

Exercise* 3.4.16*: *Suppose $f\colon S \to \mathbb{R}$ and $g\colon [0,\infty) \to [0,\infty)$ are functions, g is continuous at 0, $g(0) = 0$, and whenever x and y are in S, we have $|f(x) - f(y)| \leq g\bigl(|x-y|\bigr)$. Prove that f is uniformly continuous.*

Exercise* 3.4.17*: *Suppose $f\colon [a,b] \to \mathbb{R}$ is a function such that for every $c \in [a,b]$ there is a $K_c > 0$ and an $\varepsilon_c > 0$ for which $|f(x) - f(y)| \leq K_c\,|x-y|$ for all x and y in $(c - \varepsilon_c, c + \varepsilon_c) \cap [a,b]$. In other words, f is "locally Lipschitz."*

a) Prove that there exists a single $K > 0$ such that $|f(x) - f(y)| \leq K\,|x-y|$ for all x,y in $[a,b]$.

b) Find a counterexample to the above if the interval is open, that is, find an $f\colon (a,b) \to \mathbb{R}$ that is locally Lipschitz, but not Lipschitz.

3.5 Limits at infinity

Note: less than 1 lecture (optional, can safely be omitted unless §3.6 or §5.5 is also covered)

3.5.1 Limits at infinity

As for sequences, a continuous variable can also approach infinity. Let us make this notion precise.

Definition 3.5.1. We say ∞ is a *cluster point* of $S \subset \mathbb{R}$ if for every $M \in \mathbb{R}$, there exists an $x \in S$ such that $x \geq M$. Similarly, $-\infty$ is a *cluster point* of $S \subset \mathbb{R}$ if for every $M \in \mathbb{R}$, there exists an $x \in S$ such that $x \leq M$.

Let $f : S \to \mathbb{R}$ be a function, where ∞ is a cluster point of S. If there exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that

$$|f(x) - L| < \varepsilon$$

whenever $x \in S$ and $x \geq M$, then we say $f(x)$ *converges* to L as x goes to ∞. We call L the *limit* and write

$$\lim_{x \to \infty} f(x) := L.$$

Alternatively we write $f(x) \to L$ as $x \to \infty$.

Similarly, if $-\infty$ is a cluster point of S and there exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that

$$|f(x) - L| < \varepsilon$$

whenever $x \in S$ and $x \leq M$, then we say $f(x)$ *converges* to L as x goes to $-\infty$. We call L the *limit* and write

$$\lim_{x \to -\infty} f(x) := L.$$

Alternatively we write $f(x) \to L$ as $x \to -\infty$.

We cheated a little bit again and said *the* limit. We leave it as an exercise for the reader to prove the following proposition.

Proposition 3.5.2. *The limit at ∞ or $-\infty$ as defined above is unique if it exists.*

Example 3.5.3: Let $f(x) := \frac{1}{|x|+1}$. Then

$$\lim_{x \to \infty} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = 0.$$

Proof: Let $\varepsilon > 0$ be given. Find $M > 0$ large enough so that $\frac{1}{M+1} < \varepsilon$. If $x \geq M$, then $\frac{1}{x+1} \leq \frac{1}{M+1} < \varepsilon$. Since $\frac{1}{|x|+1} > 0$ for all x the first limit is proved. The proof for $-\infty$ is left to the reader.

Example 3.5.4: Let $f(x) := \sin(\pi x)$. Then $\lim_{x\to\infty} f(x)$ does not exist. To prove this fact note that if $x = 2n + 1/2$ for some $n \in \mathbb{N}$, then $f(x) = 1$, while if $x = 2n + 3/2$, then $f(x) = -1$. So they cannot both be within a small ε of a single real number.

We must be careful not to confuse continuous limits with limits of sequences. We could say

$$\lim_{n\to\infty} \sin(\pi n) = 0, \qquad \text{but} \qquad \lim_{x\to\infty} \sin(\pi x) \text{ does not exist.}$$

Of course the notation is ambiguous: Are we thinking of the sequence $\{\sin(\pi n)\}_{n=1}^{\infty}$ or the function $\sin(\pi x)$ of a real variable? We are simply using the convention that $n \in \mathbb{N}$, while $x \in \mathbb{R}$. When the notation is not clear, it is good to explicitly mention where the variable lives, or what kind of limit are you using. If there is possibility of confusion, one can write, for example,

$$\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \sin(\pi n).$$

There is a connection of continuous limits to limits of sequences, but we must take all sequences going to infinity, just as before in Lemma 3.1.7.

Lemma 3.5.5. *Suppose $f \colon S \to \mathbb{R}$ is a function, ∞ is a cluster point of $S \subset \mathbb{R}$, and $L \in \mathbb{R}$. Then*

$$\lim_{x\to\infty} f(x) = L$$

if and only if

$$\lim_{n\to\infty} f(x_n) = L$$

for all sequences $\{x_n\}$ in S such that $\lim\limits_{n\to\infty} x_n = \infty$.

The lemma holds for the limit as $x \to -\infty$. Its proof is almost identical and is left as an exercise.

Proof. First suppose $f(x) \to L$ as $x \to \infty$. Given an $\varepsilon > 0$, there exists an M such that for all $x \geq M$, we have $|f(x) - L| < \varepsilon$. Let $\{x_n\}$ be a sequence in S such that $\lim x_n = \infty$. Then there exists an N such that for all $n \geq N$, we have $x_n \geq M$. And thus $|f(x_n) - L| < \varepsilon$.

We prove the converse by contrapositive. Suppose $f(x)$ does not go to L as $x \to \infty$. This means that there exists an $\varepsilon > 0$, such that for every $n \in \mathbb{N}$, there exists an $x \in S$, $x \geq n$, let us call it x_n, such that $|f(x_n) - L| \geq \varepsilon$. Consider the sequence $\{x_n\}$. Clearly $\{f(x_n)\}$ does not converge to L. It remains to note that $\lim x_n = \infty$, because $x_n \geq n$ for all n. $\square$

Using the lemma, we again translate results about sequential limits into results about continuous limits as x goes to infinity. That is, we have almost immediate analogues of the corollaries in §3.1.3. We simply allow the cluster point c to be either ∞ or $-\infty$, in addition to a real number. We leave it to the student to verify these statements.

3.5.2 Infinite limit

Just as for sequences, it is often convenient to distinguish certain divergent sequences, and talk about limits being infinite almost as if the limits existed.

Definition 3.5.6. Let $f: S \to \mathbb{R}$ be a function and suppose S has ∞ as a cluster point. We say $f(x)$ *diverges to infinity* as x goes to ∞, if for every $N \in \mathbb{R}$ there exists an $M \in \mathbb{R}$ such that

$$f(x) > N$$

whenever $x \in S$ and $x \geq M$. We write

$$\lim_{x \to \infty} f(x) := \infty,$$

or we say that $f(x) \to \infty$ as $x \to \infty$.

A similar definition can be made for limits as $x \to -\infty$ or as $x \to c$ for a finite c. Also similar definitions can be made for limits being $-\infty$. Stating these definitions is left as an exercise. Note that sometimes *converges to infinity* is used. We can again use sequential limits, and an analogue of Lemma 3.1.7 is left as an exercise.

Example 3.5.7: Let us show that $\lim_{x\to\infty} \frac{1+x^2}{1+x} = \infty$.

Proof: For $x \geq 1$, we have

$$\frac{1+x^2}{1+x} \geq \frac{x^2}{x+x} = \frac{x}{2}.$$

Given $N \in \mathbb{R}$, take $M = \max\{2N+1, 1\}$. If $x \geq M$, then $x \geq 1$ and $x/2 > N$. So

$$\frac{1+x^2}{1+x} \geq \frac{x}{2} > N.$$

3.5.3 Compositions

Finally, just as for limits at finite numbers we can compose functions easily.

Proposition 3.5.8. *Suppose $f: A \to B$, $g: B \to \mathbb{R}$, $A, B \subset \mathbb{R}$, $a \in \mathbb{R} \cup \{-\infty, \infty\}$ is a cluster point of A, and $b \in \mathbb{R} \cup \{-\infty, \infty\}$ is a cluster point of B. Suppose*

$$\lim_{x \to a} f(x) = b \qquad \textit{and} \qquad \lim_{y \to b} g(y) = c$$

for some $c \in \mathbb{R} \cup \{-\infty, \infty\}$. If $b \in B$, then suppose $g(b) = c$. Then

$$\lim_{x \to a} g\big(f(x)\big) = c.$$

The proof is straightforward, and left as an exercise. We already know the proposition when $a, b, c \in \mathbb{R}$, see Exercises 3.1.9 and 3.1.14. Again the requirement that g is continuous at b, if $b \in B$, is necessary.

Example 3.5.9: Let $h(x) := e^{-x^2+x}$. Then

$$\lim_{x \to \infty} h(x) = 0.$$

Proof: The claim follows once we know

$$\lim_{x \to \infty} -x^2 + x = -\infty$$

and

$$\lim_{y \to -\infty} e^y = 0,$$

which is usually proved when the exponential function is defined.

3.5.4 Exercises

Exercise* 3.5.1*: *Prove Proposition 3.5.2.*

Exercise* 3.5.2*: *Let $f : [1,\infty) \to \mathbb{R}$ be a function. Define $g : (0,1] \to \mathbb{R}$ via $g(x) := f(1/x)$. Using the definitions of limits directly, show that $\lim_{x\to 0^+} g(x)$ exists if and only if $\lim_{x\to\infty} f(x)$ exists, in which case they are equal.*

Exercise* 3.5.3*: *Prove Proposition 3.5.8.*

Exercise* 3.5.4*: *Let us justify terminology. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to\infty} f(x) = \infty$ (diverges to infinity). Show that $f(x)$ diverges (i.e. does not converge) as $x \to \infty$.*

Exercise* 3.5.5*: *Come up with the definitions for limits of $f(x)$ going to $-\infty$ as $x \to \infty$, $x \to -\infty$, and as $x \to c$ for a finite $c \in \mathbb{R}$. Then state the definitions for limits of $f(x)$ going to ∞ as $x \to -\infty$, and as $x \to c$ for a finite $c \in \mathbb{R}$.*

Exercise* 3.5.6*: *Suppose $P(x) := x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a* monic polynomial *of degree $n \geq 1$ (monic means that the coefficient of x^n is 1).*

a) Show that if n is even, then $\lim_{x\to\infty} P(x) = \lim_{x\to-\infty} P(x) = \infty$.

b) Show that if n is odd, then $\lim_{x\to\infty} P(x) = \infty$ and $\lim_{x\to-\infty} P(x) = -\infty$ (see previous exercise).

Exercise* 3.5.7*: *Let $\{x_n\}$ be a sequence. Consider $S := \mathbb{N} \subset \mathbb{R}$, and $f : S \to \mathbb{R}$ defined by $f(n) := x_n$. Show that the two notions of limit,*

$$\lim_{n\to\infty} x_n \qquad \textit{and} \qquad \lim_{x\to\infty} f(x)$$

are equivalent. That is, show that if one exists so does the other one, and in this case they are equal.

Exercise* 3.5.8*: *Extend Lemma 3.5.5 as follows. Suppose $S \subset \mathbb{R}$ has a cluster point $c \in \mathbb{R}$, $c = \infty$, or $c = -\infty$. Let $f : S \to \mathbb{R}$ be a function and suppose $L = \infty$ or $L = -\infty$. Show that*

$$\lim_{x\to c} f(x) = L \qquad \textit{if and only if} \qquad \lim_{n\to\infty} f(x_n) = L \ \textit{ for all sequences } \{x_n\} \textit{ such that } \lim x_n = c.$$

Exercise* 3.5.9*: *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a 2-periodic function, that is $f(x+2) = f(x)$ for all x. Define $g : \mathbb{R} \to \mathbb{R}$ by*

$$g(x) := f\left(\frac{\sqrt{x^2+1}-1}{x}\right)$$

a) Find the function $\varphi : (-1,1) \to \mathbb{R}$ such that $g\big(\varphi(t)\big) = f(t)$, that is $\varphi^{-1}(x) = \frac{\sqrt{x^2+1}-1}{x}$.

b) Show that f is continuous if and only if g is continuous and

$$\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = f(1) = f(-1).$$

3.6 Monotone functions and continuity

Note: 1 lecture (optional, can safely be omitted unless §4.4 is also covered, requires §3.5)

Definition 3.6.1. Let $S \subset \mathbb{R}$. We say $f \colon S \to \mathbb{R}$ is *increasing* (resp. *strictly increasing*) if $x, y \in S$ with $x < y$ implies $f(x) \leq f(y)$ (resp. $f(x) < f(y)$). We define *decreasing* and *strictly decreasing* in the same way by switching the inequalities for f.

If a function is either increasing or decreasing, we say it is *monotone*. If it is strictly increasing or strictly decreasing, we say it is *strictly monotone*.

Sometimes *nondecreasing* (resp. *nonincreasing*) is used for increasing (resp. decreasing) function to emphasize it is not strictly increasing (resp. strictly decreasing).

If f is increasing, then $-f$ is decreasing and vice versa. Therefore, many results about monotone functions can just be proved for, say, increasing functions, and the results follow easily for decreasing functions.

3.6.1 Continuity of monotone functions

One-sided limits for monotone functions are computed by computing infima and suprema.

Proposition 3.6.2. *Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, $f \colon S \to \mathbb{R}$ be increasing, and $g \colon S \to \mathbb{R}$ be decreasing. If c is a cluster point of $S \cap (-\infty, c)$, then*

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x < c, x \in S\} \qquad \text{and} \qquad \lim_{x \to c^-} g(x) = \inf\{g(x) : x < c, x \in S\}.$$

If c is a cluster point of $S \cap (c, \infty)$, then

$$\lim_{x \to c^+} f(x) = \inf\{f(x) : x > c, x \in S\} \qquad \text{and} \qquad \lim_{x \to c^+} g(x) = \sup\{g(x) : x > c, x \in S\}.$$

If ∞ is a cluster point of S, then

$$\lim_{x \to \infty} f(x) = \sup\{f(x) : x \in S\} \qquad \text{and} \qquad \lim_{x \to \infty} g(x) = \inf\{g(x) : x \in S\}.$$

If $-\infty$ is a cluster point of S, then

$$\lim_{x \to -\infty} f(x) = \inf\{f(x) : x \in S\} \qquad \text{and} \qquad \lim_{x \to -\infty} g(x) = \sup\{g(x) : x \in S\}.$$

Namely, all the one-sided limits exist whenever they make sense. For monotone functions therefore, when we say the left-hand limit $x \to c^-$ exists, we mean that c is a cluster point of $S \cap (-\infty, c)$, and same for the right-hand limit.

Proof. Let us assume f is increasing, and we will show the first equality. The rest of the proof is very similar and is left as an exercise.

Let $a := \sup\{f(x) : x < c, x \in S\}$. If $a = \infty$, then given an $M \in \mathbb{R}$, there exists an $x_M \in S$, $x_M < c$, such that $f(x_M) > M$. As f is increasing, $f(x) \geq f(x_M) > M$ for all $x \in S$ with $x > x_M$. If we take $\delta := c - x_M > 0$, then we obtain the definition of the limit going to infinity.

Next suppose $a < \infty$. Let $\varepsilon > 0$ be given. Because a is the supremum and $S \cap (-\infty, c)$ is nonempty, $a \in \mathbb{R}$ and there exists an $x_\varepsilon \in S$, $x_\varepsilon < c$, such that $f(x_\varepsilon) > a - \varepsilon$. As f is increasing, if $x \in S$ and $x_\varepsilon < x < c$, we have $a - \varepsilon < f(x_\varepsilon) \leq f(x) \leq a$. Let $\delta := c - x_\varepsilon$. Then for $x \in S \cap (-\infty, c)$ with $|x - c| < \delta$, we have $|f(x) - a| < \varepsilon$. $\square$

Suppose $f\colon S \to \mathbb{R}$ is increasing, $c \in S$, and that both one-sided limits exist. Since $f(x) \leq f(c) \leq f(y)$ whenever $x < c < y$, taking the limits we obtain

$$\lim_{x \to c^-} f(x) \leq f(c) \leq \lim_{x \to c^+} f(x).$$

Then f is continuous at c if and only if both limits are equal to each other (and hence equal to $f(c)$). See also Proposition 3.1.17. See Figure 3.8 to get an idea of a what a discontinuity looks like.

Corollary 3.6.3. *If $I \subset \mathbb{R}$ is an interval and $f\colon I \to \mathbb{R}$ is monotone and not constant, then $f(I)$ is an interval if and only if f is continuous.*

Assuming f is not constant is to avoid the technicality that $f(I)$ is a single point: $f(I)$ is a single point if and only if f is constant. A constant function is continuous.

Proof. Without loss of generality, suppose f is increasing.

First suppose f is continuous. Take two points $f(x_1) < f(x_2)$ in $f(I)$. As f is increasing, then $x_1 < x_2$. By the intermediate value theorem, given y with $f(x_1) < y < f(x_2)$, we find a $c \in (x_1, x_2) \subset I$ such that $f(c) = y$, so $y \in f(I)$. Hence, $f(I)$ is an interval.

Let us prove the reverse direction by contrapositive. Suppose f is not continuous at $c \in I$, and that c is not an endpoint of I. Let

$$a := \lim_{x \to c^-} f(x) = \sup\bigl\{f(x) : x \in I, x < c\bigr\}, \qquad b := \lim_{x \to c^+} f(x) = \inf\bigl\{f(x) : x \in I, x > c\bigr\}.$$

As c is a discontinuity, $a < b$. If $x < c$, then $f(x) \leq a$, and if $x > c$, then $f(x) \geq b$. Therefore no point in $(a,b) \setminus \{f(c)\}$ is in $f(I)$. However there exists $x_1 \in I$, $x_1 < c$, so $f(x_1) \leq a$, and there exists $x_2 \in I$, $x_2 > c$, so $f(x_2) \geq b$. Both $f(x_1)$ and $f(x_2)$ are in $f(I)$, but there are points in between them that are not in $f(I)$. So $f(I)$ is not an interval. See Figure 3.8.

When $c \in I$ is an endpoint, the proof is similar and is left as an exercise. $\square$

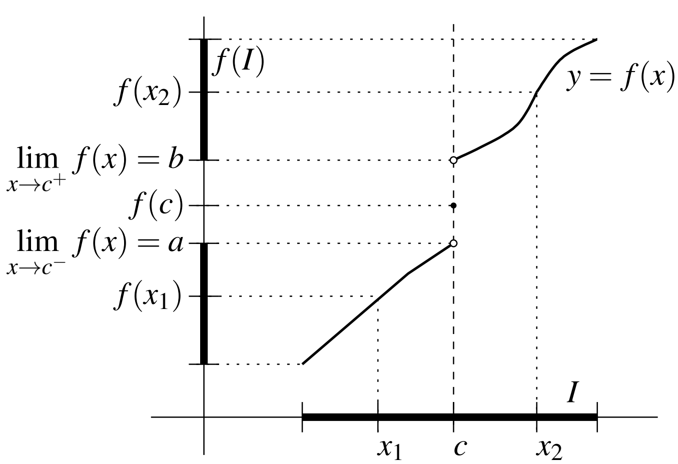

Figure 3.8: Increasing function $f\colon I \to \mathbb{R}$ discontinuity at c.

A striking property of monotone functions is that they cannot have too many discontinuities.

Corollary 3.6.4. *Let $I \subset \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ be monotone. Then f has at most countably many discontinuities.*

Proof. Let $E \subset I$ be the set of all discontinuities that are not endpoints of I. As there are only two endpoints, it is enough to show that E is countable. Without loss of generality, suppose f is increasing. We will define an injection $h: E \to \mathbb{Q}$. For each $c \in E$ the one-sided limits of f both exist as c is not an endpoint. Let

$$a := \lim_{x \to c^-} f(x) = \sup\{f(x) : x \in I, x < c\}, \qquad b := \lim_{x \to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}.$$

As c is a discontinuity, we have $a < b$. There exists a rational number $q \in (a,b)$, so let $h(c) := q$. If $d \in E$ is another discontinuity, then if $d > c$, then there exist an $x \in I$ with $c < x < d$, and so $\lim_{x \to d^-} f(x) \geq b$. Hence the rational number we choose for $h(d)$ is different from q, since $q = h(c) < b$ and $h(d) > b$. Similarly if $d < c$. So after making such a choice for every $c \in E$, we have a one-to-one (injective) function into $\mathbb{Q}$. Therefore, E is countable. $\square$

Example 3.6.5: By $\lfloor x \rfloor$ denote the largest integer less than or equal to x. Define $f: [0,1] \to \mathbb{R}$ by

$$f(x) := x + \sum_{n=0}^{\lfloor 1/(1-x) \rfloor} 2^{-n},$$

for $x < 1$ and $f(1) := 3$. It is left as an exercise to show that f is strictly increasing, bounded, and has a discontinuity at all points $1 - 1/k$ for $k \in \mathbb{N}$. In particular, there are countably many discontinuities, but the function is bounded and defined on a closed bounded interval. See Figure 3.9.

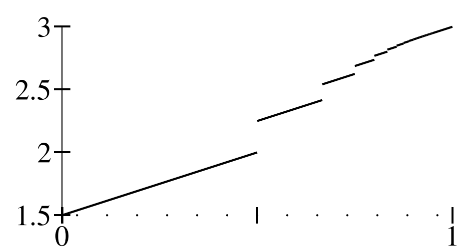

Figure 3.9: Increasing function with countably many discontinuities.

Similarly, one can find an example of a function discontinuous on a dense set such as the rational numbers. See the exercises.

3.6.2 Continuity of inverse functions

A strictly monotone function f is one-to-one (injective). To see this fact, notice that if $x \neq y$, then we can assume $x < y$. Either $f(x) < f(y)$ if f is strictly increasing or $f(x) > f(y)$ if f is strictly decreasing, so $f(x) \neq f(y)$. Hence, f must have an inverse f^{-1} defined on its range.

Proposition 3.6.6. *If $I \subset \mathbb{R}$ is an interval and $f: I \to \mathbb{R}$ is strictly monotone, then the inverse $f^{-1}: f(I) \to I$ is continuous.*

Proof. Let us suppose f is strictly increasing. The proof is almost identical for a strictly decreasing function. Since f is strictly increasing, so is f^{-1}. That is, if $f(x) < f(y)$, then we must have $x < y$ and therefore $f^{-1}\big(f(x)\big) < f^{-1}\big(f(y)\big)$.

Take $c \in f(I)$. If c is not a cluster point of $f(I)$, then f^{-1} is continuous at c automatically. So let c be a cluster point of $f(I)$. Suppose both of the following one-sided limits exist:

$$x_0 := \lim_{y \to c^-} f^{-1}(y) = \sup\big\{f^{-1}(y) : y < c, y \in f(I)\big\} = \sup\big\{x \in I : f(x) < c\big\},$$
$$x_1 := \lim_{y \to c^+} f^{-1}(y) = \inf\big\{f^{-1}(y) : y > c, y \in f(I)\big\} = \inf\big\{x \in I : f(x) > c\big\}.$$

We have $x_0 \leq x_1$ as f^{-1} is increasing. For all $x \in I$ where $x > x_0$, we have $f(x) \geq c$. As f is strictly increasing, we must have $f(x) > c$ for all $x \in I$ where $x > x_0$. Therefore,

$$\{x \in I : x > x_0\} \subset \big\{x \in I : f(x) > c\big\}.$$

The infimum of the left-hand set is x_0, and the infimum of the right-hand set is x_1, so we obtain $x_0 \geq x_1$. So $x_1 = x_0$, and f^{-1} is continuous at c.

If one of the one-sided limits does not exist, the argument is similar and is left as an exercise. $\square$

Example 3.6.7: The proposition does not require f itself to be continuous. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x < 0, \\ x+1 & \text{if } x \geq 0. \end{cases}$$

The function f is not continuous at 0. The image of $I = \mathbb{R}$ is the set $(-\infty, 0) \cup [1, \infty)$, not an interval. Then $f^{-1}\colon (-\infty, 0) \cup [1, \infty) \to \mathbb{R}$ can be written as

$$f^{-1}(y) = \begin{cases} y & \text{if } y < 0, \\ y-1 & \text{if } y \geq 1. \end{cases}$$

It is not difficult to see that f^{-1} is a continuous function. See Figure 3.10 for the graphs.

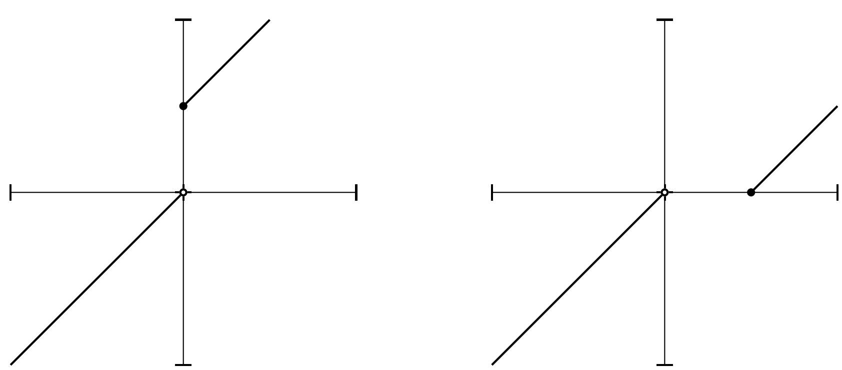

Figure 3.10: Graph of f on the left and f^{-1} on the right.

Notice what happens with the proposition if $f(I)$ is an interval. In that case, we could simply apply Corollary 3.6.3 to both f and f^{-1}. That is, if $f: I \to J$ is an onto strictly monotone function and I and J are intervals, then both f and f^{-1} are continuous. Furthermore, $f(I)$ is an interval precisely when f is continuous.

3.6.3 Exercises

Exercise* 3.6.1*: *Suppose $f: [0,1] \to \mathbb{R}$ is monotone. Prove f is bounded.*

Exercise* 3.6.2*: *Finish the proof of Proposition 3.6.2. Hint: You can halve your work by noticing that if g is decreasing, then $-g$ is increasing.*

Exercise* 3.6.3*: *Finish the proof of Corollary 3.6.3.*

Exercise* 3.6.4*: *Prove the claims in Example 3.6.5.*

Exercise* 3.6.5*: *Finish the proof of Proposition 3.6.6.*

Exercise* 3.6.6*: *Suppose $S \subset \mathbb{R}$, and $f: S \to \mathbb{R}$ is an increasing function. Prove:*

a) *If c is a cluster point of $S \cap (c,\infty)$, then $\lim\limits_{x \to c^+} f(x) < \infty$.*

b) *If c is a cluster point of $S \cap (-\infty,c)$ and $\lim\limits_{x \to c^-} f(x) = \infty$, then $S \subset (-\infty,c)$.*

Exercise* 3.6.7*: *Let $I \subset \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ a function. Suppose that for each $c \in I$, there exist $a,b \in \mathbb{R}$ with $a > 0$ such that $f(x) \geq ax+b$ for all $x \in I$ and $f(c) = ac+b$. Show that f is strictly increasing.*

Exercise* 3.6.8*: *Suppose I and J are intervals and $f: I \to J$ is a continuous, bijective (one-to-one and onto) function. Show that f is strictly monotone.*

Exercise* 3.6.9*: *Consider a monotone function $f: I \to \mathbb{R}$ on an interval I. Prove that there exists a function $g: I \to \mathbb{R}$ such that $\lim\limits_{x \to c^-} g(x) = g(c)$ for all c in I except the smaller (left) endpoint of I, and such that $g(x) = f(x)$ for all but countably many $x \in I$.*

Exercise* 3.6.10*:

a) *Let $S \subset \mathbb{R}$ be a subset. If $f: S \to \mathbb{R}$ is increasing and bounded, then show that there exists an increasing $F: \mathbb{R} \to \mathbb{R}$ such that $f(x) = F(x)$ for all $x \in S$.*

b) *Find an example of a strictly increasing bounded $f: S \to \mathbb{R}$ such that an increasing F as above is never strictly increasing.*

Exercise* 3.6.11** (Challenging):*** *Find an example of an increasing function $f: [0,1] \to \mathbb{R}$ that has a discontinuity at each rational number. Then show that the image $f([0,1])$ contains no interval. Hint: Enumerate the rational numbers and define the function with a series.*

Exercise* 3.6.12*: *Suppose I is an interval and $f: I \to \mathbb{R}$ is monotone. Show that $\mathbb{R} \setminus f(I)$ is a countable union of disjoint intervals.*

Exercise* 3.6.13*: *Suppose $f: [0,1] \to (0,1)$ is increasing. Show that for every $\varepsilon > 0$, there exists a strictly increasing $g: [0,1] \to (0,1)$ such that $g(0) = f(0)$, $f(x) \leq g(x)$ for all x, and $g(1) - f(1) < \varepsilon$.*

Exercise* 3.6.14*: *Prove that the Dirichlet function $f: [0,1] \to \mathbb{R}$ defined by $f(x) := 1$ if x is rational and $f(x) := 0$ otherwise cannot be written as a difference of two increasing functions. That is, there do not exist increasing g and h such that, $f(x) = g(x) - h(x)$.*

Exercise* 3.6.15*: *Suppose $f: (a,b) \to (c,d)$ is a strictly increasing onto function. Prove that there exists a $g: (a,b) \to (c,d)$, which is also strictly increasing and onto, and $g(x) < f(x)$ for all $x \in (a,b)$.*

Chapter 4

The Derivative

4.1 The derivative

Note: 1 lecture

The idea of a derivative is the following. If the graph of a function looks locally like a straight line, then we can then talk about the slope of this line. The slope tells us the rate at which the value of the function is changing at that particular point. Of course, we are leaving out any function that has corners or discontinuities. Let us be precise.

4.1.1 Definition and basic properties

Definition 4.1.1. Let I be an interval, let $f \colon I \to \mathbb{R}$ be a function, and let $c \in I$. If the limit

$$L := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists, then we say f is *differentiable* at c, that L is the *derivative* of f at c, and write $f'(c) := L$.

If f is differentiable at all $c \in I$, then we simply say that f is *differentiable*, and then we obtain a function $f' \colon I \to \mathbb{R}$. The derivative is sometimes written as $\frac{df}{dx}$ or $\frac{d}{dx}\bigl(f(x)\bigr)$.

The expression $\frac{f(x)-f(c)}{x-c}$ is called the *difference quotient*.

The graphical interpretation of the derivative is depicted in Figure 4.1. The left-hand plot gives the line through $\bigl(c, f(c)\bigr)$ and $\bigl(x, f(x)\bigr)$ with slope $\frac{f(x)-f(c)}{x-c}$, that is, the so-called *secant line*. When we take the limit as x goes to c, we get the right-hand plot, where we see that the derivative of the function at the point c is the slope of the line tangent to the graph of f at the point $\bigl(c, f(c)\bigr)$.

We allow I to be a closed interval and we allow c to be an endpoint of I. Some calculus books do not allow c to be an endpoint of an interval, but all the theory still works by allowing it, and it makes our work easier.

Example 4.1.2: Let $f(x) := x^2$ defined on the whole real line. Let $c \in \mathbb{R}$ be arbitrary. We find that if $x \neq c$,

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

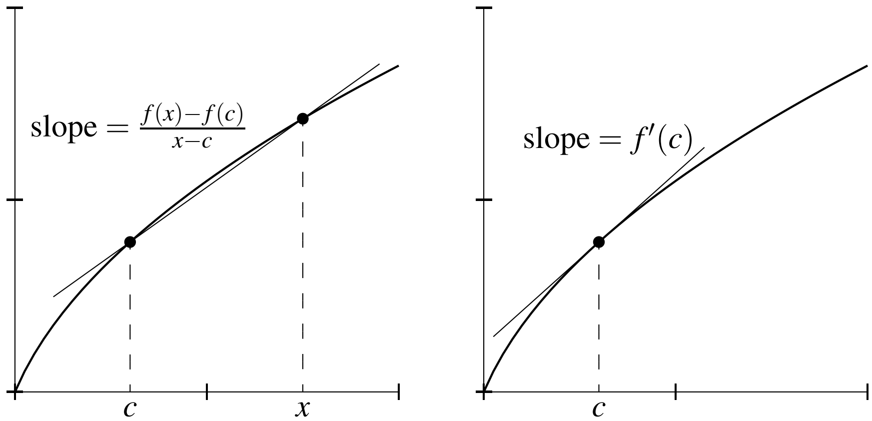

Figure 4.1: Graphical interpretation of the derivative.

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c.$$

Example 4.1.3: Let $f(x) := ax + b$ for numbers $a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$ be arbitrary. For $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} = \frac{a(x - c)}{x - c} = a.$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} a = a.$$

In fact, every differentiable function "infinitesimally" behaves like the affine function $ax + b$. You can guess many results and formulas for derivatives, if you work them out for affine functions first.

Example 4.1.4: The function $f(x) := \sqrt{x}$ is differentiable for $x > 0$. To see this fact, fix $c > 0$, and take $x \neq c$, $x > 0$. Compute

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \frac{1}{\sqrt{x} + \sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} = \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

Example 4.1.5: The function $f(x) := |x|$ is not differentiable at the origin. When $x > 0$,

$$\frac{|x| - |0|}{x - 0} = \frac{x - 0}{x - 0} = 1.$$

When $x < 0$,

$$\frac{|x| - |0|}{x - 0} = \frac{-x - 0}{x - 0} = -1.$$

A famous example of Weierstrass shows that there exists a continuous function that is not differentiable at *any* point. The construction of this function is beyond the scope of this chapter. On the other hand, a differentiable function is always continuous.

Proposition 4.1.6. *Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.*

Proof. We know the limits

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) \qquad \text{and} \qquad \lim_{x\to c}(x-c) = 0$$

exist. Furthermore,

$$f(x)-f(c) = \left(\frac{f(x)-f(c)}{x-c}\right)(x-c).$$

Therefore, the limit of $f(x)-f(c)$ exists and

$$\lim_{x\to c}\bigl(f(x)-f(c)\bigr) = \left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x\to c}(x-c)\right) = f'(c)\cdot 0 = 0.$$

Hence $\lim\limits_{x\to c} f(x) = f(c)$, and f is continuous at c. □

An important property of the derivative is linearity. The derivative is the approximation of a function by a straight line. The slope of a line through two points changes linearly when the y-coordinates are changed linearly. By taking the limit, it makes sense that the derivative is linear.

Proposition 4.1.7. *Let I be an interval, let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable at $c \in I$, and let $\alpha \in \mathbb{R}$.*

(i) Define $h : I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.

(ii) Define $h : I \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.

Proof. First, let $h(x) := \alpha f(x)$. For $x \in I$, $x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = \frac{\alpha f(x)-\alpha f(c)}{x-c} = \alpha \frac{f(x)-f(c)}{x-c}.$$

The limit as x goes to c exists on the right-hand side by Corollary 3.1.12. We get

$$\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \alpha \lim_{x\to c} \frac{f(x)-f(c)}{x-c}.$$

Therefore, h is differentiable at c, and the derivative is computed as given.

Next, define $h(x) := f(x) + g(x)$. For $x \in I$, $x \neq c$, we have

$$\frac{h(x)-h(c)}{x-c} = \frac{\bigl(f(x)+g(x)\bigr)-\bigl(f(c)+g(c)\bigr)}{x-c} = \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}.$$

The limit as x goes to c exists on the right-hand side by Corollary 3.1.12. We get

$$\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c}.$$

Therefore, h is differentiable at c, and the derivative is computed as given. □

It is not true that the derivative of a multiple of two functions is the multiple of the derivatives. Instead we get the so-called *product rule* or the *Leibniz rule**.

Proposition 4.1.8 (Product rule)**.** *Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be functions differentiable at c. If $h\colon I \to \mathbb{R}$ is defined by*

$$h(x) := f(x)g(x),$$

then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c).$$

The proof of the product rule is left as an exercise. The key to the proof is the identity $f(x)g(x) - f(c)g(c) = f(x)\bigl(g(x) - g(c)\bigr) + \bigl(f(x) - f(c)\bigr)g(c)$, which is illustrated in Figure 4.2.

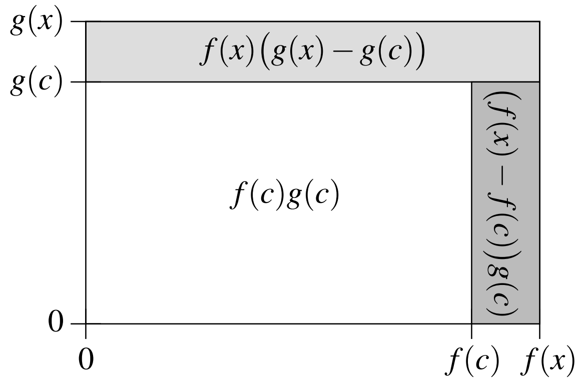

Figure 4.2: The idea of product rule. The area of the entire rectangle $f(x)g(x)$ differs from the area of the white rectangle $f(c)g(c)$ by the area of the lightly shaded rectangle $f(x)\bigl(g(x) - g(c)\bigr)$ plus the darker rectangle $\bigl(f(x) - f(c)\bigr)g(c)$. In other words, $\Delta(f \cdot g) = f \cdot \Delta g + \Delta f \cdot g$.

Proposition 4.1.9 (Quotient Rule)**.** *Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be differentiable at c and $g(x) \neq 0$ for all $x \in I$. If $h\colon I \to \mathbb{R}$ is defined by*

$$h(x) := \frac{f(x)}{g(x)},$$

then h is differentiable at c and

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{\bigl(g(c)\bigr)^2}.$$

Again, the proof is left as an exercise.

*Named for the German mathematician Gottfried Wilhelm Leibniz (1646–1716).

4.1.2 Chain rule

More complicated functions are often obtained by composition, which is differentiated via the chain rule. The rule also tells us how a derivative changes if we change variables.

Proposition 4.1.10 (Chain Rule)**.** *Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by*

$$h(x) := (f \circ g)(x) = f\bigl(g(x)\bigr),$$

then h is differentiable at c and

$$h'(c) = f'\bigl(g(c)\bigr)g'(c).$$

Proof. Let $d := g(c)$. Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

Because f is differentiable at $d = g(c)$, we find that u is continuous at d. Similarly, v is continuous at c. For any x and y,

$$f(y) - f(d) = u(y)(y-d) \qquad \text{and} \qquad g(x) - g(c) = v(x)(x-c).$$

Plug in to obtain

$$h(x) - h(c) = f\bigl(g(x)\bigr) - f\bigl(g(c)\bigr) = u\bigl(g(x)\bigr)\bigl(g(x)-g(c)\bigr) = u\bigl(g(x)\bigr)\bigl(v(x)(x-c)\bigr).$$

Therefore, if $x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = u\bigl(g(x)\bigr)v(x). \tag{4.1}$$

By continuity of u and v at d and c respectively, we find $\lim_{y\to d} u(y) = f'(d) = f'\bigl(g(c)\bigr)$ and $\lim_{x\to c} v(x) = g'(c)$. The function g is continuous at c, and so $\lim_{x\to c} g(x) = g(c)$. Hence the limit of the right-hand side of (4.1) as x goes to c exists and is equal to $f'\bigl(g(c)\bigr)g'(c)$. Thus h is differentiable at c and the limit is $f'\bigl(g(c)\bigr)g'(c)$. □

4.1.3 Exercises

Exercise* 4.1.1*: *Prove the product rule. Hint: Prove and use $f(x)g(x) - f(c)g(c) = f(x)\bigl(g(x)-g(c)\bigr) + \bigl(f(x)-f(c)\bigr)g(c)$.*

Exercise* 4.1.2*: *Prove the quotient rule. Hint: You can do this directly, but it may be easier to find the derivative of $1/x$ and then use the chain rule and the product rule.*

Exercise* 4.1.3*: *For $n \in \mathbb{Z}$, prove that x^n is differentiable and find the derivative, unless, of course, $n < 0$ and $x = 0$. Hint: Use the product rule.*

Exercise* 4.1.4*: *Prove that a polynomial is differentiable and find the derivative. Hint: Use the previous exercise.*

Exercise* 4.1.5*: *Define* $f: \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) := \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f *is differentiable at* 0*, but discontinuous at all points except* 0*.*

Exercise* 4.1.6*: *Assume the inequality* $|x - \sin(x)| \leq x^2$*. Prove that* $\sin$ *is differentiable at* 0*, and find the derivative at* 0*.*

Exercise* 4.1.7*: *Using the previous exercise, prove that* $\sin$ *is differentiable at all* x *and that the derivative is* $\cos(x)$*. Hint: Use the sum-to-product trigonometric identity as we did before.*

Exercise* 4.1.8*: *Let* $f: I \to \mathbb{R}$ *be differentiable. For* $n \in \mathbb{Z}$*, let* f^n *be the function defined by* $f^n(x) := \bigl(f(x)\bigr)^n$*. If* $n < 0$*, assume* $f(x) \neq 0$ *for all* $x \in I$*. Prove that* $(f^n)'(x) = n\bigl(f(x)\bigr)^{n-1} f'(x)$*.*

Exercise* 4.1.9*: *Suppose* $f: \mathbb{R} \to \mathbb{R}$ *is a differentiable Lipschitz continuous function. Prove that* f' *is a bounded function.*

Exercise* 4.1.10*: *Let* I_1, I_2 *be intervals. Let* $f: I_1 \to I_2$ *be a bijective function and* $g: I_2 \to I_1$ *be the inverse. Suppose that both* f *is differentiable at* $c \in I_1$ *and* $f'(c) \neq 0$ *and* g *is differentiable at* $f(c)$*. Use the chain rule to find a formula for* $g'\bigl(f(c)\bigr)$ *(in terms of* $f'(c)$*).*

Exercise* 4.1.11*: *Suppose* $f: I \to \mathbb{R}$ *is bounded,* $g: I \to \mathbb{R}$ *is differentiable at* $c \in I$*, and* $g(c) = g'(c) = 0$*. Show that* $h(x) := f(x)g(x)$ *is differentiable at* c*. Hint: You cannot apply the product rule.*

Exercise* 4.1.12*: *Suppose* $f: I \to \mathbb{R}$*,* $g: I \to \mathbb{R}$*, and* $h: I \to \mathbb{R}$*, are functions. Suppose* $c \in I$ *is such that* $f(c) = g(c) = h(c)$*,* g *and* h *are differentiable at* c*, and* $g'(c) = h'(c)$*. Furthermore suppose* $h(x) \leq f(x) \leq g(x)$ *for all* $x \in I$*. Prove* f *is differentiable at* c *and* $f'(c) = g'(c) = h'(c)$*.*

Exercise* 4.1.13*: *Suppose* $f: (-1,1) \to \mathbb{R}$ *is a function such that* $f(x) = xh(x)$ *for a bounded function* h*.*

a) *Show that* $g(x) := \bigl(f(x)\bigr)^2$ *is differentiable at the origin and* $g'(0) = 0$*.*
b) *Find an example of a continuous function* $f: (-1,1) \to \mathbb{R}$ *with* $f(0) = 0$*, but such that* $g(x) := \bigl(f(x)\bigr)^2$ *is not differentiable at the origin.*

Exercise* 4.1.14*: *Suppose* $f: I \to \mathbb{R}$ *is differentiable at* $c \in I$*. Prove there exist numbers* a *and* b *with the property that for every* $\varepsilon > 0$*, there is a* $\delta > 0$*, such that* $|a + b(x-c) - f(x)| \leq \varepsilon\,|x-c|$*, whenever* $x \in I$ *and* $|x - c| < \delta$*. In other words, show that there exists a function* $g: I \to \mathbb{R}$ *such that* $\lim_{x\to c} g(x) = 0$ *and* $|a + b(x-c) - f(x)| = g(x)\,|x-c|$*.*

Exercise* 4.1.15*: *Prove the following simple version of L'Hôpital's rule. Suppose* $f: (a,b) \to \mathbb{R}$ *and* $g: (a,b) \to \mathbb{R}$ *are differentiable functions whose derivatives* f' *and* g' *are continuous functions. Suppose that at* $c \in (a,b)$*,* $f(c) = 0$*,* $g(c) = 0$*,* $g'(x) \neq 0$ *for all* $x \in (a,b)$*, and* $g(x) \neq 0$ *whenever* $x \neq c$*. Note that the limit of* ${f'(x)}/{g'(x)}$ *as* x *goes to* c *exists. Show that*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Exercise* 4.1.16*: *Suppose* $f: (a,b) \to \mathbb{R}$ *is differentiable at* $c \in (a,b)$*,* $f(c) = 0$*, and* $f'(c) > 0$*. Prove that there is a* $\delta > 0$ *such that* $f(x) < 0$ *whenever* $c - \delta < x < c$ *and* $f(x) > 0$ *whenever* $c < x < c + \delta$*.*

4.2 Mean value theorem

Note: 2 lectures (some applications may be skipped)

4.2.1 Relative minima and maxima

We talked about absolute maxima and minima. These are the tallest peaks and lowest valleys in the whole mountain range. What about peaks of individual mountains and bottoms of individual valleys? The derivative, being a local concept, is like walking around in a fog; it can't tell you if you're on the highest peak, but it can help you find all the individual peaks.

Definition 4.2.1. Let $S \subset \mathbb{R}$ be a set and let $f: S \to \mathbb{R}$ be a function. The function f is said to have a *relative maximum* at $c \in S$ if there exists a $\delta > 0$ such that for all $x \in S$ where $|x-c| < \delta$, we have $f(x) \leq f(c)$. The definition of *relative minimum* is analogous.

Lemma 4.2.2. *Suppose $f: [a,b] \to \mathbb{R}$ is differentiable at $c \in (a,b)$, and f has a relative minimum or a relative maximum at c. Then $f'(c) = 0$.*

Proof. We prove the statement for a maximum. For a minimum the statement follows by considering the function $-f$.

Let c be a relative maximum of f. That is, there is a $\delta > 0$ such that as long as $|x-c| < \delta$, we have $f(x) - f(c) \leq 0$. We look at the difference quotient. If $c < x < c + \delta$, then

$$\frac{f(x)-f(c)}{x-c} \leq 0,$$

and if $c - \delta < y < c$, then

$$\frac{f(y)-f(c)}{y-c} \geq 0.$$

See Figure 4.3 for an illustration.

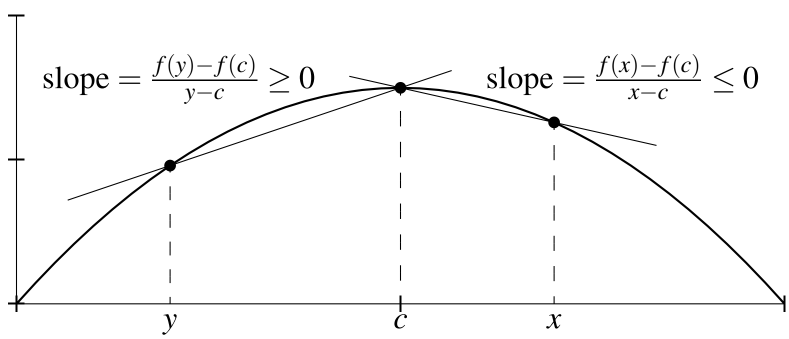

Figure 4.3: Slopes of secants at a relative maximum.

As $a < c < b$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $[a,b]$ and within δ of c, such that $x_n > c$, and $y_n < c$ for all $n \in \mathbb{N}$, and such that $\lim x_n = \lim y_n = c$. Since f is differentiable at c,

$$0 \geq \lim_{n\to\infty} \frac{f(x_n)-f(c)}{x_n-c} = f'(c) = \lim_{n\to\infty} \frac{f(y_n)-f(c)}{y_n-c} \geq 0.$$

□

For a differentiable function, a point where $f'(c) = 0$ is called a *critical point*. When f is not differentiable at some points, it is common to also say that c is a critical point if $f'(c)$ does not exist. The theorem says that a relative minimum or maximum at an interior point of an interval must be a critical point. As you remember from calculus, finding minima and maxima of a function can be done by finding all the critical points together with the endpoints of the interval and simply checking at which of these points is the function biggest or smallest.

4.2.2 Rolle's theorem

Suppose a function has the same value at both endpoints of an interval. Intuitively it ought to attain a minimum or a maximum in the interior of the interval, then at such a minimum or a maximum, the derivative should be zero. See Figure 4.4 for the geometric idea. This is the content of the so-called Rolle's theorem*.

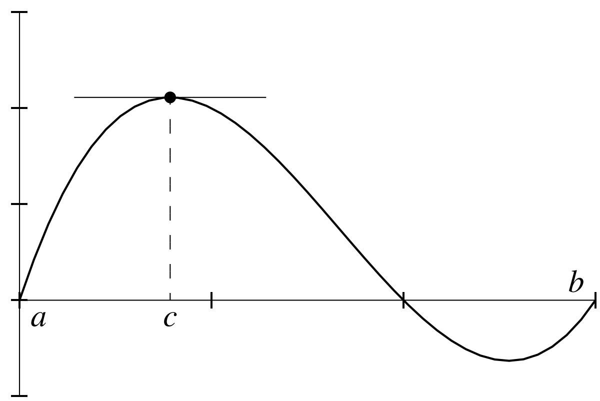

Figure 4.4: Point where the tangent line is horizontal, that is $f'(c) = 0$.

Theorem 4.2.3 (Rolle)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous function differentiable on (a,b) such that $f(a) = f(b)$. Then there exists a $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. As f is continuous on $[a,b]$, it attains an absolute minimum and an absolute maximum in $[a,b]$. We wish to apply Lemma 4.2.2, and so we need to find some $c \in (a,b)$ where f attains a minimum or a maximum. Write $K := f(a) = f(b)$. If there exists an x such that $f(x) > K$, then the absolute maximum is bigger than K and hence occurs at some $c \in (a,b)$, and therefore $f'(c) = 0$. On the other hand, if there exists an x such that $f(x) < K$, then the absolute minimum occurs at some $c \in (a,b)$, and so $f'(c) = 0$. If there is no x such that $f(x) > K$ or $f(x) < K$, then $f(x) = K$ for all x and then $f'(x) = 0$ for all $x \in [a,b]$, so any $c \in (a,b)$ works. □

It is absolutely necessary for the derivative to exist for all $x \in (a,b)$. Consider the function $f(x) := |x|$ on $[-1,1]$. Clearly $f(-1) = f(1)$, but there is no point c where $f'(c) = 0$.

*Named after the French mathematician Michel Rolle (1652–1719).

4.2.3 Mean value theorem

We extend Rolle's theorem to functions that attain different values at the endpoints.

Theorem 4.2.4 (Mean value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

For a geometric interpretation of the mean value theorem, see Figure 4.5. The idea is that the value $\frac{f(b)-f(a)}{b-a}$ is the slope of the line between the points $\bigl(a, f(a)\bigr)$ and $\bigl(b, f(b)\bigr)$. Then c is the point such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, that is, the tangent line at the point $\bigl(c, f(c)\bigr)$ has the same slope as the line between $\bigl(a, f(a)\bigr)$ and $\bigl(b, f(b)\bigr)$. The theorem follows from Rolle's theorem, by subtracting from f the affine linear function with the derivative $\frac{f(b)-f(a)}{b-a}$ with the same values at a and b as f. That is, we subtract the function whose graph is the straight line $\bigl(a, f(a)\bigr)$ and $\bigl(b, f(b)\bigr)$. Then we are looking for a point where this new function has derivative zero.

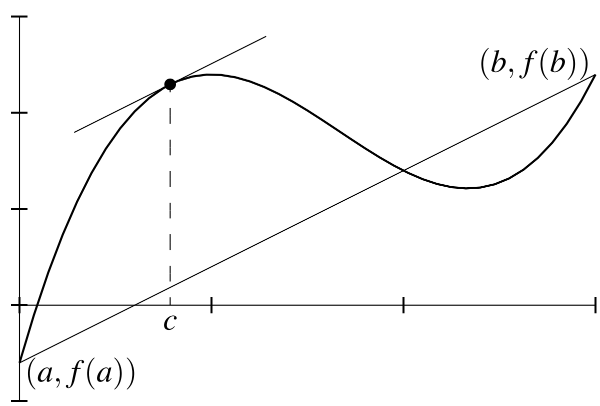

Figure 4.5: Graphical interpretation of the mean value theorem.

Proof. Define the function $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) := f(x) - f(b) - \frac{f(b)-f(a)}{b-a}(x-b).$$

The function g is differentiable on (a,b), continuous on $[a,b]$, such that $g(a) = 0$ and $g(b) = 0$. Thus there exists a $c \in (a,b)$ such that $g'(c) = 0$, that is,

$$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}.$$

In other words, $f'(c)(b-a) = f(b) - f(a)$. $\square$

The proof generalizes. By considering $g(x) := f(x) - f(b) - \frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}\bigl(\varphi(x) - \varphi(b)\bigr)$, one can prove the following version. We leave the proof as an exercise.

Theorem 4.2.5 (Cauchy's mean value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ and $\varphi\colon [a,b] \to \mathbb{R}$ be continuous functions differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that*

$$\bigl(f(b) - f(a)\bigr)\varphi'(c) = f'(c)\bigl(\varphi(b) - \varphi(a)\bigr).$$

The mean value theorem has the distinction of being one of the few theorems commonly cited in court. That is, when police measure the speed of cars by aircraft, or via cameras reading license plates, they measure the time the car takes to go between two points. The mean value theorem then says that the car must have somewhere attained the speed you get by dividing the difference in distance by the difference in time.

4.2.4 Applications

We now solve our very first differential equation.

Proposition 4.2.6. *Let I be an interval and let $f\colon I \to \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ for all $x \in I$. Then f is constant.*

Proof. Take arbitrary $x,y \in I$ with $x < y$. As I is an interval, $[x,y] \subset I$. Then f restricted to $[x,y]$ satisfies the hypotheses of the mean value theorem. Therefore, there is a $c \in (x,y)$ such that

$$f(y) - f(x) = f'(c)(y-x).$$

As $f'(c) = 0$, we have $f(y) = f(x)$. Hence, the function is constant. □

Now that we know what it means for the function to stay constant, let us look at increasing and decreasing functions. We say $f\colon I \to \mathbb{R}$ is *increasing* (resp. *strictly increasing*) if $x < y$ implies $f(x) \leq f(y)$ (resp. $f(x) < f(y)$). We define *decreasing* and *strictly decreasing* in the same way by switching the inequalities for f.

Proposition 4.2.7. *Let I be an interval and let $f\colon I \to \mathbb{R}$ be a differentiable function.*

(i) f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.

(ii) f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

Proof. Let us prove the first item. Suppose f is increasing, then for all $x,c \in I$ with $x \neq c$, we have

$$\frac{f(x) - f(c)}{x-c} \geq 0.$$

Taking a limit as x goes to c we see that $f'(c) \geq 0$.

For the other direction, suppose $f'(x) \geq 0$ for all $x \in I$. Take any $x,y \in I$ where $x < y$, and note that $[x,y] \subset I$. By the mean value theorem, there is some $c \in (x,y)$ such that

$$f(y) - f(x) = f'(c)(y-x).$$

As $f'(c) \geq 0$, and $y - x > 0$, then $f(y) - f(x) \geq 0$ or $f(x) \leq f(y)$ and so f is increasing.

We leave the decreasing part to the reader as exercise. □

A similar but weaker statement is true for strictly increasing and decreasing functions.

Proposition 4.2.8. *Let I be an interval and let $f: I \to \mathbb{R}$ be a differentiable function.*

(i) If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing.

(ii) If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing.

The proof of (i) is left as an exercise. Then (ii) follows from (i) by considering $-f$ instead. The converse of this proposition is not true. The function $f(x) := x^3$ is strictly increasing, but $f'(0) = 0$.

Another application of the mean value theorem is the following result about location of extrema, sometimes called the *first derivative test*. The theorem is stated for an absolute minimum and maximum. To apply it to find relative minima and maxima, restrict f to an interval $(c-\delta, c+\delta)$.

Proposition 4.2.9. *Let $f: (a,b) \to \mathbb{R}$ be continuous. Let $c \in (a,b)$ and suppose f is differentiable on (a,c) and (c,b).*

(i) If $f'(x) \leq 0$ for $x \in (a,c)$ and $f'(x) \geq 0$ for $x \in (c,b)$, then f has an absolute minimum at c.

(ii) If $f'(x) \geq 0$ for $x \in (a,c)$ and $f'(x) \leq 0$ for $x \in (c,b)$, then f has an absolute maximum at c.

Proof. We prove the first item and leave the second to the reader. Take $x \in (a,c)$ and a sequence $\{y_n\}$ such that $x < y_n < c$ for all n and $\lim y_n = c$. By the preceding proposition, f is decreasing on (a,c) so $f(x) \geq f(y_n)$. As f is continuous at c, we take the limit to get $f(x) \geq f(c)$ for all $x \in (a,c)$.

Similarly, take $x \in (c,b)$ and $\{y_n\}$ a sequence such that $c < y_n < x$ and $\lim y_n = c$. The function is increasing on (c,b) so $f(x) \geq f(y_n)$. By continuity of f we get $f(x) \geq f(c)$ for all $x \in (c,b)$. Thus $f(x) \geq f(c)$ for all $x \in (a,b)$. □

The converse of the proposition does not hold. See Example 4.2.12 below.

Another often used application of the mean value theorem you have possibly seen in calculus is the following result on differentiability at the end points of an interval. The proof is Exercise 4.2.13.

Proposition 4.2.10.

(i) Suppose $f: [a,b) \to \mathbb{R}$ is continuous, differentiable in (a,b), and $\lim_{x\to a} f'(x) = L$. Then f is differentiable at a and $f'(a) = L$.

(ii) Suppose $f: (a,b] \to \mathbb{R}$ is continuous, differentiable in (a,b), and $\lim_{x\to b} f'(x) = L$. Then f is differentiable at b and $f'(b) = L$.

In fact, using the extension result Proposition 3.4.6, you do not need to assume that f is defined at the end point. See Exercise 4.2.14.

4.2.5 Continuity of derivatives and the intermediate value theorem

Derivatives of functions satisfy an intermediate value property.

Theorem 4.2.11 (Darboux)**.** *Let $f: [a,b] \to \mathbb{R}$ be differentiable. Suppose $y \in \mathbb{R}$ is such that $f'(a) < y < f'(b)$ or $f'(a) > y > f'(b)$. Then there exists a $c \in (a,b)$ such that $f'(c) = y$.*

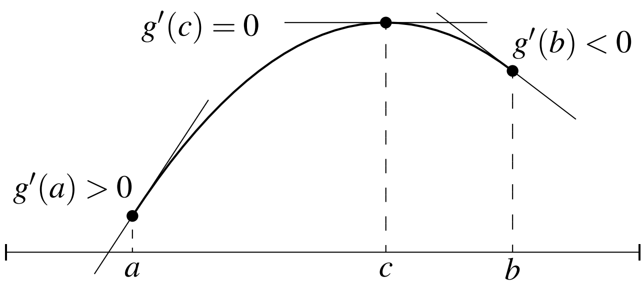

Figure 4.6: Idea of the proof of Darboux theorem.

The proof follows by subtracting f and a linear function with derivative y. The new function g reduces the problem to the case $y = 0$, where $g'(a) > 0 > g'(b)$. That is, g is increasing at a and decreasing at b, so it must attain a maximum inside (a,b), where the derivative is zero. See Figure 4.6.

Proof. Suppose $f'(a) < y < f'(b)$. Define

$$g(x) := yx - f(x).$$

The function g is continuous on $[a,b]$, and so g attains a maximum at some $c \in [a,b]$.

The function g is also differentiable on $[a,b]$. Compute $g'(x) = y - f'(x)$. Thus $g'(a) > 0$. As the derivative is the limit of difference quotients and is positive, there must be some difference quotient that is positive. That is, there must exist an $x > a$ such that

$$\frac{g(x) - g(a)}{x - a} > 0,$$

or $g(x) > g(a)$. Thus g cannot possibly have a maximum at a. Similarly, as $g'(b) < 0$, we find an $x < b$ (a different x) such that $\frac{g(x)-g(b)}{x-b} < 0$ or that $g(x) > g(b)$, thus g cannot possibly have a maximum at b. Therefore, $c \in (a,b)$, and Lemma 4.2.2 applies: As g attains a maximum at c we find $g'(c) = 0$ and so $f'(c) = y$.

Similarly, if $f'(a) > y > f'(b)$, consider $g(x) := f(x) - yx$. $\square$

We have seen already that there exist discontinuous functions that have the intermediate value property. While it is hard to imagine at first, there also exist functions that are differentiable everywhere and the derivative is not continuous.

Example 4.2.12: Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) := \begin{cases} \bigl(x \sin(1/x)\bigr)^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that f is differentiable everywhere, but $f' : \mathbb{R} \to \mathbb{R}$ is not continuous at the origin. Furthermore, f has a minimum at 0, but the derivative changes sign infinitely often near the origin. See Figure 4.7.

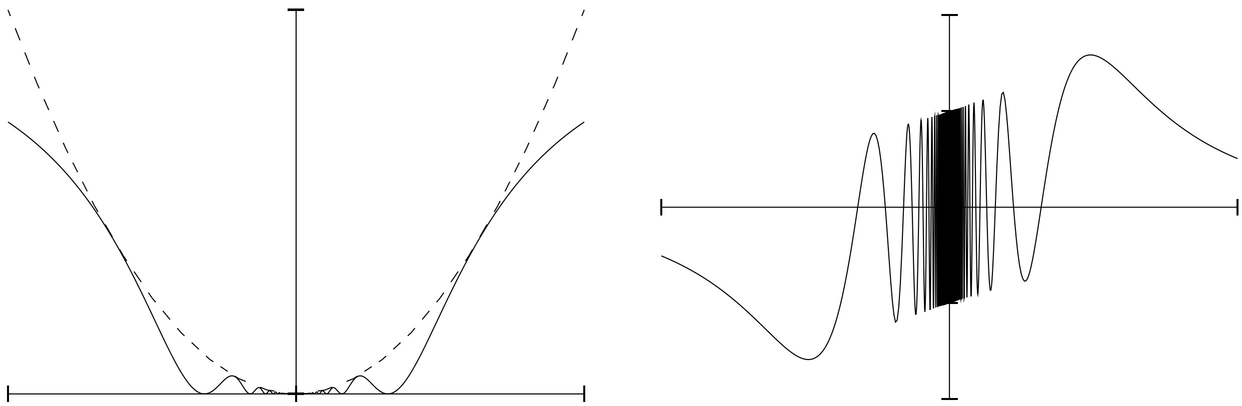

Figure 4.7: A function with a discontinuous derivative. The function f is on the left and f' is on the right. Notice that $f(x) \leq x^2$ on the left graph.

Proof: It is immediate from the definition that f has an absolute minimum at 0; we know $f(x) \geq 0$ for all x and $f(0) = 0$.

The function f is differentiable for $x \neq 0$, and the derivative is $2\sin(1/x)\bigl(x\sin(1/x) - \cos(1/x)\bigr)$. As an exercise, show that for $x_n = \frac{4}{(8n+1)\pi}$, we have $\lim f'(x_n) = -1$, and for $y_n = \frac{4}{(8n+3)\pi}$, we have $\lim f'(y_n) = 1$. Hence if f' exists at 0, then it cannot be continuous.

Let us show that f' exists at 0. We claim that the derivative is zero. In other words, $\left| \frac{f(x)-f(0)}{x-0} - 0 \right|$ goes to zero as x goes to zero. For $x \neq 0$,

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{x^2 \sin^2(1/x)}{x} \right| = \left| x \sin^2(1/x) \right| \leq |x| .$$

And, of course, as x tends to zero, $|x|$ tends to zero, and hence $\left| \frac{f(x)-f(0)}{x-0} - 0 \right|$ goes to zero. Therefore, f is differentiable at 0 and the derivative at 0 is 0. A key point in the calculation above is that $|f(x)| \leq x^2$, see also Exercises 4.1.11 and 4.1.12.

It is sometimes useful to assume the derivative of a differentiable function is continuous. If $f: I \to \mathbb{R}$ is differentiable and the derivative f' is continuous on I, then we say f is *continuously differentiable*. It is common to write $C^1(I)$ for the set of continuously differentiable functions on I.

4.2.6 Exercises

***Exercise* 4.2.1:** *Finish the proof of Proposition 4.2.7.*

***Exercise* 4.2.2:** *Finish the proof of Proposition 4.2.9.*

***Exercise* 4.2.3:** *Suppose $f: \mathbb{R} \to \mathbb{R}$ is a differentiable function such that f' is a bounded function. Prove that f is a Lipschitz continuous function.*

***Exercise* 4.2.4:** *Suppose $f: [a,b] \to \mathbb{R}$ is differentiable and $c \in [a,b]$. Show there exists a sequence $\{x_n\}$ converging to c, $x_n \neq c$ for all n, such that*

$$f'(c) = \lim_{n\to\infty} f'(x_n).$$

Do note this does not *imply that f' is continuous (why?).*

***Exercise* 4.2.5:** *Suppose $f: \mathbb{R} \to \mathbb{R}$ is a function such that $|f(x) - f(y)| \leq |x-y|^2$ for all x and y. Show that $f(x) = C$ for some constant C. Hint: Show that f is differentiable at all points and compute the derivative.*

***Exercise* 4.2.6:** *Finish the proof of Proposition 4.2.8. That is, suppose I is an interval and $f: I \to \mathbb{R}$ is a differentiable function such that $f'(x) > 0$ for all $x \in I$. Show that f is strictly increasing.*

***Exercise* 4.2.7:** *Suppose $f: (a,b) \to \mathbb{R}$ is a differentiable function such that $f'(x) \neq 0$ for all $x \in (a,b)$. Suppose there exists a point $c \in (a,b)$ such that $f'(c) > 0$. Prove $f'(x) > 0$ for all $x \in (a,b)$.*

***Exercise* 4.2.8:** *Suppose $f: (a,b) \to \mathbb{R}$ and $g: (a,b) \to \mathbb{R}$ are differentiable functions such that $f'(x) = g'(x)$ for all $x \in (a,b)$, then show that there exists a constant C such that $f(x) = g(x) + C$.*

***Exercise* 4.2.9:** *Prove the following version of L'Hôpital's rule. Suppose $f: (a,b) \to \mathbb{R}$ and $g: (a,b) \to \mathbb{R}$ are differentiable functions and $c \in (a,b)$. Suppose that $f(c) = 0$, $g(c) = 0$, $g'(x) \neq 0$ when $x \neq c$, and that the limit of $f'(x)/g'(x)$ as x goes to c exists. Show that*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Compare to Exercise 4.1.15. Note: Before you do anything else, prove that $g(x) \neq 0$ when $x \neq c$.

***Exercise* 4.2.10:** *Let $f: (a,b) \to \mathbb{R}$ be an unbounded differentiable function. Show $f': (a,b) \to \mathbb{R}$ is unbounded.*

***Exercise* 4.2.11:** *Prove the theorem Rolle actually proved in 1691:* If f is a polynomial, $f'(a) = f'(b) = 0$ for some $a < b$, and there is no $c \in (a,b)$ such that $f'(c) = 0$, then there is at most one root of f in (a,b), that is at most one $x \in (a,b)$ such that $f(x) = 0$. *In other words, between any two consecutive roots of f' is at most one root of f. Hint: Suppose there are two roots and see what happens.*

***Exercise* 4.2.12:** *Suppose $a, b \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ is differentiable, $f'(x) = a$ for all x, and $f(0) = b$. Find f and prove that it is the unique differentiable function with this property.*

***Exercise* 4.2.13:**

a) *Prove Proposition 4.2.10.*

b) *Suppose $f: (a,b) \to \mathbb{R}$ is continuous, and suppose f is differentiable everywhere except at $c \in (a,b)$ and $\lim_{x\to c} f'(x) = L$. Prove that f is differentiable at c and $f'(c) = L$.*

***Exercise* 4.2.14:** *Suppose $f: (0,1) \to \mathbb{R}$ is differentiable and f' is bounded.*

a) *Show that there exists a continuous function $g: [0,1) \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \neq 0$.*
Hint: Proposition 3.4.6 and Exercise 4.2.3.

b) *Find an example where the g is not differentiable at $x = 0$.*
Hint: Consider something based on $\sin(\ln x)$*, and assume you know basic properties of* $\sin$ *and* $\ln$ *from calculus.*

c) *Instead of assuming that f' is bounded, assume that $\lim_{x\to 0} f'(x) = L$. Prove that not only does g exist but it is differentiable at 0 and $g'(0) = L$.*

***Exercise* 4.2.15:** *Prove Theorem 4.2.5.*

4.3 Taylor's theorem

Note: less than a lecture (optional section)

4.3.1 Derivatives of higher orders

When $f: I \to \mathbb{R}$ is differentiable, we obtain a function $f': I \to \mathbb{R}$. The function f' is called the *first derivative* of f. If f' is differentiable, we denote by $f'': I \to \mathbb{R}$ the derivative of f'. The function f'' is called the *second derivative* of f. We similarly obtain f''', f'''', and so on. With a larger number of derivatives the notation would get out of hand; we denote by $f^{(n)}$ the *nth derivative* of f.

When f possesses n derivatives, we say f is *n times differentiable*.

4.3.2 Taylor's theorem

Taylor's theorem* is a generalization of the mean value theorem. Mean value theorem says that up to a small error $f(x)$ for x near x_0 can be approximated by $f(x_0)$, that is

$$f(x) = f(x_0) + f'(c)(x - x_0),$$

where the "error" is measured in terms of the first derivative at some point c between x and x_0. Taylor's theorem generalizes this result to higher derivatives. It tells us that up to a small error, any n times differentiable function can be approximated at a point x_0 by a polynomial. The error of this approximation behaves like $(x - x_0)^n$ near the point x_0. To see why this is a good approximation notice that for a big n, $(x - x_0)^n$ is very small in a small interval around x_0.

Definition 4.3.1. For an n times differentiable function f defined near a point $x_0 \in \mathbb{R}$, define the nth order *Taylor polynomial* for f at x_0 as

$$\begin{aligned} P_n^{x_0}(x) &:= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{6}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$

See Figure 4.8 for the odd degree Taylor polynomials for the sine function at $x_0 = 0$. The even degree terms are all zero, as even derivatives of sine are again a sine, which are zero at the origin.

Taylor's theorem says a function behaves like its nth Taylor polynomial. The mean value theorem is really Taylor's theorem for the first derivative.

Theorem 4.3.2 (Taylor)**.** *Suppose $f: [a, b] \to \mathbb{R}$ is a function with n continuous derivatives on $[a, b]$ and such that $f^{(n+1)}$ exists on (a, b). Given distinct points x_0 and x in $[a, b]$, we can find a point c between x_0 and x such that*

$$f(x) = P_n^{x_0}(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

*Named for the English mathematician Brook Taylor (1685–1731). It was first found by the Scottish mathematician James Gregory (1638–1675). The statement we give was proved by Joseph-Louis Lagrange (1736–1813)

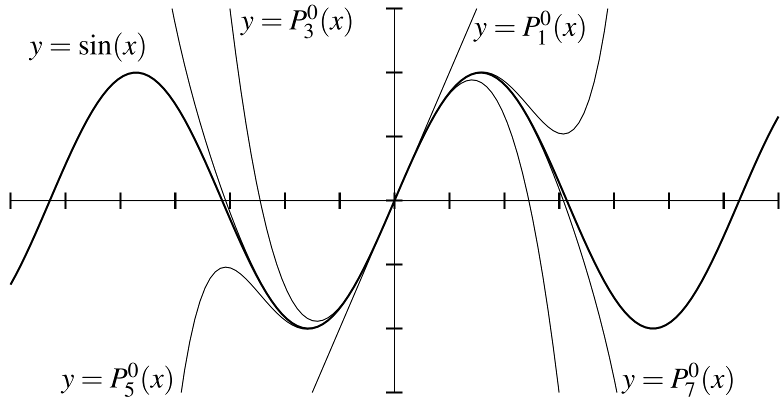

Figure 4.8: The odd degree Taylor polynomials for the sine function.

The term $R_n^{x_0}(x) := \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ is called the *remainder term*. This form of the remainder term is called the *Lagrange form* of the remainder. There are other ways to write the remainder term, but we skip those. Note that c depends on both x and x_0.

Proof. Find a number M_{x,x_0} (depending on x and x_0) solving the equation

$$f(x) = P_n^{x_0}(x) + M_{x,x_0}(x-x_0)^{n+1}.$$

Define a function $g(s)$ by

$$g(s) := f(s) - P_n^{x_0}(s) - M_{x,x_0}(s-x_0)^{n+1}.$$

We compute the kth derivative at x_0 of the Taylor polynomial $(P_n^{x_0})^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0,1,2,\ldots,n$ (the zeroth derivative of a function is the function itself). Therefore,

$$g(x_0) = g'(x_0) = g''(x_0) = \cdots = g^{(n)}(x_0) = 0.$$

In particular, $g(x_0) = 0$. On the other hand $g(x) = 0$. By the mean value theorem there exists an x_1 between x_0 and x such that $g'(x_1) = 0$. Applying the mean value theorem to g' we obtain that there exists x_2 between x_0 and x_1 (and therefore between x_0 and x) such that $g''(x_2) = 0$. We repeat the argument $n+1$ times to obtain a number x_{n+1} between x_0 and x_n (and therefore between x_0 and x) such that $g^{(n+1)}(x_{n+1}) = 0$.

Let $c := x_{n+1}$. We compute the $(n+1)$th derivative of g to find

$$g^{(n+1)}(s) = f^{(n+1)}(s) - (n+1)!\,M_{x,x_0}.$$

Plugging in c for s we obtain $M_{x,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}$, and we are done. □

In the proof, we have computed $(P_n^{x_0})^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0,1,2,\ldots,n$. Therefore, the Taylor polynomial has the same derivatives as f at x_0 up to the nth derivative. That is why the

Taylor polynomial is a good approximation to f. Notice how in Figure 4.8 the Taylor polynomials are reasonably good approximations to the sine near $x = 0$.

We do not necessarily get good approximations by the Taylor polynomial everywhere. Consider expanding the function $f(x) := \frac{x}{1-x}$ around 0, for $x < 1$, we get the graphs in Figure 4.9. The dotted lines are the first, second, and third degree approximations. The dashed line is the 20th degree polynomial, and yet the approximation only seems to get better with the degree for $x > -1$, and for smaller x, it in fact gets worse. The polynomials are the partial sums of the geometric series $\sum_{n=1}^{\infty} x^n$, and the series only converges on $(-1,1)$. See the discussion of power series §2.6.

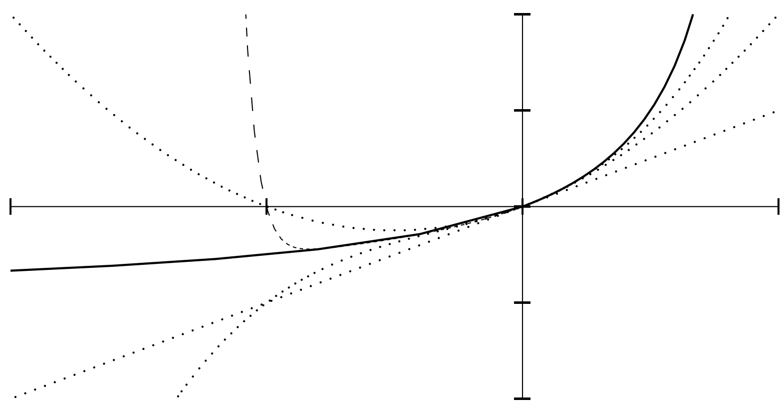

Figure 4.9: The function $\frac{x}{1-x}$, and the Taylor polynomials P_1^0, P_2^0, P_3^0 (all dotted), and the polynomial P_{20}^0 (dashed).

If f is *infinitely differentiable*, that is, if f can be differentiated any number of times, then we define the *Taylor series*:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

There is no guarantee that this series converges for any $x \neq x_0$. And even where it does converge, there is no guarantee that it converges to the function f. Functions f whose Taylor series at every point x_0 converges to f in some open interval containing x_0 are called *analytic functions*. Most functions one tends to see in practice are analytic. See Exercise 5.4.11, for an example of a non-analytic function.

The definition of derivative says that a function is differentiable if it is locally approximated by a line. We mention in passing that there exists a converse to Taylor's theorem, which we will neither state nor prove, saying that if a function is locally approximated in a certain way by a polynomial of degree d, then it has d derivatives.

Taylor's theorem gives us a quick proof of a version of the second derivative test. By a *strict relative minimum* of f at c, we mean that there exists a $\delta > 0$ such that $f(x) > f(c)$ for all $x \in (c-\delta, c+\delta)$ where $x \neq c$. A *strict relative maximum* is defined similarly. Continuity of the second derivative is not needed, but the proof is more difficult and is left as an exercise. The proof also generalizes immediately into the nth derivative test, which is also left as an exercise.

Proposition 4.3.3 (Second derivative test)**.** *Suppose $f\colon (a,b) \to \mathbb{R}$ is twice continuously differentiable, $x_0 \in (a,b)$, $f'(x_0) = 0$ and $f''(x_0) > 0$. Then f has a strict relative minimum at x_0.*

Proof. As f'' is continuous, there exists a $\delta > 0$ such that $f''(c) > 0$ for all $c \in (x_0 - \delta, x_0 + \delta)$, see Exercise 3.2.11. Take $x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$. Taylor's theorem says that for some c between x_0 and x,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c)}{2}(x - x_0)^2 = f(x_0) + \frac{f''(c)}{2}(x - x_0)^2.$$

As $f''(c) > 0$, and $(x - x_0)^2 > 0$, then $f(x) > f(x_0)$. $\square$

4.3.3 Exercises

Exercise* 4.3.1*: *Compute the nth Taylor polynomial at 0 for the exponential function.*

Exercise* 4.3.2*: *Suppose p is a polynomial of degree d. Given $x_0 \in \mathbb{R}$, show that the dth Taylor polynomial for p at x_0 is equal to p.*

Exercise* 4.3.3*: *Let $f(x) := \lvert x \rvert^3$. Compute $f'(x)$ and $f''(x)$ for all x, but show that $f^{(3)}(0)$ does not exist.*

Exercise* 4.3.4*: *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has n continuous derivatives. Show that for every $x_0 \in \mathbb{R}$, there exist polynomials P and Q of degree n and an $\varepsilon > 0$ such that $P(x) \leq f(x) \leq Q(x)$ for all $x \in [x_0, x_0 + \varepsilon]$ and $Q(x) - P(x) = \lambda (x - x_0)^n$ for some $\lambda \geq 0$.*

Exercise* 4.3.5*: *If $f\colon [a,b] \to \mathbb{R}$ has $n+1$ continuous derivatives and $x_0 \in [a,b]$, prove $\lim\limits_{x \to x_0} \frac{R_n^{x_0}(x)}{(x-x_0)^n} = 0$.*

Exercise* 4.3.6*: *Suppose $f\colon [a,b] \to \mathbb{R}$ has $n+1$ continuous derivatives and $x_0 \in (a,b)$. Prove: $f^{(k)}(x_0) = 0$ for all $k = 0, 1, 2, \ldots, n$ if and only if $\lim\limits_{x \to x_0} \frac{f(x)}{(x-x_0)^{n+1}}$ exists.*

Exercise* 4.3.7*: *Suppose $a, b, c \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, $f''(x) = a$ for all x, $f'(0) = b$, and $f(0) = c$. Find f and prove that it is the unique differentiable function with this property.*

Exercise* 4.3.8** (Challenging):*** *Show that a simple converse to Taylor's theorem does not hold. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ with no second derivative at $x = 0$ such that $\lvert f(x) \rvert \leq \lvert x^3 \rvert$, that is, f goes to zero at 0 faster than x^3, and while $f'(0)$ exists, $f''(0)$ does not.*

Exercise* 4.3.9*: *Suppose $f\colon (0,1) \to \mathbb{R}$ is differentiable and f'' is bounded.*

a) Show that there exists a once differentiable function $g\colon [0,1) \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \neq 0$. Hint: See Exercise 4.2.14.

b) Find an example where the g is not twice differentiable at $x = 0$.

Exercise* 4.3.10*: *Prove the* nth derivative test. *Suppose $n \in \mathbb{N}$, $x_0 \in (a,b)$, and $f\colon (a,b) \to \mathbb{R}$ is n times continuously differentiable, with $f^{(k)}(x_0) = 0$ for $k = 1, 2, \ldots, n-1$, and $f^{(n)}(x_0) \neq 0$. Prove:*

a) If n is odd, then f has neither a relative minimum, nor a maximum at x_0.

b) If n is even, then f has a strict relative minimum at x_0 if $f^{(n)}(x_0) > 0$ and a strict relative maximum at x_0 if $f^{(n)}(x_0) < 0$.

Exercise* 4.3.11*: *Prove the more general version of the second derivative test. Suppose $f\colon (a,b) \to \mathbb{R}$ is differentiable and $x_0 \in (a,b)$ is such that, $f'(x_0) = 0$, $f''(x_0)$ exists, and $f''(x_0) > 0$. Prove that f has a strict relative minimum at x_0. Hint: Consider the limit definition of $f''(x_0)$.*

4.4 Inverse function theorem

Note: less than 1 lecture (optional section, needed for §5.4, requires §3.6)

4.4.1 Inverse function theorem

We start with a simple example. Consider the function $f(x) := ax$ for a number $a \neq 0$. Then $f: \mathbb{R} \to \mathbb{R}$ is bijective, and the inverse is $f^{-1}(y) = \frac{1}{a}y$. In particular, $f'(x) = a$ and $(f^{-1})'(y) = \frac{1}{a}$. As differentiable functions are "infinitesimally like" linear functions, we expect the same behavior from the inverse function. The main idea of differentiating inverse functions is the following lemma.

Lemma 4.4.1. *Let $I, J \subset \mathbb{R}$ be intervals. If $f: I \to J$ is strictly monotone (hence one-to-one), onto ($f(I) = J$), differentiable at $x_0 \in I$, and $f'(x_0) \neq 0$, then the inverse f^{-1} is differentiable at $y_0 = f(x_0)$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'\big(f^{-1}(y_0)\big)} = \frac{1}{f'(x_0)}.$$

If f is continuously differentiable and f' is never zero, then f^{-1} is continuously differentiable.

Proof. By Proposition 3.6.6, f has a continuous inverse. For convenience call the inverse $g: J \to I$. Let x_0, y_0 be as in the statement. For $x \in I$ write $y := f(x)$. If $x \neq x_0$ and so $y \neq y_0$, we find

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)} = \frac{x - x_0}{f(x) - f(x_0)}.$$

See Figure 4.10 for the geometric idea.

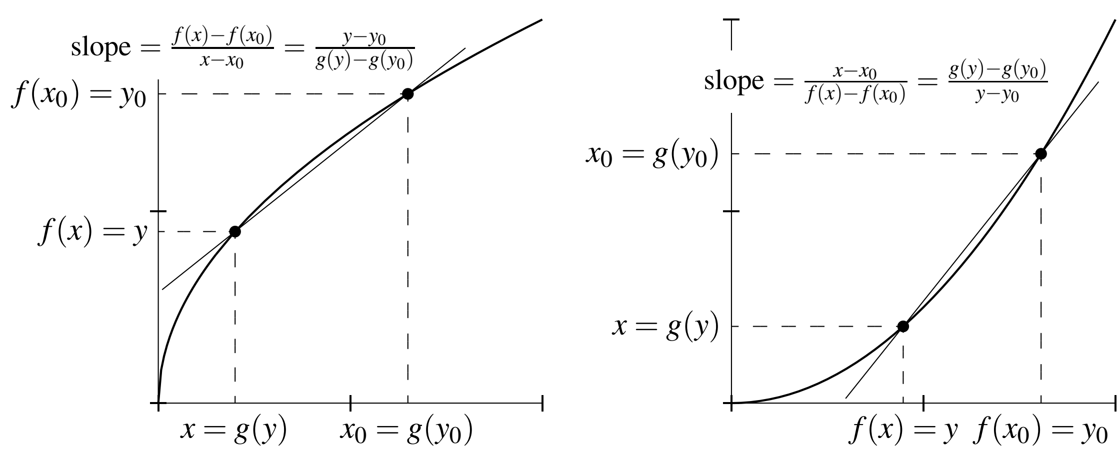

Figure 4.10: Interpretation of the derivative of the inverse function.

Let

$$Q(x) := \begin{cases} \frac{x-x_0}{f(x)-f(x_0)} & \text{if } x \neq x_0, \\ \frac{1}{f'(x_0)} & \text{if } x = x_0 \quad \text{(notice that } f'(x_0) \neq 0\text{).} \end{cases}$$

As f is differentiable at x_0,

$$\lim_{x \to x_0} Q(x) = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} = Q(x_0),$$

that is, Q is continuous at x_0. As $g(y)$ is continuous at y_0, the composition $Q\bigl(g(y)\bigr) = \frac{g(y)-g(y_0)}{y-y_0}$ is continuous at y_0 by Proposition 3.2.7. Therefore,

$$\frac{1}{f'\bigl(g(y_0)\bigr)} = Q\bigl(g(y_0)\bigr) = \lim_{y \to y_0} Q\bigl(g(y)\bigr) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0}.$$

So g is differentiable at y_0 and $g'(y_0) = \frac{1}{f'\left(g(y_0)\right)}$.

If f' is continuous and nonzero at all $x \in I$, then the lemma applies at all $x \in I$. As g is also continuous (it is differentiable), the derivative $g'(y) = \frac{1}{f'\left(g(y)\right)}$ must be continuous. □

What is usually called the inverse function theorem is the following result.

Theorem 4.4.2 (Inverse function theorem)**.** *Let $f\colon (a,b) \to \mathbb{R}$ be a continuously differentiable function, $x_0 \in (a,b)$ a point where $f'(x_0) \neq 0$. Then there exists an open interval $I \subset (a,b)$ with $x_0 \in I$, the restriction $f|_I$ is injective with a continuously differentiable inverse $g\colon J \to I$ defined on an interval $J := f(I)$, and*

$$g'(y) = \frac{1}{f'\bigl(g(y)\bigr)} \qquad \textit{for all } y \in J.$$

Proof. Without loss of generality, suppose $f'(x_0) > 0$. As f' is continuous, there must exist an open interval $I = (x_0 - \delta, x_0 + \delta)$ such that $f'(x) > 0$ for all $x \in I$. See Exercise 3.2.11.

By Proposition 4.2.8, f is strictly increasing on I, and hence the restriction $f|_I$ is bijective onto $J := f(I)$. As f is continuous, then by the Corollary 3.6.3 (or directly via the intermediate value theorem) $f(I)$ is in interval. Now apply Lemma 4.4.1. □

If you tried to prove the existence of roots directly as in Example 1.2.3, you saw how difficult that endeavor is. However, with the machinery we have built for inverse functions it becomes an almost trivial exercise, and with the lemma above we prove far more than mere existence.

Corollary 4.4.3. *Given $n \in \mathbb{N}$ and $x \geq 0$ there exists a unique number $y \geq 0$ (denoted $x^{1/n} := y$), such that $y^n = x$. Furthermore, the function $g\colon (0,\infty) \to (0,\infty)$ defined by $g(x) := x^{1/n}$ is continuously differentiable and*

$$g'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n} x^{(1-n)/n},$$

using the convention $x^{m/n} := \left(x^{1/n}\right)^m$.

Proof. For $x = 0$ the existence of a unique root is trivial.

Let $f\colon (0,\infty) \to (0,\infty)$ be defined by $f(y) := y^n$. The function f is continuously differentiable and $f'(y) = ny^{n-1}$, see Exercise 4.1.3. For $y > 0$ the derivative f' is strictly positive and so again by Proposition 4.2.8, f is strictly increasing (this can also be proved directly). Given any $M > 1$, $f(M) = M^n \geq M$, and given any $1 > \varepsilon > 0$, $f(\varepsilon) = \varepsilon^n \leq \varepsilon$. For every x with $\varepsilon < x < M$, we have, by the intermediate value theorem, that $x \in f\big([\varepsilon, M]\big) \subset f\big((0,\infty)\big)$. As M and ε were arbitrary, f is onto $(0,\infty)$, and hence f is bijective. Let g be the inverse of f, and we obtain the existence and uniqueness of positive nth roots. Lemma 4.4.1 says g has a continuous derivative and $g'(x) = \frac{1}{f'\left(g(x)\right)} = \frac{1}{n(x^{1/n})^{n-1}}$. $\square$

Example 4.4.4: The corollary provides a good example of where the inverse function theorem gives us an interval smaller than (a,b). Take $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$. Then $f'(x) \neq 0$ as long as $x \neq 0$. If $x_0 > 0$, we can take $I = (0,\infty)$, but no larger.

Example 4.4.5: Another useful example is $f(x) := x^3$. The function $f \colon \mathbb{R} \to \mathbb{R}$ is one-to-one and onto, so $f^{-1}(y) = y^{1/3}$ exists on the entire real line including zero and negative y. The function f has a continuous derivative, but f^{-1} has no derivative at the origin. The point is that $f'(0) = 0$. See Figure 4.11 for a graph, notice the vertical tangent on the cube root at the origin. See also Exercise 4.4.4.

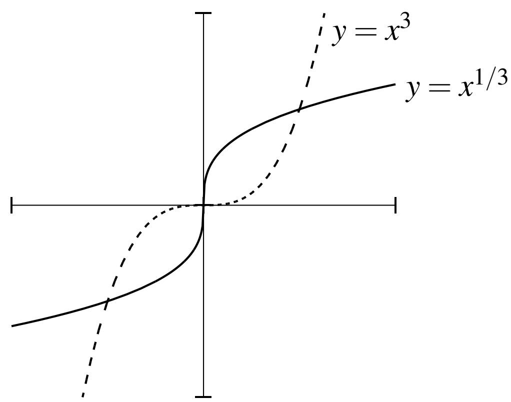

Figure 4.11: Graphs of x^3 and $x^{1/3}$.

4.4.2 Exercises

***Exercise* 4.4.1:** *Suppose $f \colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable such that $f'(x) > 0$ for all x. Show that f is invertible on the interval $J = f(\mathbb{R})$, the inverse is continuously differentiable, and $(f^{-1})'(y) > 0$ for all $y \in f(\mathbb{R})$.*

***Exercise* 4.4.2:** *Suppose I,J are intervals and a monotone onto $f \colon I \to J$ has an inverse $g \colon J \to I$. Suppose you already know that both f and g are differentiable everywhere and f' is never zero. Using chain rule but not Lemma 4.4.1 prove the formula $g'(y) = \frac{1}{f'\left(g(y)\right)}$.*

***Exercise* 4.4.3:** *Let $n \in \mathbb{N}$ be even. Prove that every $x > 0$ has a unique negative nth root. That is, there exists a negative number y such that $y^n = x$. Compute the derivative of the function $g(x) := y$.*

***Exercise* 4.4.4:** *Let $n \in \mathbb{N}$ be odd and $n \geq 3$. Prove that every x has a unique nth root. That is, there exists a number y such that $y^n = x$. Prove that the function defined by $g(x) := y$ is differentiable except at $x = 0$ and compute the derivative. Prove that g is not differentiable at $x = 0$.*

Exercise* 4.4.5** (requires §4.3):*** *Show that if in the inverse function theorem f has k continuous derivatives, then the inverse function g also has k continuous derivatives.*

Exercise* 4.4.6*: *Let $f(x) := x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) := 0$. Show that f is differentiable at all x, that $f'(0) > 0$, but that f is not invertible on any open interval containing the origin.*

Exercise* 4.4.7*:

a) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function and $k > 0$ be a number such that $f'(x) \geq k$ for all $x \in \mathbb{R}$. Show f is one-to-one and onto, and has a continuously differentiable inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$.*
b) *Find an example $f : \mathbb{R} \to \mathbb{R}$ where $f'(x) > 0$ for all x, but f is not onto.*

Exercise* 4.4.8*: *Suppose I, J are intervals and a monotone onto $f : I \to J$ has an inverse $g : J \to I$. Suppose $x \in I$ and $y := f(x) \in J$, and that g is differentiable at y. Prove:*

a) *If $g'(y) \neq 0$, then f is differentiable at x.*
b) *If $g'(y) = 0$, then f is not differentiable at x.*

Chapter 5

The Riemann Integral

5.1 The Riemann integral

Note: 1.5 lectures

An integral is a way to "sum" the values of a function. There is often confusion among students of calculus between *integral* and *antiderivative*. The integral is (informally) the area under the curve, nothing else. That we can compute an antiderivative using the integral is a nontrivial result we have to prove. In this chapter we define the *Riemann integral*[*] using the Darboux integral[†], which is technically simpler than (but equivalent to) the traditional definition of Riemann.

5.1.1 Partitions and lower and upper integrals

We want to integrate a bounded function defined on an interval $[a,b]$. We first define two auxiliary integrals that are defined for all bounded functions. Only then can we talk about the Riemann integral and the Riemann integrable functions.

Definition 5.1.1. A *partition* P of the interval $[a,b]$ is a finite set of numbers $\{x_0, x_1, x_2, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

We write

$$\Delta x_i := x_i - x_{i-1}.$$

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Let P be a partition of $[a,b]$. Define

$$\begin{aligned}
m_i &:= \inf \{ f(x) : x_{i-1} \leq x \leq x_i \}, \\
M_i &:= \sup \{ f(x) : x_{i-1} \leq x \leq x_i \}, \\
L(P,f) &:= \sum_{i=1}^{n} m_i \Delta x_i, \\
U(P,f) &:= \sum_{i=1}^{n} M_i \Delta x_i.
\end{aligned}$$

[*]Named after the German mathematician Georg Friedrich Bernhard Riemann (1826–1866).

[†]Named after the French mathematician Jean-Gaston Darboux (1842–1917).

We call $L(P,f)$ the *lower Darboux sum* and $U(P,f)$ the *upper Darboux sum*.

The geometric idea of Darboux sums is indicated in Figure 5.1. The lower sum is the area of the shaded rectangles, and the upper sum is the area of the entire rectangles, shaded plus unshaded parts. The width of the ith rectangle is Δx_i, the height of the shaded rectangle is m_i, and the height of the entire rectangle is M_i.

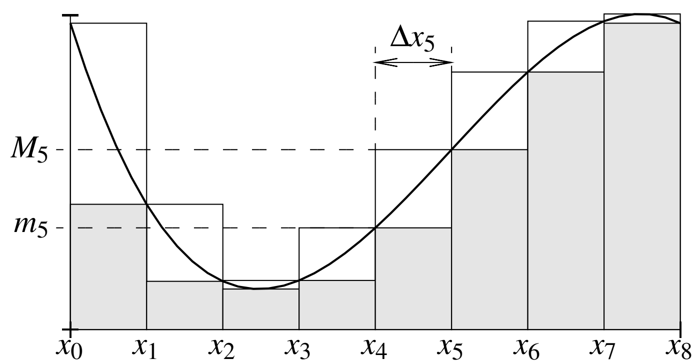

Figure 5.1: Sample Darboux sums.

Proposition 5.1.2. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in [a,b]$, we have $m \leq f(x) \leq M$. Then for every partition P of $[a,b]$,*

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a). \tag{5.1}$$

Proof. Let P be a partition. Note that $m \leq m_i$ for all i and $M_i \leq M$ for all i. Also $m_i \leq M_i$ for all i. Finally, $\sum_{i=1}^{n} \Delta x_i = (b-a)$. Therefore,

$$m(b-a) = m\left(\sum_{i=1}^{n} \Delta x_i\right) = \sum_{i=1}^{n} m\Delta x_i \leq \sum_{i=1}^{n} m_i \Delta x_i \leq$$

$$\leq \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} M \Delta x_i = M\left(\sum_{i=1}^{n} \Delta x_i\right) = M(b-a).$$

Hence we get (5.1). In particular, the set of lower and upper sums are bounded sets. $\square$

Definition 5.1.3. As the sets of lower and upper Darboux sums are bounded, we define

$$\underline{\int_a^b} f(x)\,dx := \sup\{L(P,f) : P \text{ a partition of } [a,b]\},$$

$$\overline{\int_a^b} f(x)\,dx := \inf\{U(P,f) : P \text{ a partition of } [a,b]\}.$$

We call $\underline{\int}$ the *lower Darboux integral* and $\overline{\int}$ the *upper Darboux integral*. To avoid worrying about the variable of integration, we often simply write

$$\underline{\int_a^b} f := \underline{\int_a^b} f(x)\,dx \qquad \text{and} \qquad \overline{\int_a^b} f := \overline{\int_a^b} f(x)\,dx.$$

If integration is to make sense, then the lower and upper Darboux integrals should be the same number, as we want a single number to call *the integral*. However, these two integrals may differ for some functions.

Example 5.1.4: Take the Dirichlet function $f\colon [0,1] \to \mathbb{R}$, where $f(x) := 1$ if $x \in \mathbb{Q}$ and $f(x) := 0$ if $x \notin \mathbb{Q}$. Then

$$\underline{\int_0^1} f = 0 \qquad \text{and} \qquad \overline{\int_0^1} f = 1.$$

The reason is that for every i, we have $m_i = \inf\{f(x) : x \in [x_{i-1},x_i]\} = 0$ and $M_i = \sup\{f(x) : x \in [x_{i-1},x_i]\} = 1$. Thus

$$\begin{aligned} L(P,f) &= \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0, \\ U(P,f) &= \sum_{i=1}^{n} 1 \cdot \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1. \end{aligned}$$

Remark 5.1.5. The same definition of $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ is used when f is defined on a larger set S such that $[a,b] \subset S$. In that case, we use the restriction of f to $[a,b]$ and we must ensure that the restriction is bounded on $[a,b]$.

To compute the integral, we often take a partition P and make it finer. That is, we cut intervals in the partition into yet smaller pieces.

Definition 5.1.6. Let $P = \{x_0,x_1,\ldots,x_n\}$ and $\widetilde{P} = \{\widetilde{x}_0,\widetilde{x}_1,\ldots,\widetilde{x}_\ell\}$ be partitions of $[a,b]$. We say $\widetilde{P}$ is a *refinement* of P if as sets $P \subset \widetilde{P}$.

That is, $\widetilde{P}$ is a refinement of a partition if it contains all the numbers in P and perhaps some other numbers in between. For example, $\{0,0.5,1,2\}$ is a partition of $[0,2]$ and $\{0,0.2,0.5,1,1.5,1.75,2\}$ is a refinement. The main reason for introducing refinements is the following proposition.

Proposition 5.1.7. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function, and let P be a partition of $[a,b]$. Let $\widetilde{P}$ be a refinement of P. Then*

$$L(P,f) \leq L(\widetilde{P},f) \qquad \textit{and} \qquad U(\widetilde{P},f) \leq U(P,f).$$

Proof. The tricky part of this proof is to get the notation correct. Let $\widetilde{P} = \{\widetilde{x}_0,\widetilde{x}_1,\ldots,\widetilde{x}_\ell\}$ be a refinement of $P = \{x_0,x_1,\ldots,x_n\}$. Then $x_0 = \widetilde{x}_0$ and $x_n = \widetilde{x}_\ell$. In fact, there are integers $k_0 < k_1 < \cdots < k_n$ such that $x_j = \widetilde{x}_{k_j}$ for $j = 0,1,2,\ldots,n$.

Let $\Delta\widetilde{x}_p := \widetilde{x}_p - \widetilde{x}_{p-1}$. See Figure 5.2. We get

$$\Delta x_j = x_j - x_{j-1} = \widetilde{x}_{k_j} - \widetilde{x}_{k_{j-1}} = \sum_{p=k_{j-1}+1}^{k_j} \widetilde{x}_p - \widetilde{x}_{p-1} = \sum_{p=k_{j-1}+1}^{k_j} \Delta\widetilde{x}_p.$$

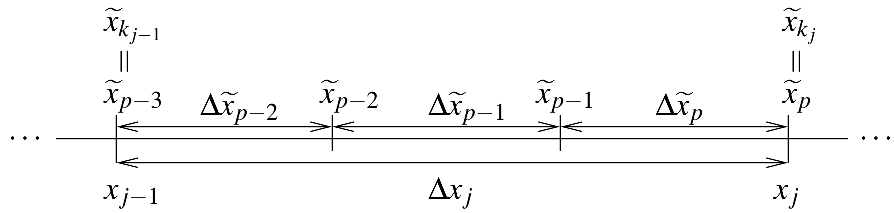

Figure 5.2: Refinement of a subinterval. Notice $\Delta x_j = \Delta\widetilde{x}_{p-2} + \Delta\widetilde{x}_{p-1} + \Delta\widetilde{x}_p$, and also $k_{j-1}+1 = p-2$ and $k_j = p$.

Let m_j be as before and correspond to the partition P. Let $\widetilde{m}_j := \inf\{f(x) : \widetilde{x}_{j-1} \leq x \leq \widetilde{x}_j\}$. Now, $m_j \leq \widetilde{m}_p$ for $k_{j-1} < p \leq k_j$. Therefore,

$$m_j \Delta x_j = m_j \sum_{p=k_{j-1}+1}^{k_j} \Delta\widetilde{x}_p = \sum_{p=k_{j-1}+1}^{k_j} m_j \Delta\widetilde{x}_p \leq \sum_{p=k_{j-1}+1}^{k_j} \widetilde{m}_p \Delta\widetilde{x}_p.$$

So

$$L(P,f) = \sum_{j=1}^{n} m_j \Delta x_j \leq \sum_{j=1}^{n} \sum_{p=k_{j-1}+1}^{k_j} \widetilde{m}_p \Delta\widetilde{x}_p = \sum_{j=1}^{\ell} \widetilde{m}_j \Delta\widetilde{x}_j = L(\widetilde{P},f).$$

The proof of $U(\widetilde{P},f) \leq U(P,f)$ is left as an exercise. □

Armed with refinements we prove the following. The key point of this next proposition is that the lower Darboux integral is less than or equal to the upper Darboux integral.

Proposition 5.1.8. *Let $f \colon [a,b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in [a,b]$, we have $m \leq f(x) \leq M$. Then*

$$m(b-a) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq M(b-a). \tag{5.2}$$

Proof. By Proposition 5.1.2, for every partition P,

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a).$$

The inequality $m(b-a) \leq L(P,f)$ implies $m(b-a) \leq \underline{\int_a^b} f$. The inequality $U(P,f) \leq M(b-a)$ implies $\overline{\int_a^b} f \leq M(b-a)$.

The middle inequality in (5.2) is the main point of this proposition. Let P_1, P_2 be partitions of $[a,b]$. Define $\widetilde{P} := P_1 \cup P_2$. The set $\widetilde{P}$ is a partition of $[a,b]$, which is a refinement of P_1 and a refinement of P_2. By Proposition 5.1.7, $L(P_1,f) \leq L(\widetilde{P},f)$ and $U(\widetilde{P},f) \leq U(P_2,f)$. So

$$L(P_1,f) \leq L(\widetilde{P},f) \leq U(\widetilde{P},f) \leq U(P_2,f).$$

In other words, for two arbitrary partitions P_1 and P_2, we have $L(P_1,f) \leq U(P_2,f)$. Recall Proposition 1.2.7, and take the supremum and infimum over all partitions:

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \text{ a partition of } [a,b]\} \leq \inf\{U(P,f) : P \text{ a partition of } [a,b]\} = \overline{\int_a^b} f. \qquad \square$$

5.1.2 Riemann integral

We can finally define the Riemann integral. However, the Riemann integral is only defined on a certain class of functions, called the Riemann integrable functions.

Definition 5.1.9. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function such that

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Then f is said to be *Riemann integrable*. The set of Riemann integrable functions on $[a,b]$ is denoted by $\mathscr{R}[a,b]$. When $f \in \mathscr{R}[a,b]$, we define

$$\int_a^b f(x)\,dx := \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

As before, we often write

$$\int_a^b f := \int_a^b f(x)\,dx.$$

The number $\int_a^b f$ is called the *Riemann integral* of f, or sometimes simply the *integral* of f.

By definition, a Riemann integrable function is bounded. Appealing to Proposition 5.1.8, we immediately obtain the following proposition. See also Figure 5.3.

Proposition 5.1.10. *Let $f\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function. Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in [a,b]$. Then*

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

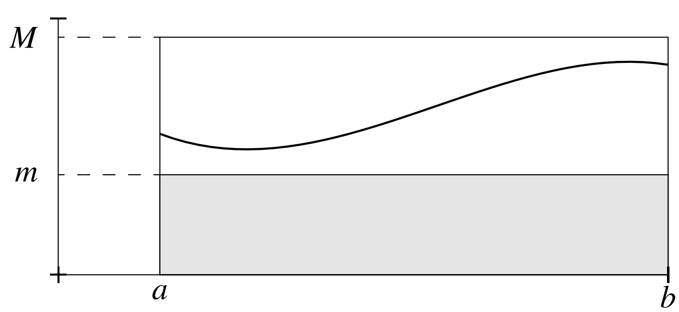

Figure 5.3: The area under the curve is bounded from above by the area of the entire rectangle, $M(b-a)$, and from below by the area of the shaded part, $m(b-a)$.

Often we use a weaker form of this proposition. That is, if $|f(x)| \leq M$ for all $x \in [a,b]$, then

$$\left| \int_a^b f \right| \leq M(b-a).$$

Example 5.1.11: We integrate constant functions using Proposition 5.1.8. If $f(x) := c$ for some constant c, then we take $m = M = c$. In inequality (5.2) all the inequalities must be equalities. Thus f is integrable on $[a,b]$ and $\int_a^b f = c(b-a)$.

Example 5.1.12: Let $f : [0,2] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 1 & \text{if } x < 1, \\ 1/2 & \text{if } x = 1, \\ 0 & \text{if } x > 1. \end{cases}$$

We claim f is Riemann integrable and $\int_0^2 f = 1$.

Proof: Let $0 < \varepsilon < 1$ be arbitrary. Let $P := \{0, 1-\varepsilon, 1+\varepsilon, 2\}$ be a partition. We use the notation from the definition of the Darboux sums. Then

$$\begin{aligned} m_1 &= \inf\{f(x) : x \in [0, 1-\varepsilon]\} = 1, & M_1 &= \sup\{f(x) : x \in [0, 1-\varepsilon]\} = 1, \\ m_2 &= \inf\{f(x) : x \in [1-\varepsilon, 1+\varepsilon]\} = 0, & M_2 &= \sup\{f(x) : x \in [1-\varepsilon, 1+\varepsilon]\} = 1, \\ m_3 &= \inf\{f(x) : x \in [1+\varepsilon, 2]\} = 0, & M_3 &= \sup\{f(x) : x \in [1+\varepsilon, 2]\} = 0. \end{aligned}$$

Furthermore, $\Delta x_1 = 1-\varepsilon$, $\Delta x_2 = 2\varepsilon$ and $\Delta x_3 = 1-\varepsilon$. See Figure 5.4.

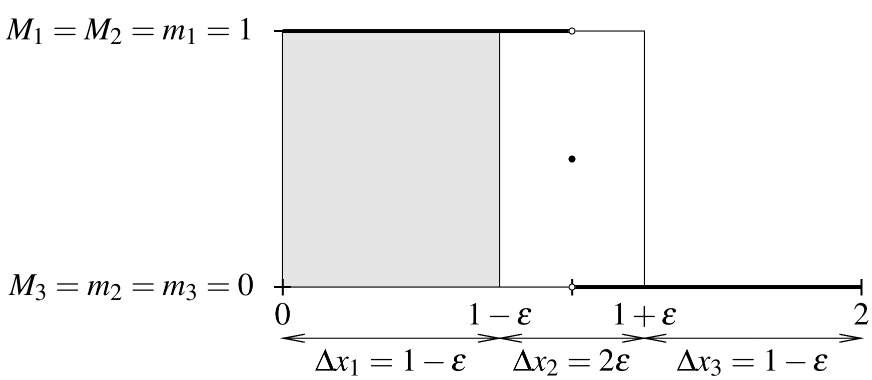

Figure 5.4: Darboux sums for the step function. $L(P,f)$ is the area of the shaded rectangle, $U(P,f)$ is the area of both rectangles, and $U(P,f) - L(P,f)$ is the area of the unshaded rectangle.

We compute

$$L(P,f) = \sum_{i=1}^{3} m_i \Delta x_i = 1 \cdot (1-\varepsilon) + 0 \cdot 2\varepsilon + 0 \cdot (1-\varepsilon) = 1-\varepsilon,$$

$$U(P,f) = \sum_{i=1}^{3} M_i \Delta x_i = 1 \cdot (1-\varepsilon) + 1 \cdot 2\varepsilon + 0 \cdot (1-\varepsilon) = 1+\varepsilon.$$

Thus,

$$\overline{\int_0^2} f - \underline{\int_0^2} f \leq U(P,f) - L(P,f) = (1+\varepsilon) - (1-\varepsilon) = 2\varepsilon.$$

By Proposition 5.1.8, we have $\underline{\int_0^2} f \leq \overline{\int_0^2} f$. As ε was arbitrary, $\overline{\int_0^2} f = \underline{\int_0^2} f$. So f is Riemann integrable. Finally,

$$1 - \varepsilon = L(P, f) \leq \int_0^2 f \leq U(P, f) = 1 + \varepsilon.$$

Hence, $\left| \int_0^2 f - 1 \right| \leq \varepsilon$. As ε was arbitrary, we conclude $\int_0^2 f = 1$.

It may be worthwhile to extract part of the technique of the example into a proposition.

Proposition 5.1.13. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof. If for every $\varepsilon > 0$ such a P exists, then

$$0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P, f) - L(P, f) < \varepsilon.$$

Therefore, $\overline{\int_a^b} f = \underline{\int_a^b} f$, and f is integrable. $\square$

Example 5.1.14: Let us show $\frac{1}{1+x}$ is integrable on $[0,b]$ for all $b > 0$. We will see later that continuous functions are integrable, but let us demonstrate how we do it directly.

Let $\varepsilon > 0$ be given. Take $n \in \mathbb{N}$ and pick $x_j := {jb}/{n}$, to form the partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[0,b]$. We have $\Delta x_j = {b}/{n}$ for all j. As f is decreasing, for every subinterval $[x_{j-1}, x_j]$, we obtain

$$m_j = \inf \left\{ \frac{1}{1+x} : x \in [x_{j-1}, x_j] \right\} = \frac{1}{1+x_j}, \qquad M_j = \sup \left\{ \frac{1}{1+x} : x \in [x_{j-1}, x_j] \right\} = \frac{1}{1+x_{j-1}}.$$

Then

$$U(P, f) - L(P, f) = \sum_{j=1}^{n} \Delta x_j (M_j - m_j) = \frac{b}{n} \sum_{j=1}^{n} \left(\frac{1}{1 + {(j-1)b}/{n}} - \frac{1}{1 + {jb}/{n}} \right) =$$

$$= \frac{b}{n} \left(\frac{1}{1 + {0b}/{n}} - \frac{1}{1 + {nb}/{n}} \right) = \frac{b^2}{n(b+1)}.$$

The sum telescopes, the terms successively cancel each other, something we have seen before. Picking n to be such that $\frac{b^2}{n(b+1)} < \varepsilon$, the proposition is satisfied, and the function is integrable.

Remark 5.1.15. A way of thinking of the integral is that it adds up (integrates) lots of local information—it sums $f(x)\,dx$ over all x. The integral sign was chosen by Leibniz to be the long S to mean summation. Unlike derivatives, which are "local," integrals show up in applications when one wants a "global" answer: total distance travelled, average temperature, total charge, etc.

5.1.3 More notation

When $f: S \to \mathbb{R}$ is defined on a larger set S and $[a,b] \subset S$, we say f is Riemann integrable on $[a,b]$ if the restriction of f to $[a,b]$ is Riemann integrable. In this case, we say $f \in \mathscr{R}[a,b]$, and we write $\int_a^b f$ to mean the Riemann integral of the restriction of f to $[a,b]$.

It is useful to define the integral $\int_a^b f$ even if $a \not< b$. Suppose $b < a$ and $f \in \mathscr{R}[b,a]$, then define

$$\int_a^b f := -\int_b^a f.$$

For any function f, define

$$\int_a^a f := 0.$$

At times, the variable x may already have some other meaning. When we need to write down the variable of integration, we may simply use a different letter. For example,

$$\int_a^b f(s)\,ds := \int_a^b f(x)\,dx.$$

5.1.4 Exercises

***Exercise* 5.1.1:** *Define $f: [0,1] \to \mathbb{R}$ by $f(x) := x^3$ and let $P := \{0, 0.1, 0.4, 1\}$. Compute $L(P,f)$ and $U(P,f)$.*

***Exercise* 5.1.2:** *Let $f: [0,1] \to \mathbb{R}$ be defined by $f(x) := x$. Show that $f \in \mathscr{R}[0,1]$ and compute $\int_0^1 f$ using the definition of the integral (but feel free to use the propositions of this section).*

***Exercise* 5.1.3:** *Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Suppose there exists a sequence of partitions $\{P_k\}$ of $[a,b]$ such that*

$$\lim_{k\to\infty} \bigl(U(P_k,f) - L(P_k,f)\bigr) = 0.$$

Show that f is Riemann integrable and that

$$\int_a^b f = \lim_{k\to\infty} U(P_k,f) = \lim_{k\to\infty} L(P_k,f).$$

***Exercise* 5.1.4:** *Finish the proof of Proposition 5.1.7.*

***Exercise* 5.1.5:** *Suppose $f: [-1,1] \to \mathbb{R}$ is defined as*

$$f(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Prove that $f \in \mathscr{R}[-1,1]$ and compute $\int_{-1}^1 f$ using the definition of the integral (but feel free to use the propositions of this section).

***Exercise* 5.1.6:** *Let $c \in (a,b)$ and let $d \in \mathbb{R}$. Define $f: [a,b] \to \mathbb{R}$ as*

$$f(x) := \begin{cases} d & \text{if } x = c, \\ 0 & \text{if } x \neq c. \end{cases}$$

Prove that $f \in \mathscr{R}[a,b]$ and compute $\int_a^b f$ using the definition of the integral (but feel free to use the propositions of this section).

***Exercise* 5.1.7:** *Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Let $\varepsilon > 0$ be given. Then show that there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that for every set of numbers $\{c_1, c_2, \ldots, c_n\}$ with $c_k \in [x_{k-1}, x_k]$ for all k, we have*

$$\left| \int_a^b f - \sum_{k=1}^{n} f(c_k) \Delta x_k \right| < \varepsilon.$$

***Exercise* 5.1.8:** *Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then define $g(x) := f(\alpha x + \beta)$ on the interval $I = [\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}]$. Show that g is Riemann integrable on I.*

***Exercise* 5.1.9:** *Suppose $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are such that for all $x \in (0,1]$, we have $f(x) = g(x)$. Suppose f is Riemann integrable. Prove g is Riemann integrable and $\int_0^1 f = \int_0^1 g$.*

***Exercise* 5.1.10:** *Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Let $P_n = \{x_0, x_1, \ldots, x_n\}$ be a uniform partition of $[0,1]$, that is, $x_j = j/n$. Is $\{L(P_n, f)\}_{n=1}^{\infty}$ always monotone? Yes/No: Prove or find a counterexample.*

***Exercise* 5.1.11** (Challenging)**:** *For a bounded function $f : [0,1] \to \mathbb{R}$, let $R_n := (1/n) \sum_{j=1}^{n} f(j/n)$ (the uniform right-hand rule).*

a) If f is Riemann integrable show $\int_0^1 f = \lim R_n$.

b) Find an f that is not Riemann integrable, but $\lim R_n$ exists.

***Exercise* 5.1.12** (Challenging)**:** *Generalize the previous exercise. Show that $f \in \mathscr{R}[a,b]$ if and only if there exists an $I \in \mathbb{R}$, such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if P is a partition with $\Delta x_i < \delta$ for all i, then $|L(P,f) - I| < \varepsilon$ and $|U(P,f) - I| < \varepsilon$. If $f \in \mathscr{R}[a,b]$, then $I = \int_a^b f$.*

***Exercise* 5.1.13:** *Using Exercise 5.1.12 and the idea of the proof in Exercise 5.1.7, show that Darboux integral is the same as the standard definition of Riemann integral, which you have most likely seen in calculus. That is, show that $f \in \mathscr{R}[a,b]$ if and only if there exists an $I \in \mathbb{R}$, such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $P = \{x_0, x_1, \ldots, x_n\}$ is a partition with $\Delta x_i < \delta$ for all i, then $\left|\sum_{i=1}^{n} f(c_i) \Delta x_i - I\right| < \varepsilon$ for every set $\{c_1, c_2, \ldots, c_n\}$ with $c_i \in [x_{i-1}, x_i]$. If $f \in \mathscr{R}[a,b]$, then $I = \int_a^b f$.*

***Exercise* 5.1.14** (Challenging)**:** *Construct functions f and g, where $f : [0,1] \to \mathbb{R}$ is Riemann integrable, $g : [0,1] \to [0,1]$ is one-to-one and onto, and such that the composition $f \circ g$ is not Riemann integrable.*

5.2 Properties of the integral

Note: 2 lectures, integrability of functions with discontinuities can safely be skipped

5.2.1 Additivity

Adding a bunch of things in two parts and then adding those two parts should be the same as adding everything all at once. The corresponding property for integral is called the additive property of the integral. First, we prove the additivity property for the lower and upper Darboux integrals.

Lemma 5.2.1. *Suppose $a < b < c$ and $f \colon [a,c] \to \mathbb{R}$ is a bounded function. Then*

$$\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f$$

and

$$\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f.$$

Proof. If we have partitions $P_1 = \{x_0, x_1, \ldots, x_k\}$ of $[a,b]$ and $P_2 = \{x_k, x_{k+1}, \ldots, x_n\}$ of $[b,c]$, then the set $P := P_1 \cup P_2 = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,c]$. We find

$$L(P,f) = \sum_{j=1}^{n} m_j \Delta x_j = \sum_{j=1}^{k} m_j \Delta x_j + \sum_{j=k+1}^{n} m_j \Delta x_j = L(P_1,f) + L(P_2,f).$$

When we take the supremum of the right-hand side over all P_1 and P_2, we are taking a supremum of the left-hand side over all partitions P of $[a,c]$ that contain b. If Q is a partition of $[a,c]$ and $P = Q \cup \{b\}$, then P is a refinement of Q and so $L(Q,f) \leq L(P,f)$. Therefore, taking a supremum only over the P that contain b is sufficient to find the supremum of $L(P,f)$ over all partitions P, see Exercise 1.1.9. Finally, recall Exercise 1.2.9 to compute

$$\begin{aligned}
\underline{\int_a^c} f &= \sup \bigl\{ L(P,f) : P \text{ a partition of } [a,c] \bigr\} \\
&= \sup \bigl\{ L(P,f) : P \text{ a partition of } [a,c], b \in P \bigr\} \\
&= \sup \bigl\{ L(P_1,f) + L(P_2,f) : P_1 \text{ a partition of } [a,b], P_2 \text{ a partition of } [b,c] \bigr\} \\
&= \sup \bigl\{ L(P_1,f) : P_1 \text{ a partition of } [a,b] \bigr\} + \sup \bigl\{ L(P_2,f) : P_2 \text{ a partition of } [b,c] \bigr\} \\
&= \underline{\int_a^b} f + \underline{\int_b^c} f.
\end{aligned}$$

Similarly, for P, P_1, and P_2 as above, we obtain

$$U(P,f) = \sum_{j=1}^{n} M_j \Delta x_j = \sum_{j=1}^{k} M_j \Delta x_j + \sum_{j=k+1}^{n} M_j \Delta x_j = U(P_1,f) + U(P_2,f).$$

We wish to take the infimum on the right over all P_1 and P_2, and so we are taking the infimum over all partitions P of $[a,c]$ that contain b. If Q is a partition of $[a,c]$ and $P = Q \cup \{b\}$, then P is a refinement of Q and so $U(Q,f) \geq U(P,f)$. Therefore, taking an infimum only over the P that contain b is sufficient to find the infimum of $U(P,f)$ for all P. We obtain

$$\overline{\int_a^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f. \qquad \square$$

Proposition 5.2.2. *Let $a < b < c$. A function $f\colon [a,c] \to \mathbb{R}$ is Riemann integrable if and only if f is Riemann integrable on $[a,b]$ and $[b,c]$. If f is Riemann integrable, then*

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Proof. Suppose $f \in \mathscr{R}[a,c]$, then $\overline{\int_a^c} f = \underline{\int_a^c} f = \int_a^c f$. We apply the lemma to get

$$\int_a^c f = \underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f \leq \overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f = \int_a^c f.$$

Thus the inequality is an equality,

$$\underline{\int_a^b} f + \underline{\int_b^c} f = \overline{\int_a^b} f + \overline{\int_b^c} f.$$

As we also know $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ and $\underline{\int_b^c} f \leq \overline{\int_b^c} f$, we conclude

$$\underline{\int_a^b} f = \overline{\int_a^b} f \qquad \text{and} \qquad \underline{\int_b^c} f = \overline{\int_b^c} f.$$

Thus f is Riemann integrable on $[a,b]$ and $[b,c]$ and the desired formula holds.

Now assume f is Riemann integrable on $[a,b]$ and on $[b,c]$. Again apply the lemma to get

$$\underline{\int_a^c} f = \underline{\int_a^b} f + \underline{\int_b^c} f = \int_a^b f + \int_b^c f = \overline{\int_a^b} f + \overline{\int_b^c} f = \overline{\int_a^c} f.$$

Therefore, f is Riemann integrable on $[a,c]$, and the integral is computed as indicated. $\square$

An easy consequence of the additivity is the following corollary. We leave the details to the reader as an exercise.

Corollary 5.2.3. *If $f \in \mathscr{R}[a,b]$ and $[c,d] \subset [a,b]$, then the restriction $f|_{[c,d]}$ is in $\mathscr{R}[c,d]$.*

5.2.2 Linearity and monotonicity

A sum is a linear function of the summands. So is the integral.

Proposition 5.2.4 (Linearity)**.** *Let f and g be in $\mathscr{R}[a,b]$ and $\alpha \in \mathbb{R}$.*

(i) αf is in $\mathscr{R}[a,b]$ and

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

(ii) $f+g$ is in $\mathscr{R}[a,b]$ and

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof. Let us prove the first item for $\alpha \geq 0$. Let P be a partition of $[a,b]$. Let $m_i := \inf\{f(x) : x \in [x_{i-1},x_i]\}$ as usual. Since α is nonnegative, we can move the multiplication by α past the infimum,

$$\inf\{\alpha f(x) : x \in [x_{i-1},x_i]\} = \alpha \inf\{f(x) : x \in [x_{i-1},x_i]\} = \alpha m_i.$$

Therefore,

$$L(P,\alpha f) = \sum_{i=1}^{n} \alpha m_i \Delta x_i = \alpha \sum_{i=1}^{n} m_i \Delta x_i = \alpha L(P,f).$$

Similarly,

$$U(P,\alpha f) = \alpha U(P,f).$$

Again, as $\alpha \geq 0$ we may move multiplication by α past the supremum. Hence,

$$\begin{aligned}
\underline{\int_a^b} \alpha f(x)\,dx &= \sup\big\{L(P,\alpha f) : P \text{ a partition of } [a,b]\big\} \\
&= \sup\big\{\alpha L(P,f) : P \text{ a partition of } [a,b]\big\} \\
&= \alpha \sup\big\{L(P,f) : P \text{ a partition of } [a,b]\big\} \\
&= \alpha \underline{\int_a^b} f(x)\,dx.
\end{aligned}$$

Similarly, we show

$$\overline{\int_a^b} \alpha f(x)\,dx = \alpha \overline{\int_a^b} f(x)\,dx.$$

The conclusion now follows for $\alpha \geq 0$.

To finish the proof of the first item, we need to show that $-f$ is Riemann integrable and $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$. The proof of this fact is left as Exercise 5.2.1.

The proof of the second item is left as Exercise 5.2.2. It is not difficult, but it is not as trivial as it may appear at first glance. □

The second item in the proposition does not hold with equality for the Darboux integrals, but we do obtain inequalities. The proof of the following proposition is Exercise 5.2.16. It follows for upper and lower sums on a fixed partition by Exercise 1.3.7, that is, supremum of a sum is less than or equal to the sum of suprema and similarly for infima.

Proposition 5.2.5. *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded functions. Then*

$$\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g, \qquad \text{and} \qquad \underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g.$$

Adding up smaller numbers should give us a smaller result. That is true for an integral as well.

Proposition 5.2.6 (Monotonicity)**.** *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded, and $f(x) \leq g(x)$ for all $x \in [a,b]$. Then*

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g \qquad \text{and} \qquad \overline{\int_a^b} f \leq \overline{\int_a^b} g.$$

Moreover, if f and g are in $\mathscr{R}[a,b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Then let

$$m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} \qquad \text{and} \qquad \widetilde{m}_i := \inf\{g(x) : x \in [x_{i-1}, x_i]\}.$$

As $f(x) \leq g(x)$, then $m_i \leq \widetilde{m}_i$. Therefore,

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} \widetilde{m}_i \Delta x_i = L(P,g).$$

We take the supremum over all P (see Proposition 1.3.7) to obtain

$$\underline{\int_a^b} f \leq \underline{\int_a^b} g.$$

Similarly, we obtain the same conclusion for the upper integrals. Finally, if f and g are Riemann integrable all the integrals are equal, and the conclusion follows. $\square$

5.2.3 Continuous functions

Let us show that continuous functions are Riemann integrable. In fact, we can even allow some discontinuities. We start with a function continuous on the whole closed interval $[a,b]$.

Lemma 5.2.7. *If $f \colon [a,b] \to \mathbb{R}$ is a continuous function, then $f \in \mathscr{R}[a,b]$.*

Proof. As f is continuous on a closed bounded interval, it is uniformly continuous. Let $\varepsilon > 0$ be given. Find a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that $\Delta x_i < \delta$ for all $i = 1, 2, \ldots, n$. For example, take n such that $\frac{b-a}{n} < \delta$, and let $x_i := \frac{i}{n}(b-a) + a$. Then for all $x, y \in [x_{i-1}, x_i]$, we have $|x-y| \leq \Delta x_i < \delta$, and so

$$f(x) - f(y) \leq |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

As f is continuous on $[x_{i-1}, x_i]$, it attains a maximum and a minimum on this interval. Let x be a point where f attains the maximum and y be a point where f attains the minimum. Then $f(x) = M_i$ and $f(y) = m_i$ in the notation from the definition of the integral. Therefore,

$$M_i - m_i = f(x) - f(y) < \frac{\varepsilon}{b-a}.$$

And so

$$\begin{aligned}
\overline{\int_a^b} f - \underline{\int_a^b} f &\leq U(P,f) - L(P,f) \\
&= \left(\sum_{i=1}^{n} M_i \Delta x_i \right) - \left(\sum_{i=1}^{n} m_i \Delta x_i \right) \\
&= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \\
&< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i \\
&= \frac{\varepsilon}{b-a}(b-a) = \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary,

$$\overline{\int_a^b} f = \underline{\int_a^b} f,$$

and f is Riemann integrable on $[a,b]$. $\square$

The second lemma says that we need the function to only be "Riemann integrable inside the interval," as long as it is bounded. It also tells us how to compute the integral.

Lemma 5.2.8. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function, $\{a_n\}$ and $\{b_n\}$ be sequences such that $a < a_n < b_n < b$ for all n, with $\lim a_n = a$ and $\lim b_n = b$. Suppose $f \in \mathscr{R}[a_n,b_n]$ for all n. Then $f \in \mathscr{R}[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty} \int_{a_n}^{b_n} f.$$

Proof. Let $M > 0$ be a real number such that $|f(x)| \leq M$. As $(b-a) \geq (b_n - a_n)$,

$$-M(b-a) \leq -M(b_n - a_n) \leq \int_{a_n}^{b_n} f \leq M(b_n - a_n) \leq M(b-a).$$

Therefore, the sequence of numbers $\left\{\int_{a_n}^{b_n} f\right\}_{n=1}^{\infty}$ is bounded and by Bolzano–Weierstrass has a convergent subsequence indexed by n_k. Let us call L the limit of the subsequence $\left\{\int_{a_{n_k}}^{b_{n_k}} f\right\}_{k=1}^{\infty}$.

Lemma 5.2.1 says that the lower and upper integral are additive and the hypothesis says that f is integrable on $[a_{n_k}, b_{n_k}]$. Therefore

$$\underline{\int_a^b} f = \underline{\int_a^{a_{n_k}}} f + \int_{a_{n_k}}^{b_{n_k}} f + \underline{\int_{b_{n_k}}^b} f \geq -M(a_{n_k} - a) + \int_{a_{n_k}}^{b_{n_k}} f - M(b - b_{n_k}).$$

We take the limit as k goes to ∞ on the right-hand side,

$$\underline{\int_a^b} f \geq -M \cdot 0 + L - M \cdot 0 = L.$$

Next we use additivity of the upper integral,

$$\overline{\int_a^b} f = \overline{\int_a^{a_{n_k}}} f + \int_{a_{n_k}}^{b_{n_k}} f + \overline{\int_{b_{n_k}}^b} f \leq M(a_{n_k} - a) + \int_{a_{n_k}}^{b_{n_k}} f + M(b - b_{n_k}).$$

We take the same subsequence $\{\int_{a_{n_k}}^{b_{n_k}} f\}_{k=1}^{\infty}$ and take the limit to obtain

$$\overline{\int_a^b} f \leq M \cdot 0 + L + M \cdot 0 = L.$$

Thus $\overline{\int_a^b} f = \underline{\int_a^b} f = L$ and hence f is Riemann integrable and $\int_a^b f = L$. In particular, no matter what subsequence we chose, the L is the same number.

To prove the final statement of the lemma we use Proposition 2.3.7. We have shown that every convergent subsequence $\left\{\int_{a_{n_k}}^{b_{n_k}} f\right\}$ converges to $L = \int_a^b f$. Therefore, the sequence $\left\{\int_{a_n}^{b_n} f\right\}$ is convergent and converges to $\int_a^b f$. $\square$

We say a function $f : [a,b] \to \mathbb{R}$ has *finitely many discontinuities* if there exists a finite set $S = \{x_1, x_2, \ldots, x_n\} \subset [a,b]$, and f is continuous at all points of $[a,b] \setminus S$.

Theorem 5.2.9. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function with finitely many discontinuities. Then $f \in \mathscr{R}[a,b]$.*

Proof. We divide the interval into finitely many intervals $[a_i, b_i]$ so that f is continuous on the interior (a_i, b_i). If f is continuous on (a_i, b_i), then it is continuous and hence integrable on $[c_i, d_i]$ whenever $a_i < c_i < d_i < b_i$. By Lemma 5.2.8 the restriction of f to $[a_i, b_i]$ is integrable. By additivity of the integral (and induction) f is integrable on the union of the intervals. $\square$

5.2.4 More on integrable functions

Sometimes it is convenient (or necessary) to change certain values of a function and then integrate. The next result says that if we change the values at finitely many points, the integral does not change.

Proposition 5.2.10. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Let $g : [a,b] \to \mathbb{R}$ be such that $f(x) = g(x)$ for all $x \in [a,b] \setminus S$, where S is a finite set. Then g is a Riemann integrable function and*

$$\int_a^b g = \int_a^b f.$$

Sketch of proof. Using additivity of the integral, we split up the interval $[a,b]$ into smaller intervals such that $f(x) = g(x)$ holds for all x except at the endpoints (details are left to the reader).

Therefore, without loss of generality suppose $f(x) = g(x)$ for all $x \in (a,b)$. The proof follows by Lemma 5.2.8, and is left as Exercise 5.2.3. $\square$

Finally, monotone (increasing or decreasing) functions are always Riemann integrable. The proof is left to the reader as part of Exercise 5.2.14.

Proposition 5.2.11. *Let $f : [a,b] \to \mathbb{R}$ be a monotone function. Then $f \in \mathscr{R}[a,b]$.*

5.2.5 Exercises

***Exercise* 5.2.1:** *Finish the proof of the first part of Proposition 5.2.4. Let f be in $\mathscr{R}[a,b]$. Prove that $-f$ is in $\mathscr{R}[a,b]$ and*

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx.$$

***Exercise* 5.2.2:** *Prove the second part of Proposition 5.2.4. Let f and g be in $\mathscr{R}[a,b]$. Prove, without using Proposition 5.2.5, that $f+g$ is in $\mathscr{R}[a,b]$ and*

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Hint: One way to do it is to use Proposition 5.1.7 to find a single partition P such that $U(P,f) - L(P,f) < \nicefrac{\varepsilon}{2}$ and $U(P,g) - L(P,g) < \nicefrac{\varepsilon}{2}$.

Exercise* 5.2.3*: *Let $f: [a,b] \to \mathbb{R}$ be Riemann integrable, and $g: [a,b] \to \mathbb{R}$ be such that $f(x) = g(x)$ for all $x \in (a,b)$. Prove that g is Riemann integrable and that*

$$\int_a^b g = \int_a^b f.$$

Exercise* 5.2.4*: *Prove the* mean value theorem for integrals*: If $f: [a,b] \to \mathbb{R}$ is continuous, then there exists a $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.*

Exercise* 5.2.5*: *Let $f: [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f = 0$. Prove that $f(x) = 0$ for all x.*

Exercise* 5.2.6*: *Let $f: [a,b] \to \mathbb{R}$ be a continuous function and $\int_a^b f = 0$. Prove that there exists a $c \in [a,b]$ such that $f(c) = 0$. (Compare with the previous exercise.)*

Exercise* 5.2.7*: *Let $f: [a,b] \to \mathbb{R}$ and $g: [a,b] \to \mathbb{R}$ be continuous functions such that $\int_a^b f = \int_a^b g$. Show that there exists a $c \in [a,b]$ such that $f(c) = g(c)$.*

Exercise* 5.2.8*: *Let $f \in \mathscr{R}[a,b]$. Let α, β, γ be arbitrary numbers in $[a,b]$ (not necessarily ordered in any way). Prove*

$$\int_\alpha^\gamma f = \int_\alpha^\beta f + \int_\beta^\gamma f.$$

Recall what $\int_a^b f$ means if $b \leq a$.

Exercise* 5.2.9*: *Prove* Corollary 5.2.3.

Exercise* 5.2.10*: *Suppose $f: [a,b] \to \mathbb{R}$ is bounded and has finitely many discontinuities. Show that as a function of x the expression $|f(x)|$ is bounded with finitely many discontinuities and is thus Riemann integrable. Then show*

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

Exercise* 5.2.11** (Hard):*** *Show that the Thomae or popcorn function (see Example 3.2.12) is Riemann integrable. Therefore, there exists a function discontinuous at all rational numbers (a dense set) that is Riemann integrable.*

In particular, define $f: [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1/k & \text{if } x = m/k \text{ where } m,k \in \mathbb{N} \text{ and } m \text{ and } k \text{ have no common divisors,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show $\int_0^1 f = 0$.

If $I \subset \mathbb{R}$ is a bounded interval, then the function

$$\varphi_I(x) := \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise,} \end{cases}$$

is called an *elementary step function*.

Exercise* 5.2.12*: *Let I be an arbitrary bounded interval (you should consider all types of intervals: closed, open, half-open) and $a < b$, then using only the definition of the integral show that the elementary step function φ_I is integrable on $[a,b]$, and find the integral in terms of a, b, and the endpoints of I.*

A function f is called a *step function* if it can be written as

$$f(x) = \sum_{k=1}^{n} \alpha_k \varphi_{I_k}(x)$$

for some real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and some bounded intervals $I_1, I_2, \dots, I_n$.

***Exercise* 5.2.13:** *Using Exercise 5.2.12, show that a step function is integrable on every interval $[a,b]$. Furthermore, find the integral in terms of a, b, the endpoints of I_k and the α_k.*

***Exercise* 5.2.14:** *Let $f : [a,b] \to \mathbb{R}$ be a function.*

a) *Show that if f is increasing, then it is Riemann integrable. Hint: Use a uniform partition; each subinterval of same length.*

b) *Use part a) to show that if f is decreasing, then it is Riemann integrable.*

c) *Suppose[*] $h = f - g$ where f and g are increasing functions on $[a,b]$. Show that h is Riemann integrable.*

***Exercise* 5.2.15** (Challenging)**:** *Suppose $f \in \mathscr{R}[a,b]$, then the function that takes x to $|f(x)|$ is also Riemann integrable on $[a,b]$. Then show the same inequality as Exercise 5.2.10.*

***Exercise* 5.2.16:** *Suppose $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are bounded.*

a) *Show $\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$ and $\underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g$.*

b) *Find example f and g where the inequality is strict. Hint: f and g should not be Riemann integrable.*

***Exercise* 5.2.17:** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Define*

$$h(x) := \int_a^b g(t-x) f(t)\, dt.$$

Prove that h is Lipschitz continuous.

[*] Such an h is said to be of *bounded variation*.

5.3 Fundamental theorem of calculus

Note: 1.5 lectures

In this chapter we discuss and prove the *fundamental theorem of calculus*. The entirety of integral calculus is built upon this theorem, ergo the name. The theorem relates the seemingly unrelated concepts of integral and derivative. It tells us how to compute the antiderivative of a function using the integral and vice versa.

5.3.1 First form of the theorem

Theorem 5.3.1. *Let $F\colon [a,b] \to \mathbb{R}$ be a continuous function, differentiable on (a,b). Let $f \in \mathscr{R}[a,b]$ be such that $f(x) = F'(x)$ for $x \in (a,b)$. Then*

$$\int_a^b f = F(b) - F(a).$$

It is not hard to generalize the theorem to allow a finite number of points in $[a,b]$ where F is not differentiable, as long as it is continuous. This generalization is left as an exercise.

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. For each interval $[x_{i-1}, x_i]$, use the mean value theorem to find a $c_i \in (x_{i-1}, x_i)$ such that

$$f(c_i)\Delta x_i = F'(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1}).$$

See Figure 5.5, and notice that the area of all three shaded rectangles is $F(x_{i+1}) - F(x_{i-2})$. The idea is that by taking smaller and smaller subintervals we prove that this area is the integral of f.

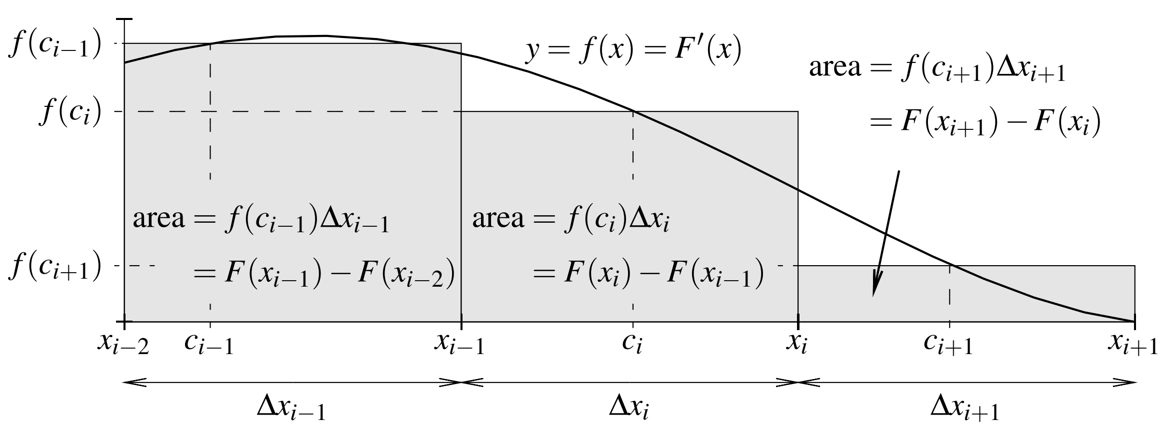

Figure 5.5: Mean value theorem on subintervals of a partition approximating area under the curve.

Using the notation from the definition of the integral, we have $m_i \leq f(c_i) \leq M_i$, and so

$$m_i \Delta x_i \leq F(x_i) - F(x_{i-1}) \leq M_i \Delta x_i.$$

We sum over $i = 1, 2, \ldots, n$ to get

$$\sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) \leq \sum_{i=1}^{n} M_i \Delta x_i.$$

In the middle sum, all the terms except the first and last cancel and we end up with $F(x_n) - F(x_0) = F(b) - F(a)$. The sums on the left and on the right are the lower and the upper sum respectively. So

$$L(P, f) \leq F(b) - F(a) \leq U(P, f).$$

We take the supremum of $L(P, f)$ over all partitions P and the left inequality yields

$$\underline{\int_a^b} f \leq F(b) - F(a).$$

Similarly, taking the infimum of $U(P, f)$ over all partitions P yields

$$F(b) - F(a) \leq \overline{\int_a^b} f.$$

As f is Riemann integrable, we have

$$\int_a^b f = \underline{\int_a^b} f \leq F(b) - F(a) \leq \overline{\int_a^b} f = \int_a^b f.$$

The inequalities must be equalities and we are done. □

The theorem is used to compute integrals. Suppose we know that the function $f(x)$ is a derivative of some other function $F(x)$, then we can find an explicit expression for $\int_a^b f$.

Example 5.3.2: To compute

$$\int_0^1 x^2 \, dx,$$

we notice x^2 is the derivative of $\frac{x^3}{3}$. The fundamental theorem says

$$\int_0^1 x^2 \, dx = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

5.3.2 Second form of the theorem

The second form of the fundamental theorem gives us a way to solve the differential equation $F'(x) = f(x)$, where f is a known function and we are trying to find an F that satisfies the equation.

Theorem 5.3.3. *Let $f \colon [a,b] \to \mathbb{R}$ be a Riemann integrable function. Define*

$$F(x) := \int_a^x f.$$

First, F is continuous on $[a,b]$. Second, if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Proof. As f is bounded, there is an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Suppose $x, y \in [a,b]$ with $x > y$. Then

$$|F(x) - F(y)| = \left| \int_a^x f - \int_a^y f \right| = \left| \int_y^x f \right| \leq M |x - y| .$$

By symmetry, the same also holds if $x < y$. So F is Lipschitz continuous and hence continuous.

Now suppose f is continuous at c. Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that for $x \in [a,b]$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. In particular, for such x, we have

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon.$$

Thus if $x > c$, then

$$\bigl(f(c) - \varepsilon\bigr)(x - c) \leq \int_c^x f \leq \bigl(f(c) + \varepsilon\bigr)(x - c).$$

When $c > x$, then the inequalities are reversed. Therefore, assuming $c \neq x$, we get

$$f(c) - \varepsilon \leq \frac{\int_c^x f}{x - c} \leq f(c) + \varepsilon.$$

As

$$\frac{F(x) - F(c)}{x - c} = \frac{\int_a^x f - \int_a^c f}{x - c} = \frac{\int_c^x f}{x - c},$$

we have

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| \leq \varepsilon.$$

The result follows. It is left to the reader to see why is it OK that we just have a non-strict inequality. $\square$

Of course, if f is continuous on $[a,b]$, then it is automatically Riemann integrable, F is differentiable on all of $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$.

Remark 5.3.4. The second form of the fundamental theorem of calculus still holds if we let $d \in [a,b]$ and define

$$F(x) := \int_d^x f.$$

That is, we can use any point of $[a,b]$ as our base point. The proof is left as an exercise.

Let us look at what a simple discontinuity can do. Take $f(x) := -1$ if $x < 0$, and $f(x) := 1$ if $x \geq 0$. Let $F(x) := \int_0^x f$. It is not difficult to see that $F(x) = |x|$. Notice that f is discontinuous at 0 and F is not differentiable at 0. However, the converse in the theorem does not hold. Let $g(x) := 0$ if $x \neq 0$, and $g(0) := 1$. Letting $G(x) := \int_0^x g$, we find that $G(x) = 0$ for all x. So g is discontinuous at 0, but $G'(0)$ exists and is equal to 0.

A common misunderstanding of the integral for calculus students is to think of integrals whose solution cannot be given in closed-form as somehow deficient. This is not the case. Most integrals we write down are not computable in closed-form. Even some integrals that we consider in closed-form are not really such. We define the natural logarithm as the antiderivative of $1/x$ such that $\ln 1 = 0$:

$$\ln x := \int_1^x \frac{1}{s}\, ds.$$

How does a computer find the value of $\ln x$? One way to do it is to numerically approximate this integral. Morally, we did not really "simplify" $\int_1^x \frac{1}{s}\,ds$ by writing down $\ln x$. We simply gave the integral a name. If we require numerical answers, it is possible we end up doing the calculation by approximating an integral anyway. In the next section, we even define the exponential using the logarithm, which we define in terms of the integral.

Another common function defined by an integral that cannot be evaluated symbolically in terms of elementary functions is the erf function, defined as

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\,ds.$$

This function comes up often in applied mathematics. It is simply the antiderivative of $(2/\sqrt{\pi})\,e^{-x^2}$ that is zero at zero. The second form of the fundamental theorem tells us that we can write the function as an integral. If we wish to compute any particular value, we numerically approximate the integral.

5.3.3 Change of variables

A theorem often used in calculus to solve integrals is the change of variables theorem, you may have called it *u-substitution*. Let us prove it now. Recall a function is continuously differentiable if it is differentiable and the derivative is continuous.

Theorem 5.3.5 (Change of variables)**.** *Let $g\colon [a,b] \to \mathbb{R}$ be a continuously differentiable function, let $f\colon [c,d] \to \mathbb{R}$ be continuous, and suppose $g\big([a,b]\big) \subset [c,d]$. Then*

$$\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(s)\,ds.$$

Proof. As g, g', and f are continuous, $f\big(g(x)\big)\,g'(x)$ is a continuous function of $[a,b]$, therefore it is Riemann integrable. Similarly, f is integrable on every subinterval of $[c,d]$.

Define $F\colon [c,d] \to \mathbb{R}$ by

$$F(y) := \int_{g(a)}^{y} f(s)\,ds.$$

By the second form of the fundamental theorem of calculus (see Remark 5.3.4 and Exercise 5.3.4), F is a differentiable function and $F'(y) = f(y)$. Apply the chain rule,

$$\big(F \circ g\big)'(x) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x).$$

Note that $F\big(g(a)\big) = 0$ and use the first form of the fundamental theorem to obtain

$$\begin{aligned}\int_{g(a)}^{g(b)} f(s)\,ds = F\big(g(b)\big) &= F\big(g(b)\big) - F\big(g(a)\big) \\ &= \int_a^b \big(F \circ g\big)'(x)\,dx = \int_a^b f\big(g(x)\big)g'(x)\,dx.\end{aligned}$$

□

The change of variables theorem is often used to solve integrals by changing them to integrals that we know or that we can solve using the fundamental theorem of calculus.

Example 5.3.6: The derivative of $\sin(x)$ is $\cos(x)$. Using $g(x) := x^2$, we solve

$$\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx = \int_0^{\pi} \frac{\cos(s)}{2}\,ds = \frac{1}{2}\int_0^{\pi} \cos(s)\,ds = \frac{\sin(\pi)-\sin(0)}{2} = 0.$$

However, beware that we must satisfy the hypotheses of the theorem. The following example demonstrates why we should not just move symbols around mindlessly. We must be careful that those symbols really make sense.

Example 5.3.7: Consider

$$\int_{-1}^{1} \frac{\ln\lvert x\rvert}{x}\,dx.$$

It may be tempting to take $g(x) := \ln\lvert x\rvert$. Compute $g'(x) = 1/x$ and try to write

$$\int_{g(-1)}^{g(1)} s\,ds = \int_0^0 s\,ds = 0.$$

This "solution" is incorrect, and it does not say that we can solve the given integral. First problem is that $\frac{\ln\lvert x\rvert}{x}$ is not continuous on $[-1,1]$. It is not defined at 0, and cannot be made continuous by defining a value at 0. Second, $\frac{\ln\lvert x\rvert}{x}$ is not even Riemann integrable on $[-1,1]$ (it is unbounded). The integral we wrote down simply does not make sense. Finally, g is not continuous on $[-1,1]$, let alone continuously differentiable.

5.3.4 Exercises

Exercise* 5.3.1*: *Compute* $\displaystyle\frac{d}{dx}\left(\int_{-x}^{x} e^{s^2}\,ds\right)$.

Exercise* 5.3.2*: *Compute* $\displaystyle\frac{d}{dx}\left(\int_{0}^{x^2} \sin(s^2)\,ds\right)$.

Exercise* 5.3.3*: *Suppose $F\colon [a,b]\to\mathbb{R}$ is continuous and differentiable on $[a,b]\setminus S$, where S is a finite set. Suppose there exists an $f \in \mathscr{R}[a,b]$ such that $f(x) = F'(x)$ for $x \in [a,b]\setminus S$. Show that $\int_a^b f = F(b)-F(a)$.*

Exercise* 5.3.4*: *Let $f\colon [a,b]\to\mathbb{R}$ be a continuous function. Let $c\in[a,b]$ be arbitrary. Define*

$$F(x) := \int_c^x f.$$

Prove that F is differentiable and that $F'(x) = f(x)$ for all $x\in[a,b]$.

Exercise* 5.3.5*: *Prove* integration by parts. *That is, suppose F and G are continuously differentiable functions on $[a,b]$. Then prove*

$$\int_a^b F(x)G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,dx.$$

Exercise* 5.3.6*: *Suppose F and G are continuously*[*] *differentiable functions defined on $[a,b]$ such that $F'(x) = G'(x)$ for all $x\in[a,b]$. Using the fundamental theorem of calculus, show that F and G differ by a constant. That is, show that there exists a $C\in\mathbb{R}$ such that $F(x)-G(x) = C$.*

[*]Compare this hypothesis to Exercise 4.2.8.

The next exercise shows how we can use the integral to "smooth out" a non-differentiable function.

Exercise* 5.3.7*: *Let $f \colon [a,b] \to \mathbb{R}$ be a continuous function. Let $\varepsilon > 0$ be a constant. For $x \in [a+\varepsilon, b-\varepsilon]$, define*

$$g(x) := \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f.$$

a) *Show that g is differentiable and find the derivative.*

b) *Let f be differentiable and fix $x \in (a,b)$ (let ε be small enough). What happens to $g'(x)$ as ε gets smaller?*

c) *Find g for $f(x) := |x|$, $\varepsilon = 1$ (you can assume $[a,b]$ is large enough).*

Exercise* 5.3.8*: *Suppose $f \colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f = \int_x^b f$ for all $x \in [a,b]$. Show that $f(x) = 0$ for all $x \in [a,b]$.*

Exercise* 5.3.9*: *Suppose $f \colon [a,b] \to \mathbb{R}$ is continuous and $\int_a^x f = 0$ for all rational x in $[a,b]$. Show that $f(x) = 0$ for all $x \in [a,b]$.*

Exercise* 5.3.10*: *A function f is an* odd function *if $f(x) = -f(-x)$, and f is an* even function *if $f(x) = f(-x)$. Let $a > 0$. Assume f is continuous. Prove:*

a) *If f is odd, then $\int_{-a}^{a} f = 0$.*

b) *If f is even, then $\int_{-a}^{a} f = 2\int_0^a f$.*

Exercise* 5.3.11*:

a) *Show that $f(x) := \sin(1/x)$ is integrable on every interval (you can define $f(0)$ to be anything).*

b) *Compute $\int_{-1}^{1} \sin(1/x)\, dx$ (mind the discontinuity).*

Exercise* 5.3.12** (uses §3.6):***

a) *Suppose $f \colon [a,b] \to \mathbb{R}$ is increasing, by Proposition 5.2.11, f is Riemann integrable. Suppose f has a discontinuity at $c \in (a,b)$, show that $F(x) := \int_a^x f$ is not differentiable at c.*

b) *In Exercise 3.6.11, you constructed an increasing function $f \colon [0,1] \to \mathbb{R}$ that is discontinuous at every $x \in [0,1] \cap \mathbb{Q}$. Use this f to construct a function $F(x)$ that is continuous on $[0,1]$, but not differentiable at every $x \in [0,1] \cap \mathbb{Q}$.*

5.4 The logarithm and the exponential

Note: 1 lecture (optional, requires the optional sections §3.5, §3.6, §4.4)

We now have the tools required to properly define the exponential and the logarithm that you know from calculus so well. We start with exponentiation. If n is a positive integer, it is obvious to define

$$x^n := \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ times}}.$$

It makes sense to define $x^0 := 1$. For negative integers, let $x^{-n} := 1/x^n$. If $x > 0$, define $x^{1/n}$ as the unique positive nth root. Finally, for a rational number n/m (in lowest terms), define

$$x^{n/m} := \left(x^{1/m}\right)^n.$$

It is not difficult to show we get the same number no matter what representation of n/m we use, so we do not need to use lowest terms.

However, what do we mean by $\sqrt{2}^{\sqrt{2}}$? Or x^y in general? In particular, what is e^x for all x? And how do we solve $y = e^x$ for x? This section answers these questions and more.

5.4.1 The logarithm

It is convenient to define the logarithm first. Let us show that a unique function with the right properties exists, and only then will we call it *the* logarithm.

Proposition 5.4.1. *There exists a unique function $L\colon (0,\infty) \to \mathbb{R}$ such that*

(i) $L(1) = 0$.

(ii) *L is differentiable and* $L'(x) = 1/x$.

(iii) *L is strictly increasing, bijective, and*

$$\lim_{x\to 0} L(x) = -\infty, \qquad \textit{and} \qquad \lim_{x\to\infty} L(x) = \infty.$$

(iv) $L(xy) = L(x) + L(y)$ *for all* $x, y \in (0,\infty)$.

(v) *If q is a rational number and $x > 0$, then* $L(x^q) = qL(x)$.

Proof. To prove existence, we define a candidate and show it satisfies all the properties. Let

$$L(x) := \int_1^x \frac{1}{t}\,dt.$$

Obviously, (i) holds. Property (ii) holds via the second form of the fundamental theorem of calculus (Theorem 5.3.3).

To prove property (iv), we change variables $u = yt$ to obtain

$$L(x) = \int_1^x \frac{1}{t}\,dt = \int_y^{xy} \frac{1}{u}\,du = \int_1^{xy} \frac{1}{u}\,du - \int_1^{y} \frac{1}{u}\,du = L(xy) - L(y).$$

Let us prove (iii). Property (ii) together with the fact that $L'(x) = 1/x > 0$ for $x > 0$, implies that L is strictly increasing and hence one-to-one. Let us show L is onto. As $1/t \geq 1/2$ when $t \in [1,2]$,

$$L(2) = \int_1^2 \frac{1}{t}\,dt \geq 1/2.$$

By induction, (iv) implies that for $n \in \mathbb{N}$

$$L(2^n) = L(2) + L(2) + \cdots + L(2) = nL(2).$$

Given $y > 0$, by the Archimedean property of the real numbers (notice $L(2) > 0$), there is an $n \in \mathbb{N}$ such that $L(2^n) > y$. By the intermediate value theorem there is an $x_1 \in (1, 2^n)$ such that $L(x_1) = y$. We get $(0,\infty)$ is in the image of L. As L is increasing, $L(x) > y$ for all $x > 2^n$, and so

$$\lim_{x\to\infty} L(x) = \infty.$$

Next $0 = L(x/x) = L(x) + L(1/x)$, and so $L(x) = -L(1/x)$. Using $x = 2^{-n}$, we obtain as above that L achieves all negative numbers. And

$$\lim_{x\to 0} L(x) = \lim_{x\to 0} -L(1/x) = \lim_{x\to\infty} -L(x) = -\infty.$$

In the limits, note that only $x > 0$ are in the domain of L.

Let us prove (v). Fix $x > 0$. As above, (iv) implies $L(x^n) = nL(x)$ for all $n \in \mathbb{N}$. We already found that $L(x) = -L(1/x)$, so $L(x^{-n}) = -L(x^n) = -nL(x)$. Then for $m \in \mathbb{N}$

$$L(x) = L\Big(\big(x^{1/m}\big)^m\Big) = mL\big(x^{1/m}\big).$$

Putting everything together for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, we have $L(x^{n/m}) = nL(x^{1/m}) = (n/m)L(x)$.

Uniqueness follows using properties (i) and (ii). Via the first form of the fundamental theorem of calculus (Theorem 5.3.1),

$$L(x) = \int_1^x \frac{1}{t}\,dt$$

is the unique function such that $L(1) = 0$ and $L'(x) = 1/x$. $\square$

Having proved that there is a unique function with these properties, we simply define the *logarithm* or sometimes called the *natural logarithm*:

$$\ln(x) := L(x).$$

Mathematicians usually write $\log(x)$ instead of $\ln(x)$, which is more familiar to calculus students. For all practical purposes, there is only one logarithm: the natural logarithm. See Exercise 5.4.2.

5.4.2 The exponential

Just as with the logarithm we define the exponential via a list of properties.

Proposition 5.4.2. *There exists a unique function $E : \mathbb{R} \to (0,\infty)$ such that*

(i) $E(0) = 1$.

(ii) *E is differentiable and $E'(x) = E(x)$.*

(iii) *E is strictly increasing, bijective, and*

$$\lim_{x\to-\infty} E(x) = 0, \qquad \text{and} \qquad \lim_{x\to\infty} E(x) = \infty.$$

(iv) *$E(x+y) = E(x)E(y)$ for all $x,y \in \mathbb{R}$.*

(v) *If $q \in \mathbb{Q}$, then $E(qx) = E(x)^q$.*

Proof. Again, we prove existence of such a function by defining a candidate and proving that it satisfies all the properties. The $L = \ln$ defined above is invertible. Let E be the inverse function of L. Property (i) is immediate.

Property (ii) follows via the inverse function theorem, in particular Lemma 4.4.1: L satisfies all the hypotheses of the lemma, and hence

$$E'(x) = \frac{1}{L'\big(E(x)\big)} = E(x).$$

Let us look at property (iii). The function E is strictly increasing since $E'(x) = E(x) > 0$. As E is the inverse of L, it must also be bijective. To find the limits, we use that E is strictly increasing and onto $(0,\infty)$. For every $M > 0$, there is an x_0 such that $E(x_0) = M$ and $E(x) \geq M$ for all $x \geq x_0$. Similarly, for every $\varepsilon > 0$, there is an x_0 such that $E(x_0) = \varepsilon$ and $E(x) < \varepsilon$ for all $x < x_0$. Therefore,

$$\lim_{n\to-\infty} E(x) = 0, \qquad \text{and} \qquad \lim_{n\to\infty} E(x) = \infty.$$

To prove property (iv), we use the corresponding property for the logarithm. Take $x,y \in \mathbb{R}$. As L is bijective, find a and b such that $x = L(a)$ and $y = L(b)$. Then

$$E(x+y) = E\big(L(a)+L(b)\big) = E\big(L(ab)\big) = ab = E(x)E(y).$$

Property (v) also follows from the corresponding property of L. Given $x \in \mathbb{R}$, let a be such that $x = L(a)$ and

$$E(qx) = E\big(qL(a)\big) = E\big(L(a^q)\big) = a^q = E(x)^q.$$

Uniqueness follows from (i) and (ii). Let E and F be two functions satisfying (i) and (ii).

$$\frac{d}{dx}\Big(F(x)E(-x)\Big) = F'(x)E(-x) - E'(-x)F(x) = F(x)E(-x) - E(-x)F(x) = 0.$$

Therefore, by Proposition 4.2.6, $F(x)E(-x) = F(0)E(-0) = 1$ for all $x \in \mathbb{R}$. Next, $1 = E(0) = E(x-x) = E(x)E(-x)$. Then

$$0 = 1 - 1 = F(x)E(-x) - E(x)E(-x) = \big(F(x) - E(x)\big)E(-x).$$

Finally, $E(-x) \neq 0$* for all $x \in \mathbb{R}$. So $F(x) - E(x) = 0$ for all x, and we are done. □

Having proved E is unique, we define the *exponential* function as

$$\exp(x) := E(x).$$

If $y \in \mathbb{Q}$ and $x > 0$, then

$$x^y = \exp\bigl(\ln(x^y)\bigr) = \exp\bigl(y \ln(x)\bigr).$$

We can now make sense of exponentiation x^y for arbitrary $y \in \mathbb{R}$; if $x > 0$ and y is irrational, define

$$x^y := \exp\bigl(y \ln(x)\bigr).$$

As exp is continuous, then x^y is a continuous function of y. Therefore, we would obtain the same result had we taken a sequence of rational numbers $\{y_n\}$ approaching y and defined $x^y = \lim x^{y_n}$.

Define the number e, sometimes called *Euler's number* or the *base of the natural logarithm*, as

$$e := \exp(1).$$

Let us justify the notation e^x for $\exp(x)$:

$$e^x = \exp\bigl(x \ln(e)\bigr) = \exp(x).$$

The properties of the logarithm and the exponential extend to irrational powers. The proof is immediate.

Proposition 5.4.3. *Let $x, y \in \mathbb{R}$.*

(i) $\exp(xy) = \bigl(\exp(x)\bigr)^y$.

(ii) If $x > 0$, then $\ln(x^y) = y \ln(x)$.

Remark 5.4.4. There are other equivalent ways to define the exponential and the logarithm. A common way is to define E as the solution to the differential equation $E'(x) = E(x)$, $E(0) = 1$. See Example 6.3.3, for a sketch of that approach. Yet another approach is to define the exponential function by power series, see Example 6.2.14.

Remark 5.4.5. We proved the uniqueness of the functions L and E from just the properties $L(1) = 0$, $L'(x) = 1/x$ and the equivalent condition for the exponential $E'(x) = E(x)$, $E(0) = 1$. Existence also follows from just these properties. Alternatively, uniqueness also follows from the laws of exponents, see the exercises.

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*E is a function into $(0, \infty)$ after all. However, $E(-x) \neq 0$ also follows from $E(x)E(-x) = 1$. Therefore, we can prove uniqueness of E given (i) and (ii), even for functions $E : \mathbb{R} \to \mathbb{R}$.

5.4.3 Exercises

Exercise* 5.4.1*: *Given a real number y and $b > 0$, define $f : (0,\infty) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ as $f(x) := x^y$ and $g(x) := b^x$. Show that f and g are differentiable and find their derivative.*

Exercise* 5.4.2*: *Let $b > 0$, $b \neq 1$ be given.*

a) *Show that for every $y > 0$, there exists a unique number x such that $y = b^x$. Define the* logarithm base *b, $\log_b : (0,\infty) \to \mathbb{R}$, by $\log_b(y) := x$.*
b) *Show that $\log_b(x) = \frac{\ln(x)}{\ln(b)}$.*
c) *Prove that if $c > 0$, $c \neq 1$, then $\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$.*
d) *Prove $\log_b(xy) = \log_b(x) + \log_b(y)$, and $\log_b(x^y) = y \log_b(x)$.*

Exercise* 5.4.3** (requires §4.3):*** *Use Taylor's theorem to study the remainder term and show that for all $x \in \mathbb{R}$*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Hint: Do not differentiate the series term by term (unless you would prove that it works).

Exercise* 5.4.4*: *Use the geometric sum formula to show (for $t \neq -1$)*

$$1 - t + t^2 - \cdots + (-1)^n t^n = \frac{1}{1+t} - \frac{(-1)^{n+1} t^{n+1}}{1+t}.$$

Using this fact show

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

for all $x \in (-1,1]$ (note that $x = 1$ is included). Finally, find the limit of the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Exercise* 5.4.5*: *Show*

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Hint: Take the logarithm.
Note: The expression $\left(1 + \frac{x}{n}\right)^n$ arises in compound interest calculations. It is the amount of money in a bank account after 1 year if 1 dollar was deposited initially at interest x and the interest was compounded n times during the year. The exponential e^x is the result of continuous compounding.

Exercise* 5.4.6*:

a) *Prove that for $n \in \mathbb{N}$,*

$$\sum_{k=2}^{n} \frac{1}{k} \leq \ln(n) \leq \sum_{k=1}^{n-1} \frac{1}{k}.$$

b) *Prove that the limit*

$$\gamma := \lim_{n\to\infty} \left(\left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln(n) \right)$$

exists. This constant is known as the Euler–Mascheroni constant*. *It is not known if this constant is rational or not. It is approximately $\gamma \approx 0.5772$.*

*Named for the Swiss mathematician Leonhard Paul Euler (1707–1783) and the Italian mathematician Lorenzo Mascheroni (1750–1800).

Exercise* 5.4.7*: *Show*

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0.$$

Exercise* 5.4.8*: *Show that e^x is* convex, *in other words, show that if $a \leq x \leq b$, then $e^x \leq e^a \frac{b-x}{b-a} + e^b \frac{x-a}{b-a}$.*

Exercise* 5.4.9*: *Using the logarithm find*

$$\lim_{n\to\infty} n^{1/n}.$$

Exercise* 5.4.10*: *Show that $E(x) = e^x$ is the unique continuous function such that $E(x+y) = E(x)E(y)$ and $E(1) = e$. Similarly, prove that $L(x) = \ln(x)$ is the unique continuous function defined on positive x such that $L(xy) = L(x) + L(y)$ and $L(e) = 1$.*

Exercise* 5.4.11** (requires §4.3):*** *Since $(e^x)' = e^x$, it is easy to see that e^x is infinitely differentiable (has derivatives of all orders). Define the function $f \colon \mathbb{R} \to \mathbb{R}$.*

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

a) Prove that for every $m \in \mathbb{N}$,

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{x^m} = 0.$$

b) Prove that f is infinitely differentiable.

c) Compute the Taylor series for f at the origin, that is,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Show that it converges, but show that it does not converge to $f(x)$ for any given $x > 0$.

5.5 Improper integrals

Note: 2–3 lectures (optional section, can safely be skipped, requires the optional §3.5)

Often it is necessary to integrate over the entire real line, or an unbounded interval of the form $[a,\infty)$ or $(-\infty,b]$. We may also wish to integrate unbounded functions defined on a open bounded interval (a,b). For such intervals or functions, the Riemann integral is not defined, but we will write down the integral anyway in the spirit of Lemma 5.2.8. These integrals are called *improper integrals* and are limits of integrals rather than integrals themselves.

Definition 5.5.1. Suppose $f\colon [a,b) \to \mathbb{R}$ is a function (not necessarily bounded) that is Riemann integrable on $[a,c]$ for all $c < b$. We define

$$\int_a^b f := \lim_{c \to b^-} \int_a^c f$$

if the limit exists.

Suppose $f\colon [a,\infty) \to \mathbb{R}$ is a function such that f is Riemann integrable on $[a,c]$ for all $c < \infty$. We define

$$\int_a^\infty f := \lim_{c \to \infty} \int_a^c f$$

if the limit exists.

If the limit exists, we say the improper integral *converges*. If the limit does not exist, we say the improper integral *diverges*.

We similarly define improper integrals for the left-hand endpoint, we leave this to the reader.

For a finite endpoint b, if f is bounded, then Lemma 5.2.8 says that we defined nothing new. What is new is that we can apply this definition to unbounded functions. The following set of examples is so useful that we state it as a proposition.

Proposition 5.5.2 (p-test for integrals)**.** *The improper integral*

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges to $\frac{1}{p-1}$ *if* $p > 1$ *and diverges if* $0 < p \leq 1$.

The improper integral

$$\int_0^1 \frac{1}{x^p}\,dx$$

converges to $\frac{1}{1-p}$ *if* $0 < p < 1$ *and diverges if* $p \geq 1$.

Proof. The proof follows by application of the fundamental theorem of calculus. Let us do the proof for $p > 1$ for the infinite right endpoint and leave the rest to the reader. Hint: You should handle $p = 1$ separately.

Suppose $p > 1$. Then using the fundamental theorem,

$$\int_1^b \frac{1}{x^p}\,dx = \int_1^b x^{-p}\,dx = \frac{b^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} = \frac{-1}{(p-1)b^{p-1}} + \frac{1}{p-1}.$$

As $p > 1$, then $p-1 > 0$. Take the limit as $b \to \infty$ to obtain that $\frac{1}{b^{p-1}}$ goes to 0. The result follows. $\square$

We state the following proposition on "tails" for just one type of improper integral, though the proof is straight forward and the same for other types of improper integrals.

Proposition 5.5.3. *Let $f\colon [a,\infty) \to \mathbb{R}$ be a function that is Riemann integrable on $[a,b]$ for all $b > a$. For every $b > a$, the integral $\int_b^\infty f$ converges if and only if $\int_a^\infty f$ converges, in which case*

$$\int_a^\infty f = \int_a^b f + \int_b^\infty f.$$

Proof. Let $c > b$. Then

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Taking the limit $c \to \infty$ finishes the proof. $\square$

Nonnegative functions are easier to work with as the following proposition demonstrates. The exercises will show that this proposition holds only for nonnegative functions. Analogues of this proposition exist for all the other types of improper limits and are left to the student.

Proposition 5.5.4. *Suppose $f\colon [a,\infty) \to \mathbb{R}$ is nonnegative ($f(x) \geq 0$ for all x) and such that f is Riemann integrable on $[a,b]$ for all $b > a$.*

(i)

$$\int_a^\infty f = \sup \left\{ \int_a^x f : x \geq a \right\}.$$

(ii) Suppose $\{x_n\}$ is a sequence with $\lim x_n = \infty$. Then $\int_a^\infty f$ converges if and only if $\lim \int_a^{x_n} f$ exists, in which case

$$\int_a^\infty f = \lim_{n\to\infty} \int_a^{x_n} f.$$

In the first item we allow for the value of ∞ in the supremum indicating that the integral diverges to infinity.

Proof. We start with the first item. As f is nonnegative, $\int_a^x f$ is increasing as a function of x. If the supremum is infinite, then for every $M \in \mathbb{R}$ we find N such that $\int_a^N f \geq M$. As $\int_a^x f$ is increasing, $\int_a^x f \geq M$ for all $x \geq N$. So $\int_a^\infty f$ diverges to infinity.

Next suppose the supremum is finite, say $A := \sup\{\int_a^x f : x \geq a\}$. For every $\varepsilon > 0$, we find an N such that $A - \int_a^N f < \varepsilon$. As $\int_a^x f$ is increasing, then $A - \int_a^x f < \varepsilon$ for all $x \geq N$ and hence $\int_a^\infty f$ converges to A.

Let us look at the second item. If $\int_a^\infty f$ converges, then every sequence $\{x_n\}$ going to infinity works. The trick is proving the other direction. Suppose $\{x_n\}$ is such that $\lim x_n = \infty$ and

$$\lim_{n\to\infty} \int_a^{x_n} f = A$$

converges. Given $\varepsilon > 0$, pick N such that for all $n \geq N$, we have $A - \varepsilon < \int_a^{x_n} f < A + \varepsilon$. Because $\int_a^x f$ is increasing as a function of x, we have that for all $x \geq x_N$

$$A - \varepsilon < \int_a^{x_N} f \leq \int_a^x f.$$

As $\{x_n\}$ goes to ∞, then for any given x, there is an x_m such that $m \geq N$ and $x \leq x_m$. Then

$$\int_a^x f \leq \int_a^{x_m} f < A + \varepsilon.$$

In particular, for all $x \geq x_N$, we have $\left|\int_a^x f - A\right| < \varepsilon$. □

Proposition 5.5.5 (Comparison test for improper integrals)**.** *Let $f\colon [a,\infty) \to \mathbb{R}$ and $g\colon [a,\infty) \to \mathbb{R}$ be functions that are Riemann integrable on $[a,b]$ for all $b > a$. Suppose that for all $x \geq a$,*

$$|f(x)| \leq g(x).$$

(i) If $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges, and in this case $\left|\int_a^\infty f\right| \leq \int_a^\infty g$.

(ii) If $\int_a^\infty f$ diverges, then $\int_a^\infty g$ diverges.

Proof. We start with the first item. For every b and c, such that $a \leq b \leq c$, we have $-g(x) \leq f(x) \leq g(x)$, and so

$$\int_b^c -g \leq \int_b^c f \leq \int_b^c g.$$

In other words, $\left|\int_b^c f\right| \leq \int_b^c g$.

Let $\varepsilon > 0$ be given. Because of Proposition 5.5.3,

$$\int_a^\infty g = \int_a^b g + \int_b^\infty g.$$

As $\int_a^b g$ goes to $\int_a^\infty g$ as b goes to infinity, $\int_b^\infty g$ goes to 0 as b goes to infinity. Choose B such that

$$\int_B^\infty g < \varepsilon.$$

As g is nonnegative, if $B \leq b < c$, then $\int_b^c g < \varepsilon$ as well. Let $\{x_n\}$ be a sequence going to infinity. Let M be such that $x_n \geq B$ for all $n \geq M$. Take $n, m \geq M$, with $x_n \leq x_m$,

$$\left|\int_a^{x_m} f - \int_a^{x_n} f\right| = \left|\int_{x_n}^{x_m} f\right| \leq \int_{x_n}^{x_m} g < \varepsilon.$$

Therefore, the sequence $\{\int_a^{x_n} f\}_{n=1}^\infty$ is Cauchy and hence converges.

We need to show that the limit is unique. Suppose $\{x_n\}$ is a sequence converging to infinity such that $\{\int_a^{x_n} f\}$ converges to L_1, and $\{y_n\}$ is a sequence converging to infinity is such that $\{\int_a^{y_n} f\}$ converges to L_2. Then there must be some n such that $\left|\int_a^{x_n} f - L_1\right| < \varepsilon$ and $\left|\int_a^{y_n} f - L_2\right| < \varepsilon$. We can also suppose $x_n \geq B$ and $y_n \geq B$. Then

$$|L_1 - L_2| \leq \left|L_1 - \int_a^{x_n} f\right| + \left|\int_a^{x_n} f - \int_a^{y_n} f\right| + \left|\int_a^{y_n} f - L_2\right| < \varepsilon + \left|\int_{x_n}^{y_n} f\right| + \varepsilon < 3\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, $L_1 = L_2$, and hence $\int_a^\infty f$ converges. Above we have shown that $\left|\int_a^c f\right| \leq \int_a^c g$ for all $c > a$. By taking the limit $c \to \infty$, the first item is proved.

The second item is simply a contrapositive of the first item. □

Example 5.5.6: The improper integral

$$\int_0^\infty \frac{\sin(x^2)(x+2)}{x^3+1}\,dx$$

converges.

Proof: Observe we simply need to show that the integral converges when going from 1 to infinity. For $x \geq 1$ we obtain

$$\left|\frac{\sin(x^2)(x+2)}{x^3+1}\right| \leq \frac{x+2}{x^3+1} \leq \frac{x+2}{x^3} \leq \frac{x+2x}{x^3} \leq \frac{3}{x^2}.$$

Then

$$\int_1^\infty \frac{3}{x^2}\,dx = 3\int_1^\infty \frac{1}{x^2}\,dx = 3.$$

So using the comparison test and the tail test, the original integral converges.

Example 5.5.7: You should be careful when doing formal manipulations with improper integrals. The integral

$$\int_2^\infty \frac{2}{x^2-1}\,dx$$

converges via the comparison test using $1/x^2$ again. However, if you succumb to the temptation to write

$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$$

and try to integrate each part separately, you will not succeed. It is *not* true that you can split the improper integral in two; you cannot split the limit.

$$\begin{aligned}
\int_2^\infty \frac{2}{x^2-1}\,dx &= \lim_{b\to\infty} \int_2^b \frac{2}{x^2-1}\,dx \\
&= \lim_{b\to\infty} \left(\int_2^b \frac{1}{x-1}\,dx - \int_2^b \frac{1}{x+1}\,dx \right) \\
&\neq \int_2^\infty \frac{1}{x-1}\,dx - \int_2^\infty \frac{1}{x+1}\,dx.
\end{aligned}$$

The last line in the computation does not even make sense. Both of the integrals there diverge to infinity, since we can apply the comparison test appropriately with $1/x$. We get $\infty - \infty$.

Now suppose we need to take limits at both endpoints.

Definition 5.5.8. Suppose $f\colon (a,b) \to \mathbb{R}$ is a function that is Riemann integrable on $[c,d]$ for all c, d such that $a < c < d < b$, then we define

$$\int_a^b f := \lim_{c\to a^+} \lim_{d\to b^-} \int_c^d f$$

if the limits exist.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that f is Riemann integrable on all bounded intervals $[a,b]$. Then we define

$$\int_{-\infty}^{\infty} f := \lim_{c \to -\infty} \lim_{d \to \infty} \int_c^d f$$

if the limits exist.

We similarly define improper integrals with one infinite and one finite improper endpoint, we leave this to the reader.

One ought to always be careful about double limits. The definition given above says that we first take the limit as d goes to b or ∞ for a fixed c, and then we take the limit in c. We will have to prove that in this case it does not matter which limit we compute first.

Example 5.5.9: Let us see an example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b \frac{1}{1+x^2}\,dx = \lim_{a \to -\infty} \lim_{b \to \infty} \bigl(\arctan(b) - \arctan(a)\bigr) = \pi.$$

In the definition the order of the limits can always be switched if they exist. Let us prove this fact only for the infinite limits.

Proposition 5.5.10. *If $f : \mathbb{R} \to \mathbb{R}$ is a function integrable on every bounded interval $[a,b]$. Then*

$$\lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f \quad \textit{converges} \qquad \textit{if and only if} \qquad \lim_{b \to \infty} \lim_{a \to -\infty} \int_a^b f \quad \textit{converges,}$$

in which case the two expressions are equal. If either of the expressions converges, then the improper integral converges and

$$\lim_{a \to \infty} \int_{-a}^{a} f = \int_{-\infty}^{\infty} f.$$

Proof. Without loss of generality, assume $a < 0$ and $b > 0$. Suppose the first expression converges. Then

$$\begin{aligned}\lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f &= \lim_{a \to -\infty} \lim_{b \to \infty} \left(\int_a^0 f + \int_0^b f \right) = \left(\lim_{a \to -\infty} \int_a^0 f \right) + \left(\lim_{b \to \infty} \int_0^b f \right) \\ &= \lim_{b \to \infty} \left(\left(\lim_{a \to -\infty} \int_a^0 f \right) + \int_0^b f \right) = \lim_{b \to \infty} \lim_{a \to -\infty} \left(\int_a^0 f + \int_0^b f \right).\end{aligned}$$

Similar computation shows the other direction. Therefore, if either expression converges, then the improper integral converges and

$$\begin{aligned}\int_{-\infty}^{\infty} f = \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f &= \left(\lim_{a \to -\infty} \int_a^0 f \right) + \left(\lim_{b \to \infty} \int_0^b f \right) \\ &= \left(\lim_{a \to \infty} \int_{-a}^0 f \right) + \left(\lim_{a \to \infty} \int_0^a f \right) = \lim_{a \to \infty} \left(\int_{-a}^0 f + \int_0^a f \right) = \lim_{a \to \infty} \int_{-a}^a f.\end{aligned}$$

$\square$

Example 5.5.11: On the other hand, you must be careful to take the limits independently before you know convergence. Let $f(x) = \frac{x}{|x|}$ for $x \neq 0$ and $f(0) = 0$. If $a < 0$ and $b > 0$, then

$$\int_a^b f = \int_a^0 f + \int_0^b f = a + b.$$

For every fixed $a < 0$, the limit as $b \to \infty$ is infinite. So even the first limit does not exist, and the improper integral $\int_{-\infty}^{\infty} f$ does not converge. On the other hand, if $a > 0$, then

$$\int_{-a}^{a} f = (-a) + a = 0.$$

Therefore,

$$\lim_{a \to \infty} \int_{-a}^{a} f = 0.$$

Example 5.5.12: An example to keep in mind for improper integrals is the so-called *sinc function**. This function comes up quite often in both pure and applied mathematics. Define

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

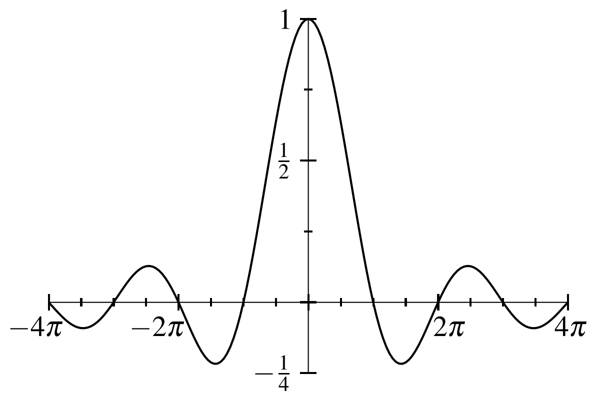

Figure 5.6: The sinc function.

It is not difficult to show that the sinc function is continuous at zero, but that is not important right now. What is important is that

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x)\, dx = \pi, \qquad \text{while} \qquad \int_{-\infty}^{\infty} |\operatorname{sinc}(x)|\, dx = \infty.$$

The integral of the sinc function is a continuous analogue of the alternating harmonic series $\sum {(-1)^n}/{n}$, while the absolute value is like the regular harmonic series $\sum {1}/{n}$. In particular, the fact that the integral converges must be done directly rather than using comparison test.

*Shortened from Latin: *sinus cardinalis*

We will not prove the first statement exactly. Let us simply prove that the integral of the sinc function converges, but we will not worry about the exact limit. Because $\frac{\sin(-x)}{-x} = \frac{\sin(x)}{x}$, it is enough to show that

$$\int_{2\pi}^{\infty} \frac{\sin(x)}{x}\, dx$$

converges. We also avoid $x = 0$ this way to make our life simpler.

For every $n \in \mathbb{N}$, we have that for $x \in [\pi 2n, \pi(2n+1)]$,

$$\frac{\sin(x)}{\pi(2n+1)} \leq \frac{\sin(x)}{x} \leq \frac{\sin(x)}{\pi 2n},$$

as $\sin(x) \geq 0$. For $x \in [\pi(2n+1), \pi(2n+2)]$,

$$\frac{\sin(x)}{\pi(2n+1)} \leq \frac{\sin(x)}{x} \leq \frac{\sin(x)}{\pi(2n+2)},$$

as $\sin(x) \leq 0$.

Via the fundamental theorem of calculus,

$$\frac{2}{\pi(2n+1)} = \int_{\pi 2n}^{\pi(2n+1)} \frac{\sin(x)}{\pi(2n+1)}\, dx \leq \int_{\pi 2n}^{\pi(2n+1)} \frac{\sin(x)}{x}\, dx \leq \int_{\pi 2n}^{\pi(2n+1)} \frac{\sin(x)}{\pi 2n}\, dx = \frac{1}{\pi n}.$$

Similarly,

$$\frac{-2}{\pi(2n+1)} \leq \int_{\pi(2n+1)}^{\pi(2n+2)} \frac{\sin(x)}{x}\, dx \leq \frac{-1}{\pi(n+1)}.$$

Adding the two together we find

$$0 = \frac{2}{\pi(2n+1)} + \frac{-2}{\pi(2n+1)} \leq \int_{2\pi n}^{2\pi(n+1)} \frac{\sin(x)}{x}\, dx \leq \frac{1}{\pi n} + \frac{-1}{\pi(n+1)} = \frac{1}{\pi n(n+1)}.$$

See Figure 5.7.

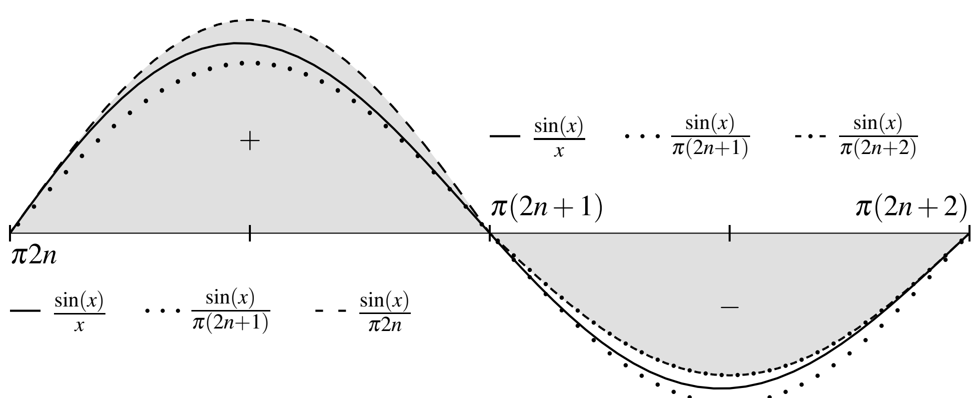

Figure 5.7: Bound of $\int_{2\pi n}^{2\pi(n+1)} \frac{\sin(x)}{x}\, dx$ using the shaded integral (signed area $\frac{1}{\pi n} + \frac{-1}{\pi(n+1)}$).

For $k \in \mathbb{N}$,

$$\int_{2\pi}^{2k\pi} \frac{\sin(x)}{x}\,dx = \sum_{n=1}^{k-1} \int_{2\pi n}^{2\pi(n+1)} \frac{\sin(x)}{x}\,dx \leq \sum_{n=1}^{k-1} \frac{1}{\pi n(n+1)}.$$

We find the partial sums of a series with positive terms. The series converges as $\sum \frac{1}{\pi n(n+1)}$ is a convergent series. Thus as a sequence,

$$\lim_{k\to\infty} \int_{2\pi}^{2k\pi} \frac{\sin(x)}{x}\,dx = L \leq \sum_{n=1}^{\infty} \frac{1}{\pi n(n+1)} < \infty.$$

Let $M > 2\pi$ be arbitrary, and let $k \in \mathbb{N}$ be the largest integer such that $2k\pi \leq M$. For $x \in [2k\pi, M]$, we have $\frac{-1}{2k\pi} \leq \frac{\sin(x)}{x} \leq \frac{1}{2k\pi}$, and so

$$\left| \int_{2k\pi}^{M} \frac{\sin(x)}{x}\,dx \right| \leq \frac{M-2k\pi}{2k\pi} \leq \frac{1}{k}.$$

As k is the largest k such that $2k\pi \leq M$, then as $M \in \mathbb{R}$ goes to infinity, so does $k \in \mathbb{N}$.

Then

$$\int_{2\pi}^{M} \frac{\sin(x)}{x}\,dx = \int_{2\pi}^{2k\pi} \frac{\sin(x)}{x}\,dx + \int_{2k\pi}^{M} \frac{\sin(x)}{x}\,dx.$$

As M goes to infinity, the first term on the right-hand side goes to L, and the second term on the right-hand side goes to zero. Hence

$$\int_{2\pi}^{\infty} \frac{\sin(x)}{x}\,dx = L.$$

The double-sided integral of sinc also exists as noted above. We leave the other statement—that the integral of the absolute value of the sinc function diverges—as an exercise.

5.5.1 Integral test for series

The fundamental theorem of calculus can be used in proving a series is summable and to estimate its sum.

Proposition 5.5.13 (Integral test)**.** *Suppose $f\colon [k,\infty) \to \mathbb{R}$ is a decreasing nonnegative function where $k \in \mathbb{Z}$. Then*

$$\sum_{n=k}^{\infty} f(n) \quad \textit{converges} \qquad \textit{if and only if} \qquad \int_k^{\infty} f \quad \textit{converges}.$$

In this case

$$\int_k^{\infty} f \leq \sum_{n=k}^{\infty} f(n) \leq f(k) + \int_k^{\infty} f.$$

See Figure 5.8, for an illustration with $k = 1$. By Proposition 5.2.11, f is integrable on every interval $[k,b]$ for all $b > k$, so the statement of the theorem makes sense without additional hypotheses of integrability.

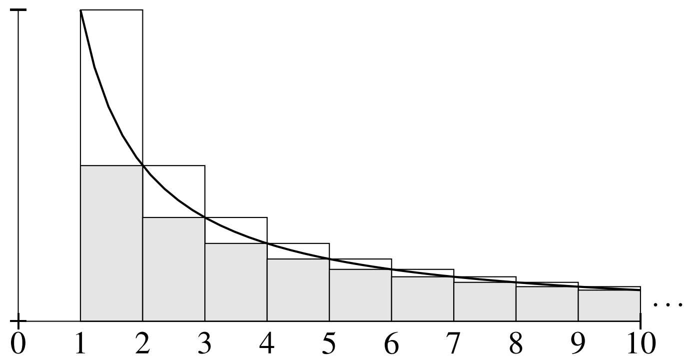

Figure 5.8: The area under the curve, $\int_1^\infty f$, is bounded below by the area of the shaded rectangles, $f(2)+f(3)+f(4)+\cdots$, and bounded above by the area entire rectangles, $f(1)+f(2)+f(3)+\cdots$.

Proof. Let $\ell, m \in \mathbb{Z}$ be such that $m > \ell \geq k$. Because f is decreasing, we have $\int_n^{n+1} f \leq f(n) \leq \int_{n-1}^{n} f$. Therefore,

$$\int_\ell^m f = \sum_{n=\ell}^{m-1} \int_n^{n+1} f \leq \sum_{n=\ell}^{m-1} f(n) \leq f(\ell) + \sum_{n=\ell+1}^{m-1} \int_{n-1}^{n} f \leq f(\ell) + \int_\ell^{m-1} f. \tag{5.3}$$

Suppose first that $\int_k^\infty f$ converges and let $\varepsilon > 0$ be given. As before, since f is positive, then there exists an $L \in \mathbb{N}$ such that if $\ell \geq L$, then $\int_\ell^m f < \varepsilon/2$ for all $m \geq \ell$. The function f must decrease to zero (why?), so make L large enough so that for $\ell \geq L$, we have $f(\ell) < \varepsilon/2$. Thus, for $m > \ell \geq L$, we have via (5.3),

$$\sum_{n=\ell}^{m} f(n) \leq f(\ell) + \int_\ell^m f < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

The series is therefore Cauchy and thus converges. The estimate in the proposition is obtained by letting m go to infinity in (5.3) with $\ell = k$.

Conversely, suppose $\int_k^\infty f$ diverges. As f is positive, then by Proposition 5.5.4, the sequence $\{\int_k^m f\}_{m=k}^\infty$ diverges to infinity. Using (5.3) with $\ell = k$, we find

$$\int_k^m f \leq \sum_{n=k}^{m-1} f(n).$$

As the left-hand side goes to infinity as $m \to \infty$, so does the right-hand side. □

Example 5.5.14: The integral test can be used not only to show that a series converges, but to estimate its sum to arbitrary precision. Let us show $\sum_{n=1}^\infty \frac{1}{n^2}$ exists and estimate its sum to within 0.01. As this series is the p-series for $p = 2$, we already proved it converges (let us pretend we do not know that), but we only roughly estimated its sum.

The fundamental theorem of calculus says that for all $k \in \mathbb{N}$,

$$\int_k^\infty \frac{1}{x^2}\,dx = \frac{1}{k}.$$

In particular, the series must converge. But we also have

$$\frac{1}{k} = \int_k^\infty \frac{1}{x^2}\,dx \leq \sum_{n=k}^\infty \frac{1}{n^2} \leq \frac{1}{k^2} + \int_k^\infty \frac{1}{x^2}\,dx = \frac{1}{k^2} + \frac{1}{k}.$$

Adding the partial sum up to $k-1$ we get

$$\frac{1}{k} + \sum_{n=1}^{k-1} \frac{1}{n^2} \leq \sum_{n=1}^\infty \frac{1}{n^2} \leq \frac{1}{k^2} + \frac{1}{k} + \sum_{n=1}^{k-1} \frac{1}{n^2}.$$

In other words, $1/k + \sum_{n=1}^{k-1} 1/n^2$ is an estimate for the sum to within $1/k^2$. Therefore, if we wish to find the sum to within 0.01, we note $1/10^2 = 0.01$. We obtain

$$1.6397\ldots \approx \frac{1}{10} + \sum_{n=1}^{9} \frac{1}{n^2} \leq \sum_{n=1}^\infty \frac{1}{n^2} \leq \frac{1}{100} + \frac{1}{10} + \sum_{n=1}^{9} \frac{1}{n^2} \approx 1.6497\ldots.$$

The actual sum is $\pi^2/6 \approx 1.6449\ldots$.

5.5.2 Exercises

***Exercise* 5.5.1:** *Finish the proof of Proposition 5.5.2.*

***Exercise* 5.5.2:** *Find out for which $a \in \mathbb{R}$ does $\sum\limits_{n=1}^\infty e^{an}$ converge. When the series converges, find an upper bound for the sum.*

***Exercise* 5.5.3:**

a) Estimate $\sum\limits_{n=1}^\infty \frac{1}{n(n+1)}$ correct to within 0.01 using the integral test.

b) Compute the limit of the series exactly and compare. Hint: The sum telescopes.

***Exercise* 5.5.4:** *Prove*

$$\int_{-\infty}^\infty \lvert \operatorname{sinc}(x) \rvert \,dx = \infty.$$

Hint: Again, it is enough to show this on just one side.

***Exercise* 5.5.5:** *Can you interpret*

$$\int_{-1}^1 \frac{1}{\sqrt{\lvert x \rvert}}\,dx$$

as an improper integral? If so, compute its value.

***Exercise* 5.5.6:** *Take $f\colon [0,\infty) \to \mathbb{R}$, Riemann integrable on every interval $[0,b]$, and such that there exist M, a, and T, such that $\lvert f(t) \rvert \leq M e^{at}$ for all $t \geq T$. Show that the* Laplace transform *of f exists. That is, for every $s > a$ the following integral converges:*

$$F(s) := \int_0^\infty f(t) e^{-st}\,dt.$$

Exercise* 5.5.7*: *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Riemann integrable function on every interval $[a,b]$, and such that $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$. Show that the* Fourier sine and cosine transforms *exist. That is, for every $\omega \geq 0$ the following integrals converge*

$$F^s(\omega) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt, \qquad F^c(\omega) := \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt.$$

Furthermore, show that F^s and F^c are bounded functions.

Exercise* 5.5.8*: *Suppose $f\colon [0,\infty) \to \mathbb{R}$ is Riemann integrable on every interval $[0,b]$. Show that $\int_0^\infty f$ converges if and only if for every $\varepsilon > 0$ there exists an M such that if $M \leq a < b$, then $\left|\int_a^b f\right| < \varepsilon$.*

Exercise* 5.5.9*: *Suppose $f\colon [0,\infty) \to \mathbb{R}$ is nonnegative and* decreasing. *Prove:*

a) If $\int_0^\infty f < \infty$, then $\lim\limits_{x\to\infty} f(x) = 0$.

b) The converse does not hold.

Exercise* 5.5.10*: *Find an example of an* unbounded *continuous function $f\colon [0,\infty) \to \mathbb{R}$ that is nonnegative and such that $\int_0^\infty f < \infty$. Note that $\lim_{x\to\infty} f(x)$ will not exist; compare previous exercise. Hint: On each interval $[k,k+1]$, $k \in \mathbb{N}$, define a function whose integral over this interval is less than say 2^{-k}.*

Exercise* 5.5.11** (More challenging):*** *Find an example of a function $f\colon [0,\infty) \to \mathbb{R}$ integrable on all intervals such that $\lim_{n\to\infty} \int_0^n f$ converges as a limit of a sequence (so $n \in \mathbb{N}$), but such that $\int_0^\infty f$ does not exist. Hint: For all $n \in \mathbb{N}$, divide $[n,n+1]$ into two halves. On one half make the function negative, on the other make the function positive.*

Exercise* 5.5.12*: *Suppose $f\colon [1,\infty) \to \mathbb{R}$ is such that $g(x) := x^2 f(x)$ is a bounded function. Prove that $\int_1^\infty f$ converges.*

It is sometimes desirable to assign a value to integrals that normally cannot be interpreted even as improper integrals, e.g. $\int_{-1}^1 1/x\,dx$. Suppose $f\colon [a,b] \to \mathbb{R}$ is a function and $a < c < b$, where f is Riemann integrable on the intervals $[a,c-\varepsilon]$ and $[c+\varepsilon,b]$ for all $\varepsilon > 0$. Define the *Cauchy principal value* of $\int_a^b f$ as

$$p.v.\int_a^b f := \lim_{\varepsilon\to 0^+}\left(\int_a^{c-\varepsilon} f + \int_{c+\varepsilon}^b f\right),$$

if the limit exists.

Exercise* 5.5.13*:

a) Compute $p.v.\int_{-1}^1 1/x\,dx$.

b) Compute $\lim_{\varepsilon\to 0^+}\left(\int_{-1}^{-\varepsilon} 1/x\,dx + \int_{2\varepsilon}^1 1/x\,dx\right)$ and show it is not equal to the principal value.

c) Show that if f is integrable on $[a,b]$, then $p.v.\int_a^b f = \int_a^b f$ (for an arbitrary $c \in (a,b)$).

d) Suppose $f\colon [-1,1] \to \mathbb{R}$ is an odd function ($f(-x) = -f(x)$) that is integrable on $[-1,-\varepsilon]$ and $[\varepsilon,1]$ for all $\varepsilon > 0$. Prove that $p.v.\int_{-1}^1 f = 0$

e) Suppose $f\colon [-1,1] \to \mathbb{R}$ is continuous and differentiable at 0. Show that $p.v.\int_{-1}^1 \frac{f(x)}{x}\,dx$ exists.

Exercise* 5.5.14*: *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions, where $g(x) = 0$ for all $x \notin [a,b]$ for some interval $[a,b]$.*

a) *Show that the* convolution
$$(g * f)(x) := \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$$
is well-defined for all $x \in \mathbb{R}$.

b) *Suppose $\int_{-\infty}^{\infty} \lvert f(x) \rvert \, dx < \infty$. Prove that*
$$\lim_{x\to-\infty} (g * f)(x) = 0, \qquad \textit{and} \qquad \lim_{x\to\infty} (g * f)(x) = 0.$$

Chapter 6

Sequences of Functions

6.1 Pointwise and uniform convergence

Note: 1–1.5 lecture

Up till now, when we talked about limits of sequences we talked about sequences of numbers. A very useful concept in analysis is a sequence of functions. For example, a solution to some differential equation might be found by finding only approximate solutions. Then the actual solution is some sort of limit of those approximate solutions.

When talking about sequences of functions, the tricky part is that there are multiple notions of a limit. Let us describe two common notions of a limit of a sequence of functions.

6.1.1 Pointwise convergence

Definition 6.1.1. For every $n \in \mathbb{N}$, let $f_n \colon S \to \mathbb{R}$ be a function. The sequence $\{f_n\}_{n=1}^{\infty}$ *converges pointwise* to $f \colon S \to \mathbb{R}$ if for every $x \in S$, we have

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Limits of sequences of numbers are unique, and so if a sequence $\{f_n\}$ converges pointwise, the limit function f is unique. It is common to say that $f_n \colon S \to \mathbb{R}$ *converges to f on $T \subset S$* for some $f \colon T \to \mathbb{R}$. In that case we mean $f(x) = \lim f_n(x)$ for every $x \in T$. In other words, the restrictions of f_n to T converge pointwise to f.

Example 6.1.2: On $[-1,1]$, the sequence of functions defined by $f_n(x) := x^{2n}$ converges pointwise to $f \colon [-1,1] \to \mathbb{R}$, where

$$f(x) = \begin{cases} 1 & \text{if } x = -1 \text{ or } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 6.1.

To see this is so, first take $x \in (-1,1)$. Then $0 \leq x^2 < 1$. We have seen before that

$$\left| x^{2n} - 0 \right| = (x^2)^n \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $\lim f_n(x) = 0$.

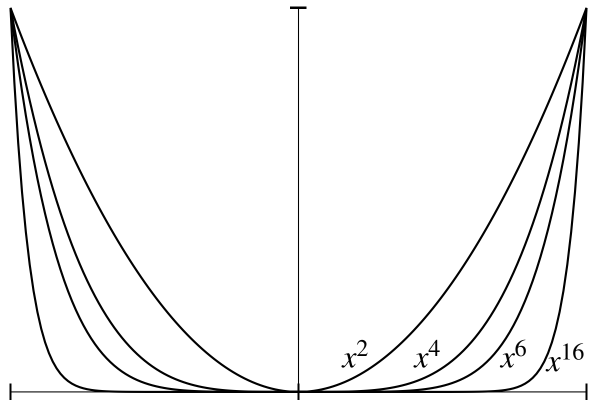

Figure 6.1: Graphs of f_1, f_2, f_3, and f_8 for $f_n(x) := x^{2n}$.

When $x = 1$ or $x = -1$, then $x^{2n} = 1$ for all n and hence $\lim f_n(x) = 1$. For all other x, the sequence $\{f_n(x)\}$ does not converge.

Often, functions are given as a series. In this case, we use the notion of pointwise convergence to find the values of the function.

Example 6.1.3: We write

$$\sum_{k=0}^{\infty} x^k$$

to denote the limit of the functions

$$f_n(x) := \sum_{k=0}^{n} x^k.$$

When studying series, we saw that for $(-1,1)$ the f_n converge pointwise to

$$\frac{1}{1-x}.$$

The subtle point here is that while $\frac{1}{1-x}$ is defined for all $x \neq 1$, and f_n are defined for all x (even at $x = 1$), convergence only happens on $(-1,1)$. Therefore, when we write

$$f(x) := \sum_{k=0}^{\infty} x^k$$

we mean that f is defined on $(-1,1)$ and is the pointwise limit of the partial sums.

Example 6.1.4: Let $f_n(x) := \sin(nx)$. Then f_n does not converge pointwise to any function on any interval. It may converge at certain points, such as when $x = 0$ or $x = \pi$. It is left as an exercise that in any interval $[a,b]$, there exists an x such that $\sin(xn)$ does not have a limit as n goes to infinity. See Figure 6.2.

Before we move to uniform convergence, let us reformulate pointwise convergence in a different way. We leave the proof to the reader—it is a simple application of the definition of convergence of a sequence of real numbers.

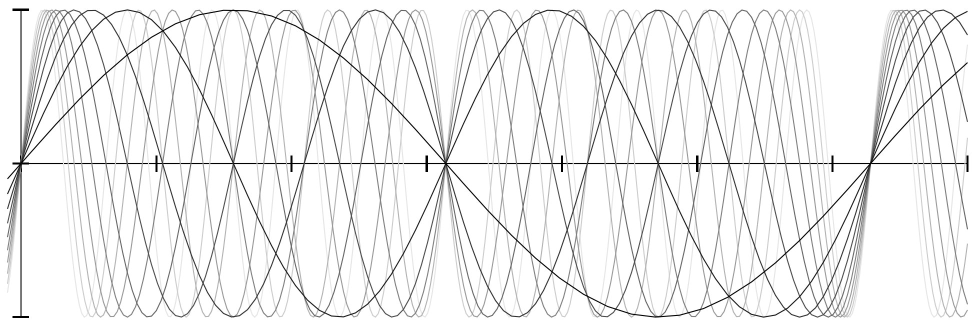

Figure 6.2: Graphs of $\sin(nx)$ for $n = 1, 2, \ldots, 10$, with higher n in lighter gray.

Proposition 6.1.5. *Let $f_n : S \to \mathbb{R}$ and $f : S \to \mathbb{R}$ be functions. Then $\{f_n\}$ converges pointwise to f if and only if for every $x \in S$, and every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have*

$$|f_n(x) - f(x)| < \varepsilon.$$

The key point here is that N can depend on x, not just on ε. That is, for each x we can pick a different N. If we could pick one N for all x, we would have what is called uniform convergence.

6.1.2 Uniform convergence

Definition 6.1.6. Let $f_n : S \to \mathbb{R}$ and $f : S \to \mathbb{R}$ be functions. The sequence $\{f_n\}$ *converges uniformly* to f if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all } x \in S.$$

In uniform convergence, N cannot depend on x. Given $\varepsilon > 0$, we must find an N that works for all $x \in S$. See Figure 6.3 for an illustration.

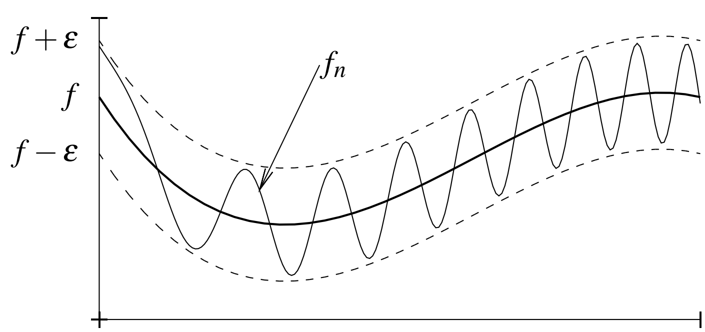

Figure 6.3: In uniform convergence, for $n \geq N$, the functions f_n are within a strip of $\pm\varepsilon$ from f.

Uniform convergence implies pointwise convergence, and the proof follows by Proposition 6.1.5:

Proposition 6.1.7. *Let $\{f_n\}$ be a sequence of functions $f_n : S \to \mathbb{R}$. If $\{f_n\}$ converges uniformly to $f : S \to \mathbb{R}$, then $\{f_n\}$ converges pointwise to f.*

The converse does not hold.

Example 6.1.8: The functions $f_n(x) := x^{2n}$ do not converge uniformly on $[-1,1]$, even though they converge pointwise. To see this, suppose for contradiction that the convergence is uniform. For $\varepsilon := 1/2$, there would have to exist an N such that $x^{2N} = \left|x^{2N} - 0\right| < 1/2$ for all $x \in (-1,1)$ (as $f_n(x)$ converges to 0 on $(-1,1)$). But that means that for every sequence $\{x_k\}$ in $(-1,1)$ such that $\lim x_k = 1$, we have $x_k^{2N} < 1/2$ for all k. On the other hand, x^{2N} is a continuous function of x (it is a polynomial). Therefore, we obtain a contradiction

$$1 = 1^{2N} = \lim_{k\to\infty} x_k^{2N} \leq 1/2.$$

However, if we restrict our domain to $[-a,a]$ where $0 < a < 1$, then $\{f_n\}$ converges uniformly to 0 on $[-a,a]$. Note that $a^{2n} \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, pick $N \in \mathbb{N}$ such that $a^{2n} < \varepsilon$ for all $n \geq N$. When $x \in [-a,a]$, we have $|x| \leq a$. So for all $n \geq N$ and all $x \in [-a,a]$,

$$\left|x^{2n}\right| = |x|^{2n} \leq a^{2n} < \varepsilon.$$

6.1.3 Convergence in uniform norm

For bounded functions, there is another more abstract way to think of uniform convergence. To every bounded function we assign a certain nonnegative number (called the uniform norm). This number measures the "distance" of the function from 0. We can then "measure" how far two functions are from each other. We then translate a statement about uniform convergence into a statement about a certain sequence of real numbers converging to zero.

Definition 6.1.9. Let $f\colon S \to \mathbb{R}$ be a bounded function. Define

$$\|f\|_u := \sup\bigl\{|f(x)| : x \in S\bigr\}.$$

We call $\|\cdot\|_u$ the *uniform norm*.

To use this notation* and this concept, the domain S must be fixed. Some authors use the notation $\|f\|_S$ to emphasize the dependence on S.

Proposition 6.1.10. *A sequence of bounded functions $f_n\colon S \to \mathbb{R}$ converges uniformly to $f\colon S \to \mathbb{R}$, if and only if*

$$\lim_{n\to\infty} \|f_n - f\|_u = 0.$$

Proof. First suppose $\lim \|f_n - f\|_u = 0$. Let $\varepsilon > 0$ be given. Then there exists an N such that for $n \geq N$, we have $\|f_n - f\|_u < \varepsilon$. As $\|f_n - f\|_u$ is the supremum of $|f_n(x) - f(x)|$, we see that for all $x \in S$, we have $|f_n(x) - f(x)| \leq \|f_n - f\|_u < \varepsilon$.

On the other hand, suppose $\{f_n\}$ converges uniformly to f. Let $\varepsilon > 0$ be given. Then find N such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$. Taking the supremum we see that $\|f_n - f\|_u \leq \varepsilon$. Hence $\lim \|f_n - f\|_u = 0$. $\square$

*The notation nor terminology is not completely standardized. The norm is also called the *sup norm* or *infinity norm*, and in addition to $\|f\|_u$ and $\|f\|_S$ it is sometimes written as $\|f\|_\infty$ or $\|f\|_{\infty,S}$.

Sometimes it is said that $\{f_n\}$ *converges to f in uniform norm* instead of *converges uniformly* if $\|f_n - f\| \to 0$. The proposition says that the two notions are the same thing.

Example 6.1.11: Let $f_n \colon [0,1] \to \mathbb{R}$ be defined by $f_n(x) := \frac{nx+\sin(nx^2)}{n}$. We claim $\{f_n\}$ converges uniformly to $f(x) := x$. Let us compute:

$$\begin{aligned}
\|f_n - f\|_u &= \sup \left\{ \left| \frac{nx+\sin(nx^2)}{n} - x \right| : x \in [0,1] \right\} \\
&= \sup \left\{ \frac{\left|\sin(nx^2)\right|}{n} : x \in [0,1] \right\} \\
&\leq \sup \{ 1/n : x \in [0,1] \} \\
&= 1/n.
\end{aligned}$$

Using uniform norm, we define Cauchy sequences in a similar way as we define Cauchy sequences of real numbers.

Definition 6.1.12. Let $f_n \colon S \to \mathbb{R}$ be bounded functions. The sequence is *Cauchy in the uniform norm* or *uniformly Cauchy* if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, k \geq N$,

$$\|f_m - f_k\|_u < \varepsilon.$$

Proposition 6.1.13. *Let $f_n \colon S \to \mathbb{R}$ be bounded functions. Then $\{f_n\}$ is Cauchy in the uniform norm if and only if there exists an $f \colon S \to \mathbb{R}$ and $\{f_n\}$ converges uniformly to f.*

Proof. Let us first suppose $\{f_n\}$ is Cauchy in the uniform norm. Let us define f. Fix x, then the sequence $\{f_n(x)\}$ is Cauchy because

$$|f_m(x) - f_k(x)| \leq \|f_m - f_k\|_u .$$

Thus $\{f_n(x)\}$ converges to some real number. Define $f \colon S \to \mathbb{R}$ by

$$f(x) := \lim_{n\to\infty} f_n(x).$$

The sequence $\{f_n\}$ converges pointwise to f. To show that the convergence is uniform, let $\varepsilon > 0$ be given. Find an N such that for all $m, k \geq N$, we have $\|f_m - f_k\|_u < \varepsilon/2$. In other words, for all x, we have $|f_m(x) - f_k(x)| < \varepsilon/2$. For any fixed x, take the limit as k goes to infinity. Then $|f_m(x) - f_k(x)|$ goes to $|f_m(x) - f(x)|$. Consequently for all x,

$$|f_m(x) - f(x)| \leq \varepsilon/2 < \varepsilon.$$

And hence $\{f_n\}$ converges uniformly.

Next, we prove the other direction. Suppose $\{f_n\}$ converges uniformly to f. Given $\varepsilon > 0$, find N such that for all $n \geq N$, we have $|f_n(x) - f(x)| < \varepsilon/4$ for all $x \in S$. Therefore, for all $m, k \geq N$,

$$|f_m(x) - f_k(x)| = |f_m(x) - f(x) + f(x) - f_k(x)| \leq |f_m(x) - f(x)| + |f(x) - f_k(x)| < \varepsilon/4 + \varepsilon/4.$$

Take supremum over all x to obtain

$$\|f_m - f_k\|_u \leq \varepsilon/2 < \varepsilon. \qquad \square$$

6.1.4 Exercises

***Exercise* 6.1.1:** *Let f and g be bounded functions on $[a,b]$. Prove*

$$\|f+g\|_u \leq \|f\|_u + \|g\|_u .$$

***Exercise* 6.1.2:**

a) *Find the pointwise limit $\dfrac{e^{x/n}}{n}$ for $x \in \mathbb{R}$.*

b) *Is the limit uniform on $\mathbb{R}$?*

c) *Is the limit uniform on $[0,1]$?*

***Exercise* 6.1.3:** *Suppose $f_n \colon S \to \mathbb{R}$ are functions that converge uniformly to $f \colon S \to \mathbb{R}$. Suppose $A \subset S$. Show that the sequence of restrictions $\{f_n|_A\}$ converges uniformly to $f|_A$.*

***Exercise* 6.1.4:** *Suppose $\{f_n\}$ and $\{g_n\}$ defined on some set A converge to f and g respectively pointwise. Show that $\{f_n + g_n\}$ converges pointwise to $f+g$.*

***Exercise* 6.1.5:** *Suppose $\{f_n\}$ and $\{g_n\}$ defined on some set A converge to f and g respectively uniformly on A. Show that $\{f_n + g_n\}$ converges uniformly to $f+g$ on A.*

***Exercise* 6.1.6:** *Find an example of a sequence of functions $\{f_n\}$ and $\{g_n\}$ that converge uniformly to some f and g on some set A, but such that $\{f_n g_n\}$ (the multiple) does not converge uniformly to fg on A. Hint: Let $A := \mathbb{R}$, let $f(x) := g(x) := x$. You can even pick $f_n = g_n$.*

***Exercise* 6.1.7:** *Suppose there exists a sequence of functions $\{g_n\}$ uniformly converging to 0 on A. Now suppose we have a sequence of functions $\{f_n\}$ and a function f on A such that*

$$|f_n(x) - f(x)| \leq g_n(x)$$

for all $x \in A$. Show that $\{f_n\}$ converges uniformly to f on A.

***Exercise* 6.1.8:** *Let $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ be sequences of functions on $[a,b]$. Suppose $\{f_n\}$ and $\{h_n\}$ converge uniformly to some function $f \colon [a,b] \to \mathbb{R}$ and suppose $f_n(x) \leq g_n(x) \leq h_n(x)$ for all $x \in [a,b]$. Show that $\{g_n\}$ converges uniformly to f.*

***Exercise* 6.1.9:** *Let $f_n \colon [0,1] \to \mathbb{R}$ be a sequence of increasing functions (that is, $f_n(x) \geq f_n(y)$ whenever $x \geq y$). Suppose $f_n(0) = 0$ and $\lim\limits_{n\to\infty} f_n(1) = 0$. Show that $\{f_n\}$ converges uniformly to 0.*

***Exercise* 6.1.10:** *Let $\{f_n\}$ be a sequence of functions defined on $[0,1]$. Suppose there exists a sequence of distinct numbers $x_n \in [0,1]$ such that*

$$f_n(x_n) = 1.$$

Prove or disprove the following statements:

a) *True or false: There exists $\{f_n\}$ as above that converges to 0 pointwise.*

b) *True or false: There exists $\{f_n\}$ as above that converges to 0 uniformly on $[0,1]$.*

***Exercise* 6.1.11:** *Fix a continuous $h \colon [a,b] \to \mathbb{R}$. Let $f(x) := h(x)$ for $x \in [a,b]$, $f(x) := h(a)$ for $x < a$ and $f(x) := h(b)$ for all $x > b$. First show that $f \colon \mathbb{R} \to \mathbb{R}$ is continuous. Now let f_n be the function g from Exercise 5.3.7 with $\varepsilon = 1/n$, defined on the interval $[a,b]$. That is,*

$$f_n(x) := \frac{n}{2} \int_{x-1/n}^{x+1/n} f .$$

Show that $\{f_n\}$ converges uniformly to h on $[a,b]$.

Exercise* 6.1.12*: *Prove that if a sequence of functions $f_n \colon S \to \mathbb{R}$ converge uniformly to a bounded function $f \colon S \to \mathbb{R}$, then there exists an N such that for all $n \geq N$, the f_n are bounded.*

Exercise* 6.1.13*: *Suppose there is a single constant B and a sequence of functions $f_n \colon S \to \mathbb{R}$ that are bounded by B, that is $|f_n(x)| \leq B$ for all $x \in S$. Suppose that $\{f_n\}$ converges pointwise to $f \colon S \to \mathbb{R}$. Prove that f is bounded.*

Exercise* 6.1.14** (requires §2.6):*** *In Example 6.1.3 we saw $\sum_{k=0}^{\infty} x^k$ converges pointwise to $\frac{1}{1-x}$ on $(-1,1)$.*

a) Show that whenever $0 \leq c < 1$, the series $\sum_{k=0}^{\infty} x^k$ converges uniformly on $[-c,c]$.

b) Show that the series $\sum_{k=0}^{\infty} x^k$ does not converge uniformly on $(-1,1)$.

6.2 Interchange of limits

Note: 1–2.5 lectures, subsections on derivatives and power series (which requires §2.6) optional.

Large parts of modern analysis deal mainly with the question of the interchange of two limiting operations. When we have a chain of two limits, we cannot always just swap the limits. For instance,

$$0 = \lim_{n\to\infty}\left(\lim_{k\to\infty}\frac{n}{n+k}\right) \neq \lim_{k\to\infty}\left(\lim_{n\to\infty}\frac{n}{n+k}\right) = 1.$$

When talking about sequences of functions, interchange of limits comes up quite often. We look at several instances: continuity of the limit, the integral of the limit, the derivative of the limit, and the convergence of power series.

6.2.1 Continuity of the limit

If we have a sequence $\{f_n\}$ of continuous functions, is the limit continuous? Suppose f is the (pointwise) limit of $\{f_n\}$. If $\lim x_k = x$, we are interested in the following interchange of limits, where the equality to prove is marked with a question mark. Equality is not always true, in fact, the limits to the left of the question mark might not even exist.

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty}\Bigl(\lim_{n\to\infty} f_n(x_k)\Bigr) \overset{?}{=} \lim_{n\to\infty}\Bigl(\lim_{k\to\infty} f_n(x_k)\Bigr) = \lim_{n\to\infty} f_n(x) = f(x).$$

We wish to find conditions on the sequence $\{f_n\}$ so that the equation above holds. If we only require pointwise convergence, then the limit of a sequence of functions need not be continuous, and the equation above need not hold.

Example 6.2.1: Define $f_n \colon [0,1] \to \mathbb{R}$ as

$$f_n(x) := \begin{cases} 1-nx & \text{if } x < 1/n, \\ 0 & \text{if } x \geq 1/n. \end{cases}$$

See Figure 6.4.

Each function f_n is continuous. Fix an $x \in (0,1]$. If $n \geq 1/x$, then $x \geq 1/n$. Therefore for $n \geq 1/x$, we have $f_n(x) = 0$, and so

$$\lim_{n\to\infty} f_n(x) = 0.$$

On the other hand, if $x = 0$, then

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1 = 1.$$

Thus the pointwise limit of f_n is the function $f \colon [0,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

The function f is not continuous at 0.

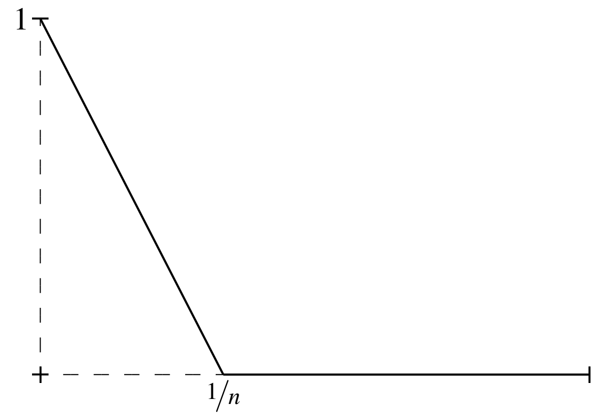

Figure 6.4: Graph of $f_n(x)$.

If we, however, require the convergence to be uniform, the limits can be interchanged.

Theorem 6.2.2. *Suppose $S \subset \mathbb{R}$. Let $\{f_n\}$ be a sequence of continuous functions $f_n \colon S \to \mathbb{R}$ converging uniformly to $f \colon S \to \mathbb{R}$. Then f is continuous.*

Proof. Let $x \in S$ be fixed. Let $\{x_n\}$ be a sequence in S converging to x.

Let $\varepsilon > 0$ be given. As $\{f_k\}$ converges uniformly to f, we find a $k \in \mathbb{N}$ such that

$$|f_k(y) - f(y)| < \varepsilon/3$$

for all $y \in S$. As f_k is continuous at x, we find an $N \in \mathbb{N}$ such that for all $m \geq N$,

$$|f_k(x_m) - f_k(x)| < \varepsilon/3.$$

Thus for all $m \geq N$,

$$\begin{aligned} |f(x_m) - f(x)| &= |f(x_m) - f_k(x_m) + f_k(x_m) - f_k(x) + f_k(x) - f(x)| \\ &\leq |f(x_m) - f_k(x_m)| + |f_k(x_m) - f_k(x)| + |f_k(x) - f(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Therefore, $\{f(x_m)\}$ converges to $f(x)$ and f is continuous at x. As x was arbitrary, f is continuous everywhere. $\square$

6.2.2 Integral of the limit

Again, if we simply require pointwise convergence, then the integral of a limit of a sequence of functions need not be equal to the limit of the integrals.

Example 6.2.3: Define $f_n \colon [0,1] \to \mathbb{R}$ as

$$f_n(x) := \begin{cases} 0 & \text{if } x = 0, \\ n - n^2x & \text{if } 0 < x < 1/n, \\ 0 & \text{if } x \geq 1/n. \end{cases}$$

See Figure 6.5.

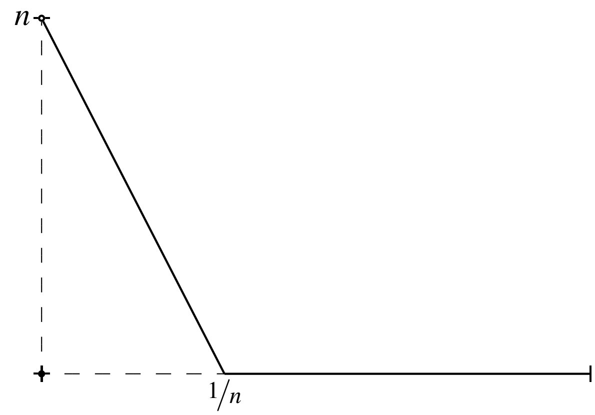

Figure 6.5: Graph of $f_n(x)$.

Each f_n is Riemann integrable (it is continuous on $(0,1]$ and bounded), and the fundamental theorem of calculus says that

$$\int_0^1 f_n = \int_0^{1/n} (n - n^2 x)\, dx = 1/2.$$

Let us compute the pointwise limit of $\{f_n\}$. Fix an $x \in (0,1]$. For $n \geq 1/x$, we have $x \geq 1/n$ and so $f_n(x) = 0$. Therefore,

$$\lim_{n\to\infty} f_n(x) = 0.$$

We also have $f_n(0) = 0$ for all n. Therefore, the pointwise limit of $\{f_n\}$ is the zero function. Thus

$$1/2 = \lim_{n\to\infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x) \right) dx = \int_0^1 0\, dx = 0.$$

But if we require the convergence to be uniform, the limits can be interchanged.*

Theorem 6.2.4. *Let $\{f_n\}$ be a sequence of Riemann integrable functions $f_n \colon [a,b] \to \mathbb{R}$ converging uniformly to $f \colon [a,b] \to \mathbb{R}$. Then f is Riemann integrable, and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. Let $\varepsilon > 0$ be given. As f_n goes to f uniformly, we find an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $|f_n(x) - f(x)| < \frac{\varepsilon}{2(b-a)}$ for all $x \in [a,b]$. In particular, by reverse triangle inequality, $|f(x)| < \frac{\varepsilon}{2(b-a)} + |f_n(x)|$ for all x. Hence f is bounded, as f_n is bounded. Note that f_n is integrable

*Weaker conditions are sufficient for this kind of theorem, but to prove such a generalization requires more sophisticated machinery than we cover here: the Lebesgue integral. In particular, the theorem holds with pointwise convergence as long as f is integrable and there is an M such that $\|f_n\|_u \leq M$ for all n.

and compute

$$\begin{aligned}
\overline{\int_a^b} f - \underline{\int_a^b} f &= \overline{\int_a^b} \big(f(x) - f_n(x) + f_n(x)\big)\,dx - \underline{\int_a^b} \big(f(x) - f_n(x) + f_n(x)\big)\,dx \\
&\leq \overline{\int_a^b} \big(f(x) - f_n(x)\big)\,dx + \overline{\int_a^b} f_n(x)\,dx - \underline{\int_a^b} \big(f(x) - f_n(x)\big)\,dx - \underline{\int_a^b} f_n(x)\,dx \\
&= \overline{\int_a^b} \big(f(x) - f_n(x)\big)\,dx + \int_a^b f_n(x)\,dx - \underline{\int_a^b} \big(f(x) - f_n(x)\big)\,dx - \int_a^b f_n(x)\,dx \\
&= \overline{\int_a^b} \big(f(x) - f_n(x)\big)\,dx - \underline{\int_a^b} \big(f(x) - f_n(x)\big)\,dx \\
&\leq \frac{\varepsilon}{2(b-a)}(b-a) + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon.
\end{aligned}$$

The first inequality is Proposition 5.2.5. The second inequality follows from Proposition 5.1.8 and the fact that for all $x \in [a,b]$, we have $\frac{-\varepsilon}{2(b-a)} < f(x) - f_n(x) < \frac{\varepsilon}{2(b-a)}$. As $\varepsilon > 0$ was arbitrary, f is Riemann integrable.

Finally, we compute $\int_a^b f$. We apply Proposition 5.1.10 in the calculation. Again, for all $n \geq M$ (where M is the same as above),

$$\begin{aligned}
\left| \int_a^b f - \int_a^b f_n \right| &= \left| \int_a^b \big(f(x) - f_n(x)\big)\,dx \right| \\
&\leq \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Therefore, $\{\int_a^b f_n\}$ converges to $\int_a^b f$. $\square$

Example 6.2.5: Suppose we wish to compute

$$\lim_{n\to\infty} \int_0^1 \frac{nx + \sin(nx^2)}{n}\,dx.$$

It is impossible to compute the integrals for any particular n using calculus as $\sin(nx^2)$ has no closed-form antiderivative. However, we can compute the limit. We have shown before that $\frac{nx+\sin(nx^2)}{n}$ converges uniformly on $[0,1]$ to x. By Theorem 6.2.4, the limit exists and

$$\lim_{n\to\infty} \int_0^1 \frac{nx + \sin(nx^2)}{n}\,dx = \int_0^1 x\,dx = 1/2.$$

Example 6.2.6: If convergence is only pointwise, the limit need not even be Riemann integrable. On $[0,1]$ define

$$f_n(x) := \begin{cases} 1 & \text{if } x = p/q \text{ in lowest terms and } q \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

The function f_n differs from the zero function at finitely many points; there are only finitely many fractions in $[0,1]$ with denominator less than or equal to n. So f_n is integrable and $\int_0^1 f_n = \int_0^1 0 = 0$.

It is an easy exercise to show that $\{f_n\}$ converges pointwise to the Dirichlet function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

which is not Riemann integrable.

Example 6.2.7: In fact, if the convergence is only pointwise, the limit of bounded functions is not even necessarily bounded. Define $f_n \colon [0,1] \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 0 & \text{if } x < 1/n, \\ 1/x & \text{else.} \end{cases}$$

For every n we get that $|f_n(x)| \leq n$ for all $x \in [0,1]$ so the functions are bounded. However f_n converge pointwise to

$$f(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1/x & \text{else,} \end{cases}$$

which is unbounded.

6.2.3 Derivative of the limit

While uniform convergence is enough to swap limits with integrals, it is not, however, enough to swap limits with derivatives, unless you also have uniform convergence of the derivatives themselves.

Example 6.2.8: Let $f_n(x) := \frac{\sin(nx)}{n}$. Then f_n converges uniformly to 0. See Figure 6.6. The derivative of the limit is 0. But $f_n'(x) = \cos(nx)$, which does not converge even pointwise, for example $f_n'(\pi) = (-1)^n$. Furthermore, $f_n'(0) = 1$ for all n, which does converge, but not to 0.

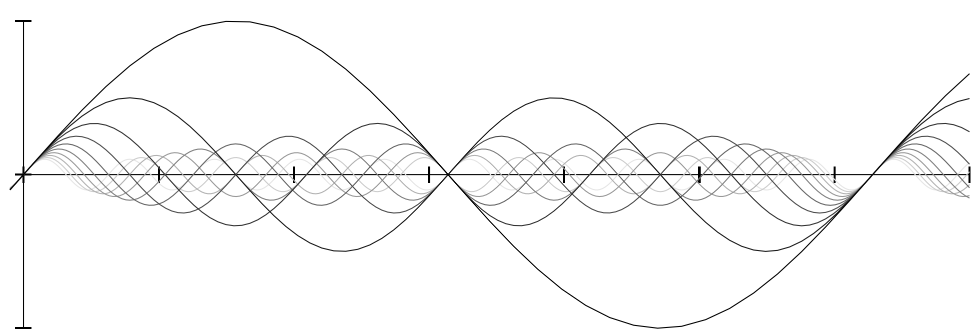

Figure 6.6: Graphs of $\frac{\sin(nx)}{n}$ for $n = 1, 2, \ldots, 10$, with higher n in lighter gray.

Example 6.2.9: Let $f_n(x) := \frac{1}{1+nx^2}$. If $x \neq 0$, then $\lim_{n\to\infty} f_n(x) = 0$, but $\lim_{n\to\infty} f_n(0) = 1$. Hence, $\{f_n\}$ converges pointwise to a function that is not continuous at 0. We compute

$$f_n'(x) = \frac{-2nx}{(1+nx^2)^2}.$$

For every x, $\lim_{n\to\infty} f_n'(x) = 0$, so the derivatives converge pointwise to 0, but the reader can check that the convergence is not uniform on any interval containing 0. The limit of f_n is not differentiable at 0—it is not even continuous at 0.

See the exercises for more examples. Using the fundamental theorem of calculus, we find an answer for continuously differentiable functions. The following theorem is true even if we do not assume continuity of the derivatives, but the proof is more difficult.

Theorem 6.2.10. *Let I be a bounded interval and let $f_n \colon I \to \mathbb{R}$ be continuously differentiable functions. Suppose $\{f_n'\}$ converges uniformly to $g \colon I \to \mathbb{R}$, and suppose $\{f_n(c)\}_{n=1}^{\infty}$ is a convergent sequence for some $c \in I$. Then $\{f_n\}$ converges uniformly to a continuously differentiable function $f \colon I \to \mathbb{R}$, and $f' = g$.*

Proof. Define $f(c) := \lim_{n\to\infty} f_n(c)$. As f_n' are continuous and hence Riemann integrable, then via the fundamental theorem of calculus, we find that for $x \in I$,

$$f_n(x) = f_n(c) + \int_c^x f_n'.$$

As $\{f_n'\}$ converges uniformly on I, it converges uniformly on $[c,x]$ (or $[x,c]$ if $x < c$). Thus, the limit as $n \to \infty$ on the right-hand side exists. Define f at the remaining points (where $x \neq c$) by this limit:

$$f(x) := \lim_{n\to\infty} f_n(c) + \lim_{n\to\infty} \int_c^x f_n' = f(c) + \int_c^x g.$$

The function g is continuous, being the uniform limit of continuous functions. Hence f is differentiable and $f'(x) = g(x)$ for all $x \in I$ by the second form of the fundamental theorem.

It remains to prove uniform convergence. Suppose I has a lower bound a and upper bound b. Let $\varepsilon > 0$ be given. Take M such that for all $n \geq M$, we have $|f(c) - f_n(c)| < \varepsilon/2$ and $|g(x) - f_n'(x)| < \frac{\varepsilon}{2(b-a)}$ for all $x \in I$. Then

$$\begin{aligned}
|f(x) - f_n(x)| &= \left| \left(f(c) + \int_c^x g \right) - \left(f_n(c) + \int_c^x f_n' \right) \right| \\
&\leq |f(c) - f_n(c)| + \left| \int_c^x g - \int_c^x f_n' \right| \\
&= |f(c) - f_n(c)| + \left| \int_c^x \bigl(g(s) - f_n'(s)\bigr)\, ds \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon.
\end{aligned}$$

□

The proof goes through without boundedness of I, except for the uniform convergence of f_n to f. As an example suppose $I = \mathbb{R}$ and let $f_n(x) := x/n$. Then $f_n'(x) = 1/n$, which converges uniformly to 0. However, $\{f_n\}$ converges to 0 only pointwise.

6.2.4 Convergence of power series

In §2.6 we saw that a power series converges absolutely inside its radius of convergence, so it converges pointwise. Let us show that it (and all its derivatives) also converges uniformly. This fact allows us to swap several types of limits. Not only is the limit continuous, we can integrate and even differentiate convergent power series term by term.

Proposition 6.2.11. *Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a convergent power series with a radius of convergence ρ, where $0 < \rho \leq \infty$. Then the series converges uniformly in $[a-r, a+r]$ whenever $0 < r < \rho$.*

In particular, the series defines a continuous function on $(a-\rho, a+\rho)$ (if $\rho < \infty$), or $\mathbb{R}$ (if $\rho = \infty$).

Proof. Let $I := (a-\rho, a+\rho)$ if $\rho < \infty$, or let $I := \mathbb{R}$ if $\rho = \infty$. Take $0 < r < \rho$. The series converges absolutely for every $x \in I$, in particular if $x = a+r$. So $\sum_{n=0}^{\infty} |c_n| r^n$ converges. Given $\varepsilon > 0$, find M such that for all $k \geq M$,

$$\sum_{n=k+1}^{\infty} |c_n| r^n < \varepsilon.$$

For all $x \in [a-r, a+r]$ and all $m > k$,

$$\begin{aligned} \left| \sum_{n=0}^{m} c_n(x-a)^n - \sum_{n=0}^{k} c_n(x-a)^n \right| &= \left| \sum_{n=k+1}^{m} c_n(x-a)^n \right| \\ &\leq \sum_{n=k+1}^{m} |c_n| \, |x-a|^n \leq \sum_{n=k+1}^{m} |c_n| r^n \leq \sum_{n=k+1}^{\infty} |c_n| r^n < \varepsilon. \end{aligned}$$

The partial sums are therefore uniformly Cauchy on $[a-r, a+r]$ and hence converge uniformly on that set.

Moreover, the partial sums are polynomials, which are continuous, and so their uniform limit on $[a-r, a+r]$ is a continuous function. As $r < \rho$ was arbitrary, the limit function is continuous on all of I. $\square$

As we said, we will show that power series can be differentiated and integrated term by term. The differentiated or integrated series is again a power series, and we will show it has the same radius of convergence. Therefore, any power series defines an infinitely differentiable function.

We first prove that we can antidifferentiate, as integration only needs uniform limits.

Corollary 6.2.12. *Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a convergent power series with a radius of convergence $0 < \rho \leq \infty$. Let $I := (a-\rho, a+\rho)$ if $\rho < \infty$ or $I := \mathbb{R}$ if $\rho = \infty$. Let $f \colon I \to \mathbb{R}$ be the limit. Then*

$$\int_a^x f = \sum_{n=1}^{\infty} \frac{c_{n-1}}{n} (x-a)^n,$$

where the radius of convergence of this series is at least ρ.

Proof. Take $0 < r < \rho$. The partial sums $\sum_{n=0}^{k} c_n(x-a)^n$ converge uniformly on $[a-r, a+r]$. For every fixed $x \in [a-r, a+r]$, the convergence is also uniform on $[a, x]$ (or $[x, a]$ if $x < a$). Hence,

$$\int_a^x f = \int_a^x \lim_{k\to\infty} \sum_{n=0}^{k} c_n(s-a)^n \, ds = \lim_{k\to\infty} \int_a^x \sum_{n=0}^{k} c_n(s-a)^n \, ds = \lim_{k\to\infty} \sum_{n=1}^{k+1} \frac{c_{n-1}}{n} (x-a)^n. \qquad \square$$

Corollary 6.2.13. *Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a convergent power series with a radius of convergence $0 < \rho \leq \infty$. Let $I := (a-\rho, a+\rho)$ if $\rho < \infty$ or $I := \mathbb{R}$ if $\rho = \infty$. Let $f \colon I \to \mathbb{R}$ be the limit. Then f is a differentiable function, and*

$$f'(x) = \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n,$$

where the radius of convergence of this series is ρ.

Proof. Take $0 < r < \rho$. The series converges uniformly on $[a-r, a+r]$, but we need uniform convergence of the derivative. Let

$$R := \limsup_{n\to\infty} \left|c_n\right|^{1/n}.$$

As the series is convergent $R < \infty$, and the radius of convergence is $1/R$ (or ∞ if $R = 0$).

Let $\varepsilon > 0$ be given. In Example 2.2.14, we saw $\lim n^{1/n} = 1$. Hence there exists an N such that for all $n \geq N$, we have $n^{1/n} < 1+\varepsilon$. So

$$R = \limsup_{n\to\infty} \left|c_n\right|^{1/n} \leq \limsup_{n\to\infty} \left|nc_n\right|^{1/n} \leq (1+\varepsilon) \limsup_{n\to\infty} \left|c_n\right|^{1/n} = (1+\varepsilon)R.$$

As ε was arbitrary, $\limsup_{n\to\infty} \left|nc_n\right|^{1/n} = R$. Therefore, $\sum_{n=1}^{\infty} nc_n(x-a)^n$ has radius of convergence ρ. By dividing by $(x-a)$, we find $\sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n$ has radius of convergence ρ as well.

Consequently, the partial sums $\sum_{n=0}^{k}(n+1)c_{n+1}(x-a)^n$, which are derivatives of the partial sums $\sum_{n=0}^{k+1} c_n(x-a)^n$, converge uniformly on $[a-r, a+r]$. Furthermore, the series clearly converges at $x = a$. We may thus apply Theorem 6.2.10, and we are done as $r < \rho$ was arbitrary. □

Example 6.2.14: We could have used this result to define the exponential function. That is, the power series

$$f(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

has radius of convergence $\rho = \infty$. Furthermore, $f(0) = 1$, and by differentiating term by term, we find that $f'(x) = f(x)$.

Example 6.2.15: The series

$$\sum_{n=1}^{\infty} nx^n$$

converges to $\frac{x}{(1-x)^2}$ on $(-1,1)$.

Proof: On $(-1,1)$, $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$. The derivative $\sum_{n=1}^{\infty} nx^{n-1}$ then converges on the same interval to $\frac{1}{(1-x)^2}$. Multiplying by x obtains the result.

6.2.5 Exercises

***Exercise* 6.2.1:** *Find an explicit example of a sequence of differentiable functions on $[-1,1]$ that converge uniformly to a function f such that f is not differentiable. Hint: There are many possibilities, simplest is perhaps to combine $|x|$ and $\frac{n}{2}x^2 + \frac{1}{2n}$, another is to consider $\sqrt{x^2 + (1/n)^2}$. Show that these functions are differentiable, converge uniformly, and then show that the limit is not differentiable.*

***Exercise* 6.2.2:** *Let $f_n(x) := \frac{x^n}{n}$. Show that $\{f_n\}$ converges uniformly to a differentiable function f on $[0,1]$ (find f). However, show that $f'(1) \neq \lim\limits_{n\to\infty} f_n'(1)$.*

***Exercise* 6.2.3:** *Let $f \colon [0,1] \to \mathbb{R}$ be a Riemann integrable (hence bounded) function. Find $\lim\limits_{n\to\infty} \int_0^1 \frac{f(x)}{n}\,dx$.*

***Exercise* 6.2.4:** *Show* $\lim\limits_{n\to\infty} \int_1^2 e^{-nx^2}\,dx = 0$. *Feel free to use what you know about the exponential function from calculus.*

***Exercise* 6.2.5:** *Find an example of a sequence of continuous functions on* $(0,1)$ *that converges pointwise to a continuous function on* $(0,1)$*, but the convergence is not uniform.*

Note: In the previous exercise, $(0,1)$ was picked for simplicity. For a more challenging exercise, replace $(0,1)$ with $[0,1]$.

***Exercise* 6.2.6:** *True/False; prove or find a counterexample to the following statement: If* $\{f_n\}$ *is a sequence of everywhere discontinuous functions on* $[0,1]$ *that converge uniformly to a function* f*, then* f *is everywhere discontinuous.*

***Exercise* 6.2.7:** *For a continuously differentiable function* $f\colon [a,b] \to \mathbb{R}$*, define*

$$\|f\|_{C^1} := \|f\|_u + \|f'\|_u .$$

Suppose $\{f_n\}$ *is a sequence of continuously differentiable functions such that for every* $\varepsilon > 0$*, there exists an* M *such that for all* $n,k \geq M$*, we have*

$$\|f_n - f_k\|_{C^1} < \varepsilon.$$

Show that $\{f_n\}$ *converges uniformly to some continuously differentiable function* $f\colon [a,b] \to \mathbb{R}$.

Suppose $f\colon [0,1] \to \mathbb{R}$ is Riemann integrable. For the following two exercises define the number

$$\|f\|_{L^1} := \int_0^1 |f(x)|\,dx.$$

It is true that $|f|$ is integrable whenever f is, see Exercise 5.2.15. The number is called the L^1*-norm* and defines another very common type of convergence called the L^1*-convergence*. It is, however, a bit more subtle.

***Exercise* 6.2.8:** *Suppose* $\{f_n\}$ *is a sequence of Riemann integrable functions on* $[0,1]$ *that converges uniformly to* 0*. Show that*

$$\lim_{n\to\infty} \|f_n\|_{L^1} = 0.$$

***Exercise* 6.2.9:** *Find a sequence* $\{f_n\}$ *of Riemann integrable functions on* $[0,1]$ *converging pointwise to* 0*, but*

$$\lim_{n\to\infty} \|f_n\|_{L^1} \text{ does not exist (is } \infty\text{).}$$

***Exercise* 6.2.10** (Hard)**:** *Prove* Dini's theorem*: Let* $f_n\colon [a,b] \to \mathbb{R}$ *be a sequence of continuous functions such that*

$$0 \leq f_{n+1}(x) \leq f_n(x) \leq \cdots \leq f_1(x) \qquad \text{for all } n \in \mathbb{N}.$$

Suppose $\{f_n\}$ *converges pointwise to* 0*. Show that* $\{f_n\}$ *converges to zero uniformly.*

***Exercise* 6.2.11:** *Suppose* $f_n\colon [a,b] \to \mathbb{R}$ *is a sequence of continuous functions that converges pointwise to a continuous* $f\colon [a,b] \to \mathbb{R}$*. Suppose that for every* $x \in [a,b]$*, the sequence* $\{|f_n(x) - f(x)|\}$ *is monotone. Show that the sequence* $\{f_n\}$ *converges uniformly.*

Exercise* 6.2.12*: *Find a sequence of Riemann integrable functions* $f_n\colon [0,1] \to \mathbb{R}$ *such that* $\{f_n\}$ *converges to zero pointwise, and such that*

a) $\left\{\int_0^1 f_n\right\}_{n=1}^{\infty}$ *increases without bound,*

b) $\left\{\int_0^1 f_n\right\}_{n=1}^{\infty}$ *is the sequence* $-1,1,-1,1,-1,1,\ldots$*.*

It is possible to define a *joint limit* of a double sequence $\{x_{n,m}\}$ of real numbers (that is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{R}$). We say L is the joint limit of $\{x_{n,m}\}$ and write

$$\lim_{\substack{n\to\infty \\ m\to\infty}} x_{n,m} = L, \qquad \text{or} \qquad \lim_{(n,m)\to\infty} x_{n,m} = L,$$

if for every $\varepsilon > 0$, there exists an M such that if $n \geq M$ and $m \geq M$, then $|x_{n,m} - L| < \varepsilon$.

Exercise* 6.2.13*: *Suppose the joint limit (see above) of* $\{x_{n,m}\}$ *is L, and suppose that for all n,* $\lim\limits_{m\to\infty} x_{n,m}$ *exists, and for all m,* $\lim\limits_{n\to\infty} x_{n,m}$ *exists. Then show* $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} x_{n,m} = \lim\limits_{m\to\infty}\lim\limits_{n\to\infty} x_{n,m} = L$*.*

Exercise* 6.2.14*: *A joint limit (see above) does not mean the iterated limits exist. Consider* $x_{n,m} := \frac{(-1)^{n+m}}{\min\{n,m\}}$*.*

a) Show that for no n does $\lim\limits_{m\to\infty} x_{n,m}$ *exist, and for no m does* $\lim\limits_{n\to\infty} x_{n,m}$ *exist. So neither* $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} x_{n,m}$ *nor* $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} x_{n,m}$ *makes any sense at all.*

b) Show that the joint limit of $\{x_{n,m}\}$ *exists and equals 0.*

Exercise* 6.2.15*: *We say that a sequence of functions* $f_n\colon \mathbb{R} \to \mathbb{R}$ converges uniformly on compact subsets *if for every* $k \in \mathbb{N}$*, the sequence* $\{f_n\}$ *converges uniformly on* $[-k,k]$*.*

a) Prove that if $f_n\colon \mathbb{R} \to \mathbb{R}$ *is a sequence of continuous functions converging uniformly on compact subsets, then the limit is continuous.*

b) Prove that if $f_n\colon \mathbb{R} \to \mathbb{R}$ *is a sequence of functions Riemann integrable on every closed and bounded interval* $[a,b]$*, and converging uniformly on compact subsets to an* $f\colon \mathbb{R} \to \mathbb{R}$*, then for every interval* $[a,b]$*, we have* $f \in \mathscr{R}[a,b]$*, and* $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$*.*

Exercise* 6.2.16** (Challenging):*** *Find a sequence of continuous functions* $f_n\colon [0,1] \to \mathbb{R}$ *that converge to the popcorn function* $f\colon [0,1] \to \mathbb{R}$*, that is the function such that* $f(p/q) := 1/q$ *(if* p/q *is in lowest terms) and* $f(x) := 0$ *if x is not rational (note that* $f(0) = f(1) = 1$*), see Example 3.2.12. So a pointwise limit of continuous functions can have a dense set of discontinuities. See also the next exercise.*

Exercise* 6.2.17** (Challenging):*** *The Dirichlet function* $f\colon [0,1] \to \mathbb{R}$*, that is the function such that* $f(x) := 1$ *if* $x \in \mathbb{Q}$ *and* $f(x) := 0$ *if* $x \notin \mathbb{Q}$*, is not the pointwise limit of continuous functions, although this is difficult to show. Prove, however, that f is a pointwise limit of functions that are themselves pointwise limits of continuous functions themselves.*

Exercise* 6.2.18*:

a) Find a sequence of Lipschitz continuous functions on $[0,1]$ *whose uniform limit is* $\sqrt{x}$*, which is a non-Lipschitz function.*

b) On the other hand, show that if $f_n\colon S \to \mathbb{R}$ *are Lipschitz with a uniform constant K (meaning all of them satisfy the definition with the same constant) and* $\{f_n\}$ *converges pointwise to* $f\colon S \to \mathbb{R}$*, then the limit f is a Lipschitz continuous function with Lipschitz constant K.*

Exercise* 6.2.19** (requires §2.6):*** *If* $\sum_{n=0}^{\infty} c_n (x-a)^n$ *has radius of convergence* ρ*, show that the term by term integral* $\sum_{n=1}^{\infty} \frac{c_{n-1}}{n} (x-a)^n$ *has radius of convergence* ρ*. Note that we only proved above that the radius of convergence was at least* ρ*.*

Exercise* 6.2.20** (requires §2.6 and §4.3):*** *Suppose* $f(x) := \sum_{n=0}^{\infty} c_n (x-a)^n$ *converges in* $(a-\rho, a+\rho)$*.*

a) Suppose that $f^{(k)}(a) = 0$ *for all* $k = 0,1,2,3,\ldots$*. Prove that* $c_n = 0$ *for all n, or in other words,* $f(x) = 0$ *for all* $x \in (a-\rho, a+\rho)$*.*

b) Using part a) prove a version of the so-called "identity theorem for analytic functions": If there exists an $\varepsilon > 0$ *such that* $f(x) = 0$ *for all* $x \in (a-\varepsilon, a+\varepsilon)$*, then* $f(x) = 0$ *for all* $x \in (a-\rho, a+\rho)$*.*

Exercise* 6.2.21*: *Let* $f_n(x) := \frac{x}{1+(nx)^2}$*. Notice that* f_n *are differentiable functions.*

a) Show that $\{f_n\}$ *converges uniformly to 0.*

b) Show that $|f_n'(x)| \leq 1$ *for all x and all n.*

c) Show that $\{f_n'\}$ *converges pointwise to a function discontinuous at the origin.*

d) Let $\{a_n\}$ *be an enumeration of the rational numbers. Define*

$$g_n(x) := \sum_{k=1}^{n} 2^{-k} f_n(x - a_k).$$

Show that $\{g_n\}$ *converges uniformly to 0.*

e) Show that $\{g_n'\}$ *converges pointwise to a function* ψ *that is discontinuous at every rational number and continuous at every irrational number. In particular,* $\lim_{n\to\infty} g_n'(x) \neq 0$ *for every rational number x.*

6.3 Picard's theorem

Note: 1–2 lectures (can be safely skipped)

A first semester course in analysis should have a *pièce de résistance* caliber theorem. We pick a theorem whose proof combines everything we have learned. It is more sophisticated than the fundamental theorem of calculus, the first highlight theorem of this course. The theorem we are talking about is Picard's theorem* on existence and uniqueness of a solution to an ordinary differential equation. Both the statement and the proof are beautiful examples of what one can do with the material we mastered so far. It is also a good example of how analysis is applied as differential equations are indispensable in science of every stripe.

6.3.1 First order ordinary differential equation

Modern science is described in the language of *differential equations*. That is, equations involving not only the unknown, but also its derivatives. The simplest nontrivial form of a differential equation is the so-called *first order ordinary differential equation*

$$y' = F(x,y).$$

Generally, we also specify an *initial condition* $y(x_0) = y_0$. The solution of the equation is a function $y(x)$ such that $y(x_0) = y_0$ and $y'(x) = F\bigl(x, y(x)\bigr)$.

When F involves only the x variable, the solution is given by the fundamental theorem of calculus. On the other hand, when F depends on both x and y we need far more firepower. It is not always true that a solution exists, and if it does, that it is the unique solution. Picard's theorem gives us certain sufficient conditions for existence and uniqueness.

6.3.2 The theorem

We need a definition of continuity in two variables. A point in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is denoted by an ordered pair (x,y). For simplicity, we give the following sequential definition of continuity.

Definition 6.3.1. Let $U \subset \mathbb{R}^2$ be a set, $F \colon U \to \mathbb{R}$ a function, and $(x,y) \in U$ a point. The function F is *continuous* at (x,y) if for every sequence $\bigl\{(x_n,y_n)\bigr\}_{n=1}^{\infty}$ of points in U such that $\lim x_n = x$ and $\lim y_n = y$, we have

$$\lim_{n\to\infty} F(x_n,y_n) = F(x,y).$$

We say F is continuous if it is continuous at all points in U.

Theorem 6.3.2 (Picard's theorem on existence and uniqueness)**.** *Let $I, J \subset \mathbb{R}$ be closed bounded intervals, let I° and J° be their interiors†, and let $(x_0,y_0) \in I^\circ \times J^\circ$. Suppose $F \colon I \times J \to \mathbb{R}$ is continuous and Lipschitz in the second variable, that is, there exists an $L \in \mathbb{R}$ such that*

$$\lvert F(x,y) - F(x,z) \rvert \leq L \lvert y - z \rvert \qquad \textit{for all } y,z \in J, x \in I.$$

Then there exists an $h > 0$ and a unique differentiable function $f \colon [x_0 - h, x_0 + h] \to J \subset \mathbb{R}$ such that

$$f'(x) = F\bigl(x, f(x)\bigr) \qquad \textit{and} \qquad f(x_0) = y_0. \tag{6.1}$$

*Named for the French mathematician Charles Émile Picard (1856–1941).

†By interior of $[a,b]$ we mean (a,b).

Proof. Suppose we could find a solution f. Using the fundamental theorem of calculus we integrate the equation $f'(x) = F\bigl(x, f(x)\bigr)$, $f(x_0) = y_0$, and write (6.1) as the integral equation

$$f(x) = y_0 + \int_{x_0}^{x} F\bigl(t, f(t)\bigr)\, dt. \tag{6.2}$$

The idea of our proof is that we try to plug in approximations to a solution to the right-hand side of (6.2) to get better approximations on the left-hand side of (6.2). We hope that in the end the sequence converges and solves (6.2) and hence (6.1). The technique below is called *Picard iteration*, and the individual functions f_k are called the *Picard iterates*.

Without loss of generality, suppose $x_0 = 0$ (exercise below). Another exercise tells us that F is bounded as it is continuous. Therefore pick some $M > 0$ so that $|F(x,y)| \leq M$ for all $(x,y) \in I \times J$. Pick $\alpha > 0$ such that $[-\alpha, \alpha] \subset I$ and $[y_0 - \alpha, y_0 + \alpha] \subset J$. Define

$$h := \min \left\{ \alpha, \frac{\alpha}{M + L\alpha} \right\}.$$

Observe $[-h,h] \subset I$.

Set $f_0(x) := y_0$. We define f_k inductively. Assuming $f_{k-1}([-h,h]) \subset [y_0 - \alpha, y_0 + \alpha]$, we see $F\bigl(t, f_{k-1}(t)\bigr)$ is a well-defined function of t for $t \in [-h,h]$. Further if f_{k-1} is continuous on $[-h,h]$, then $F\bigl(t, f_{k-1}(t)\bigr)$ is continuous as a function of t on $[-h,h]$ (left as an exercise). Define

$$f_k(x) := y_0 + \int_0^x F\bigl(t, f_{k-1}(t)\bigr)\, dt,$$

and f_k is continuous on $[-h,h]$ by the fundamental theorem of calculus. To see that f_k maps $[-h,h]$ to $[y_0 - \alpha, y_0 + \alpha]$, we compute for $x \in [-h,h]$

$$|f_k(x) - y_0| = \left| \int_0^x F\bigl(t, f_{k-1}(t)\bigr)\, dt \right| \leq M|x| \leq Mh \leq M \frac{\alpha}{M + L\alpha} \leq \alpha.$$

We now define f_{k+1} and so on, and we have defined a sequence $\{f_k\}$ of functions. We need to show that it converges to a function f that solves the equation (6.2) and therefore (6.1).

We wish to show that the sequence $\{f_k\}$ converges uniformly to some function on $[-h,h]$. First, for $t \in [-h,h]$, we have the following useful bound

$$\bigl|F\bigl(t, f_n(t)\bigr) - F\bigl(t, f_k(t)\bigr)\bigr| \leq L\,|f_n(t) - f_k(t)| \leq L\,\|f_n - f_k\|_u,$$

where $\|f_n - f_k\|_u$ is the uniform norm, that is the supremum of $|f_n(t) - f_k(t)|$ for $t \in [-h,h]$. Now note that $|x| \leq h \leq \frac{\alpha}{M+L\alpha}$. Therefore

$$\begin{aligned}
|f_n(x) - f_k(x)| &= \left| \int_0^x F\bigl(t, f_{n-1}(t)\bigr)\, dt - \int_0^x F\bigl(t, f_{k-1}(t)\bigr)\, dt \right| \\
&= \left| \int_0^x F\bigl(t, f_{n-1}(t)\bigr) - F\bigl(t, f_{k-1}(t)\bigr)\, dt \right| \\
&\leq L\,\|f_{n-1} - f_{k-1}\|_u\,|x| \\
&\leq \frac{L\alpha}{M + L\alpha}\,\|f_{n-1} - f_{k-1}\|_u.
\end{aligned}$$

Let $C := \frac{L\alpha}{M+L\alpha}$ and note that $C < 1$. Taking supremum on the left-hand side we get

$$\|f_n - f_k\|_u \le C \|f_{n-1} - f_{k-1}\|_u .$$

Without loss of generality, suppose $n \ge k$. Then by induction we can show

$$\|f_n - f_k\|_u \le C^k \|f_{n-k} - f_0\|_u .$$

For $x \in [-h,h]$, we have

$$|f_{n-k}(x) - f_0(x)| = |f_{n-k}(x) - y_0| \le \alpha.$$

Therefore,

$$\|f_n - f_k\|_u \le C^k \|f_{n-k} - f_0\|_u \le C^k \alpha.$$

As $C < 1$, $\{f_n\}$ is uniformly Cauchy and by Proposition 6.1.13 we obtain that $\{f_n\}$ converges uniformly on $[-h,h]$ to some function $f\colon [-h,h] \to \mathbb{R}$. The function f is the uniform limit of continuous functions and therefore continuous. Furthermore, since $f_n([-h,h]) \subset [y_0 - \alpha, y_0 + \alpha]$ for all n, then $f([-h,h]) \subset [y_0 - \alpha, y_0 + \alpha]$ (why?).

We now need to show that f solves (6.2). First, as before we notice

$$\left| F\bigl(t, f_n(t)\bigr) - F\bigl(t, f(t)\bigr) \right| \le L |f_n(t) - f(t)| \le L \|f_n - f\|_u .$$

As $\|f_n - f\|_u$ converges to 0, then $F\bigl(t, f_n(t)\bigr)$ converges uniformly to $F\bigl(t, f(t)\bigr)$ for $t \in [-h,h]$. Hence, for $x \in [-h,h]$ the convergence is uniform for $t \in [0,x]$ (or $[x,0]$ if $x < 0$). Therefore,

$$\begin{aligned}
y_0 + \int_0^x F(t, f(t))\, dt &= y_0 + \int_0^x F\bigl(t, \lim_{n\to\infty} f_n(t)\bigr)\, dt \\
&= y_0 + \int_0^x \lim_{n\to\infty} F\bigl(t, f_n(t)\bigr)\, dt && \text{(by continuity of } F\text{)} \\
&= \lim_{n\to\infty} \left(y_0 + \int_0^x F\bigl(t, f_n(t)\bigr)\, dt \right) && \text{(by uniform convergence)} \\
&= \lim_{n\to\infty} f_{n+1}(x) = f(x).
\end{aligned}$$

We apply the fundamental theorem of calculus (Theorem 5.3.3) to show that f is differentiable and its derivative is $F\bigl(x, f(x)\bigr)$. It is obvious that $f(0) = y_0$.

Finally, what is left to do is to show uniqueness. Suppose $g\colon [-h,h] \to J \subset \mathbb{R}$ is another solution. As before we use the fact that $\left| F\bigl(t, f(t)\bigr) - F\bigl(t, g(t)\bigr) \right| \le L \|f - g\|_u$. Then

$$\begin{aligned}
|f(x) - g(x)| &= \left| y_0 + \int_0^x F\bigl(t, f(t)\bigr)\, dt - \left(y_0 + \int_0^x F\bigl(t, g(t)\bigr)\, dt \right) \right| \\
&= \left| \int_0^x F\bigl(t, f(t)\bigr) - F\bigl(t, g(t)\bigr)\, dt \right| \\
&\le L \|f - g\|_u |x| \le Lh \|f - g\|_u \le \frac{L\alpha}{M + L\alpha} \|f - g\|_u .
\end{aligned}$$

As before, $C = \frac{L\alpha}{M+L\alpha} < 1$. By taking supremum over $x \in [-h,h]$ on the left-hand side we obtain

$$\|f - g\|_u \le C \|f - g\|_u .$$

This is only possible if $\|f - g\|_u = 0$. Therefore, $f = g$, and the solution is unique. □

6.3.3 Examples

Let us look at some examples. The proof of the theorem gives us an explicit way to find an h that works. It does not, however, give us the best h. It is often possible to find a much larger h for which the conclusion of the theorem holds.

The proof also gives us the Picard iterates as approximations to the solution. So the proof actually tells us how to obtain the solution, not just that the solution exists.

Example 6.3.3: Consider

$$f'(x) = f(x), \qquad f(0) = 1.$$

That is, we suppose $F(x,y) = y$, and we are looking for a function f such that $f'(x) = f(x)$. Let us forget for the moment that we solved this equation in §5.4.

We pick any I that contains 0 in the interior. We pick an arbitrary J that contains 1 in its interior. We can use $L = 1$. The theorem guarantees an $h > 0$ such that there exists a unique solution $f\colon [-h,h] \to \mathbb{R}$. This solution is usually denoted by

$$e^x := f(x).$$

We leave it to the reader to verify that by picking I and J large enough the proof of the theorem guarantees that we are able to pick α such that we get any h we want as long as $h < 1/2$. We omit the calculation.

Of course, we know this function exists as a function for all x, so an arbitrary h ought to work. By same reasoning as above, no matter what x_0 and y_0 are, the proof guarantees an arbitrary h as long as $h < 1/2$. Fix such an h. We get a unique function defined on $[x_0 - h, x_0 + h]$. After defining the function on $[-h,h]$ we find a solution on the interval $[0,2h]$ and notice that the two functions must coincide on $[0,h]$ by uniqueness. We thus iteratively construct the exponential for all $x \in \mathbb{R}$. Therefore Picard's theorem could be used to prove the existence and uniqueness of the exponential.

Let us compute the Picard iterates. We start with the constant function $f_0(x) := 1$. Then

$$\begin{aligned}
f_1(x) &= 1 + \int_0^x f_0(s)\,ds = 1 + x, \\
f_2(x) &= 1 + \int_0^x f_1(s)\,ds = 1 + \int_0^x (1+s)\,ds = 1 + x + \frac{x^2}{2}, \\
f_3(x) &= 1 + \int_0^x f_2(s)\,ds = 1 + \int_0^x \left(1 + s + \frac{s^2}{2}\right) ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.
\end{aligned}$$

We recognize the beginning of the Taylor series for the exponential.

Example 6.3.4: Consider the equation

$$f'(x) = \bigl(f(x)\bigr)^2 \qquad \text{and} \qquad f(0) = 1.$$

From elementary differential equations we know

$$f(x) = \frac{1}{1-x}$$

is the solution. The solution is only defined on $(-\infty,1)$. That is, we are able to use $h<1$, but never a larger h. The function that takes y to y^2 is not Lipschitz as a function on all of $\mathbb{R}$. As we approach $x=1$ from the left, the solution becomes larger and larger. The derivative of the solution grows as y^2, and so the L required has to be larger and larger as y_0 grows. If we apply the theorem with x_0 close to 1 and $y_0=\frac{1}{1-x_0}$ we find that the h that the proof guarantees is smaller and smaller as x_0 approaches 1.

The h from the proof is not the best h. By picking α correctly, the proof of the theorem guarantees $h=1-\sqrt{3}/2\approx 0.134$ (we omit the calculation) for $x_0=0$ and $y_0=1$, even though we saw above that any $h<1$ should work.

Example 6.3.5: Consider the equation

$$f'(x)=2\sqrt{|f(x)|}, \qquad f(0)=0.$$

The function $F(x,y)=2\sqrt{|y|}$ is continuous, but not Lipschitz in y (why?). The equation does not satisfy the hypotheses of the theorem. The function

$$f(x)=\begin{cases} x^2 & \text{if } x\geq 0,\\ -x^2 & \text{if } x<0,\end{cases}$$

is a solution, but $f(x)=0$ is also a solution. A solution exists, but is not unique.

Example 6.3.6: Consider $y'=\varphi(x)$ where $\varphi(x):=0$ if $x\in\mathbb{Q}$ and $\varphi(x):=1$ if $x\notin\mathbb{Q}$. In other words, the $F(x,y)=\varphi(x)$ is discontinuous. The equation has no solution regardless of the initial conditions. A solution would have derivative φ, but φ does not have the intermediate value property at any point (why?). No solution exists by Darboux's theorem.

The examples show that without the Lipschitz condition, a solution might exist but not be a unique, and without continuity of F, we may not have a solution at all. It is in fact a theorem, the Peano existence theorem, that if F is continuous a solution exists (but may not be unique).

Remark 6.3.7. It is possible to weaken what we mean by "solution to $y'=F(x,y)$" by focusing on the integral equation $f(x)=y_0+\int_{x_0}^{x}F\bigl(t,f(t)\bigr)\,dt$. For example, let H be the Heaviside function*, that is $H(t):=0$ for $t<0$ and $H(t):=1$ for $t\geq 0$. Then $y'=H(t)$, $y(0)=0$, is a common equation. The "solution" is the ramp function $f(x):=0$ if $x<0$ and $f(x):=x$ if $x\geq 0$, since this function satisfies $f(x)=\int_0^x H(t)\,dt$. Notice, however, that $f'(0)$ does not exist, so f is only a so-called *weak solution* to the differential equation.

6.3.4 Exercises

***Exercise* 6.3.1:** *Let $I,J\subset\mathbb{R}$ be intervals. Let $F\colon I\times J\to\mathbb{R}$ be a continuous function of two variables and suppose $f\colon I\to J$ be a continuous function. Show that $F\bigl(x,f(x)\bigr)$ is a continuous function on I.*

***Exercise* 6.3.2:** *Let $I,J\subset\mathbb{R}$ be closed bounded intervals. Show that if $F\colon I\times J\to\mathbb{R}$ is continuous, then F is bounded.*

---

*Named for the English engineer, mathematician, and physicist Oliver Heaviside (1850–1825).

Exercise* 6.3.3*: *We proved Picard's theorem under the assumption that $x_0 = 0$. Prove the full statement of Picard's theorem for an arbitrary x_0.*

Exercise* 6.3.4*: *Let $f'(x) = xf(x)$ be our equation. Start with the initial condition $f(0) = 2$ and find the Picard iterates f_0, f_1, f_2, f_3, f_4.*

Exercise* 6.3.5*: *Suppose $F: I \times J \to \mathbb{R}$ is a function that is continuous in the first variable, that is, for every fixed y the function that takes x to $F(x,y)$ is continuous. Further, suppose F is Lipschitz in the second variable, that is, there exists a number L such that*

$$|F(x,y) - F(x,z)| \leq L\,|y-z| \qquad \text{for all } y,z \in J, x \in I.$$

Show that F is continuous as a function of two variables. Therefore, the hypotheses in the theorem could be made even weaker.

Exercise* 6.3.6*: *A common type of equation one encounters are* linear first order differential equations, *that is equations of the form*

$$y' + p(x)y = q(x), \qquad y(x_0) = y_0.$$

Prove Picard's theorem for linear equations. Suppose I is an interval, $x_0 \in I$, and $p: I \to \mathbb{R}$ and $q: I \to \mathbb{R}$ are continuous. Show that there exists a unique differentiable $f: I \to \mathbb{R}$, such that $y = f(x)$ satisfies the equation and the initial condition. Hint: Assume existence of the exponential function and use the integrating factor formula for existence of f (prove that it works and then that it is unique):

$$f(x) := e^{-\int_{x_0}^{x} p(s)\,ds} \left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} q(t)\,dt + y_0 \right).$$

Exercise* 6.3.7*: *Consider the equation $f'(x) = f(x)$, from Example 6.3.3. Show that given any x_0, any y_0, and any positive $h < 1/2$, we can pick $\alpha > 0$ large enough that the proof of Picard's theorem guarantees a solution for the initial condition $f(x_0) = y_0$ in the interval $[x_0 - h, x_0 + h]$.*

Exercise* 6.3.8*: *Consider the equation $y' = y^{1/3}x$.*

a) Show that for the initial condition $y(1) = 1$, Picard's theorem applies. Find an $\alpha > 0$, M, L, and h that would work in the proof.

b) Show that for the initial condition $y(1) = 0$, Picard's theorem does not apply.

c) Find a solution for $y(1) = 0$ anyway.

Exercise* 6.3.9*: *Consider the equation $xy' = 2y$.*

a) Show that $y = Cx^2$ is a solution for every constant C.

b) Show that for every $x_0 \neq 0$ and every y_0, Picard's theorem applies for the initial condition $y(x_0) = y_0$.

c) Show that $y(0) = y_0$ is solvable if and only if $y_0 = 0$.

Chapter 7

Metric Spaces

7.1 Metric spaces

Note: 1.5 lectures

As mentioned in the introduction, the main idea in analysis is to take limits. In chapter 2 we learned to take limits of sequences of real numbers. And in chapter 3 we learned to take limits of functions as a real number approached some other real number.

We want to take limits in more complicated contexts. For example, we want to have sequences of points in 3-dimensional space. We wish to define continuous functions of several variables. We even want to define functions on spaces that are a little harder to describe, such as the surface of the earth. We still want to talk about limits there.

Finally, we have seen the limit of a sequence of functions in chapter 6. We wish to unify all these notions so that we do not have to reprove theorems over and over again in each context. The concept of a metric space is an elementary yet powerful tool in analysis. And while it is not sufficient to describe every type of limit we find in modern analysis, it gets us very far indeed.

Definition 7.1.1. Let X be a set, and let $d \colon X \times X \to \mathbb{R}$ be a function such that for all $x, y, z \in X$

(i) $d(x,y) \geq 0$. (*nonnegativity*)

(ii) $d(x,y) = 0$ if and only if $x = y$. (*identity of indiscernibles*)

(iii) $d(x,y) = d(y,x)$. (*symmetry*)

(iv) $d(x,z) \leq d(x,y) + d(y,z)$. (*triangle inequality*)

The pair (X,d) is called a *metric space*. The function d is called the *metric* or the *distance function*. Sometimes we write just X as the metric space instead of (X,d) if the metric is clear from context.

The geometric idea is that d is the distance between two points. Items (i)–(iii) have obvious geometric interpretation: Distance is always nonnegative, the only point that is distance 0 away from x is x itself, and finally that the distance from x to y is the same as the distance from y to x. The triangle inequality (iv) has the interpretation given in Figure 7.1.

For the purposes of drawing, it is convenient to draw figures and diagrams in the plane with the metric being the euclidean distance. However, that is only one particular metric space. Just because a certain fact seems to be clear from drawing a picture does not mean it is true in every metric space.

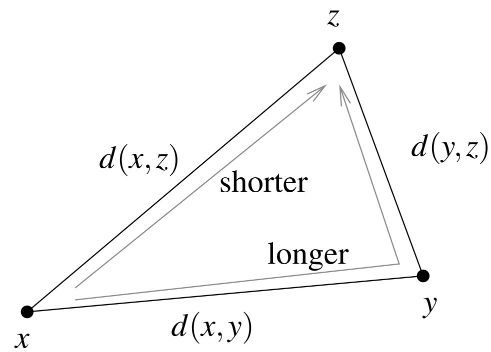

Figure 7.1: Diagram of the triangle inequality in metric spaces.

You might be getting sidetracked by intuition from euclidean geometry, whereas the concept of a metric space is a lot more general.

Let us give some examples of metric spaces.

Example 7.1.2: The set of real numbers $\mathbb{R}$ is a metric space with the metric

$$d(x,y) := |x-y| .$$

Items (i)–(iii) of the definition are easy to verify. The triangle inequality (iv) follows immediately from the standard triangle inequality for real numbers:

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z).$$

This metric is the *standard metric on* $\mathbb{R}$. If we talk about $\mathbb{R}$ as a metric space without mentioning a specific metric, we mean this particular metric.

Example 7.1.3: We can also put a different metric on the set of real numbers. For example, take the set of real numbers $\mathbb{R}$ together with the metric

$$d(x,y) := \frac{|x-y|}{|x-y|+1} .$$

Items (i)–(iii) are again easy to verify. The triangle inequality (iv) is a little bit more difficult. Note that $d(x,y) = \varphi(|x-y|)$ where $\varphi(t) = \frac{t}{t+1}$ and φ is an increasing function (positive derivative). Hence

$$\begin{aligned} d(x,z) &= \varphi(|x-z|) \\ &= \varphi(|x-y+y-z|) \\ &\leq \varphi(|x-y|+|y-z|) \\ &= \frac{|x-y|+|y-z|}{|x-y|+|y-z|+1} \\ &= \frac{|x-y|}{|x-y|+|y-z|+1} + \frac{|y-z|}{|x-y|+|y-z|+1} \\ &\leq \frac{|x-y|}{|x-y|+1} + \frac{|y-z|}{|y-z|+1} = d(x,y) + d(y,z). \end{aligned}$$

The function d is thus a metric, and we have an example of a nonstandard metric on $\mathbb{R}$. With this metric, $d(x,y) < 1$ for all $x,y \in \mathbb{R}$. That is, every two points are less than 1 unit apart.

An important metric space is the n-dimensional *euclidean space* $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. We use the following notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. We will not write $\vec{x}$ nor $\mathbf{x}$ for a vector, we simply give it a name such as x and we will remember that x is a vector. We also write simply $0 \in \mathbb{R}^n$ to mean the point $(0,0,\ldots,0)$. Before making $\mathbb{R}^n$ a metric space, we prove an important inequality, the so-called Cauchy–Schwarz inequality.

Lemma 7.1.4 (Cauchy–Schwarz inequality*)**.** *If* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$*,* $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$*, then*

$$\left(\sum_{j=1}^{n} x_j y_j \right)^2 \leq \left(\sum_{j=1}^{n} x_j^2 \right) \left(\sum_{j=1}^{n} y_j^2 \right).$$

Proof. Any square of a real number is nonnegative. Hence any sum of squares is nonnegative:

$$\begin{aligned} 0 &\leq \sum_{j=1}^{n} \sum_{k=1}^{n} (x_j y_k - x_k y_j)^2 \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \left(x_j^2 y_k^2 + x_k^2 y_j^2 - 2 x_j x_k y_j y_k \right) \\ &= \left(\sum_{j=1}^{n} x_j^2 \right) \left(\sum_{k=1}^{n} y_k^2 \right) + \left(\sum_{j=1}^{n} y_j^2 \right) \left(\sum_{k=1}^{n} x_k^2 \right) - 2 \left(\sum_{j=1}^{n} x_j y_j \right) \left(\sum_{k=1}^{n} x_k y_k \right). \end{aligned}$$

We relabel and divide by 2 to obtain

$$0 \leq \left(\sum_{j=1}^{n} x_j^2 \right) \left(\sum_{j=1}^{n} y_j^2 \right) - \left(\sum_{j=1}^{n} x_j y_j \right)^2,$$

which is precisely what we wanted. □

Example 7.1.5: Let us construct the standard metric for $\mathbb{R}^n$. Define

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}.$$

For $n = 1$, the real line, this metric agrees with what we defined above. For $n > 1$, the only tricky part of the definition to check, as before, is the triangle inequality. It is less messy to work with the

*Sometimes it is called the Cauchy–Bunyakovsky–Schwarz inequality. Karl Hermann Amandus Schwarz (1843–1921) was a German mathematician and Viktor Yakovlevich Bunyakovsky (1804–1889) was a Ukrainian mathematician. What we stated should really be called the Cauchy inequality, as Bunyakovsky and Schwarz provided proofs for infinite-dimensional versions.

square of the metric. In the following estimate, note the use of the Cauchy–Schwarz inequality.

$$\begin{aligned}
\bigl(d(x,z)\bigr)^2 &= \sum_{j=1}^{n} (x_j - z_j)^2 \\
&= \sum_{j=1}^{n} (x_j - y_j + y_j - z_j)^2 \\
&= \sum_{j=1}^{n} \Bigl((x_j - y_j)^2 + (y_j - z_j)^2 + 2(x_j - y_j)(y_j - z_j) \Bigr) \\
&= \sum_{j=1}^{n} (x_j - y_j)^2 + \sum_{j=1}^{n} (y_j - z_j)^2 + 2 \sum_{j=1}^{n} (x_j - y_j)(y_j - z_j) \\
&\leq \sum_{j=1}^{n} (x_j - y_j)^2 + \sum_{j=1}^{n} (y_j - z_j)^2 + 2 \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2 \sum_{j=1}^{n} (y_j - z_j)^2} \\
&= \left(\sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} + \sqrt{\sum_{j=1}^{n} (y_j - z_j)^2} \right)^2 = \bigl(d(x,y) + d(y,z)\bigr)^2.
\end{aligned}$$

Because the square root is an increasing function, the inequality is preserved when we take the square root of both sides, and we obtain the triangle inequality.

Example 7.1.6: The set of complex numbers $\mathbb{C}$ is the set of numbers $z = x + iy$, where x and y are in $\mathbb{R}$. By imposing $i^2 = -1$, we make $\mathbb{C}$ into a field. For the purposes of taking limits, the set $\mathbb{C}$ is regarded as the metric space $\mathbb{R}^2$, where $z = x + iy \in \mathbb{C}$ corresponds to $(x,y) \in \mathbb{R}^2$. For $z = x + iy$ define the *complex modulus* by $\lvert z \rvert := \sqrt{x^2 + y^2}$. Then for two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the distance is

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \lvert z_1 - z_2 \rvert.$$

Furthermore, when working with complex numbers it is often convenient to write the metric in terms of the so-called *complex conjugate*: The conjugate of $z = x + iy$ is $\bar{z} := x - iy$. Then $\lvert z \rvert^2 = x^2 + y^2 = z\bar{z}$, and so $\lvert z_1 - z_2 \rvert^2 = (z_1 - z_2)\overline{(z_1 - z_2)}$.

Example 7.1.7: An example to keep in mind is the so-called *discrete metric*. For any set X, define

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

That is, all points are equally distant from each other. When X is a finite set, we can draw a diagram, see for example Figure 7.2. Of course, in the diagram the distances are not the normal euclidean distances in the plane. Things become subtle when X is an infinite set such as the real numbers.

While this particular example seldom comes up in practice, it gives a useful “smell test.” If you make a statement about metric spaces, try it with the discrete metric. To show that (X,d) is indeed a metric space is left as an exercise.

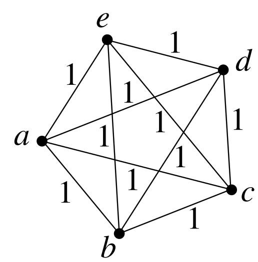

Figure 7.2: Sample discrete metric space $\{a,b,c,d,e\}$, the distance between any two points is 1.

Example 7.1.8: Let $C([a,b],\mathbb{R})$ be the set of continuous real-valued functions on the interval $[a,b]$. Define the metric on $C([a,b],\mathbb{R})$ as

$$d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)| .$$

Let us check the properties. First, $d(f,g)$ is finite as $|f(x)-g(x)|$ is a continuous function on a closed bounded interval $[a,b]$, and so is bounded. It is clear that $d(f,g) \geq 0$, it is the supremum of nonnegative numbers. If $f=g$, then $|f(x)-g(x)|=0$ for all x, and hence $d(f,g)=0$. Conversely, if $d(f,g)=0$, then for every x, we have $|f(x)-g(x)| \leq d(f,g)=0$, and hence $f(x)=g(x)$ for all x, and so $f=g$. That $d(f,g)=d(g,f)$ is equally trivial. To show the triangle inequality we use the standard triangle inequality;

$$\begin{aligned} d(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| &= \sup_{x\in[a,b]} |f(x)-h(x)+h(x)-g(x)| \\ &\leq \sup_{x\in[a,b]} \big(|f(x)-h(x)|+|h(x)-g(x)|\big) \\ &\leq \sup_{x\in[a,b]} |f(x)-h(x)| + \sup_{x\in[a,b]} |h(x)-g(x)| = d(f,h)+d(h,g). \end{aligned}$$

When treating $C([a,b],\mathbb{R})$ as a metric space without mentioning a metric, we mean this particular metric. Notice that $d(f,g) = \lVert f-g \rVert_u$, the uniform norm of Definition 6.1.9.

This example may seem esoteric at first, but it turns out that working with spaces such as $C([a,b],\mathbb{R})$ is really the meat of a large part of modern analysis. Treating sets of functions as metric spaces allows us to abstract away a lot of the grubby detail and prove powerful results such as Picard's theorem with less work.

Example 7.1.9: Another useful example of a metric space is the sphere with a metric usually called the *great circle distance*. Let S^2 be the unit sphere in $\mathbb{R}^3$, that is $S^2 := \{x \in \mathbb{R}^3 : x_1^2+x_2^2+x_3^2 = 1\}$. Take x and y in S^2, draw a line through the origin and x, and another line through the origin and y, and let θ be the angle that the two lines make. Then define $d(x,y) := \theta$. See Figure 7.3. The law of cosines from vector calculus says $d(x,y) = \arccos(x_1y_1+x_2y_2+x_3y_3)$. It is relatively easy to see that this function satisfies the first three properties of a metric. Triangle inequality is harder to prove, and requires a bit more trigonometry and linear algebra than we wish to indulge in right now, so let us leave it without proof.

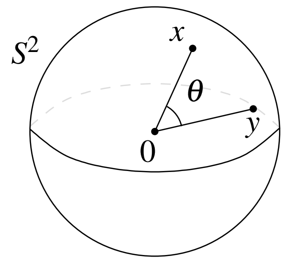

Figure 7.3: The great circle distance on the unit sphere.

This distance is the shortest distance between points on a sphere if we are allowed to travel on the sphere only. It is easy to generalize to arbitrary diameters. If we take a sphere of radius r, we let the distance be $d(x,y) := r\theta$. As an example, this is the standard distance you would use if you compute a distance on the surface of the earth, such as computing the distance a plane travels from London to Los Angeles.

Oftentimes it is useful to consider a subset of a larger metric space as a metric space itself. We obtain the following proposition, which has a trivial proof.

Proposition 7.1.10. *Let (X,d) be a metric space and $Y \subset X$. Then the restriction $d|_{Y \times Y}$ is a metric on Y.*

Definition 7.1.11. If (X,d) is a metric space, $Y \subset X$, and $d' := d|_{Y \times Y}$, then (Y,d') is said to be a *subspace* of (X,d).

It is common to simply write d for the metric on Y, as it is the restriction of the metric on X. Sometimes we say d' is the *subspace metric* and Y has the *subspace topology*.

A subset of the real numbers is bounded whenever all its elements are at most some fixed distance from 0. When dealing with an arbitrary metric space there may not be some natural fixed point 0, but for the purposes of boundedness it does not matter.

Definition 7.1.12. Let (X,d) be a metric space. A subset $S \subset X$ is said to be *bounded* if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that

$$d(p,x) \leq B \quad \text{for all } x \in S.$$

We say (X,d) is bounded if X itself is a bounded subset.

For example, the set of real numbers with the standard metric is not a bounded metric space. It is not hard to see that a subset of the real numbers is bounded in the sense of chapter 1 if and only if it is bounded as a subset of the metric space of real numbers with the standard metric.

On the other hand, if we take the real numbers with the discrete metric, then we obtain a bounded metric space. In fact, any set with the discrete metric is bounded.

There are other equivalent ways we could generalize boundedness, and are left as exercises. Suppose X is nonempty to avoid a technicality. Then $S \subset X$ being bounded is equivalent to either

(i) For every $p \in X$, there exists a $B > 0$ such that $d(p,x) \leq B$ for all $x \in S$.

(ii) $\operatorname{diam}(S) := \sup\{d(x,y) : x,y \in S\} < \infty$.

The quantity $\operatorname{diam}(S)$ is called the *diameter* of a set and is usually only defined for a nonempty set.

7.1.1 Exercises

***Exercise* 7.1.1:** *Show that for every set X, the discrete metric ($d(x,y)=1$ if $x \neq y$ and $d(x,x)=0$) does give a metric space (X,d).*

***Exercise* 7.1.2:** *Let $X := \{0\}$ be a set. Can you make it into a metric space?*

***Exercise* 7.1.3:** *Let $X := \{a,b\}$ be a set. Can you make it into two distinct metric spaces? (define two distinct metrics on it)*

***Exercise* 7.1.4:** *Let the set $X := \{A,B,C\}$ represent 3 buildings on campus. Suppose we wish our distance to be the time it takes to walk from one building to the other. It takes 5 minutes either way between buildings A and B. However, building C is on a hill and it takes 10 minutes from A and 15 minutes from B to get to C. On the other hand it takes 5 minutes to go from C to A and 7 minutes to go from C to B, as we are going downhill. Do these distances define a metric? If so, prove it, if not, say why not.*

***Exercise* 7.1.5:** *Suppose (X,d) is a metric space and $\varphi\colon [0,\infty) \to \mathbb{R}$ is an increasing function such that $\varphi(t) \geq 0$ for all t and $\varphi(t) = 0$ if and only if $t = 0$. Also suppose φ is* subadditive*, that is, $\varphi(s+t) \leq \varphi(s) + \varphi(t)$. Show that with $d'(x,y) := \varphi\bigl(d(x,y)\bigr)$, we obtain a new metric space (X,d').*

***Exercise* 7.1.6:** *Let (X,d_X) and (Y,d_Y) be metric spaces.*

a) *Show that $(X \times Y, d)$ with $d\bigl((x_1,y_1),(x_2,y_2)\bigr) := d_X(x_1,x_2) + d_Y(y_1,y_2)$ is a metric space.*
b) *Show that $(X \times Y, d)$ with $d\bigl((x_1,y_1),(x_2,y_2)\bigr) := \max\bigl\{d_X(x_1,x_2), d_Y(y_1,y_2)\bigr\}$ is a metric space.*

***Exercise* 7.1.7:** *Let X be the set of continuous functions on $[0,1]$. Let $\varphi\colon [0,1] \to (0,\infty)$ be continuous. Define*

$$d(f,g) := \int_0^1 |f(x) - g(x)|\, \varphi(x)\, dx.$$

Show that (X,d) is a metric space.

***Exercise* 7.1.8:** *Let (X,d) be a metric space. For nonempty bounded subsets A and B let*

$$d(x,B) := \inf\bigl\{d(x,b) : b \in B\bigr\} \qquad \textit{and} \qquad d(A,B) := \sup\bigl\{d(a,B) : a \in A\bigr\}.$$

Now define the Hausdorff metric *as*

$$d_H(A,B) := \max\bigl\{d(A,B), d(B,A)\bigr\}.$$

Note: d_H can be defined for arbitrary nonempty subsets if we allow the extended reals.

a) *Let $Y \subset \mathscr{P}(X)$ be the set of bounded nonempty subsets. Prove that (Y,d_H) is a so-called* pseudometric space*: d_H satisfies the metric properties (i), (iii), (iv), and further $d_H(A,A) = 0$ for all $A \in Y$.*
b) *Show by example that d itself is not symmetric, that is $d(A,B) \neq d(B,A)$.*
c) *Find a metric space X and two different nonempty bounded subsets A and B such that $d_H(A,B) = 0$.*

***Exercise* 7.1.9:** *Let (X,d) be a nonempty metric space and $S \subset X$ a subset. Prove:*

a) *S is bounded if and only if for every $p \in X$, there exists a $B > 0$ such that $d(p,x) \leq B$ for all $x \in S$.*
b) *A nonempty S is bounded if and only if* $\operatorname{diam}(S) := \sup\{d(x,y) : x,y \in S\} < \infty$.

Exercise* 7.1.10*:

a) *Working in* $\mathbb{R}$*, compute* $\operatorname{diam}\bigl([a,b]\bigr)$.

b) *Working in* $\mathbb{R}^n$*, for every* $r > 0$*, let* $B_r := \{x_1^2 + x_2^2 + \cdots + x_n^2 < r^2\}$. *Compute* $\operatorname{diam}(B_r)$.

c) *Suppose* (X,d) *is a metric space with at least two points,* d *is the discrete metric, and* $p \in X$. *Compute* $\operatorname{diam}(\{p\})$ *and* $\operatorname{diam}(X)$*, then conclude that* (X,d) *is bounded.*

Exercise* 7.1.11*:

a) *Find a metric* d *on* $\mathbb{N}$*, such that* $\mathbb{N}$ *is an unbounded set in* $(\mathbb{N},d)$.

b) *Find a metric* d *on* $\mathbb{N}$*, such that* $\mathbb{N}$ *is a bounded set in* $(\mathbb{N},d)$.

c) *Find a metric* d *on* $\mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and every* $\varepsilon > 0$*, there exists an* $m \in \mathbb{N}$ *such that* $d(n,m) < \varepsilon$.

Exercise* 7.1.12*: *Let* $C^1([a,b],\mathbb{R})$ *be the set of once continuously differentiable functions on* $[a,b]$. *Define*

$$d(f,g) := \|f-g\|_u + \|f'-g'\|_u,$$

where $\|\cdot\|_u$ *is the uniform norm. Prove that* d *is a metric.*

Exercise* 7.1.13*: *Consider* ℓ^2 *the set of sequences* $\{x_n\}$ *of real numbers such that* $\sum_{n=1}^{\infty} x_n^2 < \infty$.

a) *Prove the Cauchy–Schwarz inequality for two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* ℓ^2*: Prove that* $\sum_{n=1}^{\infty} x_n y_n$ *converges (absolutely) and*

$$\left(\sum_{n=1}^{\infty} x_n y_n\right)^2 \leq \left(\sum_{n=1}^{\infty} x_n^2\right)\left(\sum_{n=1}^{\infty} y_n^2\right).$$

b) *Prove that* ℓ^2 *is a metric space with the metric* $d(x,y) := \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}$. *Hint: Don't forget to show that the series for* $d(x,y)$ *always converges to some finite number.*

7.2 Open and closed sets

Note: 2 lectures

7.2.1 Topology

Before we get to convergence, we define the so-called *topology*. That is, we define open and closed sets in a metric space. And before doing that, we define two special open and closed sets.

Definition 7.2.1. Let (X,d) be a metric space, $x \in X$, and $\delta > 0$. Define the *open ball*, or simply *ball*, of radius δ around x as

$$B(x,\delta) := \{y \in X : d(x,y) < \delta\}.$$

Define the *closed ball* as

$$C(x,\delta) := \{y \in X : d(x,y) \leq \delta\}.$$

When dealing with different metric spaces, it is sometimes vital to emphasize which metric space the ball is in. We do this by writing $B_X(x,\delta) := B(x,\delta)$ or $C_X(x,\delta) := C(x,\delta)$.

Example 7.2.2: Take the metric space $\mathbb{R}$ with the standard metric. For $x \in \mathbb{R}$ and $\delta > 0$,

$$B(x,\delta) = (x-\delta, x+\delta) \qquad \text{and} \qquad C(x,\delta) = [x-\delta, x+\delta].$$

Example 7.2.3: Be careful when working on a subspace. Consider the metric space $[0,1]$ as a subspace of $\mathbb{R}$. Then in $[0,1]$,

$$B(0, 1/2) = B_{[0,1]}(0, 1/2) = \{y \in [0,1] : |0-y| < 1/2\} = [0, 1/2).$$

This is different from $B_{\mathbb{R}}(0, 1/2) = (-1/2, 1/2)$. The important thing to keep in mind is which metric space we are working in.

Definition 7.2.4. Let (X,d) be a metric space. A subset $V \subset X$ is *open* if for every $x \in V$, there exists a $\delta > 0$ such that $B(x,\delta) \subset V$. See Figure 7.4. A subset $E \subset X$ is *closed* if the complement $E^c = X \setminus E$ is open. When the ambient space X is not clear from context, we say *V is open in X* and *E is closed in X*.

If $x \in V$ and V is open, then we say V is an *open neighborhood* of x (or sometimes just *neighborhood*).

Intuitively, an open set V is a set that does not include its "boundary." Wherever we are in V, we are allowed to "wiggle" a little bit and stay in V. Similarly, a set E is closed if everything not in E is some distance away from E. The open and closed balls are examples of open and closed sets (this must still be proved). But not every set is either open or closed. Generally, most subsets are neither.

Example 7.2.5: The set $(0,\infty) \subset \mathbb{R}$ is open: Given any $x \in (0,\infty)$, let $\delta := x$. Then $B(x,\delta) = (0,2x) \subset (0,\infty)$.

The set $[0,\infty) \subset \mathbb{R}$ is closed: Given $x \in (-\infty,0) = [0,\infty)^c$, let $\delta := -x$. Then $B(x,\delta) = (-2x,0) \subset (-\infty,0) = [0,\infty)^c$.

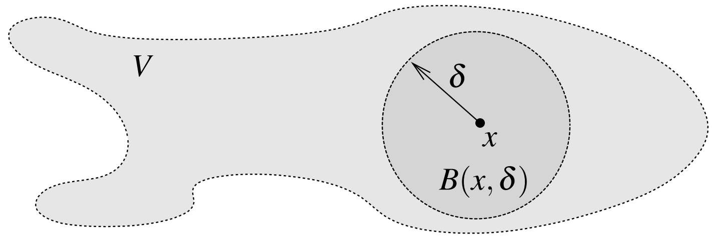

Figure 7.4: Open set in a metric space. Note that δ depends on x.

The set $[0,1) \subset \mathbb{R}$ is neither open nor closed. First, every ball in $\mathbb{R}$ around 0, $B(0,\delta) = (-\delta,\delta)$, contains negative numbers and hence is not contained in $[0,1)$. So $[0,1)$ is not open. Second, every ball in $\mathbb{R}$ around 1, $B(1,\delta) = (1-\delta, 1+\delta)$, contains numbers strictly less than 1 and greater than 0 (e.g. $1 - \delta/2$ as long as $\delta < 2$). Thus $[0,1)^c = \mathbb{R} \setminus [0,1)$ is not open, and $[0,1)$ is not closed.

Proposition 7.2.6. *Let (X,d) be a metric space.*

(i) $\emptyset$ and X are open.

(ii) If $V_1, V_2, \ldots, V_k$ are open subsets of X, then

$$\bigcap_{j=1}^{k} V_j$$

is also open. That is, a finite intersection of open sets is open.

(iii) If $\{V_\lambda\}_{\lambda \in I}$ is an arbitrary collection of open subsets of X, then

$$\bigcup_{\lambda \in I} V_\lambda$$

is also open. That is, a union of open sets is open.

The index set I in (iii) can be arbitrarily large. By $\bigcup_{\lambda \in I} V_\lambda$, we simply mean the set of all x such that $x \in V_\lambda$ for at least one $\lambda \in I$.

Proof. The sets $\emptyset$ and X are obviously open in X.

Let us prove (ii). If $x \in \bigcap_{j=1}^{k} V_j$, then $x \in V_j$ for all j. As V_j are all open, for every j there exists a $\delta_j > 0$ such that $B(x,\delta_j) \subset V_j$. Take $\delta := \min\{\delta_1, \delta_2, \ldots, \delta_k\}$ and notice $\delta > 0$. We have $B(x,\delta) \subset B(x,\delta_j) \subset V_j$ for every j and so $B(x,\delta) \subset \bigcap_{j=1}^{k} V_j$. Consequently the intersection is open.

Let us prove (iii). If $x \in \bigcup_{\lambda \in I} V_\lambda$, then $x \in V_\lambda$ for some $\lambda \in I$. As V_λ is open, there exists a $\delta > 0$ such that $B(x,\delta) \subset V_\lambda$. But then $B(x,\delta) \subset \bigcup_{\lambda \in I} V_\lambda$, and so the union is open. □

Example 7.2.7: The main thing to notice is the difference between items (ii) and (iii). Item (ii) is not true for an arbitrary intersection. For example, $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$, which is not open.

The proof of the following analogous proposition for closed sets is left as an exercise.

Proposition 7.2.8. *Let (X,d) be a metric space.*

(i) *$\emptyset$ and X are closed.*

(ii) *If $\{E_\lambda\}_{\lambda \in I}$ is an arbitrary collection of closed subsets of X, then*

$$\bigcap_{\lambda \in I} E_\lambda$$

is also closed. That is, an intersection of closed sets is closed.

(iii) *If $E_1, E_2, \ldots, E_k$ are closed subsets of X, then*

$$\bigcup_{j=1}^{k} E_j$$

is also closed. That is, a finite union of closed sets is closed.

Despite the naming, we have not yet shown that the open ball is open and the closed ball is closed. Let us show these facts now to justify the terminology.

Proposition 7.2.9. *Let (X,d) be a metric space, $x \in X$, and $\delta > 0$. Then $B(x,\delta)$ is open and $C(x,\delta)$ is closed.*

Proof. Let $y \in B(x,\delta)$. Let $\alpha := \delta - d(x,y)$. As $\alpha > 0$, consider $z \in B(y,\alpha)$. Then

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \alpha = d(x,y) + \delta - d(x,y) = \delta.$$

Therefore, $z \in B(x,\delta)$ for every $z \in B(y,\alpha)$. So $B(y,\alpha) \subset B(x,\delta)$ and $B(x,\delta)$ is open. See Figure 7.5.

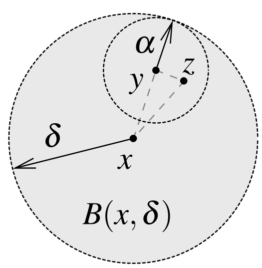

Figure 7.5: Proof that $B(x,\delta)$ is open: $B(y,\alpha) \subset B(x,\delta)$ with the triangle inequality illustrated.

The proof that $C(x,\delta)$ is closed is left as an exercise. $\square$

Again, be careful about which metric space we are in. The set $[0, 1/2)$ is an open ball in $[0,1]$, and so $[0, 1/2)$ is an open set in $[0,1]$. On the other hand, $[0, 1/2)$ is neither open nor closed in $\mathbb{R}$.

Proposition 7.2.10. *Let a,b be two real numbers, $a < b$. Then (a,b), (a,∞), and $(-\infty,b)$ are open in $\mathbb{R}$. Also $[a,b]$, $[a,\infty)$, and $(-\infty,b]$ are closed in $\mathbb{R}$.*

The proof is left as an exercise. Keep in mind that there are many other open and closed sets in the set of real numbers.

Proposition 7.2.11. *Suppose (X,d) is a metric space, and $Y \subset X$. Then $U \subset Y$ is open in Y (in the subspace topology) if and only if there exists an open set $V \subset X$ (so open in X), such that $V \cap Y = U$.*

For example, let $X := \mathbb{R}$, $Y := [0,1]$, $U := [0, 1/2)$. We saw that U is an open set in Y. We may take $V := (-1/2, 1/2)$.

Proof. Suppose $V \subset X$ is open and $x \in V \cap Y$. Let $U := V \cap Y$. As V is open, there exists a $\delta > 0$ such that $B_X(x,\delta) \subset V$. Then

$$B_Y(x,\delta) = B_X(x,\delta) \cap Y \subset V \cap Y = U.$$

The proof of the opposite direction, that is, that if $U \subset Y$ is open in the subspace topology there exists a V is left as Exercise 7.2.12. $\square$

A hint for finshing the proof (the exercise) is that a useful way to think about an open set is as a union of open balls. If U is open, then for each $x \in U$, there is a $\delta_x > 0$ (depending on x) such that $B(x,\delta_x) \subset U$. Then $U = \bigcup_{x \in U} B(x,\delta_x)$.

In case of an open subset of an open set or a closed subset of a closed set, matters are simpler.

Proposition 7.2.12. *Suppose (X,d) is a metric space, $V \subset X$ is open, and $E \subset X$ is closed.*

(i) $U \subset V$ is open in the subspace topology if and only if U is open in X.

(ii) $F \subset E$ is closed in the subspace topology if and only if F is closed in X.

Proof. Let us prove (i) and leave (ii) to an exercise.

If $U \subset V$ is open in the subspace topology, by Proposition 7.2.11, there exists a set $W \subset X$ open in X, such that $U = W \cap V$. Intersection of two open sets is open so U is open in X.

Now suppose U is open in X. Then $U = U \cap V$. So U is open in V again by Proposition 7.2.11. $\square$

7.2.2 Connected sets

Let us generalize the idea of an interval to general metric spaces. One of the main features of an interval in $\mathbb{R}$ is that it is connected—that we can continuously move from one point of it to another point without jumping. For example, in $\mathbb{R}$ we usually study functions on intervals, and in more general metric spaces we usually study functions on connected sets.

Definition 7.2.13. A nonempty* metric space (X,d) is *connected* if the only subsets of X that are both open and closed (so-called *clopen* subsets) are $\emptyset$ and X itself. If a nonempty (X,d) is not connected we say it is *disconnected*.

When we apply the term *connected* to a nonempty subset $A \subset X$, we mean that A with the subspace topology is connected.

*Some authors do not exclude the empty set from the definition, and the empty set would then be connected. We avoid the empty set for essentially the same reason why 1 is neither a prime nor a composite number: Our connected sets have exactly two clopen subsets and disconnected sets have more than two. The empty set has exactly one.

In other words, a nonempty X is connected if whenever we write $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and X_1 and X_2 are open, then either $X_1 = \emptyset$ or $X_2 = \emptyset$. So to show X is disconnected, we need to find nonempty disjoint open sets X_1 and X_2 whose union is X. For subsets, we state this idea as a proposition. The proposition is illustrated in Figure 7.6.

Proposition 7.2.14. *Let (X,d) be a metric space. A nonempty set $S \subset X$ is disconnected if and only if there exist open sets U_1 and U_2 in X, such that $U_1 \cap U_2 \cap S = \emptyset$, $U_1 \cap S \neq \emptyset$, $U_2 \cap S \neq \emptyset$, and*

$$S = \bigl(U_1 \cap S\bigr) \cup \bigl(U_2 \cap S\bigr).$$

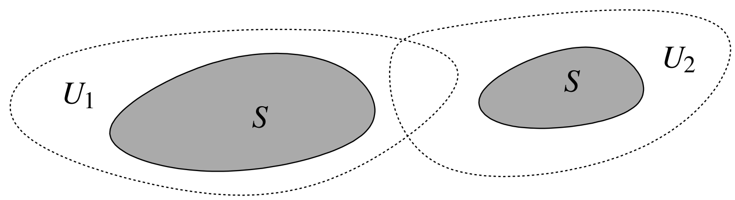

Figure 7.6: Disconnected subset. Notice that $U_1 \cap U_2$ need not be empty, but $U_1 \cap U_2 \cap S = \emptyset$.

Proof. First suppose S is disconnected: There are nonempty disjoint S_1 and S_2 that are open in S and $S = S_1 \cup S_2$. Proposition 7.2.11 says there exist U_1 and U_2 that are open in X such that $U_1 \cap S = S_1$ and $U_2 \cap S = S_2$.

For the other direction start with the U_1 and U_2. Then $U_1 \cap S$ and $U_2 \cap S$ are open in S by Proposition 7.2.11. Via the discussion before the proposition, S is disconnected. □

Example 7.2.15: Let $S \subset \mathbb{R}$ be such that $x < z < y$ with $x, y \in S$ and $z \notin S$. Claim: *S is disconnected.* Proof: Notice

$$\bigl((-\infty,z) \cap S\bigr) \cup \bigl((z,\infty) \cap S\bigr) = S.$$

Proposition 7.2.16. *A nonempty set $S \subset \mathbb{R}$ is connected if and only if it is an interval or a single point.*

Proof. Suppose S is connected. If S is a single point, then we are done. So suppose $x < y$ and $x, y \in S$. If $z \in \mathbb{R}$ is such that $x < z < y$, then $(-\infty,z) \cap S$ is nonempty and $(z,\infty) \cap S$ is nonempty. The two sets are disjoint. As S is connected, we must have they their union is not S, so $z \in S$. By Proposition 1.4.1, S is an interval.

If S is a single point, it is connected. Therefore, suppose S is an interval. Consider open subsets U_1 and U_2 of $\mathbb{R}$, such that $U_1 \cap S$ and $U_2 \cap S$ are nonempty, and $S = \bigl(U_1 \cap S\bigr) \cup \bigl(U_2 \cap S\bigr)$. We will show that $U_1 \cap S$ and $U_2 \cap S$ contain a common point, so they are not disjoint, proving that S is connected. Suppose $x \in U_1 \cap S$ and $y \in U_2 \cap S$. Without loss of generality, assume $x < y$. As S is an interval, $[x,y] \subset S$. Note that $U_2 \cap [x,y] \neq \emptyset$, and let $z := \inf(U_2 \cap [x,y])$. We wish to show that $z \in U_1$. If $z = x$, then $z \in U_1$. If $z > x$, then for every $\varepsilon > 0$, the ball $B(z,\varepsilon) = (z-\varepsilon, z+\varepsilon)$ contains points of $[x,y]$ not in U_2, as z is the infimum of such points. So $z \notin U_2$ as U_2 is open. Therefore, $z \in U_1$ as every point of $[x,y]$ is in U_1 or U_2. As U_1 is open, $B(z,\delta) \subset U_1$ for a small enough $\delta > 0$. As z is the infimum of the nonempty set $U_2 \cap [x,y]$, there must exist some $w \in U_2 \cap [x,y]$ such that $w \in [z, z+\delta) \subset B(z,\delta) \subset U_1$. Therefore, $w \in U_1 \cap U_2 \cap [x,y]$. So $U_1 \cap S$ and $U_2 \cap S$ are not disjoint, and S is connected. See Figure 7.7. □

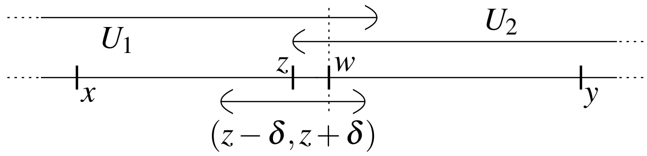

Figure 7.7: Proof that an interval is connected.

Example 7.2.17: Oftentimes a ball $B(x,\delta)$ is connected, but this is not necessarily true in every metric space. For a simplest example, take a two point space $\{a,b\}$ with the discrete metric. Then $B(a,2) = \{a,b\}$, which is not connected as $B(a,1) = \{a\}$ and $B(b,1) = \{b\}$ are open and disjoint.

7.2.3 Closure and boundary

Sometimes we wish to take a set and throw in everything that we can approach from the set. This concept is called the closure.

Definition 7.2.18. Let (X,d) be a metric space and $A \subset X$. The *closure* of A is the set

$$\bar{A} := \bigcap \{E \subset X : E \text{ is closed and } A \subset E\}.$$

That is, $\bar{A}$ is the intersection of all closed sets that contain A.

Proposition 7.2.19. *Let (X,d) be a metric space and $A \subset X$. The closure $\bar{A}$ is closed, and $A \subset \bar{A}$. Furthermore, if A is closed, then $\bar{A} = A$.*

Proof. The closure is an intersection of closed sets, so $\bar{A}$ is closed. There is at least one closed set containing A, namely X itself, so $A \subset \bar{A}$. If A is closed, then A is a closed set that contains A. So $\bar{A} \subset A$, and thus $A = \bar{A}$. □

Example 7.2.20: The closure of $(0,1)$ in $\mathbb{R}$ is $[0,1]$. Proof: If E is closed and contains $(0,1)$, then E contains 0 and 1 (why?). Thus $[0,1] \subset E$. But $[0,1]$ is also closed. Hence, the closure $\overline{(0,1)} = [0,1]$.

Example 7.2.21: Be careful to notice what ambient metric space you are working with. If $X = (0,\infty)$, then the closure of $(0,1)$ in $(0,\infty)$ is $(0,1]$. Proof: Similarly as above, $(0,1]$ is closed in $(0,\infty)$ (why?). Any closed set E that contains $(0,1)$ must contain 1 (why?). Therefore, $(0,1] \subset E$, and hence $\overline{(0,1)} = (0,1]$ when working in $(0,\infty)$.

Let us justify the statement that the closure is everything that we can "approach" from the set.

Proposition 7.2.22. *Let (X,d) be a metric space and $A \subset X$. Then $x \in \bar{A}$ if and only if for every $\delta > 0$, $B(x,\delta) \cap A \neq \emptyset$.*

Proof. Let us prove the two contrapositives. Let us show that $x \notin \bar{A}$ if and only if there exists a $\delta > 0$ such that $B(x,\delta) \cap A = \emptyset$.

First suppose $x \notin \bar{A}$. We know $\bar{A}$ is closed. Thus there is a $\delta > 0$ such that $B(x,\delta) \subset \bar{A}^c$. As $A \subset \bar{A}$ we see that $B(x,\delta) \subset \bar{A}^c \subset A^c$ and hence $B(x,\delta) \cap A = \emptyset$.

On the other hand, suppose there is a $\delta > 0$, such that $B(x,\delta) \cap A = \emptyset$. In other words, $A \subset B(x,\delta)^c$. As $B(x,\delta)^c$ is a closed set, as $x \notin B(x,\delta)^c$, and as $\bar{A}$ is the intersection of closed sets containing A, we have $x \notin \bar{A}$. □

We can also talk about the interior of a set (points we cannot approach from the complement), and the boundary of a set (points we can approach both from the set and its complement).

Definition 7.2.23. Let (X,d) be a metric space and $A \subset X$. The *interior* of A is the set

$$A^\circ := \{x \in A : \text{there exists a } \delta > 0 \text{ such that } B(x,\delta) \subset A\}.$$

The *boundary* of A is the set

$$\partial A := \bar{A} \setminus A^\circ.$$

Alternatively, the interior is the union of open sets lying in A, see Exercise 7.2.14. By definition, $A^\circ \subset A$; however, the points of the boundary may or may not be in A.

Example 7.2.24: Suppose $A := (0,1]$ and $X := \mathbb{R}$. Then $\bar{A} = [0,1]$, $A^\circ = (0,1)$, and $\partial A = \{0,1\}$.

Example 7.2.25: Consider $X := \{a,b\}$ with the discrete metric, and let $A := \{a\}$. Then $\bar{A} = A^\circ = A$ and $\partial A = \emptyset$.

Proposition 7.2.26. *Let (X,d) be a metric space and $A \subset X$. Then A° is open and ∂A is closed.*

Proof. Given $x \in A^\circ$, there is a $\delta > 0$ such that $B(x,\delta) \subset A$. If $z \in B(x,\delta)$, then as open balls are open, there is an $\varepsilon > 0$ such that $B(z,\varepsilon) \subset B(x,\delta) \subset A$. So $z \in A^\circ$. Therefore, $B(x,\delta) \subset A^\circ$, and A° is open.

As A° is open, then $\partial A = \bar{A} \setminus A^\circ = \bar{A} \cap (A^\circ)^c$ is closed. □

The boundary is the set of points that are close to both the set and its complement. See Figure 7.8 for the a diagram of the next proposition.

Proposition 7.2.27. *Let (X,d) be a metric space and $A \subset X$. Then $x \in \partial A$ if and only if for every $\delta > 0$, $B(x,\delta) \cap A$ and $B(x,\delta) \cap A^c$ are both nonempty.*

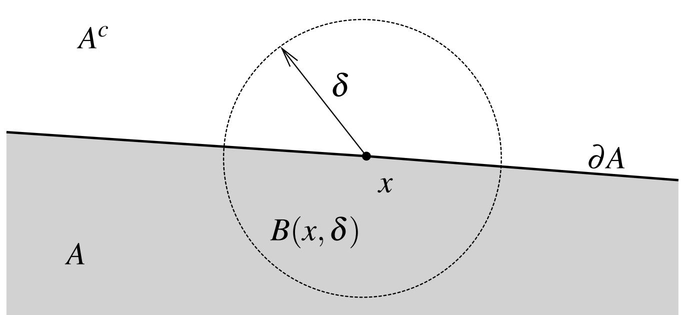

Figure 7.8: Boundary is the set where every ball contains points in the set and also its complement.

Proof. Suppose $x \in \partial A = \bar{A} \setminus A^\circ$ and let $\delta > 0$ be arbitrary. By Proposition 7.2.22, $B(x,\delta)$ contains a point of A. If $B(x,\delta)$ contained no points of A^c, then x would be in A°. Hence $B(x,\delta)$ contains a point of A^c as well.

Let us prove the other direction by contrapositive. Suppose $x \notin \partial A$, so $x \notin \bar{A}$ or $x \in A^\circ$. If $x \notin \bar{A}$, then $B(x,\delta) \subset \bar{A}^c$ for some $\delta > 0$ as $\bar{A}$ is closed. So $B(x,\delta) \cap A$ is empty, because $\bar{A}^c \subset A^c$. If $x \in A^\circ$, then $B(x,\delta) \subset A$ for some $\delta > 0$, so $B(x,\delta) \cap A^c$ is empty. □

We obtain the following immediate corollary about closures of A and A^c. We simply apply Proposition 7.2.22.

Corollary 7.2.28. *Let (X,d) be a metric space and $A \subset X$. Then $\partial A = \overline{A} \cap \overline{A^c}$.*

7.2.4 Exercises

***Exercise* 7.2.1:** *Prove Proposition 7.2.8. Hint: Apply Proposition 7.2.6 to the the complements of the sets.*

***Exercise* 7.2.2:** *Finish the proof of Proposition 7.2.9 by proving that $C(x,\delta)$ is closed.*

***Exercise* 7.2.3:** *Prove Proposition 7.2.10.*

***Exercise* 7.2.4:** *Suppose (X,d) is a nonempty metric space with the discrete topology. Show that X is connected if and only if it contains exactly one element.*

***Exercise* 7.2.5:** *Take $\mathbb{Q}$ with the standard metric, $d(x,y) = \lvert x-y \rvert$, as our metric space. Prove that $\mathbb{Q}$ is* totally disconnected, *that is, show that for every $x,y \in \mathbb{Q}$ with $x \neq y$, there exists an two open sets U and V, such that $x \in U$, $y \in V$, $U \cap V = \emptyset$, and $U \cup V = \mathbb{Q}$.*

***Exercise* 7.2.6:** *Show that in a metric space, every open set can be written as a union of closed sets.*

***Exercise* 7.2.7:** *Prove that in a metric space,*

a) E is closed if and only if $\partial E \subset E$.

b) U is open if and only if $\partial U \cap U = \emptyset$.

***Exercise* 7.2.8:** *Prove that in a metric space,*

a) A is open if and only if $A^\circ = A$.

b) $U \subset A^\circ$ for every open set U such that $U \subset A$.

***Exercise* 7.2.9:** *Let X be a set and d, d' be two metrics on X. Suppose there exists an $\alpha > 0$ and $\beta > 0$ such that $\alpha d(x,y) \leq d'(x,y) \leq \beta d(x,y)$ for all $x,y \in X$. Show that U is open in (X,d) if and only if U is open in (X,d'). That is, the topologies of (X,d) and (X,d') are the same.*

***Exercise* 7.2.10:** *Suppose $\{S_i\}$, $i \in \mathbb{N}$, is a collection of connected subsets of a metric space (X,d), and there exists an $x \in X$ such that $x \in S_i$ for all $i \in \mathbb{N}$. Show that $\bigcup_{i=1}^{\infty} S_i$ is connected.*

***Exercise* 7.2.11:** *Let A be a connected set in a metric space.*

a) Is $\overline{A}$ connected? Prove or find a counterexample.

b) Is A° connected? Prove or find a counterexample.

Hint: Think of sets in $\mathbb{R}^2$.

***Exercise* 7.2.12:** *Finish the proof of Proposition 7.2.11. Suppose (X,d) is a metric space and $Y \subset X$. Show that with the subspace metric on Y, if a set $U \subset Y$ is open (in Y), then there exists an open set $V \subset X$ such that $U = V \cap Y$.*

***Exercise* 7.2.13:** *Let (X,d) be a metric space.*

a) For every $x \in X$ and $\delta > 0$, show $\overline{B(x,\delta)} \subset C(x,\delta)$.

b) Is it always true that $\overline{B(x,\delta)} = C(x,\delta)$? Prove or find a counterexample.

Exercise* 7.2.14*: *Let (X,d) be a metric space and $A \subset X$. Show that $A^\circ = \bigcup\{V : V$ is open and $V \subset A\}$.*

Exercise* 7.2.15*: *Finish the proof of Proposition 7.2.12.*

Exercise* 7.2.16*: *Let (X,d) be a metric space. Show that there exists a bounded metric d' such that (X,d') has the same open sets, that is, the topology is the same.*

Exercise* 7.2.17*: *Let (X,d) be a metric space.*

a) Prove that for every $x \in X$, there either exists a $\delta > 0$ such that $B(x,\delta) = \{x\}$, or $B(x,\delta)$ is infinite for every $\delta > 0$.

b) Find an explicit example of (X,d), X infinite, where for every $\delta > 0$ and every $x \in X$, the ball $B(x,\delta)$ is finite.

c) Find an explicit example of (X,d) where for every $\delta > 0$ and every $x \in X$, the ball $B(x,\delta)$ is countably infinite.

d) Prove that if X is uncountable, then there exists an $x \in X$ and a $\delta > 0$ such that $B(x,\delta)$ is uncountable.

Exercise* 7.2.18*: *For every $x \in \mathbb{R}^n$ and every $\delta > 0$ define the "rectangle" $R(x,\delta) := (x_1-\delta, x_1+\delta) \times (x_2-\delta, x_2+\delta) \times \cdots \times (x_n-\delta, x_n+\delta)$. Show that these sets generate the same open sets as the balls in standard metric. That is, show that a set $U \subset \mathbb{R}^n$ is open in the sense of the standard metric if and only if for every point $x \in U$, there exists a $\delta > 0$ such that $R(x,\delta) \subset U$.*

7.3 Sequences and convergence

Note: 1 lecture

7.3.1 Sequences

The notion of a sequence in a metric space is very similar to a sequence of real numbers. The related definitions are essentially the same as those for real numbers in the sense of chapter 2, where $\mathbb{R}$ with the standard metric $d(x,y) = |x-y|$ is replaced by an arbitrary metric space (X,d).

Definition 7.3.1. A *sequence* in a metric space (X,d) is a function $x\colon \mathbb{N} \to X$. As before we write x_n for the nth element in the sequence and use the notation $\{x_n\}$, or more precisely

$$\{x_n\}_{n=1}^{\infty}.$$

A sequence $\{x_n\}$ is *bounded* if there exists a point $p \in X$ and $B \in \mathbb{R}$ such that

$$d(p,x_n) \leq B \qquad \text{for all } n \in \mathbb{N}.$$

In other words, the sequence $\{x_n\}$ is bounded whenever the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

If $\{n_j\}_{j=1}^{\infty}$ is a sequence of natural numbers such that $n_{j+1} > n_j$ for all j, then the sequence $\{x_{n_j}\}_{j=1}^{\infty}$ is said to be a *subsequence* of $\{x_n\}$.

Similarly we define convergence. Again, we cheat a little and use the definite article in front of the word *limit* before we prove that the limit is unique. See Figure 7.9, for an idea of the definition.

Definition 7.3.2. A sequence $\{x_n\}$ in a metric space (X,d) is said to *converge* to a point $p \in X$ if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that $d(x_n,p) < \varepsilon$ for all $n \geq M$. The point p is said to be the *limit* of $\{x_n\}$. We write

$$\lim_{n\to\infty} x_n := p.$$

A sequence that converges is *convergent*. Otherwise, the sequence is *divergent*.

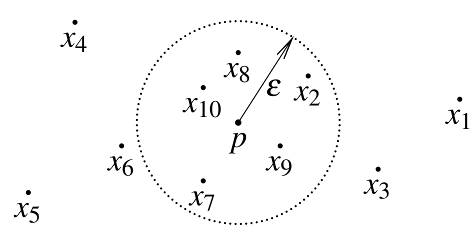

Figure 7.9: Sequence converging to p. The first 10 points are shown and $M = 7$ for this ε.

Let us prove that the limit is unique. The proof is almost identical (word for word) to the proof of the same fact for sequences of real numbers, Proposition 2.1.6. Proofs of many results we know for sequences of real numbers can be adapted to the more general settings of metric spaces. We must replace $|x-y|$ with $d(x,y)$ in the proofs and apply the triangle inequality correctly.

Proposition 7.3.3. *A convergent sequence in a metric space has a unique limit.*

Proof. Suppose the sequence $\{x_n\}$ has limits x and y. Take an arbitrary $\varepsilon > 0$. From the definition find an M_1 such that for all $n \geq M_1$, $d(x_n, x) < \varepsilon/2$. Similarly find an M_2 such that for all $n \geq M_2$, we have $d(x_n, y) < \varepsilon/2$. Now take an n such that $n \geq M_1$ and also $n \geq M_2$, and estimate

$$\begin{aligned} d(y,x) &\leq d(y,x_n) + d(x_n,x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As $d(y,x) < \varepsilon$ for all $\varepsilon > 0$, then $d(x,y) = 0$ and $y = x$. Hence the limit (if it exists) is unique. $\square$

The proofs of the following propositions are left as exercises.

Proposition 7.3.4. *A convergent sequence in a metric space is bounded.*

Proposition 7.3.5. *A sequence $\{x_n\}$ in a metric space (X,d) converges to $p \in X$ if and only if there exists a sequence $\{a_n\}$ of real numbers such that*

$$d(x_n, p) \leq a_n \quad \textit{for all } n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} a_n = 0.$$

Proposition 7.3.6. *Let $\{x_n\}$ be a sequence in a metric space (X,d).*

(i) If $\{x_n\}$ converges to $p \in X$, then every subsequence $\{x_{n_k}\}$ converges to p.

(ii) If for some $K \in \mathbb{N}$ the K-tail $\{x_n\}_{n=K+1}^{\infty}$ converges to $p \in X$, then $\{x_n\}$ converges to p.

Example 7.3.7: Take $C([0,1],\mathbb{R})$ be the set of continuous functions with the metric being the uniform metric. We saw that we obtain a metric space. If we look at the definition of convergence, we notice that it is identical to uniform convergence. That is, $\{f_n\}$ converges uniformly if and only if it converges in the metric space sense.

Remark 7.3.8. It is perhaps surprising that on the set of functions $f\colon [a,b] \to \mathbb{R}$ (continuous or not) there is no metric that gives pointwise convergence. Although the proof of this fact is beyond the scope of this book.

7.3.2 Convergence in euclidean space

In the euclidean space $\mathbb{R}^n$, a sequence converges if and only if every component converges:

Proposition 7.3.9. *Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{R}^n$, where we write $x_j = \big(x_{j,1}, x_{j,2}, \ldots, x_{j,n}\big) \in \mathbb{R}^n$. Then $\{x_j\}_{j=1}^{\infty}$ converges if and only if $\{x_{j,k}\}_{j=1}^{\infty}$ converges for every $k = 1,2,\ldots,n$, in which case*

$$\lim_{j\to\infty} x_j = \Big(\lim_{j\to\infty} x_{j,1}, \lim_{j\to\infty} x_{j,2}, \ldots, \lim_{j\to\infty} x_{j,n}\Big).$$

Proof. Suppose the sequence $\{x_j\}_{j=1}^\infty$ converges to $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. Given $\varepsilon > 0$, there exists an M, such that for all $j \geq M$, we have

$$d(y, x_j) < \varepsilon.$$

Fix some $k = 1, 2, \dots, n$. For all $j \geq M$,

$$\left|y_k - x_{j,k}\right| = \sqrt{\left(y_k - x_{j,k}\right)^2} \leq \sqrt{\sum_{\ell=1}^n \left(y_\ell - x_{j,\ell}\right)^2} = d(y, x_j) < \varepsilon.$$

Hence the sequence $\{x_{j,k}\}_{j=1}^\infty$ converges to y_k.

For the other direction, suppose $\{x_{j,k}\}_{j=1}^\infty$ converges to y_k for every $k = 1, 2, \dots, n$. Given $\varepsilon > 0$, pick an M, such that if $j \geq M$, then $\left|y_k - x_{j,k}\right| < \varepsilon/\sqrt{n}$ for all $k = 1, 2, \dots, n$. Then

$$d(y, x_j) = \sqrt{\sum_{k=1}^n \left(y_k - x_{j,k}\right)^2} < \sqrt{\sum_{k=1}^n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2} = \sqrt{\sum_{k=1}^n \frac{\varepsilon^2}{n}} = \varepsilon.$$

That is, the sequence $\{x_j\}$ converges to $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. $\square$

Example 7.3.10: As we said, the set $\mathbb{C}$ of complex numbers $z = x + iy$ is considered as the metric space $\mathbb{R}^2$. The proposition says that the sequence $\{z_j\}_{j=1}^\infty = \{x_j + iy_j\}_{j=1}^\infty$ converges to $z = x + iy$ if and only if $\{x_j\}$ converges to x and $\{y_j\}$ converges to y.

7.3.3 Convergence and topology

The topology—the set of open sets of a space—encodes which sequences converge.

Proposition 7.3.11. *Let (X, d) be a metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to $x \in X$ if and only if for every open neighborhood U of x, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n \in U$.*

Proof. Suppose $\{x_n\}$ converges to x. Let U be an open neighborhood of x, then there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. As the sequence converges, find an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $d(x, x_n) < \varepsilon$, or in other words $x_n \in B(x, \varepsilon) \subset U$.

Let us prove the other direction. Given $\varepsilon > 0$, let $U := B(x, \varepsilon)$ be the neighborhood of x. Then there is an $M \in \mathbb{N}$ such that for $n \geq M$, we have $x_n \in U = B(x, \varepsilon)$, or in other words, $d(x, x_n) < \varepsilon$. $\square$

A closed set contains the limits of its convergent sequences.

Proposition 7.3.12. *Let (X, d) be a metric space, $E \subset X$ a closed set, and $\{x_n\}$ a sequence in E that converges to some $x \in X$. Then $x \in E$.*

Proof. Let us prove the contrapositive. Suppose $\{x_n\}$ is a sequence in X that converges to $x \in E^c$. As E^c is open, Proposition 7.3.11 says that there is an M such that for all $n \geq M$, $x_n \in E^c$. So $\{x_n\}$ is not a sequence in E. $\square$

To take a closure of a set A, we take A, and we throw in points that are limits of sequences in A.

Proposition 7.3.13. *Let (X,d) be a metric space and $A \subset X$. Then $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ of elements in A such that $\lim x_n = x$.*

Proof. Let $x \in \overline{A}$. For every $n \in \mathbb{N}$, Proposition 7.2.22 guarantees a point $x_n \in B(x, 1/n) \cap A$. As $d(x,x_n) < 1/n$, we have $\lim x_n = x$.

For the other direction, see Exercise 7.3.1. □

7.3.4 Exercises

Exercise* 7.3.1*: *Finish the proof of Proposition 7.3.13: Let (X,d) be a metric space and $A \subset X$. Let $x \in X$ be such that there exists a sequence $\{x_n\}$ in A that converges to x. Prove that $x \in \overline{A}$.*

Exercise* 7.3.2*:

a) Show that $d(x,y) := \min\{1, |x-y|\}$ defines a metric on $\mathbb{R}$.

b) Show that a sequence converges in $(\mathbb{R},d)$ if and only if it converges in the standard metric.

c) Find a bounded sequence in $(\mathbb{R},d)$ that contains no convergent subsequence.

Exercise* 7.3.3*: *Prove Proposition 7.3.4.*

Exercise* 7.3.4*: *Prove Proposition 7.3.5.*

Exercise* 7.3.5*: *Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x. Suppose $f \colon \mathbb{N} \to \mathbb{N}$ is a one-to-one function. Show that $\{x_{f(n)}\}_{n=1}^{\infty}$ converges to x.*

Exercise* 7.3.6*: *Let (X,d) be a metric space where d is the discrete metric. Suppose $\{x_n\}$ is a convergent sequence in X. Show that there exists a $K \in \mathbb{N}$ such that for all $n \geq K$, we have $x_n = x_K$.*

Exercise* 7.3.7*: *A set $S \subset X$ is said to be* dense *in X if $X \subset \overline{S}$ or in other words if for every $x \in X$, there exists a sequence $\{x_n\}$ in S that converges to x. Prove that $\mathbb{R}^n$ contains a countable dense subset.*

Exercise* 7.3.8** (Tricky):*** *Suppose $\{U_n\}_{n=1}^{\infty}$ is a decreasing ($U_{n+1} \subset U_n$ for all n) sequence of open sets in a metric space (X,d) such that $\bigcap_{n=1}^{\infty} U_n = \{p\}$ for some $p \in X$. Suppose $\{x_n\}$ is a sequence of points in X such that $x_n \in U_n$. Does $\{x_n\}$ necessarily converge to p? Prove or construct a counterexample.*

Exercise* 7.3.9*: *Let $E \subset X$ be closed and let $\{x_n\}$ be a sequence in X converging to $p \in X$. Suppose $x_n \in E$ for infinitely many $n \in \mathbb{N}$. Show $p \in E$.*

Exercise* 7.3.10*: *Take $\mathbb{R}^* = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ be the extended reals. Define $d(x,y) := \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$ if $x,y \in \mathbb{R}$, define $d(\infty,x) := \left|1 - \frac{x}{1+|x|}\right|$, $d(-\infty,x) := \left|1 + \frac{x}{1+|x|}\right|$ for all $x \in \mathbb{R}$, and let $d(\infty,-\infty) := 2$.*

a) Show that $(\mathbb{R}^*,d)$ is a metric space.

b) Suppose $\{x_n\}$ is a sequence of real numbers such that for every $M \in \mathbb{R}$, there exists an N such that $x_n \geq M$ for all $n \geq N$. Show that $\lim x_n = \infty$ in $(\mathbb{R}^*,d)$.

c) Show that a sequence of real numbers converges to a real number in $(\mathbb{R}^*,d)$ if and only if it converges in $\mathbb{R}$ with the standard metric.

Exercise* 7.3.11*: *Suppose $\{V_n\}_{n=1}^{\infty}$ is a sequence of open sets in (X,d) such that $V_{n+1} \supset V_n$ for all n. Let $\{x_n\}$ be a sequence such that $x_n \in V_{n+1} \setminus V_n$ and suppose $\{x_n\}$ converges to $p \in X$. Show that $p \in \partial V$ where $V = \bigcup_{n=1}^{\infty} V_n$.*

Exercise* 7.3.12*: *Prove Proposition 7.3.6.*

Exercise* 7.3.13*: *Let (X,d) be a metric space and $\{x_n\}$ a sequence in X. Prove that $\{x_n\}$ converges to $p \in X$ if and only if every subsequence of $\{x_n\}$ has a subsequence that converges to p.*

Exercise* 7.3.14*: *Consider $\mathbb{R}^n$, and let d be the standard euclidean metric. Let $d'(x,y) := \sum_{\ell=1}^{n} \lvert x_\ell - y_\ell \rvert$ and $d''(x,y) := \max\{\lvert x_1 - y_1 \rvert, \lvert x_2 - y_2 \rvert, \cdots, \lvert x_n - y_n \rvert\}$.*

a) Use Exercise 7.1.6, to show that $(\mathbb{R}^n, d')$ and $(\mathbb{R}^n, d'')$ are metric spaces.

b) Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{R}^n$ and $p \in \mathbb{R}^n$. Prove that the following statements are equivalent:

(1) $\{x_j\}$ converges to p in $(\mathbb{R}^n, d)$.

(2) $\{x_j\}$ converges to p in $(\mathbb{R}^n, d')$.

(3) $\{x_j\}$ converges to p in $(\mathbb{R}^n, d'')$.

7.4 Completeness and compactness

Note: 2 lectures

7.4.1 Cauchy sequences and completeness

Just like with sequences of real numbers, we define Cauchy sequences.

Definition 7.4.1. Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $k \geq M$, we have

$$d(x_n,x_k) < \varepsilon.$$

The definition is again simply a translation of the concept from the real numbers to metric spaces. A sequence of real numbers is Cauchy in the sense of chapter 2 if and only if it is Cauchy in the sense above, provided we equip the real numbers with the standard metric $d(x,y) = |x-y|$.

Proposition 7.4.2. *A convergent sequence in a metric space is Cauchy.*

Proof. Suppose $\{x_n\}$ converges to x. Given $\varepsilon > 0$, there is an M such that for all $n \geq M$, we have $d(x,x_n) < \varepsilon/2$. Hence for all $n,k \geq M$, we have $d(x_n,x_k) \leq d(x_n,x) + d(x,x_k) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Definition 7.4.3. Let (X,d) be a metric space. We say X is *complete* or *Cauchy-complete* if every Cauchy sequence $\{x_n\}$ in X converges to an $x \in X$.

Proposition 7.4.4. *The space $\mathbb{R}^n$ with the standard metric is a complete metric space.*

For $\mathbb{R} = \mathbb{R}^1$, completeness was proved in chapter 2. The proof of completeness in $\mathbb{R}^n$ is a reduction to the one-dimensional case.

Proof. Let $\{x_j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathbb{R}^n$, where we write $x_j = \bigl(x_{j,1}, x_{j,2}, \ldots, x_{j,n}\bigr) \in \mathbb{R}^n$. As the sequence is Cauchy, given $\varepsilon > 0$, there exists an M such that for all $i, j \geq M$,

$$d(x_i,x_j) < \varepsilon.$$

Fix some $k = 1,2,\ldots,n$. For $i, j \geq M$,

$$\left|x_{i,k} - x_{j,k}\right| = \sqrt{\bigl(x_{i,k} - x_{j,k}\bigr)^2} \leq \sqrt{\sum_{\ell=1}^{n} \bigl(x_{i,\ell} - x_{j,\ell}\bigr)^2} = d(x_i,x_j) < \varepsilon.$$

Hence the sequence $\{x_{j,k}\}_{j=1}^\infty$ is Cauchy. As $\mathbb{R}$ is complete the sequence converges; there exists a $y_k \in \mathbb{R}$ such that $y_k = \lim_{j\to\infty} x_{j,k}$. Write $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. By Proposition 7.3.9, $\{x_j\}$ converges to $y \in \mathbb{R}^n$, and hence $\mathbb{R}^n$ is complete. □

A subset of $\mathbb{R}^n$ with the subspace metric need not be complete. For example, $(0,1]$ with the subspace metric is not complete as $\{1/n\}$ is a Cauchy sequence in $(0,1]$ with no limit in $(0,1]$.

In the language of metric spaces, the results on continuity of section §6.2, say that the metric space $C([a,b],\mathbb{R})$ of Example 7.1.8 is complete. The proof follows by "unrolling the definitions," and is left as Exercise 7.4.7.

Proposition 7.4.5. *The space of continuous functions $C([a,b],\mathbb{R})$ with the uniform norm as metric is a complete metric space.*

As we saw above, a subspace of a complete metric space need not be complete. However, a closed subspace of a complete metric space is complete. After all, one way to think of a closed set is that it contains all points reachable from the set via a sequence. The proof is Exercise 7.4.16.

Proposition 7.4.6. *Suppose (X,d) is a complete metric space and $E \subset X$ is closed, then E is a complete metric space with the subspace topology.*

7.4.2 Compactness

Definition 7.4.7. Let (X,d) be a metric space and $K \subset X$. The set K is said to be *compact* if for every collection of open sets $\{U_\lambda\}_{\lambda \in I}$ such that

$$K \subset \bigcup_{\lambda \in I} U_\lambda,$$

there exists a finite subset $\{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset I$ such that

$$K \subset \bigcup_{j=1}^{k} U_{\lambda_j}.$$

A collection of open sets $\{U_\lambda\}_{\lambda \in I}$ as above is said to be an *open cover* of K. A way to say that K is compact is to say that *every open cover of K has a finite subcover*.

Example 7.4.8: Let $\mathbb{R}$ be the metric space with the standard metric.

The set $\mathbb{R}$ is not compact. Proof: Take the sets $U_j := (-j, j)$. Any $x \in \mathbb{R}$ is in some U_j (by the Archimedean property), so we have an open cover. Suppose we have a finite subcover $\mathbb{R} \subset U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_k}$, and suppose $j_1 < j_2 < \cdots < j_k$. Then $\mathbb{R} \subset U_{j_k}$, but that is a contradiction as $j_k \in \mathbb{R}$ on one hand and $j_k \notin U_{j_k} = (-j_k, j_k)$ on the other.

The set $(0,1) \subset \mathbb{R}$ is also not compact. Proof: Take the sets $U_j := (1/j, 1 - 1/j)$ for $j = 3, 4, 5, \ldots$. As above $(0,1) = \bigcup_{j=3}^{\infty} U_j$. And similarly as above, if there exists a finite subcover, then there is one U_j such that $(0,1) \subset U_j$, which again leads to a contradiction.

The set $\{0\} \subset \mathbb{R}$ is compact. Proof: Given an open cover $\{U_\lambda\}_{\lambda \in I}$, there must exist a λ_0 such that $0 \in U_{\lambda_0}$ as it is a cover. But then U_{λ_0} gives a finite subcover.

We will prove below that $[0,1]$, and in fact every closed and bounded interval $[a,b]$, is compact.

Proposition 7.4.9. *Let (X,d) be a metric space. A compact set $K \subset X$ is closed and bounded.*

Proof. First, we prove that a compact set is bounded. Fix $p \in X$. We have the open cover

$$K \subset \bigcup_{n=1}^{\infty} B(p,n) = X.$$

If K is compact, then there exists some set of indices $n_1 < n_2 < \ldots < n_k$ such that

$$K \subset \bigcup_{j=1}^{k} B(p,n_j) = B(p,n_k).$$

As K is contained in a ball, K is bounded. See the left-hand side of Figure 7.10.

Next, we show a set that is not closed is not compact. Suppose $\overline{K} \neq K$, that is, there is a point $x \in \overline{K} \setminus K$. If $y \neq x$, then $y \notin C(x, 1/n)$ for $n \in \mathbb{N}$ such that $1/n < d(x,y)$. Furthermore, $x \notin K$, so

$$K \subset \bigcup_{n=1}^{\infty} C(x, 1/n)^c.$$

A closed ball is closed, so its complement $C(x, 1/n)^c$ is open, and we have an open cover. If we take any finite collection of indices $n_1 < n_2 < \ldots < n_k$, then

$$\bigcup_{j=1}^{k} C(x, 1/n_j)^c = C(x, 1/n_k)^c$$

As x is in the closure of K, then $C(x, 1/n_k) \cap K \neq \emptyset$. So there is no finite subcover and K is not compact. See the right-hand side of Figure 7.10. □

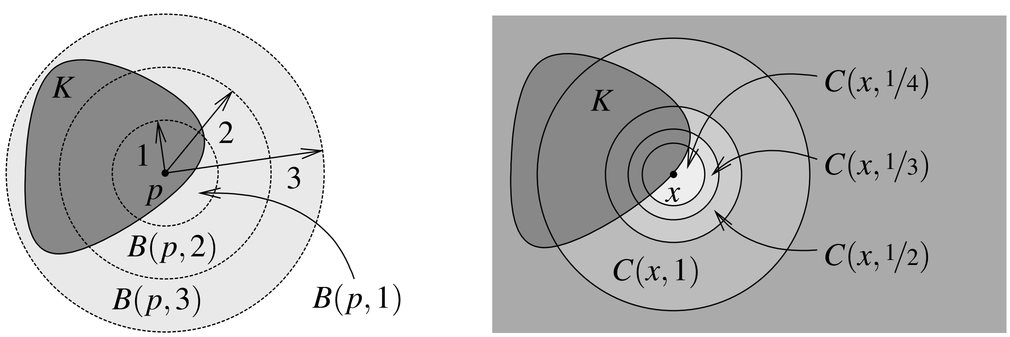

Figure 7.10: Proving compact set is bounded (left) and closed (right).

We prove below that in a finite-dimensional euclidean space, every closed bounded set is compact. So closed bounded sets of $\mathbb{R}^n$ are examples of compact sets. It is not true that in every metric space, closed and bounded is equivalent to compact. A simple example is an incomplete metric space such as $(0,1)$ with the subspace metric from $\mathbb{R}$. There are many complete and very useful metric spaces where closed and bounded is not enough to give compactness: $C([a,b],\mathbb{R})$ is a complete metric space, but the closed unit ball $C(0,1)$ is not compact, see Exercise 7.4.8. However, see also Exercise 7.4.12.

A useful property of compact sets in a metric space is that every sequence in the set has a convergent subsequence converging to a point in the set. Such sets are called *sequentially compact*. Let us prove that in the context of metric spaces, a set is compact if and only if it is sequentially compact. First we prove a lemma.

Lemma 7.4.10 (Lebesgue covering lemma*). *Let (X,d) be a metric space and $K \subset X$. Suppose every sequence in K has a subsequence convergent in K. Given an open cover $\{U_\lambda\}_{\lambda \in I}$ of K, there exists a $\delta > 0$ such that for every $x \in K$, there exists a $\lambda \in I$ with $B(x,\delta) \subset U_\lambda$.*

*Named after the French mathematician Henri Léon Lebesgue (1875–1941). The number δ is sometimes called the Lebesgue number of the cover.

Proof. We prove the lemma by contrapositive. If the conclusion is not true, then there is an open cover $\{U_\lambda\}_{\lambda \in I}$ of K with the following property. For every $n \in \mathbb{N}$, there exists an $x_n \in K$ such that $B(x_n, 1/n)$ is not a subset of any U_λ. Take any $x \in K$. There is a $\lambda \in I$ such that $x \in U_\lambda$. As U_λ is open, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U_\lambda$. Take M such that $1/M < \varepsilon/2$. If $y \in B(x, \varepsilon/2)$ and $n \geq M$, then

$$B(y, 1/n) \subset B(y, 1/M) \subset B(y, \varepsilon/2) \subset B(x, \varepsilon) \subset U_\lambda,$$

where $B(y, \varepsilon/2) \subset B(x, \varepsilon)$ follows by triangle inequality. See Figure 7.11. Thus $y \neq x_n$. In other words, for all $n \geq M$, $x_n \notin B(x, \varepsilon/2)$. The sequence cannot have a subsequence converging to x. As $x \in K$ was arbitrary we are done. □

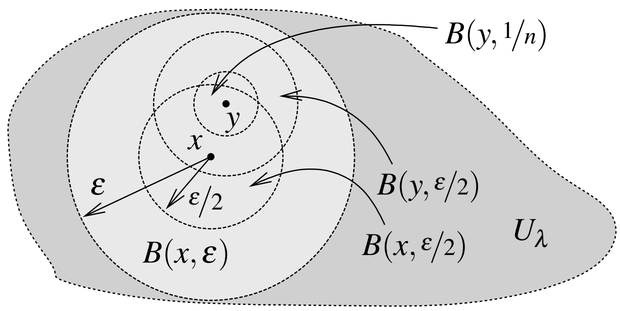

Figure 7.11: Proof of Lebesgue covering lemma. Note that $B(y, \varepsilon/2) \subset B(x, \varepsilon)$ by triangle inequality.

It is important to recognize what the lemma says. It says that if K is sequentially compact, then given any cover there is a single $\delta > 0$. The δ depends on the cover, but, of course, it does not depend on x.

For example, let $K := [-10, 10]$ and for $n \in \mathbb{Z}$ let $U_n := (n, n+2)$ define sets in an open cover. Take $x \in K$. There is an $n \in \mathbb{Z}$, such that $n \leq x < n+1$. If $n \leq x < n + 1/2$, then $B\big(x, 1/2\big) \subset U_{n-1}$. If $n + 1/2 \leq x < n+1$, then $B\big(x, 1/2\big) \subset U_n$. So $\delta = 1/2$. If instead we let $U_n' := \big(\frac{n}{2}, \frac{n+2}{2}\big)$, then we again obtain an open cover, but now the best δ we can find is $1/4$.

On the other hand, $\mathbb{N} \subset \mathbb{R}$ is not sequentially compact. It is an exercise to find a cover for which no $\delta > 0$ works.

Theorem 7.4.11. *Let (X, d) be a metric space. Then $K \subset X$ is compact if and only if every sequence in K has a subsequence converging to a point in K.*

Proof. Claim: *Let $K \subset X$ be a subset of X and $\{x_n\}$ a sequence in K. Suppose that for each $x \in K$, there is a ball $B(x, \alpha_x)$ for some $\alpha_x > 0$ such that $x_n \in B(x, \alpha_x)$ for only finitely many $n \in \mathbb{N}$. Then K is not compact.*

Proof of the claim: Notice

$$K \subset \bigcup_{x \in K} B(x, \alpha_x).$$

Any finite collection of these balls contains at most finitely many elements of $\{x_n\}$, and so there must be an $x_n \in K$ not in their union. Therefore, K is not compact and the claim is proved.

So suppose that K is compact and $\{x_n\}$ is a sequence in K. Then there exists an $x \in K$ such that for all $\delta > 0$, $B(x,\delta)$ contains x_k for infinitely many $k \in \mathbb{N}$. We define the subsequence inductively. The ball $B(x,1)$ contains some x_k, so let $n_1 := k$. Suppose n_{j-1} is defined. There must exist a $k > n_{j-1}$ such that $x_k \in B(x, 1/j)$. Define $n_j := k$. We now posses a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. Since $d(x,x_{n_j}) < 1/j$, Proposition 7.3.5 says $\lim x_{n_j} = x$.

For the other direction, suppose every sequence in K has a subsequence converging in K. Take an open cover $\{U_\lambda\}_{\lambda \in I}$ of K. Using the Lebesgue covering lemma above, find a $\delta > 0$ such that for every $x \in K$, there is a $\lambda \in I$ with $B(x,\delta) \subset U_\lambda$.

Pick $x_1 \in K$ and find $\lambda_1 \in I$ such that $B(x_1,\delta) \subset U_{\lambda_1}$. If $K \subset U_{\lambda_1}$, we stop as we have found a finite subcover. Otherwise, there must be a point $x_2 \in K \setminus U_{\lambda_1}$. Note that $d(x_2,x_1) \geq \delta$. There must exist some $\lambda_2 \in I$ such that $B(x_2,\delta) \subset U_{\lambda_2}$. We work inductively. Suppose λ_{n-1} is defined. Either $U_{\lambda_1} \cup U_{\lambda_2} \cup \cdots \cup U_{\lambda_{n-1}}$ is a finite cover of K, in which case we stop, or there must be a point $x_n \in K \setminus \left(U_{\lambda_1} \cup U_{\lambda_2} \cup \cdots \cup U_{\lambda_{n-1}}\right)$. Note that $d(x_n,x_j) \geq \delta$ for all $j = 1,2,\ldots,n-1$. Next, there must be some $\lambda_n \in I$ such that $B(x_n,\delta) \subset U_{\lambda_n}$. See Figure 7.12.

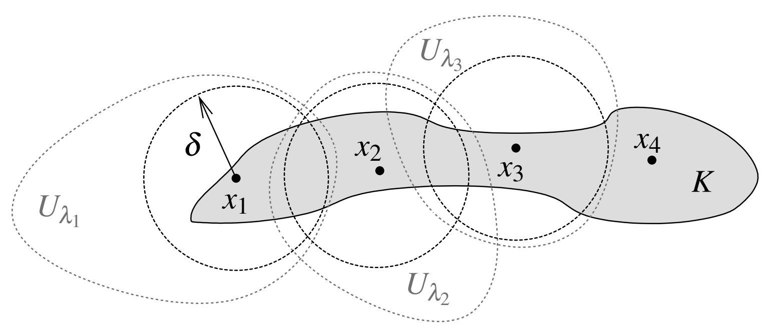

Figure 7.12: Covering K by U_λ. The points x_1,x_2,x_3,x_4, the three sets U_{λ_1}, U_{λ_2}, U_{λ_2}, and the first three balls of radius δ are drawn.

Either at some point we obtain a finite subcover of K, or we obtain an infinite sequence $\{x_n\}$ as above. For contradiction, suppose that there is no finite subcover and we have the sequence $\{x_n\}$. For all n and k, $n \neq k$, we have $d(x_n,x_k) \geq \delta$, so no subsequence of $\{x_n\}$ can be Cauchy. Hence, no subsequence of $\{x_n\}$ can be convergent, which is a contradiction. □

Example 7.4.12: The Bolzano–Weierstrass theorem for sequences of real numbers (Theorem 2.3.8) says that every bounded sequence in $\mathbb{R}$ has a convergent subsequence. Therefore, every sequence in a closed interval $[a,b] \subset \mathbb{R}$ has a convergent subsequence. The limit is also in $[a,b]$ as limits preserve non-strict inequalities. Hence a closed bounded interval $[a,b] \subset \mathbb{R}$ is (sequentially) compact.

Proposition 7.4.13. *Let (X,d) be a metric space and let $K \subset X$ be compact. If $E \subset K$ is a closed set, then E is compact.*

Because K is closed, E is closed in K if and only if it is closed in X. See Proposition 7.2.12.

Proof. Let $\{x_n\}$ be a sequence in E. It is also a sequence in K. Therefore, it has a convergent subsequence $\{x_{n_j}\}$ that converges to some $x \in K$. As E is closed the limit of a sequence in E is also in E and so $x \in E$. Thus E must be compact. □

Theorem 7.4.14 (Heine–Borel*). *A closed bounded subset $K \subset \mathbb{R}^n$ is compact.*

So subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded, a condition that is much easier to check. Let us reiterate that the Heine–Borel theorem only holds for $\mathbb{R}^n$ and not for metric spaces in general. In general, compact implies closed and bounded, but not vice versa.

Proof. For $\mathbb{R} = \mathbb{R}^1$ if $K \subset \mathbb{R}$ is closed and bounded, then every sequence $\{x_k\}$ in K is bounded, so it has a convergent subsequence by Bolzano–Weierstrass theorem (Theorem 2.3.8). As K is closed, the limit of the subsequence must be an element of K. So K is compact.

Let us carry out the proof for $n = 2$ and leave arbitrary n as an exercise. As $K \subset \mathbb{R}^2$ is bounded, there exists a set $B = [a,b] \times [c,d] \subset \mathbb{R}^2$ such that $K \subset B$. We will show that B is compact. Then K, being a closed subset of a compact B, is also compact.

Let $\big\{(x_k, y_k)\big\}_{k=1}^{\infty}$ be a sequence in B. That is, $a \leq x_k \leq b$ and $c \leq y_k \leq d$ for all k. A bounded sequence of real numbers has a convergent subsequence so there is a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ that is convergent. The subsequence $\{y_{k_j}\}_{j=1}^{\infty}$ is also a bounded sequence so there exists a subsequence $\{y_{k_{j_i}}\}_{i=1}^{\infty}$ that is convergent. A subsequence of a convergent sequence is still convergent, so $\{x_{k_{j_i}}\}_{i=1}^{\infty}$ is convergent. Let

$$x := \lim_{i\to\infty} x_{k_{j_i}} \qquad \text{and} \qquad y := \lim_{i\to\infty} y_{k_{j_i}}.$$

By Proposition 7.3.9, $\big\{(x_{k_{j_i}}, y_{k_{j_i}})\big\}_{i=1}^{\infty}$ converges to (x,y). Furthermore, as $a \leq x_k \leq b$ and $c \leq y_k \leq d$ for all k, we know that $(x,y) \in B$. $\square$

Example 7.4.15: The discrete metric provides interesting counterexamples again. Let (X,d) be a metric space with the discrete metric, that is, $d(x,y) = 1$ if $x \neq y$. Suppose X is an infinite set. Then

(i) (X,d) is a complete metric space.

(ii) Any subset $K \subset X$ is closed and bounded.

(iii) A subset $K \subset X$ is compact if and only if it is a finite set.

(iv) The conclusion of the Lebesgue covering lemma is always satisfied with e.g. $\delta = 1/2$, even for noncompact $K \subset X$.

The proofs of the statements above are either trivial or are relegated to the exercises below.

Remark 7.4.16. A subtle issue with Cauchy sequences, completeness, compactness, and convergence is that compactness and convergence only depend on the topology, that is, on which sets are the open sets. On the other hand, Cauchy sequences and completeness depend on the actual metric. See Exercise 7.4.19.

7.4.3 Exercises

***Exercise* 7.4.1:** *Let (X,d) be a metric space and A a finite subset of X. Show that A is compact.*

***Exercise* 7.4.2:** *Let $A := \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$.*

a) Show that A is not compact directly using the definition.

b) Show that $A \cup \{0\}$ is compact directly using the definition.

*Named after the German mathematician Heinrich Eduard Heine (1821–1881), and the French mathematician Félix Édouard Justin Émile Borel (1871–1956).

Exercise* 7.4.3*: *Let (X,d) be a metric space with the discrete metric.*

a) Prove that X is complete.

b) Prove that X is compact if and only if X is a finite set.

Exercise* 7.4.4*:

a) Show that the union of finitely many compact sets is a compact set.

b) Find an example where the union of infinitely many compact sets is not compact.

Exercise* 7.4.5*: *Prove Theorem 7.4.14 for arbitrary dimension. Hint: The trick is to use the correct notation.*

Exercise* 7.4.6*: *Show that a compact set K (in any metric space) is itself a complete metric space (using the subspace metric).*

Exercise* 7.4.7*: *Let $C([a,b],\mathbb{R})$ be the metric space as in Example 7.1.8. Show that $C([a,b],\mathbb{R})$ is a complete metric space.*

Exercise* 7.4.8** (Challenging):*** *Let $C([0,1],\mathbb{R})$ be the metric space of Example 7.1.8. Let 0 denote the zero function. Then show that the closed ball $C(0,1)$ is not compact (even though it is closed and bounded). Hints: Construct a sequence of distinct continuous functions $\{f_n\}$ such that $d(f_n,0)=1$ and $d(f_n,f_k)=1$ for all $n\neq k$. Show that the set $\{f_n : n\in\mathbb{N}\}\subset C(0,1)$ is closed but not compact. See chapter 6 for inspiration.*

Exercise* 7.4.9** (Challenging):*** *Show that there exists a metric on $\mathbb{R}$ that makes $\mathbb{R}$ into a compact set.*

Exercise* 7.4.10*: *Suppose (X,d) is complete and suppose we have a countably infinite collection of nonempty compact sets $E_1\supset E_2\supset E_3\supset\cdots$. Prove $\bigcap_{j=1}^{\infty} E_j\neq\emptyset$.*

Exercise* 7.4.11** (Challenging):*** *Let $C([0,1],\mathbb{R})$ be the metric space of Example 7.1.8. Let K be the set of $f\in C([0,1],\mathbb{R})$ such that f is equal to a quadratic polynomial, i.e. $f(x)=a+bx+cx^2$, and such that $|f(x)|\leq 1$ for all $x\in[0,1]$, that is $f\in C(0,1)$. Show that K is compact.*

Exercise* 7.4.12** (Challenging):*** *Let (X,d) be a complete metric space. Show that $K\subset X$ is compact if and only if K is closed and such that for every $\varepsilon>0$ there exists a finite set of points $x_1,x_2,\ldots,x_n$ with $K\subset\bigcup_{j=1}^{n} B(x_j,\varepsilon)$. Note: Such a set K is said to be* totally bounded*, so in a complete metric space a set is compact if and only if it is closed and totally bounded.*

Exercise* 7.4.13*: *Take $\mathbb{N}\subset\mathbb{R}$ using the standard metric. Find an open cover of $\mathbb{N}$ such that the conclusion of the Lebesgue covering lemma does not hold.*

Exercise* 7.4.14*: *Prove the general Bolzano–Weierstrass theorem: Any bounded sequence $\{x_k\}$ in $\mathbb{R}^n$ has a convergent subsequence.*

Exercise* 7.4.15*: *Let X be a metric space and $C\subset\mathscr{P}(X)$ the set of nonempty compact subsets of X. Using the Hausdorff metric from Exercise 7.1.8, show that (C,d_H) is a metric space. That is, show that if L and K are nonempty compact subsets, then $d_H(L,K)=0$ if and only if $L=K$.*

Exercise* 7.4.16*: *Prove Proposition 7.4.6. That is, let (X,d) be a complete metric space and $E\subset X$ a closed set. Show that E with the subspace metric is a complete metric space.*

Exercise* 7.4.17*: *Let (X,d) be an incomplete metric space. Show that there exists a closed and bounded set $E\subset X$ that is not compact.*

Exercise* 7.4.18*: *Let (X,d) be a metric space and $K \subset X$. Prove that K is compact as a subset of (X,d) if and only if K is compact as a subset of itself with the subspace metric.*

Exercise* 7.4.19*: *Consider two metrics on $\mathbb{R}$. Let $d(x,y) := |x-y|$ be the standard metric, and let $d'(x,y) := \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$.*

a) Show that $(\mathbb{R},d')$ is a metric space (if you have done Exercise 7.3.10, the computation is the same).

b) Show that the topology is the same, that is, a set is open in $(\mathbb{R},d)$ if and only if it is open in $(\mathbb{R},d')$.

c) Show that a set is compact in $(\mathbb{R},d)$ if and only if it is compact in $(\mathbb{R},d')$.

d) Show that a sequence converges in $(\mathbb{R},d)$ if and only if it converges in $(\mathbb{R},d')$.

e) Find a sequence of real numbers that is Cauchy in $(\mathbb{R},d')$ but not Cauchy in $(\mathbb{R},d)$.

f) While $(\mathbb{R},d)$ is complete, show that $(\mathbb{R},d')$ is not complete.

Exercise* 7.4.20*: *Let (X,d) be a complete metric space. We say a set $S \subset X$ is* relatively compact *if the closure $\bar{S}$ is compact. Prove that $S \subset X$ is relatively compact if and only if given any sequence $\{x_n\}$ in S, there exists a subsequence $\{x_{n_k}\}$ that converges (in X).*

7.5 Continuous functions

Note: 1.5–2 lectures

7.5.1 Continuity

Definition 7.5.1. Let (X,d_X) and (Y,d_Y) be metric spaces and $c \in X$. Then $f: X \to Y$ is *continuous at c* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x \in X$ and $d_X(x,c) < \delta$, then $d_Y\big(f(x), f(c)\big) < \varepsilon$.

When $f: X \to Y$ is continuous at all $c \in X$, then we simply say that f is a *continuous function*.

The definition agrees with the definition from chapter 3 when f is a real-valued function on the real line—as long as we take the standard metric on $\mathbb{R}$, of course.

Proposition 7.5.2. *Let (X,d_X) and (Y,d_Y) be metric spaces. Then $f: X \to Y$ is continuous at $c \in X$ if and only if for every sequence $\{x_n\}$ in X converging to c, the sequence $\{f(x_n)\}$ converges to $f(c)$.*

Proof. Suppose f is continuous at c. Let $\{x_n\}$ be a sequence in X converging to c. Given $\varepsilon > 0$, there is a $\delta > 0$ such that $d_X(x,c) < \delta$ implies $d_Y\big(f(x), f(c)\big) < \varepsilon$. So take M such that for all $n \geq M$, we have $d_X(x_n,c) < \delta$, then $d_Y\big(f(x_n), f(c)\big) < \varepsilon$. Hence $\{f(x_n)\}$ converges to $f(c)$.

On the other hand, suppose f is not continuous at c. Then there exists an $\varepsilon > 0$, such that for every $n \in \mathbb{N}$ there exists an $x_n \in X$, with $d_X(x_n,c) < 1/n$ such that $d_Y\big(f(x_n), f(c)\big) \geq \varepsilon$. Then $\{x_n\}$ converges to c, but $\{f(x_n)\}$ does not converge to $f(c)$. $\square$

Example 7.5.3: Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is a polynomial. That is,

$$f(x,y) = \sum_{j=0}^{d} \sum_{k=0}^{d-j} a_{jk}\, x^j y^k = a_{0\,0} + a_{1\,0}\, x + a_{0\,1}\, y + a_{2\,0}\, x^2 + a_{1\,1}\, xy + a_{0\,2}\, y^2 + \cdots + a_{0\,d}\, y^d,$$

for some $d \in \mathbb{N}$ (the degree) and $a_{jk} \in \mathbb{R}$. Then we claim f is continuous. Let $\big\{(x_n,y_n)\big\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^2$ that converges to $(x,y) \in \mathbb{R}^2$. We proved that this means $\lim x_n = x$ and $\lim y_n = y$. By Proposition 2.2.5, we have

$$\lim_{n\to\infty} f(x_n,y_n) = \lim_{n\to\infty} \sum_{j=0}^{d} \sum_{k=0}^{d-j} a_{jk}\, x_n^j y_n^k = \sum_{j=0}^{d} \sum_{k=0}^{d-j} a_{jk}\, x^j y^k = f(x,y).$$

So f is continuous at (x,y), and as (x,y) was arbitrary f is continuous everywhere. Similarly, a polynomial in n variables is continuous.

Be careful about taking limits separately. Consider the function defined by $f(x,y) := \frac{xy}{x^2+y^2}$ outside the origin and $f(0,0) := 0$. See Figure 7.13. In Exercise 7.5.2, you are asked to prove that f is not continuous at the origin. However, for every y, the function $g(x) := f(x,y)$ is continuous, and for every x, the function $h(y) := f(x,y)$ is continuous.

Example 7.5.4: Let X be a metric space and $f: X \to \mathbb{C}$ a complex-valued function. We write $f(p) = g(p) + ih(p)$, where $g: X \to \mathbb{R}$ and $h: X \to \mathbb{R}$ are the real and imaginary parts of f. Then f is continuous at $c \in X$ if and only if its real and imaginary parts are continuous at c. This fact follows because $\big\{f(p_n) = g(p_n) + ih(p_n)\big\}_{n=1}^{\infty}$ converges to $f(p) = g(p) + ih(p)$ if and only if $\big\{g(p_n)\big\}$ converges to $g(p)$ and $\big\{h(p_n)\big\}$ converges to $h(p)$.

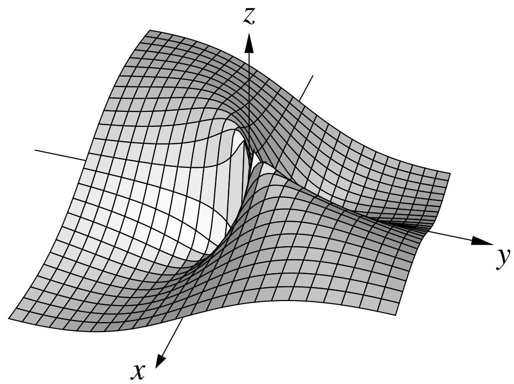

Figure 7.13: Graph of $\frac{xy}{x^2+y^2}$.

7.5.2 Compactness and continuity

Continuous maps do not map closed sets to closed sets. For example, $f\colon (0,1) \to \mathbb{R}$ defined by $f(x) := x$ takes the set $(0,1)$, which is closed in $(0,1)$, to the set $(0,1)$, which is not closed in $\mathbb{R}$. On the other hand, continuous maps do preserve compact sets.

Lemma 7.5.5. *Let (X,d_X) and (Y,d_Y) be metric spaces and $f\colon X \to Y$ a continuous function. If $K \subset X$ is a compact set, then $f(K)$ is a compact set.*

Proof. A sequence in $f(K)$ can be written as $\{f(x_n)\}_{n=1}^{\infty}$, where $\{x_n\}_{n=1}^{\infty}$ is a sequence in K. The set K is compact and therefore there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ that converges to some $x \in K$. By continuity,

$$\lim_{j\to\infty} f(x_{n_j}) = f(x) \in f(K).$$

So every sequence in $f(K)$ has a subsequence convergent to a point in $f(K)$, and $f(K)$ is compact by Theorem 7.4.11. $\square$

As before, $f\colon X \to \mathbb{R}$ achieves an *absolute minimum* at $c \in X$ if

$$f(x) \geq f(c) \qquad \text{for all } x \in X.$$

On the other hand, f achieves an *absolute maximum* at $c \in X$ if

$$f(x) \leq f(c) \qquad \text{for all } x \in X.$$

Theorem 7.5.6. *Let (X,d) be a nonempty compact metric space and let $f\colon X \to \mathbb{R}$ be continuous. Then f is bounded and in fact f achieves an absolute minimum and an absolute maximum on X.*

Proof. As X is compact and f is continuous, $f(X) \subset \mathbb{R}$ is compact. Hence $f(X)$ is closed and bounded. In particular, $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$, because both the sup and the inf can be achieved by sequences in $f(X)$ and $f(X)$ is closed. Therefore, there is some $x \in X$ such that $f(x) = \sup f(X)$ and some $y \in X$ such that $f(y) = \inf f(X)$. $\square$

7.5.3 Continuity and topology

Let us see how to define continuity in terms of the topology, that is, the open sets. We have already seen that topology determines which sequences converge, and so it is no wonder that the topology also determines continuity of functions.

Lemma 7.5.7. *Let (X,d_X) and (Y,d_Y) be metric spaces. A function $f\colon X \to Y$ is continuous at $c \in X$ if and only if for every open neighborhood U of $f(c)$ in Y, the set $f^{-1}(U)$ contains an open neighborhood of c in X. See Figure 7.14.*

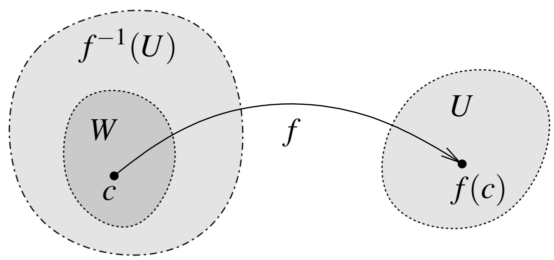

Figure 7.14: For every neighborhood U of $f(c)$, the set $f^{-1}(U)$ contains an open neighborhood W of c.

Proof. First suppose that f is continuous at c. Let U be an open neighborhood of $f(c)$ in Y, then $B_Y\bigl(f(c),\varepsilon\bigr) \subset U$ for some $\varepsilon > 0$. By continuity of f, there exists a $\delta > 0$ such that whenever x is such that $d_X(x,c) < \delta$, then $d_Y\bigl(f(x),f(c)\bigr) < \varepsilon$. In other words,

$$B_X(c,\delta) \subset f^{-1}\bigl(B_Y\bigl(f(c),\varepsilon\bigr)\bigr) \subset f^{-1}(U),$$

and $B_X(c,\delta)$ is an open neighborhood of c.

For the other direction, let $\varepsilon > 0$ be given. If $f^{-1}\bigl(B_Y\bigl(f(c),\varepsilon\bigr)\bigr)$ contains an open neighborhood W of c, it contains a ball. That is, there is some $\delta > 0$ such that

$$B_X(c,\delta) \subset W \subset f^{-1}\bigl(B_Y\bigl(f(c),\varepsilon\bigr)\bigr).$$

That means precisely that if $d_X(x,c) < \delta$, then $d_Y\bigl(f(x),f(c)\bigr) < \varepsilon$, and so f is continuous at c. $\square$

Theorem 7.5.8. *Let (X,d_X) and (Y,d_Y) be metric spaces. A function $f\colon X \to Y$ is continuous if and only if for every open $U \subset Y$, $f^{-1}(U)$ is open in X.*

The proof follows from Lemma 7.5.7 and is left as an exercise.

Example 7.5.9: Let $f\colon X \to Y$ be a continuous function. Theorem 7.5.8 tells us that if $E \subset Y$ is closed, then $f^{-1}(E) = X \setminus f^{-1}(E^c)$ is also closed. Therefore, if we have a continuous function $f\colon X \to \mathbb{R}$, then the *zero set* of f, that is, $f^{-1}(0) = \bigl\{x \in X : f(x) = 0\bigr\}$, is closed. We have just proved the most basic result in *algebraic geometry*, the study of zero sets of polynomials: The zero set of a polynomial is closed.

Similarly the set where f is nonnegative, that is, $f^{-1}\bigl([0,\infty)\bigr) = \bigl\{x \in X : f(x) \geq 0\bigr\}$, is closed. On the other hand the set where f is positive, $f^{-1}\bigl((0,\infty)\bigr) = \bigl\{x \in X : f(x) > 0\bigr\}$, is open.

7.5.4 Uniform continuity

As for continuous functions on the real line, in the definition of continuity it is sometimes convenient to be able to pick one δ for all points.

Definition 7.5.10. Let (X,d_X) and (Y,d_Y) be metric spaces. Then $f\colon X \to Y$ is *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $p,q \in X$ and $d_X(p,q) < \delta$, we have $d_Y\bigl(f(p),f(q)\bigr) < \varepsilon$.

A uniformly continuous function is continuous, but not necessarily vice versa as we have seen.

Theorem 7.5.11. *Let (X,d_X) and (Y,d_Y) be metric spaces. Suppose $f\colon X \to Y$ is continuous and X is compact. Then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$ be given. For each $c \in X$, pick $\delta_c > 0$ such that $d_Y\bigl(f(x),f(c)\bigr) < \nicefrac{\varepsilon}{2}$ whenever $x \in B(c,\delta_c)$. The balls $B(c,\delta_c)$ cover X, and the space X is compact. Apply the Lebesgue covering lemma to obtain a $\delta > 0$ such that for every $x \in X$, there is a $c \in X$ for which $B(x,\delta) \subset B(c,\delta_c)$.

If $p,q \in X$ where $d_X(p,q) < \delta$, find a $c \in X$ such that $B(p,\delta) \subset B(c,\delta_c)$. Then $q \in B(c,\delta_c)$. By the triangle inequality and the definition of δ_c, we have

$$d_Y\bigl(f(p),f(q)\bigr) \leq d_Y\bigl(f(p),f(c)\bigr) + d_Y\bigl(f(c),f(q)\bigr) < \nicefrac{\varepsilon}{2} + \nicefrac{\varepsilon}{2} = \varepsilon. \qquad \square$$

As an application of uniform continuity, let us prove a useful criterion for continuity of functions defined by integrals. Let $f(x,y)$ be a function of two variables and define

$$g(y) := \int_a^b f(x,y)\,dx.$$

Question is, is g continuous? We are really asking when do two limiting operations commute, which is not always possible, so some extra hypothesis is necessary. A useful sufficient (but not necessary) condition is that f is continuous on a closed rectangle.

Proposition 7.5.12. *If $f\colon [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function, then $g\colon [c,d] \to \mathbb{R}$ defined by*

$$g(y) := \int_a^b f(x,y)\,dx \qquad \textit{is continuous.}$$

Proof. Fix $y \in [c,d]$, and let $\{y_n\}$ be a sequence in $[c,d]$ converging to y. Let $\varepsilon > 0$ be given. As f is continuous on $[a,b] \times [c,d]$, which is compact, f is uniformly continuous. In particular, there exists a $\delta > 0$ such that whenever $\widetilde{y} \in [c,d]$ and $\lvert \widetilde{y} - y \rvert < \delta$, we have $\lvert f(x,\widetilde{y}) - f(x,y) \rvert < \frac{\varepsilon}{b-a}$ for all $x \in [a,b]$. So suppose $\lvert \widetilde{y} - y \rvert < \delta$. Then

$$\begin{aligned} \lvert g(\widetilde{y}) - g(y) \rvert &= \left\lvert \int_a^b f(x,\widetilde{y})\,dx - \int_a^b f(x,y)\,dx \right\rvert \\ &= \left\lvert \int_a^b \bigl(f(x,\widetilde{y}) - f(x,y) \bigr)\,dx \right\rvert \leq (b-a)\frac{\varepsilon}{b-a} = \varepsilon. \qquad \square \end{aligned}$$

In applications, if we are interested in continuity at y_0, we just need to apply the proposition in $[a,b] \times [y_0-\varepsilon, y_0+\varepsilon]$ for some small $\varepsilon > 0$. For example, if f is continuous in $[a,b] \times \mathbb{R}$, then g is continuous on $\mathbb{R}$.

Example 7.5.13: Useful examples of uniformly continuous functions are again the so-called *Lipschitz continuous* functions. That is, if (X,d_X) and (Y,d_Y) are metric spaces, then $f\colon X \to Y$ is called Lipschitz or K-Lipschitz if there exists a $K \in \mathbb{R}$ such that

$$d_Y\big(f(p),f(q)\big) \leq K d_X(p,q) \qquad \text{for all } p,q \in X.$$

A Lipschitz function is uniformly continuous: Take $\delta = \varepsilon/K$. A function can be uniformly continuous but not Lipschitz, as we already saw: $\sqrt{x}$ on $[0,1]$ is uniformly continuous but not Lipschitz.

It is worth mentioning that, if a function is Lipschitz, it tends to be easiest to simply show it is Lipschitz even if we are only interested in knowing continuity.

7.5.5 Cluster points and limits of functions

While we haven't started the discussion of continuity with them and we won't need them until volume II, let us also translate the idea of a limit of a function from the real line to metric spaces. Again we need to start with cluster points.

Definition 7.5.14. Let (X,d) be a metric space and $S \subset X$. A point $p \in X$ is called a *cluster point* of S if for every $\varepsilon > 0$, the set $B(p,\varepsilon) \cap S \setminus \{p\}$ is not empty.

It is not enough that p is in the closure of S, it must be in the closure of $S \setminus \{p\}$ (exercise). So, p is a cluster point if and only if there exists a sequence in $S \setminus \{p\}$ that converges to p.

Definition 7.5.15. Let (X,d_X), (Y,d_Y) be metric spaces, $S \subset X$, $p \in X$ a cluster point of S, and $f\colon S \to Y$ a function. Suppose there exists an $L \in Y$ and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in S \setminus \{p\}$ and $d_X(x,p) < \delta$, then

$$d_Y\big(f(x),L\big) < \varepsilon.$$

Then we say $f(x)$ *converges* to L as x goes to p, and L is the *limit* of $f(x)$ as x goes to p. We write

$$\lim_{x \to p} f(x) := L.$$

If $f(x)$ does not converge as x goes to p, we say f *diverges* at p.

As usual, we used the definite article without showing that the limit is unique. The proof is a direct translation of the proof from chapter 3, so we leave it as an exercise.

Proposition 7.5.16. *Let (X,d_X) and (Y,d_Y) be metric spaces, $S \subset X$, $p \in X$ a cluster point of S, and let $f\colon S \to Y$ be a function such that $f(x)$ converges as x goes to p. Then the limit of $f(x)$ as x goes to p is unique.*

In any metric space, just like in $\mathbb{R}$, continuous limits may be replaced by sequential limits. The proof is again a direct translation of the proof from chapter 3, and we leave it as an exercise. The upshot is that we really only need to prove things for sequential limits.

Lemma 7.5.17. *Let (X,d_X) and (Y,d_Y) be metric spaces, $S \subset X$, $p \in X$ a cluster point of S, and let $f\colon S \to Y$ be a function.*

Then $f(x)$ converges to $L \in Y$ as x goes to p if and only if for every sequence $\{x_n\}$ in $S \setminus \{p\}$ such that $\lim x_n = p$, *the sequence $\{f(x_n)\}$ converges to L.*

By applying Proposition 7.5.2 or the definition directly we find (exercise) as in chapter 3, that for cluster points p of $S \subset X$, the function $f\colon S \to Y$ is continuous at p if and only if

$$\lim_{x \to p} f(x) = f(p).$$

7.5.6 Exercises

***Exercise* 7.5.1:** *Consider $\mathbb{N} \subset \mathbb{R}$ with the standard metric. Let (X,d) be a metric space and $f\colon X \to \mathbb{N}$ a continuous function.*

a) Prove that if X is connected, then f is constant (the range of f is a single value).

b) Find an example where X is disconnected and f is not constant.

***Exercise* 7.5.2:** *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) := 0$, and $f(x,y) := \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$.*

a) Show that for every fixed x, the function that takes y to $f(x,y)$ is continuous. Similarly for every fixed y, the function that takes x to $f(x,y)$ is continuous.

b) Show that f is not continuous.

***Exercise* 7.5.3:** *Suppose (X,d_X), (Y,d_Y) are metric spaces and $f\colon X \to Y$ is continuous. Let $A \subset X$.*

a) Show that $f(\bar{A}) \subset \overline{f(A)}$.

b) Show that the subset can be proper.

***Exercise* 7.5.4:** *Prove Theorem 7.5.8. Hint: Use Lemma 7.5.7.*

***Exercise* 7.5.5:** *Suppose $f\colon X \to Y$ is continuous for metric spaces (X,d_X) and (Y,d_Y). Show that if X is connected, then $f(X)$ is connected.*

***Exercise* 7.5.6:** *Prove the following version of the intermediate value theorem. Let (X,d) be a connected metric space and $f\colon X \to \mathbb{R}$ a continuous function. Suppose that there exist $x_0, x_1 \in X$ and $y \in \mathbb{R}$ such that $f(x_0) < y < f(x_1)$. Then prove that there exists a $z \in X$ such that $f(z) = y$. Hint: See Exercise 7.5.5.*

***Exercise* 7.5.7:** *A continuous $f\colon X \to Y$ between metric spaces (X,d_X) and (Y,d_Y) is said to be* proper *if for every compact set $K \subset Y$, the set $f^{-1}(K)$ is compact. Suppose a continuous $f\colon (0,1) \to (0,1)$ is proper and $\{x_n\}$ is a sequence in $(0,1)$ converging to 0. Show that $\{f(x_n)\}$ has no subsequence that converges in $(0,1)$.*

***Exercise* 7.5.8:** *Let (X,d_X) and (Y,d_Y) be metric spaces and $f\colon X \to Y$ be a one-to-one and onto continuous function. Suppose X is compact. Prove that the inverse $f^{-1}\colon Y \to X$ is continuous.*

***Exercise* 7.5.9:** *Take the metric space of continuous functions $C([0,1],\mathbb{R})$. Let $k\colon [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Given $f \in C([0,1],\mathbb{R})$ define*

$$\varphi_f(x) := \int_0^1 k(x,y) f(y)\, dy.$$

a) Show that $T(f) := \varphi_f$ defines a function $T\colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$.

b) Show that T is continuous.

***Exercise* 7.5.10:** *Let (X,d) be a metric space.*

a) *If $p \in X$, show that $f: X \to \mathbb{R}$ defined by $f(x) := d(x,p)$ is continuous.*

b) *Define a metric on $X \times X$ as in Exercise 7.1.6 part b, and show that $g: X \times X \to \mathbb{R}$ defined by $g(x,y) := d(x,y)$ is continuous.*

c) *Show that if K_1 and K_2 are compact subsets of X, then there exists a $p \in K_1$ and $q \in K_2$ such that $d(p,q)$ is minimal, that is, $d(p,q) = \inf\{d(x,y) : x \in K_1, y \in K_2\}$.*

***Exercise* 7.5.11:** *Let (X,d) be a compact metric space, let $C(X,\mathbb{R})$ be the set of real-valued continuous functions. Define*

$$d(f,g) := \|f-g\|_u := \sup_{x\in X} |f(x)-g(x)|.$$

a) *Show that d makes $C(X,\mathbb{R})$ into a metric space.*

b) *Show that for every $x \in X$, the evaluation function $E_x: C(X,\mathbb{R}) \to \mathbb{R}$ defined by $E_x(f) := f(x)$ is a continuous function.*

***Exercise* 7.5.12:** *Let $C([a,b],\mathbb{R})$ be the set of continuous functions and $C^1([a,b],\mathbb{R})$ the set of once continuously differentiable functions on $[a,b]$. Define*

$$d_C(f,g) := \|f-g\|_u \qquad \text{and} \qquad d_{C^1}(f,g) := \|f-g\|_u + \|f'-g'\|_u,$$

where $\|\cdot\|_u$ is the uniform norm. By Example 7.1.8 and Exercise 7.1.12 we know that $C([a,b],\mathbb{R})$ with d_C is a metric space and so is $C^1([a,b],\mathbb{R})$ with d_{C^1}.

a) *Prove that the derivative operator $D: C^1([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ defined by $D(f) := f'$ is continuous.*

b) *On the other hand if we consider the metric d_C on $C^1([a,b],\mathbb{R})$, then prove the derivative operator is no longer continuous. Hint: Consider $\sin(nx)$.*

***Exercise* 7.5.13:** *Let (X,d) be a metric space, $S \subset X$, and $p \in X$. Prove that p is a cluster point of S if and only if $p \in \overline{S \setminus \{p\}}$.*

***Exercise* 7.5.14:** *Prove Proposition 7.5.16.*

***Exercise* 7.5.15:** *Prove Lemma 7.5.17.*

***Exercise* 7.5.16:** *Let (X,d_X) and (Y,d_Y) be metric spaces, $S \subset X$, $p \in X$ a cluster point of S, and let $f: S \to Y$ be a function. Prove that $f: S \to Y$ is continuous at p if and only if*

$$\lim_{x\to p} f(x) = f(p).$$

***Exercise* 7.5.17:** *Define*

$$f(x,y) := \begin{cases} \frac{2xy}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

a) *Show that for every fixed y the function that takes x to $f(x,y)$ is continuous and hence Riemann integrable.*

b) *For every fixed x, the function that takes y to $f(x,y)$ is continuous.*

c) *Show that f is not continuous at $(0,0)$.*

d) *Now show that $g(y) := \int_0^1 f(x,y)\,dx$ is not continuous at $y = 0$.*

Note: Feel free to use what you know about arctan *from calculus, in particular that $\frac{d}{ds}\big[\arctan(s)\big] = \frac{1}{1+s^2}$.*

Exercise* 7.5.18*: *Prove a stronger version of Proposition 7.5.12: If* $f: (a,b) \times (c,d) \to \mathbb{R}$ *is a bounded continuous function, then* $g: (c,d) \to \mathbb{R}$ *defined by*

$$g(y) := \int_a^b f(x,y)\, dx \qquad \text{is continuous.}$$

Hint: First integrate over $[a + 1/n, b - 1/n]$.

7.6 Fixed point theorem and Picard's theorem again

Note: 1 lecture (optional, does not require §6.3)

In this section we prove the fixed point theorem for contraction mappings. As an application we prove Picard's theorem, which we proved without metric spaces in §6.3. The proof we present here is similar, but the proof goes a lot smoother with metric spaces and the fixed point theorem.

7.6.1 Fixed point theorem

Definition 7.6.1. Let (X,d_X) and (Y,d_Y) be metric spaces. A mapping $f\colon X \to Y$ is said to be a *contraction* (or a contractive map) if it is a k-Lipschitz map for some $k < 1$, i.e. if there exists a $k < 1$ such that

$$d_Y\bigl(f(p),f(q)\bigr) \leq k\, d_X(p,q) \qquad \text{for all } p,q \in X.$$

Given a map $f\colon X \to X$, a point $x \in X$ is called a *fixed point* if $f(x) = x$.

Theorem 7.6.2 (Contraction mapping principle or Banach fixed point theorem*)**.** *Let (X,d) be a nonempty complete metric space and $f\colon X \to X$ a contraction. Then f has a unique fixed point.*

The words *complete* and *contraction* are necessary. See Exercise 7.6.6.

Proof. Pick $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_{n+1} := f(x_n)$. Then

$$d(x_{n+1},x_n) = d\bigl(f(x_n),f(x_{n-1})\bigr) \leq k d(x_n,x_{n-1}).$$

Repeating n times, we get $d(x_{n+1},x_n) \leq k^n d(x_1,x_0)$. For $m > n$,

$$\begin{aligned} d(x_m,x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1},x_i) \\ &\leq \sum_{i=n}^{m-1} k^i d(x_1,x_0) \\ &= k^n d(x_1,x_0) \sum_{i=0}^{m-n-1} k^i \\ &\leq k^n d(x_1,x_0) \sum_{i=0}^{\infty} k^i = k^n d(x_1,x_0) \frac{1}{1-k}. \end{aligned}$$

In particular, the sequence is Cauchy (why?). Since X is complete, we let $x := \lim x_n$, and we claim that x is our unique fixed point.

Fixed point? The function f is a contraction, so it is Lipschitz continuous:

$$f(x) = f(\lim x_n) = \lim f(x_n) = \lim x_{n+1} = x.$$

Unique? Let x and y be fixed points.

$$d(x,y) = d\bigl(f(x),f(y)\bigr) \leq k\, d(x,y).$$

As $k < 1$ this means that $d(x,y) = 0$ and hence $x = y$. The theorem is proved. □

*Named after the Polish mathematician Stefan Banach (1892–1945) who first stated the theorem in 1922.

The proof is constructive. Not only do we know a unique fixed point exists. We also know how to find it. Start with any point $x_0 \in X$, then iterate $f(x_0)$, $f(f(x_0))$, $f(f(f(x_0)))$, etc. We can even find how far away from the fixed point we are, see the exercises. The idea of the proof is therefore useful in real-world applications.

7.6.2 Picard's theorem

Let us start with the metric space to which we will apply the fixed point theorem. That is, the space $C([a,b],\mathbb{R})$ of Example 7.1.8, the space of continuous functions $f\colon [a,b] \to \mathbb{R}$ with the metric

$$d(f,g) := \|f-g\|_u = \sup_{x\in[a,b]} |f(x)-g(x)| .$$

Convergence in this metric is convergence in uniform norm, or in other words, uniform convergence. Therefore, $C([a,b],\mathbb{R})$ is a complete metric space, see Proposition 7.4.5.

Consider now the ordinary differential equation

$$\frac{dy}{dx} = F(x,y).$$

Given some x_0, y_0, we are looking for a function $y = f(x)$ such that $f(x_0) = y_0$ and such that

$$f'(x) = F\bigl(x, f(x)\bigr).$$

To avoid having to come up with many names, we often simply write $y' = F(x,y)$ for the equation and $y(x)$ for the solution.

The simplest example is the equation $y' = y$, $y(0) = 1$. The solution is the exponential $y(x) = e^x$. A somewhat more complicated example is $y' = -2xy$, $y(0) = 1$, whose solution is the Gaussian $y(x) = e^{-x^2}$.

A subtle issue is how long does the solution exist. Consider the equation $y' = y^2$, $y(0) = 1$. Then $y(x) = \frac{1}{1-x}$ is a solution. While F is a reasonably "nice" function and in particular it exists for all x and y, the solution "blows up" at $x = 1$. For more examples related to Picard's theorem, see §6.3.

It may be strange that we are looking in $C([a,b],\mathbb{R})$ for a differentiable function, but the idea is to consider the corresponding integral equation

$$f(x) = y_0 + \int_0^x F\bigl(t, f(t)\bigr)\, dt.$$

To solve this integral equation we only need a continuous function, and in some sense our task should be easier—we have more candidate functions to try. This way of thinking is quite typical when solving differential equations.

Theorem 7.6.3 (Picard's theorem on existence and uniqueness)**.** *Let $I, J \subset \mathbb{R}$ be closed and bounded intervals, let I° and J° be their interiors, and let $(x_0, y_0) \in I^\circ \times J^\circ$. Suppose $F\colon I \times J \to \mathbb{R}$ is continuous and Lipschitz in the second variable, that is, there exists an $L \in \mathbb{R}$ such that*

$$|F(x,y) - F(x,z)| \leq L\,|y-z| \qquad \textit{for all } y,z \in J, x \in I.$$

Then there exists an $h > 0$ and a unique differentiable function $f\colon [x_0-h, x_0+h] \to J \subset \mathbb{R}$ such that

$$f'(x) = F\bigl(x, f(x)\bigr) \qquad \textit{and} \qquad f(x_0) = y_0.$$

Proof. Without loss of generality assume $x_0 = 0$ (exercise). As $I \times J$ is compact and $F(x,y)$ is continuous, it is bounded. So find an $M > 0$, such that $|F(x,y)| \leq M$ for all $(x,y) \in I \times J$. Pick $\alpha > 0$ such that $[-\alpha,\alpha] \subset I$ and $[y_0 - \alpha, y_0 + \alpha] \subset J$. Let

$$h := \min\left\{\alpha, \frac{\alpha}{M + L\alpha}\right\}.$$

Note $[-h,h] \subset I$. Let

$$Y := \left\{ f \in C([-h,h],\mathbb{R}) : f([-h,h]) \subset J \right\}.$$

That is, Y is the space of continuous functions on $[-h,h]$ with values in J, in other words, exactly those functions where $F\bigl(x, f(x)\bigr)$ makes sense. The metric used is the standard metric given above.

It is left as an exercise to show that Y is closed (because J is closed). The space $C([-h,h],\mathbb{R})$ is complete, and a closed subset of a complete metric space is a complete metric space with the subspace metric, see Proposition 7.4.6. So Y with the subspace metric is a complete metric space.

Define a mapping $T : Y \to C([-h,h],\mathbb{R})$ by

$$T(f)(x) := y_0 + \int_0^x F\bigl(t, f(t)\bigr)\, dt.$$

It is an exercise to check that T is well-defined, and that $T(f)$ really is in $C([-h,h],\mathbb{R})$.

Let $f \in Y$ and $|x| \leq h$. As F is bounded by M, we have

$$\begin{aligned} |T(f)(x) - y_0| &= \left| \int_0^x F\bigl(t, f(t)\bigr)\, dt \right| \\ &\leq |x| M \leq hM \leq \frac{\alpha M}{M + L\alpha} \leq \alpha. \end{aligned}$$

So $T(f)([-h,h]) \subset [y_0 - \alpha, y_0 + \alpha] \subset J$, and $T(f) \in Y$. In other words, $T(Y) \subset Y$. From now on, we consider T as a mapping of Y to Y.

We claim $T : Y \to Y$ is a contraction. First, for $x \in [-h,h]$ and $f, g \in Y$, we have

$$\left| F\bigl(x, f(x)\bigr) - F\bigl(x, g(x)\bigr) \right| \leq L |f(x) - g(x)| \leq L\, d(f,g).$$

Therefore,

$$\begin{aligned} |T(f)(x) - T(g)(x)| &= \left| \int_0^x F\bigl(t, f(t)\bigr) - F\bigl(t, g(t)\bigr)\, dt \right| \\ &\leq |x| L\, d(f,g) \leq hL\, d(f,g) \leq \frac{L\alpha}{M + L\alpha} d(f,g). \end{aligned}$$

We chose $M > 0$ and so $\frac{L\alpha}{M+L\alpha} < 1$. The claim that T is a contraction is proved by taking supremum over $x \in [-h,h]$ of the left-hand side above to obtain $d\bigl(T(f), T(g)\bigr) \leq \frac{L\alpha}{M+L\alpha} d(f,g)$.

We apply the fixed point theorem (Theorem 7.6.2) to find a unique $f \in Y$ such that $T(f) = f$, that is,

$$f(x) = y_0 + \int_0^x F\bigl(t, f(t)\bigr)\, dt.$$

By the fundamental theorem of calculus (Theorem 5.3.3), $T(f) = f$ is differentiable, its derivative is $F\bigl(x, f(x)\bigr)$ and $T(f)(0) = y_0$. Differentiable functions are continuous, so f is the unique differentiable function $f\colon [-h,h] \to J$ such that $f'(x) = F\bigl(x, f(x)\bigr)$ and $f(0) = y_0$. $\square$

7.6.3 Exercises

For more exercises related to Picard's theorem see §6.3.

Exercise* 7.6.1*: *Suppose J is a closed and bounded interval, and let $Y := \{ f \in C([-h,h],\mathbb{R}) : f([-h,h]) \subset J \}$. Show that $Y \subset C([-h,h],\mathbb{R})$ is closed. Hint: J is closed.*

Exercise* 7.6.2*: *In the proof of Picard's theorem, show that if $f\colon [-h,h] \to J$ is continuous, then $F\bigl(t, f(t)\bigr)$ is continuous on $[-h,h]$ as a function of t. Use this to show that*

$$T(f)(x) := y_0 + \int_0^x F\bigl(t, f(t)\bigr)\, dt$$

is well-defined and that $T(f) \in C([-h,h],\mathbb{R})$.

Exercise* 7.6.3*: *Prove that in the proof of Picard's theorem, the statement "Without loss of generality assume $x_0 = 0$" is justified. That is, prove that if we know the theorem with $x_0 = 0$, the theorem is true as stated.*

Exercise* 7.6.4*: *Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) := kx + b$ where $0 < k < 1$, $b \in \mathbb{R}$.*

a) *Show that F is a contraction.*

b) *Find the fixed point and show directly that it is unique.*

Exercise* 7.6.5*: *Let $f\colon [0, 1/4] \to [0, 1/4]$ be defined by $f(x) := x^2$.*

a) *Show that f is a contraction, and find the best (smallest) k from the definition that works.*

b) *Find the fixed point and show directly that it is unique.*

Exercise* 7.6.6*:

a) *Find an example of a contraction $f\colon X \to X$ of a non-complete metric space X with no fixed point.*

b) *Find a 1-Lipschitz map $f\colon X \to X$ of a complete metric space X with no fixed point.*

Exercise* 7.6.7*: *Consider $y' = y^2$, $y(0) = 1$. Use the iteration scheme from the proof of the contraction mapping principle. Start with $f_0(x) = 1$. Find a few iterates (at least up to f_2). Prove that the pointwise limit of f_n is $\frac{1}{1-x}$, that is, for every x with $|x| < h$ for some $h > 0$, prove that $\lim\limits_{n\to\infty} f_n(x) = \frac{1}{1-x}$.*

Exercise* 7.6.8*: *Suppose $f\colon X \to X$ is a contraction for $k < 1$. Suppose you use the iteration procedure with $x_{n+1} := f(x_n)$ as in the proof of the fixed point theorem. Suppose x is the fixed point of f.*

a) *Show that $d(x,x_n) \leq k^n d(x_1,x_0) \frac{1}{1-k}$ for all $n \in \mathbb{N}$.*

b) *Suppose $d(y_1,y_2) \leq 16$ for all $y_1,y_2 \in X$, and $k = 1/2$. Find an N such that starting at any given point $x_0 \in X$, $d(x,x_n) \leq 2^{-16}$ for all $n \geq N$.*

Exercise* 7.6.9*: *Let $f(x) := x - \frac{x^2-2}{2x}$ (you may recognize Newton's method for $\sqrt{2}$).*

a) *Prove $f\bigl([1,\infty)\bigr) \subset [1,\infty)$.*

b) *Prove that $f\colon [1,\infty) \to [1,\infty)$ is a contraction.*

c) *Apply the fixed point theorem to find an $x \geq 1$ such that $f(x) = x$, and show that $x = \sqrt{2}$.*

Exercise* 7.6.10*: *Suppose $f\colon X \to X$ is a contraction, and (X,d) is a metric space with the discrete metric, that is, $d(x,y) = 1$ whenever $x \neq y$. Show that f is constant, that is, there exists a $c \in X$ such that $f(x) = c$ for all $x \in X$.*

Exercise* 7.6.11*: *Suppose (X,d) is a nonempty complete metric space, $f\colon X \to X$ is a mapping, and denote by f^n the nth iterate of f. Suppose for every n there exists a $k_n > 0$ such that $d\bigl(f^n(x), f^n(y)\bigr) \leq k_n\, d(x,y)$ for all $x,y \in X$, where $\sum_{j=1}^{\infty} k_n < \infty$. Prove that f has a unique fixed point in X.*

Further Reading

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[R1] Maxwell Rosenlicht, *Introduction to Analysis*, Dover Publications Inc., New York, 1986. Reprint of the 1968 edition.

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List of Notation

Basic Analysis II

Introduction to Real Analysis, Volume II

by Jiří Lebl

May 16, 2022
(version 2.6)

Typeset in LaTeX.

During the writing of these notes, the author was in part supported by NSF grant DMS-1362337.

The date is the main identifier of version. The major version / edition number is raised only if there have been substantial changes. Each volume has its own version number.

See https://www.jirka.org/ra/ for more information (including contact information, possible updates and errata).

The LaTeX source for the book is available for possible modification and customization at github: https://github.com/jirilebl/ra

Contents

Introduction

About this book

This second volume of "Basic Analysis" is meant to be a seamless continuation. The chapters are numbered to start where the first volume left off. The book started with my notes for a second-semester undergraduate analysis at University of Wisconsin—Madison in 2012, where I used my notes together with Rudin's book. The choice of some of the material and some of the proofs are very similar to Rudin, though I do try to provide more detail and context. In 2016, I taught a second-semester undergraduate analysis at Oklahoma State University and heavily modified and cleaned up the notes, this time using them as the main text. In 2018, I taught this course again, this time adding chapter 11 (which I originally wrote for the Wisconsin course).

I plan to eventually add some more topics. I will try to preserve the current numbering in subsequent editions as always. The new topics I have planned would add chapters onto the end of the book, or add sections to end of existing chapters, and I will try as hard as possible to leave exercise numbers unchanged.

For the most part, this second volume depends on the non-optional parts of volume I, while some of the optional parts are also used. Higher order derivatives (but not Taylor's theorem itself) are used in 8.6, 9.3, 10.6. Exponentials, logarithms, and improper integrals are used in a few examples and exercises, and they are heavily used in chapter 11.

My own plan for a two-semester course is that some bits of the first volume, such as metric spaces, are covered in the second semester, while some of the optional topics of volume I are covered in the first semester. Leaving metric spaces for the second semester makes more sense as then the second semester is the "multivariable" part of the course. Another possibility for a faster course is to leave out some of the optional parts, go quicker in the first semester including metric spaces and then arrive at chapter 11.

Several possibilities for things to cover after metric spaces, depending on time are:

1) 8.1–8.5, 10.1–10.5, 10.7 (multivariable calculus, focus on multivariable integral).
2) Chapter 8, chapter 9, 10.1 and 10.2 (multivariable calculus, focus on path integrals).
3) Chapters 8, 9, and 10 (multivariable calculus, path integrals, multivariable integrals).
4) Chapters 8, (maybe 9), and 11 (multivariable differential calculus, some advanced analysis).
5) Chapter 8, chapter 9, 11.1, 11.6, 11.7 (a simpler variation of the above).

Chapter 8

Several Variables and Partial Derivatives

8.1 Vector spaces, linear mappings, and convexity

Note: 2–3 lectures

8.1.1 Vector spaces

The euclidean space $\mathbb{R}^n$ has already made an appearance in the metric space chapter. In this chapter, we extend the differential calculus we created for one variable to several variables. The key idea in differential calculus is to approximate differentiable functions by linear functions (approximating the graph by a straight line). In several variables, we must introduce a little bit of linear algebra before we can move on. We start with vector spaces and linear mappings on vector spaces.

While it is common to use $\vec{v}$ or the bold $\mathbf{v}$ for elements of $\mathbb{R}^n$, especially in the applied sciences, we use just plain old v, which is common in mathematics. That is, $v \in \mathbb{R}^n$ is a *vector*, which means $v = (v_1, v_2, \ldots, v_n)$ is an n-tuple of real numbers.* It is common to write and treat vectors as *column vectors*, that is, n-by-1 matrices:

$$v = (v_1, v_2, \ldots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

We will do so when convenient. We call real numbers *scalars* to distinguish them from vectors.

We often think of vectors as a direction and a magnitude and draw the vector as an arrow. The vector $(v_1, v_2, \ldots, v_n)$ is represented by an arrow from the origin to the point $(v_1, v_2, \ldots, v_n)$, see Figure 8.1 in the plane $\mathbb{R}^2$. When we think of vectors as arrows, they are not based at the origin necessarily; a vector is simply the direction and the magnitude, and it does not know where it starts.

On the other hand, each vector also represents a point in $\mathbb{R}^n$. Usually, we think of $v \in \mathbb{R}^n$ as a point if we are thinking of $\mathbb{R}^n$ as a metric space, and we think of it as an arrow if we think of the so-called *vector space structure* on $\mathbb{R}^n$ (addition and scalar multiplication). Let us define the abstract notion of a vector space, as there are many other vector spaces than just $\mathbb{R}^n$.

*Subscripts are used for many purposes, so sometimes we may have several vectors that may also be identified by subscript, such as a finite or infinite sequence of vectors $y_1, y_2, \ldots$.

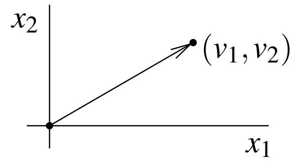

Figure 8.1: Vector as an arrow.

Definition 8.1.1. Let X be a set together with the operations of addition, $+\colon X \times X \to X$, and multiplication, $\cdot\colon \mathbb{R} \times X \to X$, (we usually write ax instead of $a \cdot x$). X is called a *vector space* (or a *real vector space*) if the following conditions are satisfied:

(i) (Addition is associative) If $u, v, w \in X$, then $u + (v + w) = (u + v) + w$.

(ii) (Addition is commutative) If $u, v \in X$, then $u + v = v + u$.

(iii) (Additive identity) There is a $0 \in X$ such that $v + 0 = v$ for all $v \in X$.

(iv) (Additive inverse) For every $v \in X$, there is a $-v \in X$, such that $v + (-v) = 0$.

(v) (Distributive law) If $a \in \mathbb{R}$, $u, v \in X$, then $a(u + v) = au + av$.

(vi) (Distributive law) If $a, b \in \mathbb{R}$, $v \in X$, then $(a + b)v = av + bv$.

(vii) (Multiplication is associative) If $a, b \in \mathbb{R}$, $v \in X$, then $(ab)v = a(bv)$.

(viii) (Multiplicative identity) $1v = v$ for all $v \in X$.

Elements of a vector space are usually called *vectors*, even if they are not elements of $\mathbb{R}^n$ (vectors in the "traditional" sense).

If $Y \subset X$ is a subset that is a vector space itself using the same operations, then Y is called a *subspace* or a *vector subspace* of X.

Multiplication by scalars works as one would expect. For example, $2v = (1 + 1)v = 1v + 1v = v + v$, similarly $3v = v + v + v$, and so on. One particular fact we often use is that $0v = 0$, where the zero on the left is $0 \in \mathbb{R}$ and the zero on the right is $0 \in X$. To see this start with $0v = (0 + 0)v = 0v + 0v$, and add $-(0v)$ to both sides to obtain $0 = 0v$. Similarly $-v = (-1)v$, which follows by $(-1)v + v = (-1)v + 1v = (-1 + 1)v = 0v = 0$. These algebraic facts which follow quickly from the definition we will take for granted from now on.

Example 8.1.2: An example vector space is $\mathbb{R}^n$, where addition and multiplication by a scalar is done componentwise: If $a \in \mathbb{R}$, $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, and $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$, then

$$\begin{aligned} v + w &:= (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n), \\ av &:= a(v_1, v_2, \ldots, v_n) = (av_1, av_2, \ldots, av_n). \end{aligned}$$

In this book, we mostly deal with vector spaces that can be often regarded as subsets of $\mathbb{R}^n$, but there are other vector spaces useful in analysis. Let us give a couple of examples.

Example 8.1.3: A trivial example of a vector space is just $X := \{0\}$. The operations are defined in the obvious way: $0 + 0 := 0$ and $a0 := 0$. A zero vector must always exist, so all vector spaces are nonempty sets, and this X is the smallest possible vector space.

Example 8.1.4: The space $C([0,1],\mathbb{R})$ of continuous functions on the interval $[0,1]$ is a vector space. For two functions f and g in $C([0,1],\mathbb{R})$ and $a \in \mathbb{R}$, we make the obvious definitions of $f+g$ and af:

$$(f+g)(x) := f(x)+g(x), \qquad (af)(x) := a\bigl(f(x)\bigr).$$

The 0 is the function that is identically zero. We leave it as an exercise to check that all the vector space conditions are satisfied.

The space $C^1([0,1],\mathbb{R})$ of continuously differentiable functions is a subspace of $C([0,1],\mathbb{R})$.

Example 8.1.5: The space of polynomials $c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m$ (of arbitrary degree m) is a vector space. Let us denote it by $\mathbb{R}[t]$ (coefficients are real and the variable is t). The operations are defined in the same way as for functions above. Suppose there are two polynomials, one of degree m and one of degree n. Assume $n \geq m$ for simplicity. Then

$$\begin{aligned}(c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m) + (d_0 + d_1 t + d_2 t^2 + \cdots + d_n t^n) = \\ (c_0+d_0) + (c_1+d_1)t + (c_2+d_2)t^2 + \cdots + (c_m+d_m)t^m + d_{m+1}t^{m+1} + \cdots + d_n t^n\end{aligned}$$

and

$$a(c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m) = (ac_0) + (ac_1)t + (ac_2)t^2 + \cdots + (ac_m)t^m.$$

Despite what it looks like, $\mathbb{R}[t]$ is not equivalent to $\mathbb{R}^n$ for any n. In particular, it is not "finite-dimensional." We will make this notion precise in just a little bit. One can make a finite-dimensional vector subspace by restricting the degree. For example, if $\mathscr{P}_n$ is the set of polynomials of degree n or less, then $\mathscr{P}_n$ is a finite-dimensional vector space, and we could identify it with $\mathbb{R}^{n+1}$.

In the above, the variable t is really just a formal placeholder. By setting t equal to a real number we obtain a function. So the space $\mathbb{R}[t]$ can be thought of as a subspace of $C(\mathbb{R},\mathbb{R})$. If we restrict the range of t to $[0,1]$, $\mathbb{R}[t]$ can be identified with a subspace of $C([0,1],\mathbb{R})$.

Remark 8.1.6. If X is a vector space, to check that a subset $S \subset X$ is a vector subspace, we only need

1) $0 \in S$,

2) S is closed under addition, adding two vectors in S gets us a vector in S, and

3) S is closed under scalar multiplication, multiplying a vector in S by a scalar gets us a vector in S.

Items 2) and 3) make sure that the addition and scalar multiplication are indeed defined on S. Item 1) is required to fulfill item (iii) from the definition of vector space. Existence of additive inverse $-v$ follows because $-v = (-1)v$ and item 3) says that $-v \in S$ if $v \in S$. All other properties are certain equalities that are already satisfied in X and thus must be satisfied in a subset.

It is often better to think of even the simpler "finite-dimensional" vector spaces using the abstract notion rather than always as $\mathbb{R}^n$. It is possible to use other fields than $\mathbb{R}$ in the definition (for example it is common to use the complex numbers $\mathbb{C}$), but let us stick with the real numbers*.

*If you want a very funky vector space over a different field, $\mathbb{R}$ itself is a vector space over the rational numbers.

8.1.2 Linear combinations and dimension

Definition 8.1.7. Suppose X is a vector space, $x_1, x_2, \ldots, x_k \in X$ are vectors, and $a_1, a_2, \ldots, a_k \in \mathbb{R}$ are scalars. Then

$$a_1 x_1 + a_2 x_2 + \cdots + a_k x_k$$

is called a *linear combination* of the vectors $x_1, x_2, \ldots, x_k$.

If $Y \subset X$ is a set, then the *span* of Y, or in notation $\operatorname{span}(Y)$, is the set of all linear combinations of all finite subsets of Y. We say Y *spans* $\operatorname{span}(Y)$. By convention, define $\operatorname{span}(\emptyset) := \{0\}$.

Example 8.1.8: Let $Y := \bigl\{(1,1)\bigr\} \subset \mathbb{R}^2$. Then

$$\operatorname{span}(Y) = \bigl\{(x,x) \in \mathbb{R}^2 : x \in \mathbb{R}\bigr\}.$$

That is, $\operatorname{span}(Y)$ is the line through the origin and the point $(1,1)$.

Example 8.1.9: Let $Y := \bigl\{(1,1),(0,1)\bigr\} \subset \mathbb{R}^2$. Then

$$\operatorname{span}(Y) = \mathbb{R}^2,$$

as every point $(x,y) \in \mathbb{R}^2$ can be written as a linear combination

$$(x,y) = x(1,1) + (y-x)(0,1).$$

Example 8.1.10: Let $Y := \{1, t, t^2, t^3, \ldots\} \subset \mathbb{R}[t]$, and $E := \{1, t^2, t^4, t^6, \ldots\} \subset \mathbb{R}[t]$.

The span of Y is all polynomials,

$$\operatorname{span}(Y) = \mathbb{R}[t].$$

The span of E is the set of polynomials with even powers of t only.

Suppose we have two linear combinations of vectors from Y. One linear combination uses the vectors $\{x_1, x_2, \ldots, x_k\}$, and the other uses $\{\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_\ell\}$. Then clearly we can write both linear combinations using vectors from the union $\{x_1, x_2, \ldots, x_k\} \cup \{\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_\ell\}$, by just taking zero multiples of the vectors we do not need, e.g. $x_1 = x_1 + 0\widetilde{x}_1$. Suppose we have two linear combinations, we can without loss of generality write them as a linear combination of $x_1, x_2, \ldots, x_k$. Then their sum is also a linear combination of vectors from Y:

$$\begin{aligned}(a_1 x_1 + a_2 x_2 + \cdots + a_k x_k) &+ (b_1 x_1 + b_2 x_2 + \cdots + b_k x_k) \\ &= (a_1 + b_1) x_1 + (a_2 + b_2) x_2 + \cdots + (a_k + b_k) x_k.\end{aligned}$$

Similarly, a scalar multiple of a linear combination of vectors from Y is a linear combination of vectors from Y:

$$b(a_1 x_1 + a_2 x_2 + \cdots + a_k x_k) = b a_1 x_1 + b a_2 x_2 + \cdots + b a_k x_k.$$

Finally, $0 \in Y$; if Y is nonempty, $0 = 0v$ for some $v \in Y$. We have proved the following proposition.

Proposition 8.1.11. *Let X be a vector space. For every $Y \subset X$, the set* $\operatorname{span}(Y)$ *is a vector space. That is,* $\operatorname{span}(Y)$ *is a subspace of X.*

If Y is already a vector space, then $\operatorname{span}(Y) = Y$.

Definition 8.1.12. A set of vectors $\{x_1, x_2, \ldots, x_k\} \subset X$ is *linearly independent*[*] if the equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = 0 \tag{8.1}$$

has only the trivial solution $a_1 = a_2 = \cdots = a_k = 0$. A set that is not linearly independent is *linearly dependent*. A linearly independent set of vectors B such that $\operatorname{span}(B) = X$ is called a *basis* of X.

If a vector space X contains a linearly independent set of d vectors, but no linearly independent set of $d+1$ vectors, then we say the *dimension* of X is d, and we write $\dim X := d$. If for all $d \in \mathbb{N}$ the vector space X contains a set of d linearly independent vectors, we say X is infinite-dimensional and write $\dim X := \infty$. For the trivial vector space $\{0\}$, we define $\dim \{0\} := 0$.

A subset of a linear independent set is clearly linearly independent, so in the definition of dimension, notice that if a set does not have $d+1$ linearly independent vectors, no set of more than $d+1$ vectors is linearly independent either. Also note that no element of a linear independent set can be zero. In particular, $\{0\}$ is the only vector space of dimension 0. By convention, the empty set is linearly independent and thus a basis of $\{0\}$.

As an example, the set Y of the two vectors in Example 8.1.9 is a basis of $\mathbb{R}^2$, and so $\dim \mathbb{R}^2 \geq 2$. We will see in a moment that every vector subspace of $\mathbb{R}^n$ has a finite dimension, and that dimension is less than or equal to n. So every set of 3 vectors in $\mathbb{R}^2$ is linearly dependent, and $\dim \mathbb{R}^2 = 2$.

If a set is linearly dependent, then one of the vectors is a linear combination of the others. In other words, in (8.1) if $a_j \neq 0$, then we solve for x_j:

$$x_j = \frac{-a_1}{a_j} x_1 + \cdots + \frac{-a_{j-1}}{a_j} x_{j-1} + \frac{-a_{j+1}}{a_j} x_{j+1} + \cdots + \frac{-a_k}{a_k} x_k.$$

The vector x_j has at least two different representations as linear combinations of $\{x_1, x_2, \ldots, x_k\}$. The one above and x_j itself. For example, the set $\bigl\{(0,1), (2,3), (5,0)\bigr\}$ in $\mathbb{R}^2$ is linearly dependent:

$$3(0,1) - (2,3) + 2(1,0) = 0, \qquad \text{so} \qquad (2,3) = 3(0,1) + 2(1,0).$$

Proposition 8.1.13. *Suppose a vector space X has basis $B = \{x_1, x_2, \ldots, x_k\}$. Then every $y \in X$ has a unique representation of the form*

$$y = \sum_{j=1}^{k} a_j x_j$$

for some scalars $a_1, a_2, \ldots, a_k$.

Proof. As X is the span of B, every $y \in X$ is a linear combination of elements of B. Suppose

$$y = \sum_{j=1}^{k} a_j x_j = \sum_{j=1}^{k} b_j x_j.$$

Then

$$\sum_{j=1}^{k} (a_j - b_j) x_j = 0.$$

By linear independence of the basis $a_j = b_j$ for all j, and so the representation is unique. $\square$

[*]For an infinite set $Y \subset X$, we would say Y is linearly independent if every finite subset of Y is linearly independent in the sense given. However, this situation only comes up in infinitely many dimensions and we will not require it.

For $\mathbb{R}^n$ we define the *standard basis* of $\mathbb{R}^n$:

$$e_1 := (1,0,0,\ldots,0), \quad e_2 := (0,1,0,\ldots,0), \quad \ldots, \quad e_n := (0,0,0,\ldots,1),$$

We use the same letters e_j for any $\mathbb{R}^n$, and which space $\mathbb{R}^n$ we are working in is understood from context. A direct computation shows that $\{e_1,e_2,\ldots,e_n\}$ is really a basis of $\mathbb{R}^n$; it spans $\mathbb{R}^n$ and is linearly independent. In fact,

$$a = (a_1,a_2,\ldots,a_n) = \sum_{j=1}^{n} a_j e_j.$$

Proposition 8.1.14. *Let X be a vector space and d a nonnegative integer.*

(i) If X is spanned by d vectors, then $\dim X \leq d$.

(ii) $\dim X = d$ *if and only if X has a basis of d vectors (and so every basis has d vectors).*

(iii) In particular, $\dim \mathbb{R}^n = n$.

(iv) If $Y \subset X$ is a vector subspace and $\dim X = d$, *then* $\dim Y \leq d$.

(v) If $\dim X = d$ *and a set T of d vectors spans X, then T is linearly independent.*

(vi) If $\dim X = d$ *and a set T of m vectors is linearly independent, then there is a set S of $d-m$ vectors such that $T \cup S$ is a basis of X.*

Proof. All statements hold trivially when $d = 0$, so assume $d \geq 1$.

Let us start with (i). Suppose $S = \{x_1,x_2,\ldots,x_d\}$ spans X, and $T = \{y_1,y_2,\ldots,y_m\}$ is a set of linearly independent vectors of X. We wish to show that $m \leq d$. Write

$$y_1 = \sum_{k=1}^{d} a_{k,1} x_k,$$

for some numbers $a_{1,1},a_{2,1},\ldots,a_{d,1}$, which we can do as S spans X. One of the $a_{k,1}$ is nonzero (otherwise y_1 would be zero), so suppose without loss of generality that this is $a_{1,1}$. Then we solve

$$x_1 = \frac{1}{a_{1,1}} y_1 - \sum_{k=2}^{d} \frac{a_{k,1}}{a_{1,1}} x_k.$$

In particular, $\{y_1,x_2,\ldots,x_d\}$ span X, since x_1 can be obtained from $\{y_1,x_2,\ldots,x_d\}$. Therefore, there are some numbers for some numbers $a_{1,2},a_{2,2},\ldots,a_{d,2}$, such that

$$y_2 = a_{1,2} y_1 + \sum_{k=2}^{d} a_{k,2} x_k.$$

As T is linearly independent—and so $\{y_1,y_2\}$ is linearly independent—one of the $a_{k,2}$ for $k \geq 2$ must be nonzero. Without loss of generality suppose $a_{2,2} \neq 0$. Proceed to solve for

$$x_2 = \frac{1}{a_{2,2}} y_2 - \frac{a_{1,2}}{a_{2,2}} y_1 - \sum_{k=3}^{d} \frac{a_{k,2}}{a_{2,2}} x_k.$$

In particular, $\{y_1,y_2,x_3,\ldots,x_d\}$ spans X.

We continue this procedure. If $m < d$, then we are done. So suppose $m \geq d$. After d steps, we obtain that $\{y_1, y_2, \dots, y_d\}$ spans X. Any other vector v in X is a linear combination of $\{y_1, y_2, \dots, y_d\}$, and hence cannot be in T as T is linearly independent. So $m = d$.

Let us look at (ii). First a short claim. If T is a set of linearly independent vectors that do not span X, that is, $X \setminus \operatorname{span}(T) \neq \emptyset$, then for any vector $v \in X \setminus \operatorname{span}(T)$, the set $T \cup \{v\}$ is linearly independent. Indeed, a nonzero linear combination of elements of $T \cup \{v\}$ would either produce v as a combination of T, or it would be a combination of elements of T, and neither option is possible.

If $\dim X = d$, then there must exist some linearly independent set T of d vectors, and T must span X, otherwise we could choose a larger set of linearly independent vectors via the claim. So we have a basis of d vectors. On the other hand, if we have a basis of d vectors, the dimension is at least d as a basis is linearly independent. On the other hand a basis also spans X, and so by (i) we know that dimension is at most d. Hence the dimension of X must equal d.

For (iii) notice that $\{e_1, e_2, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.

To see (iv), suppose $Y \subset X$ is a vector subspace, where $\dim X = d$. As X cannot contain $d+1$ linearly independent vectors, neither can Y.

For (v) suppose T is a set of m vectors that is linearly dependent and spans X, we will show that $m > d$. One of the vectors is a linear combination of the others. If we remove it from T we obtain a set of $m-1$ vectors that still span X and hence $d = \dim X \leq m-1$ by (i).

For (vi) suppose $T = \{x_1, x_2, \dots, x_m\}$ is a linearly independent set. Firstly, $m \leq d$ by definition of dimension. If $m = d$, we are done. Otherwise, we follow the procedure above in the proof of (ii) to add a vector v not in the span of T. The set $T \cup \{v\}$ is linearly independent, whose span has dimension $m+1$. Therefore, we repeat this procedure $d-m$ times to find a set of d linearly independent vectors. They must span X otherwise we could add yet another vector. $\square$

8.1.3 Linear mappings

A function $f \colon X \to Y$, when Y is not $\mathbb{R}$, is often called a *mapping* or a *map* rather than a *function*.

Definition 8.1.15. A mapping $A \colon X \to Y$ of vector spaces X and Y is *linear* (we also say A is a *linear transformation* or a *linear operator*) if for all $a \in \mathbb{R}$ and all $x, y \in X$,

$$A(ax) = aA(x), \qquad \text{and} \qquad A(x+y) = A(x) + A(y).$$

We usually write Ax instead of $A(x)$ if A is linear. If A is one-to-one and onto, then we say A is *invertible*, and we denote the inverse by A^{-1}. If $A \colon X \to X$ is linear, then we say A is a *linear operator on* X.

We write $L(X,Y)$ for the set of all linear transformations from X to Y, and just $L(X)$ for the set of linear operators on X. If $a \in \mathbb{R}$ and $A, B \in L(X,Y)$, define the transformations aA and $A+B$ by

$$(aA)(x) := aAx, \qquad (A+B)(x) := Ax + Bx.$$

If $A \in L(Y,Z)$ and $B \in L(X,Y)$, define the transformation AB as the composition $A \circ B$, that is,

$$ABx := A(Bx).$$

Finally, denote by $I \in L(X)$ the *identity*: the linear operator such that $Ix = x$ for all x.

It is not hard to see that $aA \in L(X,Y)$ and $A+B \in L(X,Y)$, and that $AB \in L(X,Z)$. In particular, $L(X,Y)$ is a vector space (0 is the linear map that takes everything to 0). As the set $L(X)$ is not only a vector space, but also admits a product (composition of operators), it is often called an *algebra*.

An immediate consequence of the definition of a linear mapping is: If A is linear, then $A0 = 0$.

Proposition 8.1.16. *If $A \in L(X,Y)$ is invertible, then A^{-1} is linear.*

Proof. Let $a \in \mathbb{R}$ and $y \in Y$. As A is onto, then there is an x such that $y = Ax$, and further as it is also one-to-one $A^{-1}(Az) = z$ for all $z \in X$. So

$$A^{-1}(ay) = A^{-1}(aAx) = A^{-1}\big(A(ax)\big) = ax = aA^{-1}(y).$$

Similarly let $y_1, y_2 \in Y$, and $x_1, x_2 \in X$ such that $Ax_1 = y_1$ and $Ax_2 = y_2$, then

$$A^{-1}(y_1 + y_2) = A^{-1}(Ax_1 + Ax_2) = A^{-1}\big(A(x_1 + x_2)\big) = x_1 + x_2 = A^{-1}(y_1) + A^{-1}(y_2). \qquad \square$$

Proposition 8.1.17. *If $A \in L(X,Y)$ is linear, then it is completely determined by its values on a basis of X. Furthermore, if B is a basis of X, then every function $\widetilde{A} \colon B \to Y$ extends to a linear function A on X.*

We will only prove this proposition for finite-dimensional spaces, as we do not need infinite-dimensional spaces. For infinite-dimensional spaces, the proof is essentially the same, but a little trickier to write, so let us stick with finitely many dimensions.

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of X, and let $y_j := Ax_j$. Every $x \in X$ has a unique representation

$$x = \sum_{j=1}^{n} b_j x_j$$

for some numbers $b_1, b_2, \ldots, b_n$. By linearity

$$Ax = A \sum_{j=1}^{n} b_j x_j = \sum_{j=1}^{n} b_j Ax_j = \sum_{j=1}^{n} b_j y_j.$$

The "furthermore" follows by setting $y_j := \widetilde{A}(x_j)$, and then for $x = \sum_{j=1}^{n} b_j x_j$, defining the extension as $Ax := \sum_{j=1}^{n} b_j y_j$. The function is well-defined by uniqueness of the representation of x. We leave it to the reader to check that A is linear. $\square$

The next proposition only works for finite-dimensional vector spaces. It is a special case of the so-called rank-nullity theorem from linear algebra.

Proposition 8.1.18. *If X is a finite-dimensional vector space and $A \in L(X)$, then A is one-to-one if and only if it is onto.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for X. First suppose A is one-to-one. Let $c_1, c_2, \ldots, c_n$ be such that

$$0 = \sum_{j=1}^{n} c_j Ax_j = A \sum_{j=1}^{n} c_j x_j.$$

As A is one-to-one, the only vector that is taken to 0 is 0 itself. Hence,

$$0 = \sum_{j=1}^{n} c_j x_j$$

and $c_j = 0$ for all j. So $\{Ax_1, Ax_2, \dots, Ax_n\}$ is a linearly independent set. By Proposition 8.1.14 and the fact that the dimension is n, we conclude $\{Ax_1, Ax_2, \dots, Ax_n\}$ spans X. Any point $x \in X$ can be written as

$$x = \sum_{j=1}^{n} a_j A x_j = A \sum_{j=1}^{n} a_j x_j,$$

so A is onto.

For the other direction, suppose A is onto. As A is determined by the action on the basis, every element of X is in the span of $\{Ax_1, Ax_2, \dots, Ax_n\}$. Suppose that for some $c_1, c_2, \dots, c_n$,

$$0 = A \sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} c_j A x_j.$$

By Proposition 8.1.14 as $\{Ax_1, Ax_2, \dots, Ax_n\}$ span X, the set is linearly independent, and hence $c_j = 0$ for all j. In other words, if $Ax = 0$, then $x = 0$. This means that A is one-to-one: If $Ax = Ay$, then $A(x-y) = 0$ and so $x = y$. $\square$

We leave the proof of the next proposition as an exercise.

Proposition 8.1.19. *If X and Y are finite-dimensional vector spaces, then $L(X,Y)$ is also finite-dimensional.*

We often identify a finite-dimensional vector space X of dimension n with $\mathbb{R}^n$, provided we fix a basis $\{x_1, x_2, \dots, x_n\}$ in X. That is, we define a bijective linear map $A \in L(X, \mathbb{R}^n)$ by $Ax_j := e_j$, where $\{e_1, e_2, \dots, e_n\}$ is the standard basis in $\mathbb{R}^n$. Then we have the correspondence

$$\sum_{j=1}^{n} c_j x_j \in X \quad \overset{A}{\mapsto} \quad (c_1, c_2, \dots, c_n) \in \mathbb{R}^n.$$

8.1.4 Convexity

A subset U of a vector space is *convex* if whenever $x, y \in U$, the line segment from x to y lies in U. That is, if the *convex combination* $(1-t)x + ty$ is in U for all $t \in [0,1]$. Sometimes we write $[x,y]$ for this line segment. See Figure 8.2.

In $\mathbb{R}$, convex sets are precisely the intervals, which are also precisely the connected sets. In two or more dimensions there are lots of nonconvex connected sets. For example, the set $\mathbb{R}^2 \setminus \{0\}$ is not convex, but it is connected. To see this, take any $x \in \mathbb{R}^2 \setminus \{0\}$ and let $y := -x$. Then $(1/2)x + (1/2)y = 0$, which is not in the set. Balls in $\mathbb{R}^n$ are convex. We use this result often enough we state it as a proposition, and leave the proof as an exercise.

Proposition 8.1.20. *Let $x \in \mathbb{R}^n$ and $r > 0$. The ball $B(x,r) \subset \mathbb{R}^n$ (using the standard metric on $\mathbb{R}^n$) is convex.*

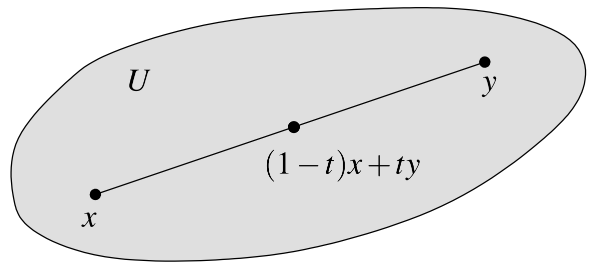

Figure 8.2: Convexity.

Example 8.1.21: As a convex combination is, in particular, a linear combination, so every subspace V of a vector space X is convex.

Example 8.1.22: Let $C([0,1],\mathbb{R})$ be the vector space of continuous real-valued functions on $\mathbb{R}$. Let $X \subset C([0,1],\mathbb{R})$ be the set of those f such that

$$\int_0^1 f(x)\,dx \leq 1 \qquad \text{and} \qquad f(x) \geq 0 \text{ for all } x \in [0,1].$$

Then X is convex. Take $t \in [0,1]$, and note that if $f,g \in X$, then $tf(x)+(1-t)g(x) \geq 0$ for all x. Furthermore,

$$\int_0^1 \bigl(tf(x)+(1-t)g(x)\bigr)\,dx = t\int_0^1 f(x)\,dx + (1-t)\int_0^1 g(x)\,dx \leq 1.$$

Note that X is not a vector subspace of $C([0,1],\mathbb{R})$. The function $f(x) := 1$ is in X, but $2f$ and $-f$ is not.

Proposition 8.1.23. *The intersection of two convex sets is convex. In fact, if $\{C_\lambda\}_{\lambda \in I}$ is an arbitrary collection of convex sets, then*

$$C := \bigcap_{\lambda \in I} C_\lambda$$

is convex.

Proof. If $x,y \in C$, then $x,y \in C_\lambda$ for all $\lambda \in I$, and hence if $t \in [0,1]$, then $tx+(1-t)y \in C_\lambda$ for all $\lambda \in I$. Therefore, $tx+(1-t)y \in C$ and C is convex. $\square$

Proposition 8.1.24. *Let $T \colon V \to W$ be a linear mapping between two vector spaces and let $C \subset V$ be a convex set. Then $T(C)$ is convex.*

Proof. Take two points $p,q \in T(C)$. Pick $x,y \in C$ such that $Tx = p$ and $Ty = q$. As C is convex, then $tx+(1-t)y \in C$ for all $t \in [0,1]$, so

$$tp+(1-t)q = tTx+(1-t)Ty = T\bigl(tx+(1-t)y\bigr) \in T(C). \qquad \square$$

For completeness, a very useful construction is the *convex hull*. Given a subset $S \subset V$ of a vector space, define the convex hull of S as the intersection of all convex sets containing S:

$$\operatorname{co}(S) := \bigcap \{C \subset V : S \subset C, \text{ and } C \text{ is convex}\}.$$

That is, the convex hull is the smallest convex set containing S. By a proposition above, the intersection of convex sets is convex and hence, the convex hull is convex.

Example 8.1.25: The convex hull of $\{0,1\}$ in $\mathbb{R}$ is $[0,1]$. Proof: Any convex set containing 0 and 1 must contain $[0,1]$, so $[0,1] \subset \operatorname{co}(\{0,1\})$. The set $[0,1]$ is convex and contains $\{0,1\}$, so $\operatorname{co}(\{0,1\}) \subset [0,1]$.

8.1.5 Exercises

Exercise* 8.1.1*: *Show that in $\mathbb{R}^n$ (with the standard euclidean metric), for every $x \in \mathbb{R}^n$ and every $r > 0$, the ball $B(x,r)$ is convex.*

Exercise* 8.1.2*: *Verify that $\mathbb{R}^n$ is a vector space.*

Exercise* 8.1.3*: *Let X be a vector space. Prove that a finite set of vectors $\{x_1, x_2, \ldots, x_n\} \subset X$ is linearly independent if and only if for every $j = 1, 2, \ldots, n$*

$$\operatorname{span}\bigl(\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}\bigr) \subsetneq \operatorname{span}\bigl(\{x_1, x_2, \ldots, x_n\}\bigr).$$

That is, the span of the set with one vector removed is strictly smaller.

Exercise* 8.1.4*: *Show that the set $X \subset C([0,1],\mathbb{R})$ of those functions such that $\int_0^1 f = 0$ is a vector subspace.*

Exercise* 8.1.5** (Challenging):*** *Prove $C([0,1],\mathbb{R})$ is an infinite-dimensional vector space where the operations are defined in the obvious way: $s = f + g$ and $m = af$ are defined as $s(x) := f(x) + g(x)$ and $m(x) := af(x)$. Hint: For the dimension, think of functions that are only nonzero on the interval $(1/n+1, 1/n)$.*

Exercise* 8.1.6*: *Let $k\colon [0,1]^2 \to \mathbb{R}$ be continuous. Show that $L\colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by*

$$Lf(y) := \int_0^1 k(x,y) f(x)\, dx$$

is a linear operator. That is, first show that L is well-defined by showing that Lf is continuous whenever f is, and then showing that L is linear.

Exercise* 8.1.7*: *Let $\mathscr{P}_n$ be the vector space of polynomials in one variable of degree n or less. Show that $\mathscr{P}_n$ is a vector space of dimension $n+1$.*

Exercise* 8.1.8*: *Let $\mathbb{R}[t]$ be the vector space of polynomials in one variable t. Let $D\colon \mathbb{R}[t] \to \mathbb{R}[t]$ be the derivative operator (derivative in t). Show that D is a linear operator.*

Exercise* 8.1.9*: *Let us show that Proposition 8.1.18 only works in finite dimensions. Take the space of polynomials $\mathbb{R}[t]$ and define the operator $A\colon \mathbb{R}[t] \to \mathbb{R}[t]$ by $A\bigl(P(t)\bigr) := tP(t)$. Show that A is linear and one-to-one, but show that it is not onto.*

***Exercise* 8.1.10:** *Finish the proof of Proposition 8.1.17 in the finite-dimensional case. That is, suppose $\{x_1, x_2, \ldots x_n\}$ is a basis of X, $\{y_1, y_2, \ldots y_n\} \subset Y$, and we define a function*

$$Ax := \sum_{j=1}^{n} b_j y_j, \qquad \text{if} \quad x = \sum_{j=1}^{n} b_j x_j.$$

Then prove that $A \colon X \to Y$ is linear.

***Exercise* 8.1.11:** *Prove Proposition 8.1.19. Hint: A linear transformation is determined by its action on a basis. So given two bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ for X and Y respectively, consider the linear operators A_{jk} that send $A_{jk} x_j = y_k$, and $A_{jk} x_\ell = 0$ if $\ell \neq j$.*

***Exercise* 8.1.12** (Easy)**:** *Suppose X and Y are vector spaces and $A \in L(X,Y)$ is a linear operator.*

a) Show that the nullspace $N := \{x \in X : Ax = 0\}$ is a vector space.

b) Show that the range $R := \{y \in Y : Ax = y \text{ for some } x \in X\}$ is a vector space.

***Exercise* 8.1.13** (Easy)**:** *Show by example that a union of convex sets need not be convex.*

***Exercise* 8.1.14:** *Compute the convex hull of the set of 3 points $\{(0,0),(0,1),(1,1)\}$ in $\mathbb{R}^2$.*

***Exercise* 8.1.15:** *Show that the set $\{(x,y) \in \mathbb{R}^2 : y > x^2\}$ is a convex set.*

***Exercise* 8.1.16:** *Show that the set $X \subset C([0,1],\mathbb{R})$ of those functions such that $\int_0^1 f = 1$ is a convex set, but not a vector subspace.*

***Exercise* 8.1.17:** *Show that every convex set in $\mathbb{R}^n$ is connected using the standard topology on $\mathbb{R}^n$.*

***Exercise* 8.1.18:** *Suppose $K \subset \mathbb{R}^2$ is a convex set such that the only point of the form $(x,0)$ in K is the point $(0,0)$. Further suppose that there $(0,1) \in K$ and $(1,1) \in K$. Then show that if $(x,y) \in K$, then $y > 0$ unless $x = 0$.*

***Exercise* 8.1.19:** *Prove that an arbitrary intersection of vector subspaces is a vector subspace. That is, if X is a vector space and $\{V_\lambda\}_{\lambda \in I}$ is an arbitrary collection of vector subspaces of X, then $\bigcap_{\lambda \in I} V_\lambda$ is a vector subspace of X.*

8.2 Analysis with vector spaces

Note: 3 lectures

8.2.1 Norms

Let us start measuring the size of vectors and hence distance.

Definition 8.2.1. If X is a vector space, then we say a function $\lVert\cdot\rVert \colon X \to \mathbb{R}$ is a *norm* if

(i) $\lVert x\rVert \geq 0$, with $\lVert x\rVert = 0$ if and only if $x = 0$.

(ii) $\lVert cx\rVert = \lvert c\rvert \, \lVert x\rVert$ for all $c \in \mathbb{R}$ and $x \in X$.

(iii) $\lVert x+y\rVert \leq \lVert x\rVert + \lVert y\rVert$ for all $x, y \in X$. (*triangle inequality*)

A vector space equipped with a norm is called a *normed vector space*.

Given a norm (any norm) on a vector space X, we define a distance $d(x,y) := \lVert x-y\rVert$, and this d makes X into a metric space (exercise). So everything you know about metric spaces applies to normed vector spaces.

Before defining the standard norm on $\mathbb{R}^n$, let us define the standard scalar *dot product* on $\mathbb{R}^n$. For vectors $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ define

$$x \cdot y := \sum_{j=1}^{n} x_j y_j .$$

The dot product is linear in each variable separately, or in more fancy language it is *bilinear*. That is, if y is fixed, the map $x \mapsto x \cdot y$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$. Similarly, if x is fixed, then $y \mapsto x \cdot y$ is linear. It is also *symmetric* in the sense that $x \cdot y = y \cdot x$. The *euclidean norm* is defined as

$$\lVert x\rVert := \lVert x\rVert_{\mathbb{R}^n} := \sqrt{x \cdot x} = \sqrt{(x_1)^2 + (x_2)^2 + \cdots + (x_n)^2}.$$

We normally just use $\lVert x\rVert$, only in the rare instance when it is necessary to emphasize that we are talking about the euclidean norm will we use $\lVert x\rVert_{\mathbb{R}^n}$. It is easy to see that the euclidean norm satisfies (i) and (ii). To prove that (iii) holds, the key inequality is the so-called Cauchy–Schwarz inequality we saw before. As this inequality is so important, we restate and reprove a slightly stronger version using the notation of this chapter.

Theorem 8.2.2 (Cauchy–Schwarz inequality)**.** *Let $x, y \in \mathbb{R}^n$, then*

$$\lvert x \cdot y\rvert \leq \lVert x\rVert \, \lVert y\rVert = \sqrt{x \cdot x}\,\sqrt{y \cdot y},$$

with equality if and only if $x = \lambda y$ or $y = \lambda x$ for some $\lambda \in \mathbb{R}$.

Proof. If $x = 0$ or $y = 0$, then the theorem holds trivially. So assume $x \neq 0$ and $y \neq 0$.

If x is a scalar multiple of y, that is $x = \lambda y$ for some $\lambda \in \mathbb{R}$, then the theorem holds with equality:

$$\lvert x \cdot y\rvert = \lvert \lambda y \cdot y\rvert = \lvert\lambda\rvert \, \lvert y \cdot y\rvert = \lvert\lambda\rvert \, \lVert y\rVert^2 = \lVert \lambda y\rVert \, \lVert y\rVert = \lVert x\rVert \, \lVert y\rVert .$$

Fixing x and y, as a function of t, $\|x+ty\|^2$ is a quadratic polynomial:

$$\|x+ty\|^2 = (x+ty)\cdot(x+ty) = x\cdot x + x\cdot ty + ty\cdot x + ty\cdot ty = \|x\|^2 + 2t(x\cdot y) + t^2\|y\|^2.$$

If x is not a scalar multiple of y, then $\|x+ty\|^2 > 0$ for all t. So the polynomial $\|x+ty\|^2$ is never zero. Elementary algebra says that the discriminant must be negative:

$$4(x\cdot y)^2 - 4\|x\|^2\|y\|^2 < 0,$$

or in other words, $(x\cdot y)^2 < \|x\|^2\|y\|^2$. □

Item (iii), the triangle inequality in $\mathbb{R}^n$, follows from:

$$\|x+y\|^2 = x\cdot x + y\cdot y + 2(x\cdot y) \leq \|x\|^2 + \|y\|^2 + 2\big(\|x\|\,\|y\|\big) = \big(\|x\| + \|y\|\big)^2.$$

The distance $d(x,y) := \|x-y\|$ is the standard distance (standard metric) on $\mathbb{R}^n$ that we used when we talked about metric spaces.

Definition 8.2.3. Let $A \in L(X,Y)$. Define

$$\|A\| := \sup\big\{\|Ax\| : x \in X \text{ with } \|x\| = 1\big\}.$$

The number $\|A\|$ (possibly ∞) is called the *operator norm*. We will see below that it is indeed a norm for finite-dimensional spaces. Again, when necessary to emphasize which norm we are talking about, we may write it as $\|A\|_{L(X,Y)}$.

For example, if $X = \mathbb{R}^1$ with norm $\|x\| = |x|$, we think of elements of $L(X)$ as multiplication by scalars: $x \mapsto ax$. If $\|x\| = |x| = 1$, then $|ax| = |a|$, so the operator norm of a is $|a|$.

By linearity, $\left\|A\frac{x}{\|x\|}\right\| = \frac{\|Ax\|}{\|x\|}$ for all nonzero $x \in X$. The vector $\frac{x}{\|x\|}$ is of norm 1. Therefore,

$$\|A\| = \sup\big\{\|Ax\| : x \in X \text{ with } \|x\| = 1\big\} = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Ax\|}{\|x\|}.$$

This implies, assuming $\|A\|$ is not infinity, that for every $x \in X$,

$$\|Ax\| \leq \|A\|\,\|x\|.$$

We will use this inequality a lot. It is not hard to see from the definition that $\|A\| = 0$ if and only if $A = 0$, where by $A = 0$ we mean that A takes every vector to the zero vector. It is also not difficult to compute the operator norm of the identity operator:

$$\|I\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Ix\|}{\|x\|} = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|x\|}{\|x\|} = 1.$$

The operator norm is not always a norm on $L(X,Y)$, in particular, $\|A\|$ is not always finite for $A \in L(X,Y)$. We prove below that $\|A\|$ is finite when X is finite-dimensional. The operator norm being finite is equivalent to A being continuous. For infinite-dimensional spaces, neither statement needs to be true. For an example, consider the vector space of continuously differentiable functions on $[0,2\pi]$ using the uniform norm. The functions $t \mapsto \sin(nt)$ have norm 1, but their derivatives have norm n. So differentiation, which is a linear operator valued in the space of continuous functions, has infinite operator norm on this space. We will stick to finite-dimensional spaces.

When we talk about a finite-dimensional vector space X, we often think of $\mathbb{R}^n$, although if we have a norm on X, the norm might not be the standard euclidean norm. In the exercises, you can prove that every norm is "equivalent" to the euclidean norm in that the topology it generates is the same. For simplicity, we only prove the following proposition for the euclidean space, and the proof for a general finite-dimensional space is left as an exercise.

Proposition 8.2.4. *Let X and Y be normed vector spaces and $A \in L(X,Y)$. Suppose that X is finite-dimensional. Then $\|A\| < \infty$, and A is uniformly continuous (Lipschitz with constant $\|A\|$).*

Proof. As we said we only prove the proposition for euclidean space, so suppose that $X = \mathbb{R}^n$ and the norm is the standard euclidean norm. The general case is left as an exercise.

Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Write $x \in \mathbb{R}^n$, with $\|x\| = 1$, as

$$x = \sum_{j=1}^{n} c_j e_j.$$

Since $e_j \cdot e_\ell = 0$ whenever $j \neq \ell$ and $e_j \cdot e_j = 1$, then $c_j = x \cdot e_j$ and by Cauchy–Schwarz

$$|c_j| = |x \cdot e_j| \leq \|x\| \, \|e_j\| = 1.$$

Then

$$\|Ax\| = \left\| \sum_{j=1}^{n} c_j A e_j \right\| \leq \sum_{j=1}^{n} |c_j| \, \|Ae_j\| \leq \sum_{j=1}^{n} \|Ae_j\|.$$

The right-hand side does not depend on x. We found a finite upper bound for $\|Ax\|$ independent of x, so $\|A\| < \infty$.

For any normed vector spaces X and Y, and $A \in L(X,Y)$, suppose that $\|A\| < \infty$. For $v, w \in X$,

$$\|Av - Aw\| = \|A(v-w)\| \leq \|A\| \, \|v-w\|.$$

As $\|A\| < \infty$, then this says A is Lipschitz with constant $\|A\|$. □

Proposition 8.2.5. *Let X, Y, and Z be finite-dimensional normed vector spaces*[*]*.*

(i) If $A, B \in L(X,Y)$ and $c \in \mathbb{R}$, then

$$\|A+B\| \leq \|A\| + \|B\|, \qquad \|cA\| = |c| \, \|A\|.$$

In particular, the operator norm is a norm on the vector space $L(X,Y)$.

(ii) If $A \in L(X,Y)$ and $B \in L(Y,Z)$, then

$$\|BA\| \leq \|B\| \, \|A\|.$$

[*]If we strike the "In particular" part and interpret the algebra with infinite operator norms properly, namely decree that 0 times ∞ is 0, then this result also holds for infinite-dimensional spaces.

Proof. First, since all the spaces are finite-dimensional, then all the operator norms are finite, and the statements make sense to begin with.

For (i),

$$\|(A+B)x\| = \|Ax+Bx\| \leq \|Ax\| + \|Bx\| \leq \|A\| \, \|x\| + \|B\| \, \|x\| = \big(\|A\| + \|B\|\big) \|x\|.$$

So $\|A+B\| \leq \|A\| + \|B\|$.

Similarly,

$$\|(cA)x\| = |c| \, \|Ax\| \leq \big(|c| \, \|A\|\big) \|x\|.$$

Thus $\|cA\| \leq |c| \, \|A\|$. Next,

$$|c| \, \|Ax\| = \|cAx\| \leq \|cA\| \, \|x\|.$$

Hence $|c| \, \|A\| \leq \|cA\|$.

For (ii) write

$$\|BAx\| \leq \|B\| \, \|Ax\| \leq \|B\| \, \|A\| \, \|x\|.$$ □

As a norm defines a metric, there is a metric space topology on $L(X,Y)$ for finite-dimensional vector spaces, so we can talk about open/closed sets, continuity, and convergence.

Proposition 8.2.6. *Let X be a finite-dimensional normed vector space. Let $GL(X) \subset L(X)$ be the set of invertible linear operators.[*]*

(i) If $A \in GL(X)$, $B \in L(X)$, and

$$\|A-B\| < \frac{1}{\|A^{-1}\|}, \tag{8.2}$$

then B is invertible.

(ii) $GL(X)$ is an open subset, and $A \mapsto A^{-1}$ is a continuous function on $GL(X)$.

Let us make sense of this proposition on a simple example. Consider $X = \mathbb{R}^1$, where linear operators are just numbers a and the operator norm of a is $|a|$. The operator a is invertible ($a^{-1} = 1/a$) whenever $a \neq 0$. The condition $|a-b| < \frac{1}{|a^{-1}|}$ does indeed imply that b is not zero. And $a \mapsto 1/a$ is a continuous map. When the dimension is bigger than 1, there are other noninvertible operators than just zero, and in general things are a bit more difficult.

Proof. Let us prove (i). We know something about A^{-1} and $A-B$; they are linear operators. So apply them to a vector:

$$A^{-1}(A-B)x = x - A^{-1}Bx.$$

Therefore,

$$\begin{aligned} \|x\| &= \|A^{-1}(A-B)x + A^{-1}Bx\| \\ &\leq \|A^{-1}\| \, \|A-B\| \, \|x\| + \|A^{-1}\| \, \|Bx\|. \end{aligned}$$

Assume $x \neq 0$ and so $\|x\| \neq 0$. Using (8.2), we obtain

$$\|x\| < \|x\| + \|A^{-1}\| \, \|Bx\|.$$

[*] $GL(X)$ is called the *general linear group*, that is where the acronym GL comes from.

In other words, $\|Bx\| \neq 0$ for all nonzero x, and hence $Bx \neq 0$ for all nonzero x. This is enough to see that B is one-to-one (if $Bx = By$, then $B(x-y) = 0$, so $x = y$). As B is one-to-one operator from X to X, which is finite-dimensional and hence also onto by Proposition 8.1.18, B is invertible.

Let us prove (ii). Item (i) immediately implies that $GL(X)$ is open. Let us show that the inverse is continuous. Fix some $A \in GL(X)$. Let B be near A, specifically $\|A-B\| < \frac{1}{2\|A^{-1}\|}$. Then (8.2) is satisfied and B is invertible. A similar computation as above (using $B^{-1}y$ instead of x) gives

$$\|B^{-1}y\| \leq \|A^{-1}\| \, \|A-B\| \, \|B^{-1}y\| + \|A^{-1}\| \, \|y\| \leq \frac{1}{2}\|B^{-1}y\| + \|A^{-1}\| \, \|y\|,$$

or

$$\|B^{-1}y\| \leq 2\|A^{-1}\| \, \|y\|.$$

So $\|B^{-1}\| \leq 2\|A^{-1}\|$.

Now

$$A^{-1}(A-B)B^{-1} = A^{-1}(AB^{-1} - I) = B^{-1} - A^{-1},$$

and

$$\|B^{-1} - A^{-1}\| = \|A^{-1}(A-B)B^{-1}\| \leq \|A^{-1}\| \, \|A-B\| \, \|B^{-1}\| \leq 2\|A^{-1}\|^2 \|A-B\|.$$

Therefore, as B tends to A, $\|B^{-1} - A^{-1}\|$ tends to 0, and so the inverse operation is a continuous function at A. $\square$

8.2.2 Matrices

Once we fix a basis in a finite-dimensional vector space X, we can represent a vector of X as an n-tuple of numbers—a vector in $\mathbb{R}^n$. Same can be done with $L(X,Y)$, bringing us to matrices, which are a convenient way to represent finite-dimensional linear transformations. Suppose $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ are bases for vector spaces X and Y respectively. A linear operator is determined by its values on the basis. Given $A \in L(X,Y)$, Ax_j is an element of Y. Define the numbers $a_{i,j}$ as follows

$$Ax_j = \sum_{i=1}^{m} a_{i,j} \, y_i, \tag{8.3}$$

and write them as a *matrix*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

We sometimes write A as $[a_{i,j}]$. We say A is an m-by-n matrix. The jth *column* of the matrix gives precisely the coefficients that represent Ax_j in terms of the basis $\{y_1, y_2, \ldots, y_m\}$. If we know the numbers $a_{i,j}$, then via the formula (8.3) we find the corresponding linear operator, as it is determined by the action on a basis. Hence, once we fix a basis on X and on Y, we have a one-to-one correspondence between $L(X,Y)$ and the m-by-n matrices.

When

$$z = \sum_{j=1}^{n} c_j \, x_j,$$

then

$$Az = \sum_{j=1}^{n} c_j A x_j = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} a_{i,j} y_i \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} c_j \right) y_i,$$

which gives rise to the familiar rule for matrix multiplication.

More generally, if B is an n-by-r matrix with entries $b_{j,k}$, then the matrix for $C = AB$ is an m-by-r matrix whose (i,k)th entry $c_{i,k}$ is

$$c_{i,k} = \sum_{j=1}^{n} a_{i,j} b_{j,k}.$$

A way to remember it is if you order the indices as we do, that is *row, column*, and put the elements in the same order as the matrices, then it is the "middle index" that is "summed-out."

There is a one-to-one correspondence between matrices and linear operators in $L(X,Y)$, once we fix a basis in X and in Y. If we choose a different basis, we get different matrices. This is an important distinction, the operator A acts on elements of X, the matrix is something that works with n-tuples of numbers, that is, vectors of $\mathbb{R}^n$. By convention, we use standard bases in $\mathbb{R}^n$ unless otherwise specified, and we identify $L(\mathbb{R}^n, \mathbb{R}^m)$ with the set of m-by-n matrices.

A linear mapping changing one basis to another is represented by a square matrix in which the columns represent vectors of the second basis in terms of the first basis. We call such a linear mapping a *change of basis*. So for two choices of a basis in an n-dimensional vector space, there is a linear mapping (a change of basis) taking one basis to the other, and this corresponds to an n-by-n matrix which does the corresponding operation on $\mathbb{R}^n$.

Suppose $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and all the bases are just the standard bases. Using the Cauchy–Schwarz inequality compute

$$\|Az\|^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} c_j \right)^2 \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (c_j)^2 \right) \left(\sum_{j=1}^{n} (a_{i,j})^2 \right) = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{i,j})^2 \right) \|z\|^2.$$

In other words, we have a bound on the operator norm (note that equality rarely happens)

$$\|A\| \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{i,j})^2}.$$

The right hand side is the euclidean norm on $\mathbb{R}^{nm}$, the space of all the entries of the matrix. If the entries go to zero, then $\|A\|$ goes to zero. Conversely,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{i,j})^2 = \sum_{j=1}^{n} \|Ae_j\|^2 \leq \sum_{j=1}^{n} \|A\|^2 = n\|A\|^2.$$

So if the operator norm of A goes to zero, so do the entries. In particular, if A is fixed and B is changing, then the entries of B go to the entries of A if and only if B goes to A in operator norm ($\|A - B\|$ goes to zero). We have proved:

Proposition 8.2.7. *The topology (the set of open sets) on $L(\mathbb{R}^n, \mathbb{R}^m)$ is the same whether we consider $L(\mathbb{R}^n, \mathbb{R}^m)$ as a metric space using the operator norm, or with the euclidean metric of $\mathbb{R}^{nm}$.*

In particular, let S be a metric space and let $\pi \colon L(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{nm}$ identify an operator with the nm-tuple of entries of the corresponding matrix. Then $f \colon S \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous if and only if $\pi \circ f \colon S \to \mathbb{R}^{nm}$ is continuous. Similarly for $g \colon L(\mathbb{R}^n, \mathbb{R}^m) \to S$ and $g \circ \pi^{-1} \colon \mathbb{R}^{nm} \to S$.

8.2.3 Determinants

A certain number can be assigned to square matrices that measures how the corresponding linear mapping stretches space. In particular, this number, called the determinant, can be used to test for invertibility of a matrix.

Define the symbol $\operatorname{sgn}(x)$ (read "sign of x") for a number x by

$$\operatorname{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Suppose $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a *permutation* of the integers $(1, 2, \ldots, n)$, that is, a reordering of $(1, 2, \ldots, n)$. Define

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma_1, \ldots, \sigma_n) := \prod_{p<q} \operatorname{sgn}(\sigma_q - \sigma_p). \tag{8.4}$$

Here $\prod$ stands for multiplication, similarly to how $\sum$ stands for summation.

Any permutation can be obtained by a sequence of transpositions (switchings of two elements). A permutation is *even* (resp. *odd*) if it takes an even (resp. odd) number of transpositions to get from $(1, 2, \ldots, n)$ to σ. For instance, $(2, 4, 3, 1)$ is two transpositions away from $(1, 2, 3, 4)$ and is therefore even: $(1, 2, 3, 4) \to (2, 1, 3, 4) \to (2, 4, 3, 1)$. Being even or odd is well-defined: $\operatorname{sgn}(\sigma)$ is 1 if σ is even and -1 if σ is odd (exercise). This fact can be proved by noting that applying a transposition changes the sign, and computing that $\operatorname{sgn}(1, 2, \ldots, n) = 1$.

Let S_n be the set of all permutations on n elements (the *symmetric group*). Let $A = [a_{i,j}]$ be a square n-by-n matrix. Define the *determinant* of A as

$$\det(A) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}.$$

Proposition 8.2.8.

(i) $\det(I) = 1$.

(ii) For every $j = 1, 2, \ldots, n$, the function $x_j \mapsto \det\bigl([x_1 \quad x_2 \quad \cdots \quad x_n]\bigr)$ is linear.

(iii) If two columns of a matrix are interchanged, then the determinant changes sign.

(iv) If two columns of A are equal, then $\det(A) = 0$.

(v) If a column is zero, then $\det(A) = 0$.

(vi) $A \mapsto \det(A)$ is a continuous function on $L(\mathbb{R}^n)$.

(vii) $\det\left(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\right) = ad - bc$, *and* $\det\bigl([a]\bigr) = a$.

In fact, the determinant is the unique function that satisfies (i), (ii), and (iii). But we digress. By (ii), we mean that if we fix all the vectors $x_1, \ldots, x_n$ except for x_j, and let $v, w \in \mathbb{R}^n$ be two vectors, and $a, b \in \mathbb{R}$ be scalars, then

$$\begin{aligned} &\det\bigl([x_1 \quad \cdots \quad x_{j-1} \quad (av+bw) \quad x_{j+1} \quad \cdots \quad x_n]\bigr) = \\ &\qquad a \det\bigl([x_1 \quad \cdots \quad x_{j-1} \quad v \quad x_{j+1} \quad \cdots \quad x_n]\bigr) + b \det\bigl([x_1 \quad \cdots \quad x_{j-1} \quad w \quad x_{j+1} \quad \cdots \quad x_n]\bigr). \end{aligned}$$

Proof. We go through the proof quickly, as you have likely seen it before. Item (i) is trivial. For (ii), notice that each term in the definition of the determinant contains exactly one factor from each column. Item (iii) follows by noting that switching two columns is like switching the two corresponding numbers in every element in S_n. Hence, all the signs are changed. Item (iv) follows because if two columns are equal, and we switch them, we get the same matrix back. So item (iii) says the determinant must be 0. Item (v) follows because the product in each term in the definition includes one element from the zero column. Item (vi) follows as det is a polynomial in the entries of the matrix and hence continuous (as a function of the entries of the matrix). Two matrices are A function defined on matrices is continuous in the operator norm if and only if it is continuous as a function of the entries (Proposition 8.2.7). Finally, item (vii) is a direct computation. $\square$

The determinant tells us about areas and volumes, and how they change. For example, in the 1-by-1 case, a matrix is just a number, and the determinant is exactly this number. It says how the linear mapping "stretches" the space. Similarly for $\mathbb{R}^2$. Suppose $A \in L(\mathbb{R}^2)$ is a linear transformation. It can be checked directly that the area of the image of the unit square $A\big([0,1]^2\big)$ is precisely $|\det(A)|$. This works with arbitrary figures, not just the unit square: The absolute value of the determinant tells us the stretch in the area. The sign of the determinant tells us if the image is flipped (changes orientation) or not. In $\mathbb{R}^3$ it tells us about the 3-dimensional volume, and in n dimensions about the n-dimensional volume. We claim this without proof.

Proposition 8.2.9. *If A and B are n-by-n matrices, then* $\det(AB) = \det(A)\det(B)$. *Furthermore, A is invertible if and only if* $\det(A) \neq 0$ *and in this case,* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof. Let $b_1, b_2, \dots, b_n$ be the columns of B. Then

$$AB = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_n].$$

That is, the columns of AB are $Ab_1, Ab_2, \dots, Ab_n$.

Let $b_{j,k}$ denote the elements of B and a_j the columns of A. By linearity of the determinant,

$$\begin{aligned}
\det(AB) = \det\big([Ab_1 \quad Ab_2 \quad \cdots \quad Ab_n]\big) &= \det\left(\left[\sum_{j=1}^n b_{j,1} a_j \quad Ab_2 \quad \cdots \quad Ab_n\right]\right) \\
&= \sum_{j=1}^n b_{j,1} \det\big([a_j \quad Ab_2 \quad \cdots \quad Ab_n]\big) \\
&= \sum_{1 \leq j_1, j_2, \dots, j_n \leq n} b_{j_1,1} b_{j_2,2} \cdots b_{j_n,n} \det\big([a_{j_1} \quad a_{j_2} \quad \cdots \quad a_{j_n}]\big) \\
&= \left(\sum_{(j_1,j_2,\dots,j_n) \in S_n} b_{j_1,1} b_{j_2,2} \cdots b_{j_n,n} \operatorname{sgn}(j_1, j_2, \dots, j_n)\right) \det\big([a_1 \quad a_2 \quad \cdots \quad a_n]\big).
\end{aligned}$$

In the last equality, we can sum over just the elements of S_n instead of all n-tuples for integers between 1 and n by noting that when two columns in the determinant are the same, then the determinant is zero. Then we reordered the columns to the original ordering to obtain the sgn.

The conclusion that $\det(AB) = \det(A)\det(B)$ follows by recognizing that the expression in parentheses above is the determinant of B. We obtain this by plugging in $A = I$. The expression we get for the determinant of B has rows and columns swapped, so as a side note, we have also just proved that the determinant of a matrix and its transpose are equal.

Let us prove the "Furthermore" part. If A is invertible, then $A^{-1}A = I$ and consequently $\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$. If A is not invertible, then there must be a nonzero vector that A takes to zero as A is not one-to-one. In other words, the columns of A are linearly dependent. Suppose

$$\sum_{j=1}^{n} \gamma_j a_j = 0,$$

where not all γ_j are equal to 0. Without loss of generality suppose $\gamma_1 \neq 0$. Take

$$B := \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & 0 \\ \gamma_2 & 1 & 0 & \cdots & 0 \\ \gamma_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_n & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Using the definition of the determinant (there is only a single permutation σ for which $\prod_{i=1}^{n} b_{i,\sigma_i}$ is nonzero) we find $\det(B) = \gamma_1 \neq 0$. Then $\det(AB) = \det(A)\det(B) = \gamma_1 \det(A)$. The first column of AB is zero, and hence $\det(AB) = 0$. We conclude $\det(A) = 0$. $\square$

Proposition 8.2.10. *Determinant is independent of the basis. In other words, if B is invertible, then*

$$\det(A) = \det(B^{-1}AB).$$

The proof is to compute $\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \frac{1}{\det(B)}\det(A)\det(B) = \det(A)$.

If in one basis A is the matrix representing a linear operator, then for another basis we can find a matrix B such that the matrix $B^{-1}AB$ takes us to the first basis, applies A in the first basis, and takes us back to the basis we started with. Let X be a finite-dimensional vector space. Let $\Phi \in L(X, \mathbb{R}^n)$ take a basis $\{x_1, \ldots, x_n\}$ to the standard basis $\{e_1, \ldots, e_n\}$ and let $\Psi \in L(X, \mathbb{R}^n)$ take another basis $\{y_1, \ldots, y_n\}$ to the standard basis. Let $T \in L(X)$ be a linear operator and let a matrix A represent the operator in the basis $\{x_1, \ldots, x_n\}$. Then B would be such that we have the following diagram[*]:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "B^{-1}AB"] \arrow[dd, bend right=60, "B"'] & \mathbb{R}^n \arrow[d, "\Psi^{-1}"'] \\
X \arrow[u, "\Psi"'] \arrow[r, "T"] \arrow[d, "\Phi"] & X \\
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \arrow[u, "\Phi^{-1}"] \arrow[uu, bend right=60, "B^{-1}"']
\end{tikzcd}$$

The two $\mathbb{R}^n$s on the bottom row represent X in the first basis, and the $\mathbb{R}^n$s on top represent X in the second basis.

[*]This is a so-called *commutative diagram*. Following arrows in any way should end up with the same result.

If we compute the determinant of the matrix A, we obtain the same determinant if we use any other basis; in the other basis the matrix would be $B^{-1}AB$. Consequently,

$$\det\colon L(X) \to \mathbb{R}$$

is a well-defined function without the need to fix a basis. That is, det is defined on $L(X)$, not just on matrices.

There are three types of so-called *elementary matrices*. Let $e_1, e_2, \ldots, e_n$ be the standard basis on $\mathbb{R}^n$ as usual.

First, for $j = 1, 2, \ldots, n$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$, define the first type of an elementary matrix, an n-by-n matrix E by

$$Ee_i := \begin{cases} e_i & \text{if } i \neq j, \\ \lambda e_i & \text{if } i = j. \end{cases}$$

Given any n-by-m matrix M the matrix EM is the same matrix as M except with the jth row multiplied by λ. It is an easy computation (exercise) that $\det(E) = \lambda$.

Next, for j and k with $j \neq k$, and $\lambda \in \mathbb{R}$, define the second type of an elementary matrix E by

$$Ee_i := \begin{cases} e_i & \text{if } i \neq j, \\ e_i + \lambda e_k & \text{if } i = j. \end{cases}$$

Given any n-by-m matrix M the matrix EM is the same matrix as M except with λ times the kth row added to the jth row. It is an easy computation (exercise) that $\det(E) = 1$.

Finally, for j and k with $j \neq k$, define the third type of an elementary matrix E by

$$Ee_i := \begin{cases} e_i & \text{if } i \neq j \text{ and } i \neq k, \\ e_k & \text{if } i = j, \\ e_j & \text{if } i = k. \end{cases}$$

Given any n-by-m matrix M the matrix EM is the same matrix with jth and kth rows swapped. It is an easy computation (exercise) that $\det(E) = -1$.

Proposition 8.2.11. *Let T be an n-by-n invertible matrix. Then there exists a finite sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that*

$$T = E_1 E_2 \cdots E_k,$$

and

$$\det(T) = \det(E_1)\det(E_2)\cdots\det(E_k).$$

The proof is left as an exercise. The proposition says that we can compute the determinant by doing elementary row operations. For computing the determinant, one does not have to factor the matrix into a product of elementary matrices completely. One only does row operations until one finds an *upper triangular matrix*, that is, a matrix $[a_{i,j}]$ where $a_{i,j} = 0$ if $i > j$. Computing determinant of such a matrix is not difficult (exercise).

Factorization into elementary matrices (or variations on elementary matrices) is useful in proofs involving an arbitrary linear operator, by reducing to a proof for an elementary matrix, similarly as the computation of the determinant.

8.2.4 Exercises

Exercise* 8.2.1*: *For a vector space X with a norm $\lVert\cdot\rVert$, show that $d(x,y) := \lVert x-y \rVert$ makes X a metric space.*

Exercise* 8.2.2** (Easy):*** *Show that for square matrices A and B,* $\det(AB) = \det(BA)$.

Exercise* 8.2.3*: *For $x \in \mathbb{R}^n$, define*

$$\lVert x \rVert_\infty := \max\bigl\{ \lvert x_1 \rvert, \lvert x_2 \rvert, \ldots, \lvert x_n \rvert \bigr\},$$

sometimes called the sup or the max norm.

a) *Show that $\lVert\cdot\rVert_\infty$ is a norm on $\mathbb{R}^n$ (defining a different distance).*
b) *What is the unit ball $B(0,1)$ in this norm?*

Exercise* 8.2.4*: *For $x \in \mathbb{R}^n$, define*

$$\lVert x \rVert_1 := \sum_{j=1}^{n} \lvert x_j \rvert,$$

sometimes called the 1*-norm (or L^1 norm).*

a) *Show that $\lVert\cdot\rVert_1$ is a norm on $\mathbb{R}^n$ (defining a different distance, sometimes called the taxicab distance).*
b) *What is the unit ball $B(0,1)$ in this norm? Think about what it is in $\mathbb{R}^2$ and $\mathbb{R}^3$. Hint: It is, for example, a convex hull of a finite number of points.*

Exercise* 8.2.5*: *Using the euclidean norm on $\mathbb{R}^2$, compute the operator norm of the operators in $L(\mathbb{R}^2)$ given by the matrices:*

a) $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ *b)* $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ *c)* $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ *d)* $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$

Exercise* 8.2.6*: *Using the standard euclidean norm $\mathbb{R}^n$, show:*

a) *Suppose $A \in L(\mathbb{R},\mathbb{R}^n)$ is defined for $x \in \mathbb{R}$ by $Ax := xa$ for a vector $a \in \mathbb{R}^n$. Then the operator norm $\lVert A \rVert_{L(\mathbb{R},\mathbb{R}^n)} = \lVert a \rVert_{\mathbb{R}^n}$. (That is, the operator norm of A is the euclidean norm of a.)*
b) *Suppose $B \in L(\mathbb{R}^n,\mathbb{R})$ is defined for $x \in \mathbb{R}^n$ by $Bx := b \cdot x$ for a vector $b \in \mathbb{R}^n$. Then the operator norm $\lVert B \rVert_{L(\mathbb{R}^n,\mathbb{R})} = \lVert b \rVert_{\mathbb{R}^n}$.*

Exercise* 8.2.7*: *Suppose $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a permutation of $(1,2,\ldots,n)$.*

a) *Show that we can make a finite number of transpositions (switching of two elements) to get to $(1,2,\ldots,n)$.*
b) *Using the definition* (8.4) *show that σ is even if* $\operatorname{sgn}(\sigma) = 1$ *and σ is odd if* $\operatorname{sgn}(\sigma) = -1$. *In particular, showing that being odd or even is well-defined.*

Exercise* 8.2.8*: *Verify the computation of the determinant for the three types of elementary matrices.*

Exercise* 8.2.9*: *Prove Proposition 8.2.11.*

Exercise* 8.2.10*:

a) *Suppose $D = [d_{i,j}]$ is an n-by-n* diagonal matrix*, that is, $d_{i,j} = 0$ whenever $i \neq j$. Show that* $\det(D) = d_{1,1} d_{2,2} \cdots d_{n,n}$.
b) *Suppose A is a diagonalizable matrix. That is, there exists a matrix B such that $B^{-1}AB = D$ for a diagonal matrix $D = [d_{i,j}]$. Show that* $\det(A) = d_{1,1} d_{2,2} \cdots d_{n,n}$.

Exercise* 8.2.11*: *Take the vector space of polynomials $\mathbb{R}[t]$ and the linear operator $D \in L(\mathbb{R}[t])$ that is the differentiation (we proved in an earlier exercise that D is a linear operator). Given $P(t) = c_0 + c_1 t + \cdots + c_n t^n \in \mathbb{R}[t]$ define $\|P\| := \sup\{|c_j| : j = 0,1,2,\ldots,n\}$.*

a) *Show that $\|P\|$ is a norm on $\mathbb{R}[t]$.*

b) *Show that D does not have bounded operator norm, that is $\|D\| = \infty$. Hint: Consider the polynomials t^n as n tends to infinity.*

Exercise* 8.2.12*: *We finish the proof of Proposition 8.2.4. Let X be a finite-dimensional normed vector space with basis $\{x_1, x_2, \ldots, x_n\}$.*

a) *Show that $f : \mathbb{R}^n \to \mathbb{R}$,*
$$f(c_1, c_2, \ldots, c_n) := \|c_1 x_1 + c_2 x_2 + \cdots + c_n x_n\|,$$
is continuous (the norm is the one on X).

b) *Show that there exist numbers m and M such that if $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ with $\|c\| = 1$ (standard euclidean norm), then $m \leq \|c_1 x_1 + c_2 x_2 + \cdots + c_n x_n\| \leq M$ (here the norm is on X).*

c) *Show that there exists a number B such that if $\|c_1 x_1 + c_2 x_2 + \cdots + c_n x_n\| = 1$, then $|c_j| \leq B$.*

d) *Use part c) to show that if X is finite-dimensional vector spaces and $A \in L(X,Y)$, then $\|A\| < \infty$.*

Exercise* 8.2.13*: *Let X be a finite-dimensional vector space with norm $\|\cdot\|$ and basis $\{x_1, x_2, \ldots, x_n\}$. Let $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ and $\|c\|$ be the standard euclidean norm on $\mathbb{R}^n$.*

a) *Prove that there exist positive numbers $m, M > 0$ such that for all $c \in \mathbb{R}^n$,*
$$m\|c\| \leq \|c_1 x_1 + c_2 x_2 + \cdots + c_n x_n\| \leq M\|c\|.$$
Hint: See previous exercise.

b) *Use part a) to show that of $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X, then there exist positive numbers $m, M > 0$ (perhaps different from above) such that for all $x \in X$, we have*
$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1.$$

c) *Show that $U \subset X$ is open in the metric defined by $\|x - y\|_1$ if and only if U is open in the metric defined by $\|x - y\|_2$. In particular, convergence of sequences and continuity of functions is the same in either norm.*

Exercise* 8.2.14*: *Let A be an upper triangular matrix. Find a formula for the determinant of A in terms of the diagonal entries, and prove that your formula works.*

Exercise* 8.2.15*: *Given an n-by-n matrix A, prove that $|\det(A)| \leq \|A\|^n$ (the norm on A is the operator norm). Hint: One way to do it is to first prove it in the case $\|A\| = 1$, which means that all columns are of norm 1 or less, then prove that this means that $|\det(A)| \leq 1$ using linearity.*

Exercise* 8.2.16*: *Consider Proposition 8.2.6 where $X = \mathbb{R}^n$ (for all n) using the euclidean norm.*

a) *Prove that the estimate $\|A - B\| < \frac{1}{\|A^{-1}\|}$ is the best possible: For every $A \in GL(\mathbb{R}^n)$, find a B where equality is satisfied and B is not invertible. Hint: Difficulty is that $\|A\|\|A^{-1}\|$ is not always 1. Prove that a vector x_1 can be completed to a basis $\{x_1, \ldots, x_n\}$ such that $x_1 \cdot x_j = 0$ for $j \geq 2$. For the right x_1, make it so that $(A - B)x_j = 0$ for $j \geq 2$.*

b) *For every fixed $A \in GL(\mathbb{R}^n)$, let $\mathscr{M}$ denote the set of matrices B such that $\|A - B\| < \frac{1}{\|A^{-1}\|}$. Prove that while every $B \in \mathscr{M}$ is invertible, $\|B^{-1}\|$ is unbounded as a function of B on $\mathscr{M}$.*

Let A be an n-by-n matrix. A number $\lambda \in \mathbb{C}$ (possibly complex even for a real matrix) is called an *eigenvalue* of A if there is a nonzero (possibly complex) vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$ (the multiplication by complex vectors is the same as for real vectors. In particular, if $x = a + ib$ for real vectors a and b, and A is a real matrix, then $Ax = Aa + iAb$). The number

$$\rho(A) := \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

is called the *spectral radius* of A. Here $|\lambda|$ is the complex modulus. We state without proof that at least one eigenvalue always exists, and there are no more than n distinct eigenvalues of A. You can therefore assume that $0 \leq \rho(A) < \infty$. The exercises below hold for complex matrices, but feel free to assume they are real matrices.

Exercise* 8.2.17*: *Let A, S be n-by-n matrices, where S is invertible. Prove that λ is an eigenvalue of A, if and only if it is an eigenvalue of $S^{-1}AS$. Then prove that $\rho(S^{-1}AS) = \rho(S)$. In particular, ρ is a well-defined function on $L(X)$ for every finite-dimensional vector space X.*

Exercise* 8.2.18*: *Let A be an n-by-n matrix A.*

a) Prove $\rho(A) \leq \|A\|$.

b) For every $k \in \mathbb{N}$, prove $\rho(A) \leq \|A^k\|^{1/k}$.

c) Suppose $\lim\limits_{k\to\infty} A^k = 0$ (limit in the operator norm). Prove that $\rho(A) < 1$.

Exercise* 8.2.19*: *We say a set $C \subset \mathbb{R}^n$ is* symmetric *if $x \in C$ implies $-x \in C$.*

a) Let $\|\cdot\|$ be any given norm on $\mathbb{R}^n$. Show that the closed unit ball $C(0,1)$ (using the metric induced by this norm) is a compact symmetric convex set.

b) (Challenging) Let $C \subset \mathbb{R}^n$ be a compact symmetric convex set and $0 \in C$. Show that

$$\|x\| := \inf\left\{\lambda : \lambda > 0 \text{ and } \frac{x}{\lambda} \in C\right\}$$

is a norm on $\mathbb{R}^n$, and $C = C(0,1)$ (the closed unit ball) in the metric induced by this norm.

Hint: Feel free to the result of Exercise 8.2.13 part c).

8.3 The derivative

Note: 2–3 lectures

8.3.1 The derivative

For a function $f\colon \mathbb{R} \to \mathbb{R}$, we defined the derivative at x as

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

In other words, there is a number a (the derivative of f at x) such that

$$\lim_{h\to 0} \left\lvert \frac{f(x+h)-f(x)}{h} - a \right\rvert = \lim_{h\to 0} \left\lvert \frac{f(x+h)-f(x)-ah}{h} \right\rvert = \lim_{h \to 0} \frac{\lvert f(x+h)-f(x)-ah\rvert}{\lvert h \rvert} = 0.$$

Multiplying by a is a linear map in one dimension: $h \mapsto ah$. Namely, we think of $a \in L(\mathbb{R}^1,\mathbb{R}^1)$, which is the best linear approximation of how f changes near x. We use this interpretation to extend differentiation to more variables.

Definition 8.3.1. Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^m$ a function. We say f is *differentiable* at $x \in U$ if there exists an $A \in L(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$\lim_{\substack{h\to 0\\ h \in \mathbb{R}^n}} \frac{\lVert f(x+h)-f(x)-Ah \rVert}{\lVert h \rVert} = 0.$$

We write $Df(x) := A$, or $f'(x) := A$, and we say A is the *derivative* of f at x. When f is differentiable at every $x \in U$, we say simply that f is *differentiable*. See Figure 8.3 for an illustration.

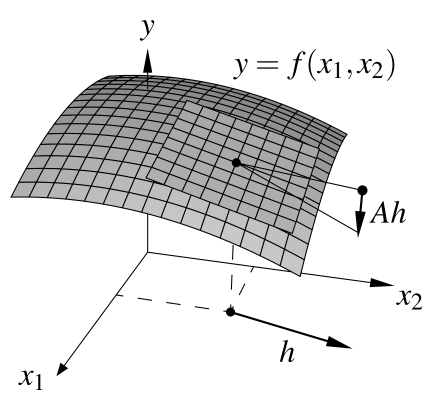

Figure 8.3: Illustration of a derivative for a function $f\colon \mathbb{R}^2 \to \mathbb{R}$. The vector h is shown in the x_1x_2-plane based at (x_1,x_2), and the vector $Ah \in \mathbb{R}^1$ is shown along the y direction.

For a differentiable function, the derivative of f is a function from U to $L(\mathbb{R}^n,\mathbb{R}^m)$. Compare to the one-dimensional case, where the derivative is a function from U to $\mathbb{R}$, but we really want to think of $\mathbb{R}$ here as $L(\mathbb{R}^1,\mathbb{R}^1)$. As in one dimension, the idea is that a differentiable mapping is "infinitesimally close" to a linear mapping, and this linear mapping is the derivative.

Notice which norms are being used in the definition. The norm in the numerator is on $\mathbb{R}^m$, and the norm in the denominator is on $\mathbb{R}^n$ where h lives. Normally it is understood that $h \in \mathbb{R}^n$ from context (the formula makes no sense otherwise). We will not explicitly say so from now on.

We have again cheated somewhat and said that A is *the* derivative. We have not shown yet that there is only one, let us do that now.

Proposition 8.3.2. *Let $U \subset \mathbb{R}^n$ be an open subset and $f\colon U \to \mathbb{R}^m$ a function. Suppose $x \in U$ and there exist $A,B \in L(\mathbb{R}^n,\mathbb{R}^m)$ such that*

$$\lim_{h\to 0} \frac{\lVert f(x+h)-f(x)-Ah\rVert}{\lVert h\rVert} = 0 \qquad \textit{and} \qquad \lim_{h\to 0} \frac{\lVert f(x+h)-f(x)-Bh\rVert}{\lVert h\rVert} = 0.$$

Then $A = B$.

Proof. Suppose $h \in \mathbb{R}^n$, $h \neq 0$. Compute

$$\begin{aligned} \frac{\lVert (A-B)h\rVert}{\lVert h\rVert} &= \frac{\left\lVert -\bigl(f(x+h)-f(x)-Ah\bigr) + f(x+h)-f(x)-Bh\right\rVert}{\lVert h\rVert} \\ &\leq \frac{\lVert f(x+h)-f(x)-Ah\rVert}{\lVert h\rVert} + \frac{\lVert f(x+h)-f(x)-Bh\rVert}{\lVert h\rVert}. \end{aligned}$$

So $\frac{\lVert (A-B)h\rVert}{\lVert h\rVert} \to 0$ as $h \to 0$. Given $\varepsilon > 0$, for all nonzero h in some δ-ball around the origin we have

$$\varepsilon > \frac{\lVert (A-B)h\rVert}{\lVert h\rVert} = \left\lVert (A-B)\frac{h}{\lVert h\rVert}\right\rVert.$$

For any given $v \in \mathbb{R}^n$ with $\lVert v\rVert = 1$, let $h = (\delta/2)\,v$, then $\lVert h\rVert < \delta$ and $\frac{h}{\lVert h\rVert} = v$. So $\lVert (A-B)v\rVert < \varepsilon$. Taking the supremum over all v with $\lVert v\rVert = 1$, we get the operator norm $\lVert A-B\rVert \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary, $\lVert A-B\rVert = 0$, or in other words $A = B$. ☐

Example 8.3.3: If $f(x) = Ax$ for a linear mapping A, then $f'(x) = A$:

$$\frac{\lVert f(x+h)-f(x)-Ah\rVert}{\lVert h\rVert} = \frac{\lVert A(x+h)-Ax-Ah\rVert}{\lVert h\rVert} = \frac{0}{\lVert h\rVert} = 0.$$

Example 8.3.4: Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y) = \bigl(f_1(x,y), f_2(x,y)\bigr) := (1+x+2y+x^2, 2x+3y+xy).$$

Let us show that f is differentiable at the origin and let us compute the derivative, directly using the definition. If the derivative exists, it is in $L(\mathbb{R}^2,\mathbb{R}^2)$, so it can be represented by a 2-by-2 matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Suppose $h = (h_1,h_2)$. We need the following expression to go to zero.

$$\frac{\lVert f(h_1,h_2) - f(0,0) - (ah_1+bh_2, ch_1+dh_2)\rVert}{\lVert (h_1,h_2)\rVert} = \\ \frac{\sqrt{\bigl((1-a)h_1+(2-b)h_2+h_1^2\bigr)^2 + \bigl((2-c)h_1+(3-d)h_2+h_1h_2\bigr)^2}}{\sqrt{h_1^2+h_2^2}}.$$

If we choose $a = 1$, $b = 2$, $c = 2$, $d = 3$, the expression becomes

$$\frac{\sqrt{h_1^4 + h_1^2 h_2^2}}{\sqrt{h_1^2 + h_2^2}} = |h_1| \frac{\sqrt{h_1^2 + h_2^2}}{\sqrt{h_1^2 + h_2^2}} = |h_1|.$$

This expression does indeed go to zero as $h \to 0$. The function f is differentiable at the origin and the derivative $f'(0)$ is represented by the matrix $\left[\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right]$.

Proposition 8.3.5. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be differentiable at $p \in U$. Then f is continuous at p.*

Proof. Another way to write the differentiability of f at p is to consider

$$r(h) := f(p+h) - f(p) - f'(p)h.$$

The function f is differentiable at p if $\frac{\|r(h)\|}{\|h\|}$ goes to zero as $h \to 0$, so $r(h)$ itself goes to zero. The mapping $h \mapsto f'(p)h$ is a linear mapping between finite-dimensional spaces, hence continuous and $f'(p)h \to 0$ as $h \to 0$. Thus, $f(p+h)$ must go to $f(p)$ as $h \to 0$. That is, f is continuous at p. $\square$

The derivative is itself a linear operator on the space of differentiable functions.

Proposition 8.3.6. *Suppose $U \subset \mathbb{R}^n$ is open, $f : U \to \mathbb{R}^m$ and $g : U \to \mathbb{R}^m$ are differentiable at p, and $\alpha \in \mathbb{R}$. Then the functions $f + g$ and αf are differentiable at p and*

$$(f+g)'(p) = f'(p) + g'(p), \qquad \text{and} \qquad (\alpha f)'(p) = \alpha f'(p).$$

Proof. Let $h \in \mathbb{R}^n$, $h \neq 0$. Then

$$\frac{\left\| f(p+h) + g(p+h) - \big(f(p) + g(p)\big) - \big(f'(p) + g'(p)\big)h \right\|}{\|h\|} \\ \leq \frac{\|f(p+h) - f(p) - f'(p)h\|}{\|h\|} + \frac{\|g(p+h) - g(p) - g'(p)h\|}{\|h\|},$$

and

$$\frac{\|\alpha f(p+h) - \alpha f(p) - \alpha f'(p)h\|}{\|h\|} = |\alpha| \frac{\|f(p+h)) - f(p) - f'(p)h\|}{\|h\|}.$$

The limits as h goes to zero of the right-hand sides are zero by hypothesis. The result follows. $\square$

If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$ are linear maps, then they are their own derivative. The composition $BA \in L(\mathbb{R}^n, \mathbb{R}^k)$ is also its own derivative, and so the derivative of the composition is the composition of the derivatives. As differentiable maps are "infinitesimally close" to linear maps, they have the same property:

Theorem 8.3.7 (Chain rule)**.** *Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be differentiable at $p \in U$. Let $V \subset \mathbb{R}^m$ be open, $f(U) \subset V$ and let $g : V \to \mathbb{R}^\ell$ be differentiable at $f(p)$. Then*

$$F(x) = g\big(f(x)\big)$$

is differentiable at p and

$$F'(p) = g'\big(f(p)\big) f'(p).$$

Without the points where things are evaluated, this is sometimes written as $F' = (g \circ f)' = g' f'$. The way to understand it is that the derivative of the composition $g \circ f$ is the composition of the derivatives of g and f. If $f'(p) = A$ and $g'\big(f(p)\big) = B$, then $F'(p) = BA$, just as for linear maps.

Proof. Let $A := f'(p)$ and $B := g'\big(f(p)\big)$. Take a nonzero $h \in \mathbb{R}^n$ and write $q := f(p)$, $k := f(p+h) - f(p)$. Let

$$r(h) := f(p+h) - f(p) - Ah.$$

Then $r(h) = k - Ah$ or $Ah = k - r(h)$, and $f(p+h) = q+k$. We look at the quantity we need to go to zero:

$$\begin{aligned}
\frac{\|F(p+h) - F(p) - BAh\|}{\|h\|} &= \frac{\|g\big(f(p+h)\big) - g\big(f(p)\big) - BAh\|}{\|h\|} \\
&= \frac{\|g(q+k) - g(q) - B\big(k - r(h)\big)\|}{\|h\|} \\
&\leq \frac{\|g(q+k) - g(q) - Bk\|}{\|h\|} + \|B\| \frac{\|r(h)\|}{\|h\|} \\
&= \frac{\|g(q+k) - g(q) - Bk\|}{\|k\|} \frac{\|f(p+h) - f(p)\|}{\|h\|} + \|B\| \frac{\|r(h)\|}{\|h\|}.
\end{aligned}$$

First, $\|B\|$ is a constant and f is differentiable at p, so the term $\|B\| \frac{\|r(h)\|}{\|h\|}$ goes to 0. Next because f is continuous at p, then as h goes to 0, so k goes to 0. Thus $\frac{\|g(q+k)-g(q)-Bk\|}{\|k\|}$ goes to 0, because g is differentiable at q. Finally,

$$\frac{\|f(p+h) - f(p)\|}{\|h\|} \leq \frac{\|f(p+h) - f(p) - Ah\|}{\|h\|} + \frac{\|Ah\|}{\|h\|} \leq \frac{\|f(p+h) - f(p) - Ah\|}{\|h\|} + \|A\|.$$

As f is differentiable at p, for small enough h, the quantity $\frac{\|f(p+h)-f(p)-Ah\|}{\|h\|}$ is bounded. Hence, the term $\frac{\|f(p+h)-f(p)\|}{\|h\|}$ stays bounded as h goes to 0. Therefore, $\frac{\|F(p+h)-F(p)-BAh\|}{\|h\|}$ goes to zero, and $F'(p) = BA$, which is what was claimed. $\square$

8.3.2 Partial derivatives

There is another way to generalize the derivative from one dimension. We hold all but one variable constant and take the regular one-variable derivative.

Definition 8.3.8. Let $f \colon U \to \mathbb{R}$ be a function on an open set $U \subset \mathbb{R}^n$. If the following limit exists, we write

$$\frac{\partial f}{\partial x_j}(x) := \lim_{h \to 0} \frac{f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) - f(x)}{h} = \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}.$$

We call $\frac{\partial f}{\partial x_j}(x)$ the *partial derivative* of f with respect to x_j. See Figure 8.4. Here h is a number, not a vector.

For a mapping $f \colon U \to \mathbb{R}^m$ we write $f = (f_1, f_2, \ldots, f_m)$, where f_k are real-valued functions. We then take partial derivatives of the components, $\frac{\partial f_k}{\partial x_j}$.

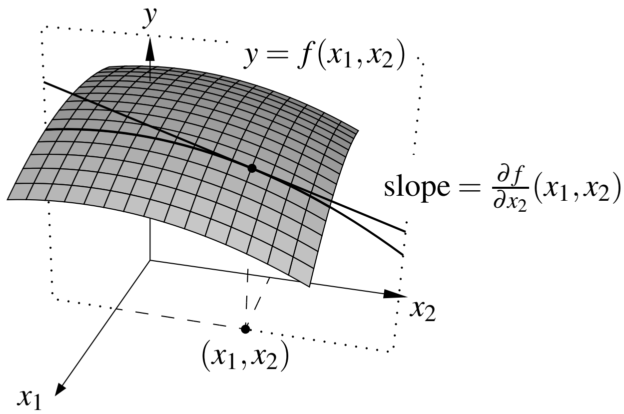

Figure 8.4: Illustration of a partial derivative for a function $f\colon \mathbb{R}^2 \to \mathbb{R}$. The yx_2-plane where x_1 is fixed is marked in dotted line, and the slope of the tangent line in the yx_2-plane is $\frac{\partial f}{\partial x_2}(x_1,x_2)$.

Partial derivatives are easier to compute with all the machinery of calculus, and they provide a way to compute the derivative of a function.

Proposition 8.3.9. *Let $U \subset \mathbb{R}^n$ be open and let $f\colon U \to \mathbb{R}^m$ be differentiable at $p \in U$. Then all the partial derivatives at p exist and, in terms of the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, $f'(p)$ is represented by the matrix*

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) & \ldots & \frac{\partial f_1}{\partial x_n}(p) \\ \frac{\partial f_2}{\partial x_1}(p) & \frac{\partial f_2}{\partial x_2}(p) & \ldots & \frac{\partial f_2}{\partial x_n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \frac{\partial f_m}{\partial x_2}(p) & \ldots & \frac{\partial f_m}{\partial x_n}(p) \end{bmatrix}.$$

In other words,

$$f'(p)\,e_j = \sum_{k=1}^{m} \frac{\partial f_k}{\partial x_j}(p)\,e_k.$$

If $v = \sum_{j=1}^{n} c_j\,e_j = (c_1,c_2,\ldots,c_n)$, then

$$f'(p)\,v = \sum_{j=1}^{n}\sum_{k=1}^{m} c_j \frac{\partial f_k}{\partial x_j}(p)\,e_k = \sum_{k=1}^{m}\left(\sum_{j=1}^{n} c_j \frac{\partial f_k}{\partial x_j}(p)\right) e_k.$$

Proof. Fix a j and note that for nonzero h,

$$\begin{aligned} \left\lVert \frac{f(p+he_j)-f(p)}{h} - f'(p)\,e_j \right\rVert &= \left\lVert \frac{f(p+he_j)-f(p)-f'(p)\,he_j}{h} \right\rVert \\ &= \frac{\lVert f(p+he_j)-f(p)-f'(p)\,he_j \rVert}{\lVert he_j \rVert}. \end{aligned}$$

As h goes to 0, the right-hand side goes to zero by differentiability of f, and hence

$$\lim_{h\to 0} \frac{f(p+he_j)-f(p)}{h} = f'(p)\,e_j.$$

Let us represent f by components $f = (f_1, f_2, \ldots, f_m)$, since it is vector-valued. Taking a limit in $\mathbb{R}^m$ is the same as taking the limit in each component separately. For every k, the partial derivative

$$\frac{\partial f_k}{\partial x_j}(p) = \lim_{h \to 0} \frac{f_k(p + he_j) - f_k(p)}{h}$$

exists and is equal to the kth component of $f'(p)\, e_j$, and we are done. $\square$

The converse of the proposition is not true. Just because the partial derivatives exist, does not mean that the function is differentiable. See the exercises. However, when the partial derivatives are continuous, we will prove that the converse holds. One of the consequences of the proposition is that if f is differentiable on U, then $f' \colon U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a continuous function if and only if all the $\frac{\partial f_k}{\partial x_j}$ are continuous functions.

8.3.3 Gradients, curves, and directional derivatives

Let $U \subset \mathbb{R}^n$ be open and $f \colon U \to \mathbb{R}$ a differentiable function. We define the *gradient* as

$$\nabla f(x) := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x)\, e_j.$$

The gradient gives a way to represent the action of the derivative as a dot product: $f'(x)\, v = \nabla f(x) \cdot v$.

Suppose $\gamma \colon (a,b) \subset \mathbb{R} \to \mathbb{R}^n$ is differentiable. Such a function and its image is sometimes called a *curve*, or a *differentiable curve*. Write $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. For the purposes of computation, we identify $L(\mathbb{R}^1)$ and $\mathbb{R}$ as we did when we defined the derivative in one variable. We also identify $L(\mathbb{R}^1, \mathbb{R}^n)$ with $\mathbb{R}^n$. We treat $\gamma'(t)$ both as an operator in $L(\mathbb{R}^1, \mathbb{R}^n)$ and the vector $\bigl(\gamma_1'(t), \gamma_2'(t), \ldots, \gamma_n'(t)\bigr)$ in $\mathbb{R}^n$. Using Proposition 8.3.9, if $v \in \mathbb{R}^n$ is $\gamma'(t)$ acting as a vector, then $h \mapsto h\, v$ (for $h \in \mathbb{R}^1 = \mathbb{R}$) is $\gamma'(t)$ acting as an operator in $L(\mathbb{R}^1, \mathbb{R}^n)$. We often use this slight abuse of notation when dealing with curves. The vector $\gamma'(t)$ is called a *tangent vector*. See Figure 8.5.

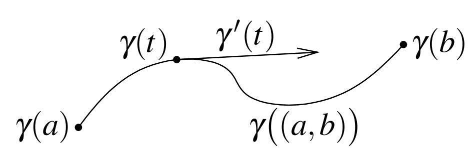

Figure 8.5: Differentiable curve and its derivative as a vector (for clarity assuming γ defined on $[a,b]$). The tangent vector $\gamma'(t)$ points along the curve.

Suppose $\gamma\bigl((a,b)\bigr) \subset U$ and let

$$g(t) := f\bigl(\gamma(t)\bigr).$$

The function g is differentiable. Treating $g'(t)$ as a number,

$$g'(t) = f'\bigl(\gamma(t)\bigr)\gamma'(t) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\bigl(\gamma(t)\bigr)\frac{d\gamma_j}{dt}(t) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{d\gamma_j}{dt}.$$

For convenience, we often leave out the points where we are evaluating, such as above on the far right-hand side. With the notation of the gradient and the dot product the equation becomes

$$g'(t) = (\nabla f)\big(\gamma(t)\big) \cdot \gamma'(t) = \nabla f \cdot \gamma'.$$

We use this idea to define derivatives in a specific direction. A direction is simply a vector pointing in that direction. Pick a vector $u \in \mathbb{R}^n$ such that $\|u\| = 1$, and fix $x \in U$. We define the *directional derivative* as

$$D_u f(x) := \frac{d}{dt}\Big|_{t=0} \big[f(x+tu)\big] = \lim_{h \to 0} \frac{f(x+hu) - f(x)}{h},$$

where the notation $\frac{d}{dt}\big|_{t=0}$ represents the derivative evaluated at $t = 0$. Taking the standard basis vector e_j we find $\frac{\partial f}{\partial x_j} = D_{e_j} f$. For this reason, sometimes the notation $\frac{\partial f}{\partial u}$ is used instead of $D_u f$.

Let γ be defined by

$$\gamma(t) := x + tu.$$

Then $\gamma'(t) = u$ for all t. Let us see what happens to f when we travel along γ:

$$D_u f(x) = \frac{d}{dt}\Big|_{t=0} \big[f(x+tu)\big] = (\nabla f)\big(\gamma(0)\big) \cdot \gamma'(0) = (\nabla f)(x) \cdot u.$$

In fact, this computation holds whenever γ is any curve such that $\gamma(0) = x$ and $\gamma'(0) = u$.

Suppose $(\nabla f)(x) \neq 0$. By the Cauchy–Schwarz inequality,

$$|D_u f(x)| \leq \|(\nabla f)(x)\|.$$

Equality is achieved when u is a scalar multiple of $(\nabla f)(x)$. That is, when

$$u = \frac{(\nabla f)(x)}{\|(\nabla f)(x)\|},$$

we get $D_u f(x) = \|(\nabla f)(x)\|$. The gradient points in the direction in which the function grows fastest, in other words, in the direction in which $D_u f(x)$ is maximal.

8.3.4 The Jacobian

Definition 8.3.10. Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a differentiable mapping. Define the *Jacobian**, or the *Jacobian determinant*†, of f at x as

$$J_f(x) := \det\big(f'(x)\big).$$

Sometimes J_f is written as

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}.$$

*Named after the Italian mathematician Carl Gustav Jacob Jacobi (1804–1851).

†The matrix from Proposition 8.3.9 representing $f'(x)$ is sometimes called the *Jacobian matrix*.

This last piece of notation may seem somewhat confusing, but it is quite useful when we need to specify the exact variables and function components used, as we will do, for example, in the implicit function theorem.

The Jacobian J_f is a real-valued function, and when $n = 1$ it is simply the derivative. From the chain rule and the fact that $\det(AB) = \det(A)\det(B)$, it follows that:

$$J_{f\circ g}(x) = J_f\big(g(x)\big)J_g(x).$$

The determinant of a linear mapping tells us what happens to area/volume under the mapping. Similarly, the Jacobian measures how much a differentiable mapping stretches things locally, and if it flips orientation. In particular, if the Jacobian is non-zero than we would assume that locally the mapping is invertible (and we would be correct as we will later see).

8.3.5 Exercises

Exercise* 8.3.1*: *Suppose* $\gamma\colon (-1,1) \to \mathbb{R}^n$ *and* $\alpha\colon (-1,1) \to \mathbb{R}^n$ *are two differentiable curves such that* $\gamma(0) = \alpha(0)$ *and* $\gamma'(0) = \alpha'(0)$. *Suppose* $F\colon \mathbb{R}^n \to \mathbb{R}$ *is a differentiable function. Show that*

$$\frac{d}{dt}\Big|_{t=0} F\big(\gamma(t)\big) = \frac{d}{dt}\Big|_{t=0} F\big(\alpha(t)\big).$$

Exercise* 8.3.2*: *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be given by* $f(x,y) := \sqrt{x^2+y^2}$, *see* *Figure 8.6*. *Show that* f *is not differentiable at the origin.*

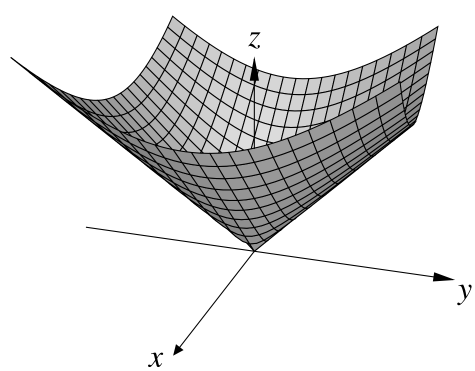

Figure 8.6: Graph of $\sqrt{x^2+y^2}$.

Exercise* 8.3.3*: *Using only the definition of the derivative, show that the following* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ *are differentiable at the origin and find their derivative.*

a) $f(x,y) := (1+x+xy,x)$,

b) $f(x,y) := \big(y-y^{10},x\big)$,

c) $f(x,y) := \big((x+y+1)^2,(x-y+2)^2\big)$.

Exercise* 8.3.4*: *Suppose* $f\colon \mathbb{R} \to \mathbb{R}$ *and* $g\colon \mathbb{R} \to \mathbb{R}$ *are differentiable functions. Using only the definition of the derivative, show that* $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $h(x,y) := \big(f(x),g(y)\big)$ *is a differentiable function, and find the derivative, at all points* (x,y).

Exercise* 8.3.5*: *Define a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *by (see Figure 8.7)*

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

a) *Show that the partial derivatives* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *exist at all points (including the origin).*

b) *Show that* f *is not continuous at the origin (and hence not differentiable).*

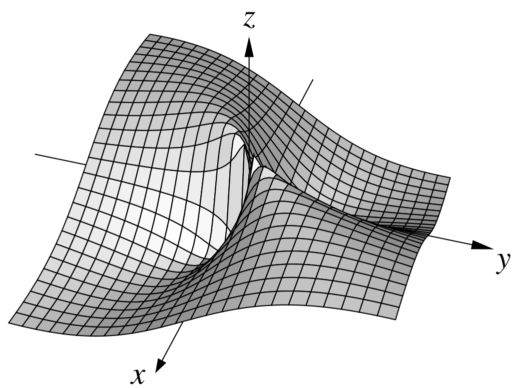

Figure 8.7: Graph of $\frac{xy}{x^2+y^2}$.

Exercise* 8.3.6*: *Define a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *by (see Figure 8.8)*

$$f(x,y) := \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

a) *Show that the partial derivatives* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *exist at all points.*

b) *Show that for all* $u \in \mathbb{R}^2$ *with* $\|u\| = 1$*, the directional derivative* $D_u f$ *exists at all points.*

c) *Show that* f *is continuous at the origin.*

d) *Show that* f *is not differentiable at the origin.*

Exercise* 8.3.7*: *Suppose* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is one-to-one, onto, differentiable at all points, and such that* f^{-1} *is also differentiable at all points.*

a) *Show that* $f'(p)$ *is invertible at all points* p *and compute* $(f^{-1})'\big(f(p)\big)$*. Hint: Consider* $x = f^{-1}\big(f(x)\big)$*.*

b) *Let* $g : \mathbb{R}^n \to \mathbb{R}^n$ *be a function differentiable at* $q \in \mathbb{R}^n$ *and such that* $g(q) = q$*. Suppose* $f(p) = q$ *for some* $p \in \mathbb{R}^n$*. Show* $J_g(q) = J_{f^{-1} \circ g \circ f}(p)$ *where* J_g *is the Jacobian determinant.*

Exercise* 8.3.8*: *Suppose* $f : \mathbb{R}^2 \to \mathbb{R}$ *is differentiable and such that* $f(x,y) = 0$ *if and only if* $y = 0$ *and such that* $\nabla f(0,0) = (0,1)$*. Prove that* $f(x,y) > 0$ *whenever* $y > 0$*, and* $f(x,y) < 0$ *whenever* $y < 0$*.*

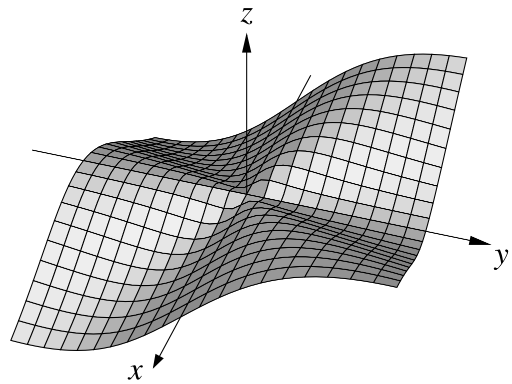

Figure 8.8: Graph of $\frac{x^2y}{x^2+y^2}$.

As for functions of one variable, $f: U \to \mathbb{R}$ has a *relative maximum* at $p \in U$ if there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in B(p,\delta) \cap U$. Similarly for *relative minimum*.

Exercise* 8.3.9*: *Suppose $U \subset \mathbb{R}^n$ is open and $f: U \to \mathbb{R}$ is differentiable. Suppose f has a relative maximum at $p \in U$. Show that $f'(p) = 0$, that is the zero mapping in $L(\mathbb{R}^n,\mathbb{R})$. That is p is a* critical point *of f.*

Exercise* 8.3.10*: *Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $f(x,y) = 0$ whenever $x^2 + y^2 = 1$. Prove that there exists at least one point (x_0,y_0) such that $\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0$.*

Exercise* 8.3.11*: *Define $f(x,y) := (x-y^2)(2y^2-x)$. The graph of f is called the* Peano surface.*

a) *Show that $(0,0)$ is a critical point, that is $f'(0,0) = 0$, that is the zero linear map in $L(\mathbb{R}^2,\mathbb{R})$.*
b) *Show that for every direction the restriction of f to a line through the origin in that direction has a relative maximum at the origin. In other words, for every (x,y) such that $x^2+y^2 = 1$, the function $g(t) := f(tx,ty)$, has a relative maximum at $t = 0$.*
   Hint: While not necessary §4.3 of volume I makes this part easier.
c) *Show that f does not have a relative maximum at $(0,0)$.*

Exercise* 8.3.12*: *Suppose $f: \mathbb{R} \to \mathbb{R}^n$ is differentiable and $\|f(t)\| = 1$ for all t (that is, we have a curve in the unit sphere). Show that $f'(t) \cdot f(t) = 0$ (treating f' as a vector) for all t.*

Exercise* 8.3.13*: *Define $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) := \bigl(x, y+\varphi(x)\bigr)$ for some differentiable function φ of one variable. Show f is differentiable and find f'.*

Exercise* 8.3.14*: *Suppose $U \subset \mathbb{R}^n$ is open, $p \in U$, and $f: U \to \mathbb{R}$, $g: U \to \mathbb{R}$, $h: U \to \mathbb{R}$ are functions such that $f(p) = g(p) = h(p)$, f and h are differentiable at p, $f'(p) = h'(p)$, and*

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in U$. Show that g is differentiable at p and $g'(p) = f'(p) = h'(p)$.

*Named after the Italian mathematician Giuseppe Peano (1858–1932).

8.4 Continuity and the derivative

Note: 1–2 lectures

8.4.1 Bounding the derivative

Let us prove a "mean value theorem" for vector-valued functions.

Lemma 8.4.1. *If $\varphi \colon [a,b] \to \mathbb{R}^n$ is differentiable on (a,b) and continuous on $[a,b]$, then there exists a $t_0 \in (a,b)$ such that*

$$\|\varphi(b)-\varphi(a)\| \leq (b-a)\|\varphi'(t_0)\|.$$

Proof. By the mean value theorem on the scalar-valued function $t \mapsto \bigl(\varphi(b)-\varphi(a)\bigr)\cdot\varphi(t)$, where the dot is the dot product, we obtain that there is a $t_0 \in (a,b)$ such that

$$\begin{aligned} \|\varphi(b)-\varphi(a)\|^2 &= \bigl(\varphi(b)-\varphi(a)\bigr)\cdot\bigl(\varphi(b)-\varphi(a)\bigr) \\ &= \bigl(\varphi(b)-\varphi(a)\bigr)\cdot\varphi(b) - \bigl(\varphi(b)-\varphi(a)\bigr)\cdot\varphi(a) \\ &= (b-a)\bigl(\varphi(b)-\varphi(a)\bigr)\cdot\varphi'(t_0), \end{aligned}$$

where we treat φ' as a vector in $\mathbb{R}^n$ by the abuse of notation we mentioned in the previous section. If we think of $\varphi'(t)$ as a vector, then by Exercise 8.2.6, $\|\varphi'(t)\|_{L(\mathbb{R},\mathbb{R}^n)} = \|\varphi'(t)\|_{\mathbb{R}^n}$. That is, the euclidean norm of the vector is the same as the operator norm of $\varphi'(t)$.

By the Cauchy–Schwarz inequality

$$\|\varphi(b)-\varphi(a)\|^2 = (b-a)\bigl(\varphi(b)-\varphi(a)\bigr)\cdot\varphi'(t_0) \leq (b-a)\|\varphi(b)-\varphi(a)\|\,\|\varphi'(t_0)\|. \qquad \square$$

Recall that a set U is convex if whenever $x,y \in U$, the line segment from x to y lies in U.

Proposition 8.4.2. *Let $U \subset \mathbb{R}^n$ be a convex open set, $f \colon U \to \mathbb{R}^m$ be a differentiable function, and an M be such that*

$$\|f'(x)\| \leq M \qquad \textit{for all } x \in U.$$

Then f is Lipschitz with constant M, that is

$$\|f(x)-f(y)\| \leq M\|x-y\| \qquad \textit{for all } x,y \in U.$$

Proof. Fix x and y in U and note that $(1-t)x+ty \in U$ for all $t \in [0,1]$ by convexity. Next

$$\frac{d}{dt}\Bigl[f\bigl((1-t)x+ty\bigr)\Bigr] = f'\bigl((1-t)x+ty\bigr)(y-x).$$

By Lemma 8.4.1 there is some $t_0 \in (0,1)$ such that

$$\begin{aligned} \|f(x)-f(y)\| &\leq \left\| \frac{d}{dt}\Big|_{t=t_0} \Bigl[f\bigl((1-t)x+ty\bigr)\Bigr] \right\| \\ &\leq \bigl\|f'\bigl((1-t_0)x+t_0y\bigr)\bigr\|\,\|y-x\| \leq M\|y-x\|. \end{aligned} \qquad \square$$

Example 8.4.3: If U is not convex the proposition is not true: Consider the set

$$U := \{(x,y) : 0.5 < x^2 + y^2 < 2\} \setminus \{(x,0) : x < 0\}.$$

For $(x,y) \in U$, let $f(x,y)$ be the angle that the line from the origin to (x,y) makes with the positive x axis. We even have a formula for f:

$$f(x,y) = 2\arctan\left(\frac{y}{x + \sqrt{x^2+y^2}}\right).$$

Think a spiral staircase with room in the middle. See Figure 8.9.

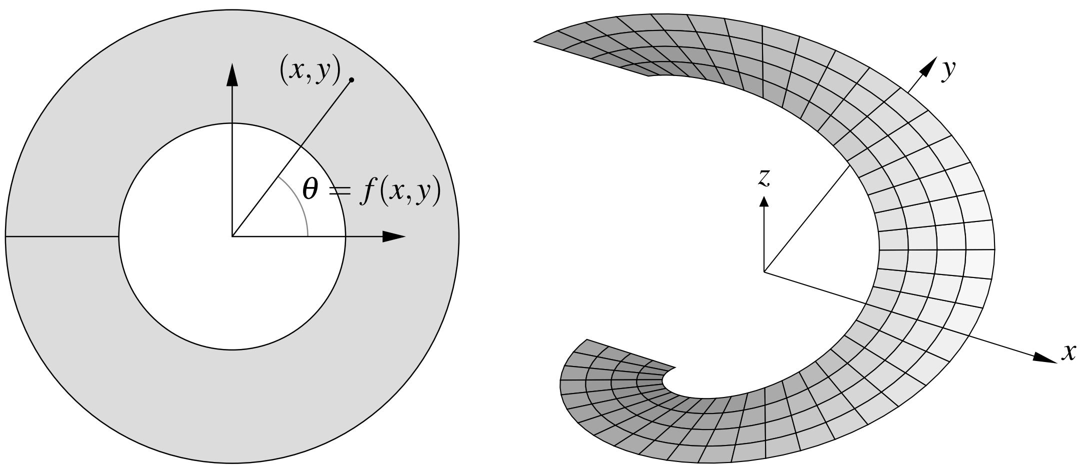

Figure 8.9: A non-Lipschitz function with uniformly bounded derivative.

The function is differentiable, and the derivative is bounded on U, which is not hard to see. Now think of what happens near where the negative x-axis cuts the annulus in half. As we approach this cut from positive y, $f(x,y)$ approaches π. From negative y, $f(x,y)$ approaches $-\pi$. So for small $\varepsilon > 0$, $|f(-1,\varepsilon) - f(-1,-\varepsilon)|$ approaches 2π, but $\|(-1,\varepsilon) - (-1,-\varepsilon)\| = 2\varepsilon$, which is arbitrarily small. The conclusion of the proposition does not hold for this nonconvex U.

Let us solve the differential equation $f' = 0$.

Corollary 8.4.4. *If $U \subset \mathbb{R}^n$ is open and connected, $f : U \to \mathbb{R}^m$ is differentiable, and $f'(x) = 0$ for all $x \in U$, then f is constant.*

Proof. For any given $x \in U$, there is a ball $B(x,\delta) \subset U$. The ball $B(x,\delta)$ is convex. Since $\|f'(y)\| \leq 0$ for all $y \in B(x,\delta)$, then by the proposition, $\|f(x) - f(y)\| \leq 0\|x-y\| = 0$. So $f(x) = f(y)$ for all $y \in B(x,\delta)$.

This means that $f^{-1}(c)$ is open for all $c \in \mathbb{R}^m$. Suppose $f^{-1}(c)$ is nonempty. The two sets

$$U' = f^{-1}(c), \qquad U'' = f^{-1}\big(\mathbb{R}^m \setminus \{c\}\big)$$

are open and disjoint, and further $U = U' \cup U''$. As U' is nonempty and U is connected, then $U'' = \emptyset$. So $f(x) = c$ for all $x \in U$. $\square$

8.4.2 Continuously differentiable functions

Definition 8.4.5. Let $U \subset \mathbb{R}^n$ be open. We say $f: U \to \mathbb{R}^m$ is *continuously differentiable*, or $C^1(U)$, if f is differentiable and $f': U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

Proposition 8.4.6. *Let $U \subset \mathbb{R}^n$ be open and $f: U \to \mathbb{R}^m$. The function f is continuously differentiable if and only if the partial derivatives $\frac{\partial f_j}{\partial x_\ell}$ exist for all j and ℓ and are continuous.*

Without continuity the theorem does not hold. Just because partial derivatives exist does not mean that f is differentiable, in fact, f may not even be continuous. See the exercises for the last section and also for this section.

Proof. We proved that if f is differentiable, then the partial derivatives exist. The partial derivatives are the entries of the matrix of $f'(x)$. If $f': U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous, then the entries are continuous, and hence the partial derivatives are continuous.

To prove the opposite direction, suppose the partial derivatives exist and are continuous. Fix $x \in U$. If we show that $f'(x)$ exists we are done, because the entries of the matrix $f'(x)$ are the partial derivatives and if the entries are continuous functions, the matrix-valued function f' is continuous.

We do induction on dimension. First, the conclusion is true when $n = 1$. In this case the derivative is just the regular derivative (exercise, noting that f is vector-valued).

Suppose the conclusion is true for $\mathbb{R}^{n-1}$, that is, if we restrict to the first $n-1$ variables, the function is differentiable. It is easy to see that the first $n-1$ partial derivatives of f restricted to the set where the last coordinate is fixed are the same as those for f. In the following, by a slight abuse of notation, we think of $\mathbb{R}^{n-1}$ as a subset of $\mathbb{R}^n$, that is the set in $\mathbb{R}^n$ where $x_n = 0$. In other words, we identify the vectors $(x_1, x_2, \ldots, x_{n-1})$ and $(x_1, x_2, \ldots, x_{n-1}, 0)$. Let

$$A := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \ldots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}, \qquad A' := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_{n-1}}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \ldots & \frac{\partial f_m}{\partial x_{n-1}}(x) \end{bmatrix}, \qquad v := \begin{bmatrix} \frac{\partial f_1}{\partial x_n}(x) \\ \vdots \\ \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

Let $\varepsilon > 0$ be given. By the induction hypothesis, there is a $\delta > 0$ such that for every $k \in \mathbb{R}^{n-1}$ with $\lVert k \rVert < \delta$, we have

$$\frac{\lVert f(x+k) - f(x) - A'k \rVert}{\lVert k \rVert} < \varepsilon.$$

By continuity of the partial derivatives, suppose δ is small enough so that

$$\left| \frac{\partial f_j}{\partial x_n}(x+h) - \frac{\partial f_j}{\partial x_n}(x) \right| < \varepsilon$$

for all j and all $h \in \mathbb{R}^n$ with $\lVert h \rVert < \delta$.

Suppose $h = k + te_n$ is a vector in $\mathbb{R}^n$, where $k \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, such that $\lVert h \rVert < \delta$. Then $\lVert k \rVert \leq \lVert h \rVert < \delta$. Note that $Ah = A'k + tv$.

$$\begin{aligned} \lVert f(x+h) - f(x) - Ah \rVert &= \lVert f(x+k+te_n) - f(x+k) - tv + f(x+k) - f(x) - A'k \rVert \\ &\leq \lVert f(x+k+te_n) - f(x+k) - tv \rVert + \lVert f(x+k) - f(x) - A'k \rVert \\ &\leq \lVert f(x+k+te_n) - f(x+k) - tv \rVert + \varepsilon \lVert k \rVert. \end{aligned}$$

As all the partial derivatives exist, by the mean value theorem, for each j there is some $\theta_j \in [0,t]$ (or $[t,0]$ if $t<0$), such that

$$f_j(x+k+te_n) - f_j(x+k) = t\frac{\partial f_j}{\partial x_n}(x+k+\theta_j e_n).$$

Note that if $\|h\| < \delta$, then $\|k+\theta_j e_n\| \leq \|h\| < \delta$. We finish the estimate

$$\begin{aligned}
\|f(x+h)-f(x)-Ah\| &\leq \|f(x+k+te_n)-f(x+k)-tv\| + \varepsilon\|k\| \\
&\leq \sqrt{\sum_{j=1}^{m}\left(t\frac{\partial f_j}{\partial x_n}(x+k+\theta_j e_n) - t\frac{\partial f_j}{\partial x_n}(x)\right)^2} + \varepsilon\|k\| \\
&\leq \sqrt{m}\,\varepsilon|t| + \varepsilon\|k\| \\
&\leq (\sqrt{m}+1)\varepsilon\|h\|.
\end{aligned}$$

□

A common application is to prove that a certain function is differentiable. For example, let us show that all polynomials are differentiable, and in fact continuously differentiable by computing the partial derivatives.

Corollary 8.4.7. *A polynomial $p\colon \mathbb{R}^n \to \mathbb{R}$ in several variables*

$$p(x_1,x_2,\ldots,x_n) = \sum_{0\leq j_1+j_2+\cdots+j_n\leq d} c_{j_1,j_2,\ldots,j_n} x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$$

is continuously differentiable.

Proof. Consider the partial derivative of p in the x_n variable. Write p as

$$p(x) = \sum_{j=0}^{d} p_j(x_1,\ldots,x_{n-1})\,x_n^j,$$

where p_j are polynomials in one less variable. Then

$$\frac{\partial p}{\partial x_n}(x) = \sum_{j=1}^{d} p_j(x_1,\ldots,x_{n-1})\,jx_n^{j-1},$$

which is again a polynomial. So the partial derivatives of polynomials exist and are again polynomials. By the continuity of algebraic operations, polynomials are continuous functions. Therefore p is continuously differentiable. □

8.4.3 Exercises

***Exercise* 8.4.1:** *Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ as*

$$f(x,y) := \begin{cases} (x^2+y^2)\sin\bigl((x^2+y^2)^{-1}\bigr) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at the origin, but that it is not continuously differentiable.
Note: Feel free to use what you know about sine and cosine from calculus.

Exercise* 8.4.2*: *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be the function from Exercise 8.3.5, that is,*

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at all points and show that these are not continuous functions.

Exercise* 8.4.3*: *Let $B(0,1) \subset \mathbb{R}^2$ be the unit ball, that is, the set given by $x^2+y^2 < 1$. Suppose $f: B(0,1) \to \mathbb{R}$ is a differentiable function such that $|f(0,0)| \leq 1$, and $\left|\frac{\partial f}{\partial x}\right| \leq 1$ and $\left|\frac{\partial f}{\partial y}\right| \leq 1$ for all points in $B(0,1)$.*

a) Find an $M \in \mathbb{R}$ such that $\|f'(x,y)\| \leq M$ for all $(x,y) \in B(0,1)$.

b) Find a $B \in \mathbb{R}$ such that $|f(x,y)| \leq B$ for all $(x,y) \in B(0,1)$.

Exercise* 8.4.4*: *Define $\varphi: [0,2\pi] \to \mathbb{R}^2$ by $\varphi(t) = \big(\sin(t), \cos(t)\big)$. Compute $\varphi'(t)$ for all t. Compute $\|\varphi'(t)\|$ for all t. Notice that $\varphi'(t)$ is never zero, yet $\varphi(0) = \varphi(2\pi)$, therefore, Rolle's theorem is not true in more than one dimension.*

Exercise* 8.4.5*: *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at all points and there exists an $M \in \mathbb{R}$ such that $\left|\frac{\partial f}{\partial x}\right| \leq M$ and $\left|\frac{\partial f}{\partial y}\right| \leq M$ at all points. Show that f is continuous.*

Exercise* 8.4.6*: *Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function and $M \in R$, such that for every $(x,y) \in \mathbb{R}^2$, the function $g(t) := f(xt, yt)$ is differentiable and $|g'(t)| \leq M$ for all t.*

a) Show that f is continuous at $(0,0)$.

b) Find an example of such an f that is discontinuous at every other point of $\mathbb{R}^2$.
Hint: Think back to how we constructed a nowhere continuous function on $[0,1]$.

Exercise* 8.4.7*: *Suppose $r: \mathbb{R}^n \setminus X \to \mathbb{R}$ is a rational function, that is, let $p: \mathbb{R}^n \to \mathbb{R}$ and $q: \mathbb{R}^n \to \mathbb{R}$ be polynomials, q not identically zero, where $X = q^{-1}(0)$, and $r = \frac{p}{q}$. Show that r is continuously differentiable.*

Exercise* 8.4.8*: *Suppose $f: \mathbb{R}^n \to \mathbb{R}$ and $h: \mathbb{R}^n \to \mathbb{R}$ are two differentiable functions such that $f'(x) = h'(x)$ for all $x \in \mathbb{R}^n$. Prove that if $f(0) = h(0)$, then $f(x) = h(x)$ for all $x \in \mathbb{R}^n$.*

Exercise* 8.4.9*: *Prove the base case in Proposition 8.4.6. That is, prove that if $n = 1$ and "the partials exist and are continuous," then the function is continuously differentiable. Note that f is vector-valued.*

Exercise* 8.4.10*: *Suppose $g: \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $h: \mathbb{R}^2 \to \mathbb{R}$ is continuous. Show that*

$$F(x,y) := g(x) + \int_0^y h(x,s)\,ds$$

is continuously differentiable, and that it is the solution of the partial differential equation $\frac{\partial F}{\partial y} = h$, with the initial condition $F(x,0) = g(x)$ for all $x \in \mathbb{R}$.

8.5 Inverse and implicit function theorems

Note: 2–3 lectures

To prove the inverse function theorem we use the contraction mapping principle from chapter 7, where we used it to prove Picard's theorem. Recall that a mapping $f: X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called a contraction if there exists a $k < 1$ such that

$$d_Y\bigl(f(p), f(q)\bigr) \leq k\, d_X(p,q) \qquad \text{for all } p,q \in X.$$

The contraction mapping principle says that if $f: X \to X$ is a contraction and X is a complete metric space, then there exists a unique fixed point, that is, there exists a unique $x \in X$ such that $f(x) = x$.

Intuitively, if a function is continuously differentiable, then it locally "behaves like" the derivative (which is a linear function). The idea of the inverse function theorem is that if a function is continuously differentiable and the derivative is invertible, the function is (locally) invertible.

Theorem 8.5.1 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open set and let $f: U \to \mathbb{R}^n$ be a continuously differentiable function. Suppose $p \in U$ and $f'(p)$ is invertible (that is, $J_f(p) \neq 0$). Then there exist open sets $V, W \subset \mathbb{R}^n$ such that $p \in V \subset U$, $f(V) = W$ and $f|_V$ is one-to-one. Hence a function $g: W \to V$ exists such that $g(y) := (f|_V)^{-1}(y)$. See Figure 8.10. Furthermore, g is continuously differentiable and*

$$g'(y) = \bigl(f'(x)\bigr)^{-1}, \qquad \textit{for all } x \in V, y = f(x).$$

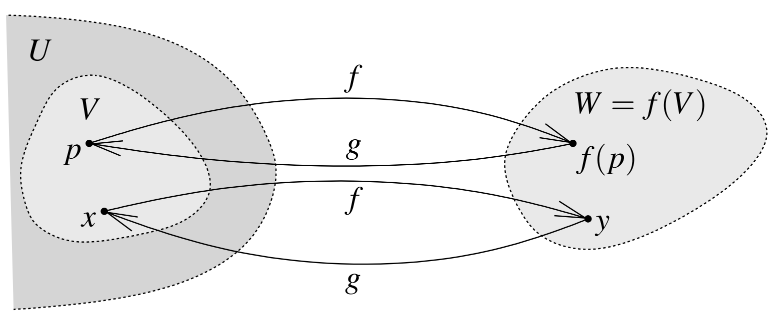

Figure 8.10: Setup of the inverse function theorem in $\mathbb{R}^n$.

Proof. Write $A = f'(p)$. As f' is continuous, there exists an open ball V around p such that

$$\|A - f'(x)\| < \frac{1}{2\|A^{-1}\|} \qquad \text{for all } x \in V.$$

Consequently, the derivative $f'(x)$ is invertible for all $x \in V$ by Proposition 8.2.6.

Given $y \in \mathbb{R}^n$, we define $\varphi_y: V \to \mathbb{R}^n$ by

$$\varphi_y(x) := x + A^{-1}\bigl(y - f(x)\bigr).$$

As A^{-1} is one-to-one, $\varphi_y(x) = x$ (x is a fixed point) if only if $y - f(x) = 0$, or in other words $f(x) = y$. Using the chain rule we obtain

$$\varphi_y'(x) = I - A^{-1} f'(x) = A^{-1}\bigl(A - f'(x)\bigr).$$

So for $x \in V$, we have

$$\|\varphi_y'(x)\| \leq \|A^{-1}\| \, \|A - f'(x)\| < 1/2.$$

As V is a ball, it is convex. Hence

$$\|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\| \qquad \text{for all } x_1, x_2 \in V.$$

In other words, φ_y is a contraction defined on V, though we so far do not know what is the range of φ_y. We cannot yet apply the fixed point theorem, but we can say that φ_y has at most one fixed point in V: If $\varphi_y(x_1) = x_1$ and $\varphi_y(x_2) = x_2$, then $\|x_1 - x_2\| = \|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|$, so $x_1 = x_2$. That is, there exists at most one $x \in V$ such that $f(x) = y$, and so $f|_V$ is one-to-one.

Let $W := f(V)$ and let $g \colon W \to V$ be the inverse of $f|_V$. We need to show that W is open. Take a $y_0 \in W$. There is a unique $x_0 \in V$ such that $f(x_0) = y_0$. Let $r > 0$ be small enough such that the closed ball $C(x_0, r) \subset V$ (such $r > 0$ exists as V is open).

Suppose y is such that

$$\|y - y_0\| < \frac{r}{2\|A^{-1}\|}.$$

If we show that $y \in W$, then we have shown that W is open. If $x_1 \in C(x_0, r)$, then

$$\begin{aligned}
\|\varphi_y(x_1) - x_0\| &\leq \|\varphi_y(x_1) - \varphi_y(x_0)\| + \|\varphi_y(x_0) - x_0\| \\
&\leq \frac{1}{2}\|x_1 - x_0\| + \|A^{-1}(y - y_0)\| \\
&\leq \frac{1}{2}r + \|A^{-1}\| \, \|y - y_0\| \\
&< \frac{1}{2}r + \|A^{-1}\| \frac{r}{2\|A^{-1}\|} = r.
\end{aligned}$$

So φ_y takes $C(x_0, r)$ into $B(x_0, r) \subset C(x_0, r)$. It is a contraction on $C(x_0, r)$ and $C(x_0, r)$ is complete (closed subset of $\mathbb{R}^n$ is complete). Apply the contraction mapping principle to obtain a fixed point x, i.e., $\varphi_y(x) = x$. That is, $f(x) = y$, and $y \in f\bigl(C(x_0, r)\bigr) \subset f(V) = W$. Therefore W is open.

Next we need to show that g is continuously differentiable and compute its derivative. First, let us show that it is differentiable. Let $y \in W$ and $k \in \mathbb{R}^n$, $k \neq 0$, such that $y + k \in W$. Because $f|_V$ is a one-to-one and onto mapping of V onto W, there are unique $x \in V$ and $h \in \mathbb{R}^n$, $h \neq 0$ and $x + h \in V$, such that $f(x) = y$ and $f(x + h) = y + k$. In other words, $g(y) = x$ and $g(y + k) = x + h$. See Figure 8.11.

We can still squeeze some information from the fact that φ_y is a contraction.

$$\varphi_y(x + h) - \varphi_y(x) = h + A^{-1}\bigl(f(x) - f(x + h)\bigr) = h - A^{-1}k.$$

So

$$\|h - A^{-1}k\| = \|\varphi_y(x + h) - \varphi_y(x)\| \leq \frac{1}{2}\|x + h - x\| = \frac{\|h\|}{2}.$$

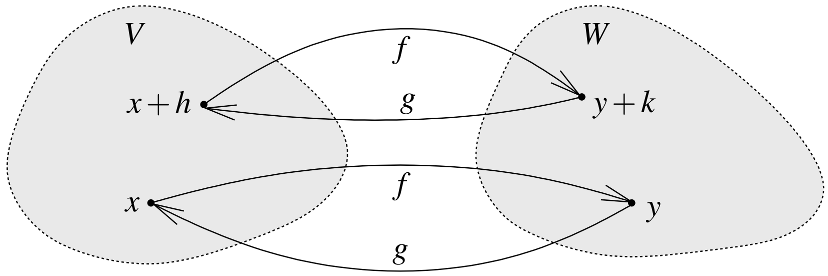

Figure 8.11: Proving that g is differentiable.

By the inverse triangle inequality, $\lVert h \rVert - \lVert A^{-1}k \rVert \leq \frac{1}{2}\lVert h \rVert$. So

$$\lVert h \rVert \leq 2\lVert A^{-1}k \rVert \leq 2\lVert A^{-1} \rVert \, \lVert k \rVert.$$

In particular, as k goes to 0, so does h.

As $x \in V$, then $f'(x)$ is invertible. Let $B := \bigl(f'(x)\bigr)^{-1}$, which is what we think the derivative of g at y is. Then

$$\begin{aligned}
\frac{\lVert g(y+k) - g(y) - Bk \rVert}{\lVert k \rVert} &= \frac{\lVert h - Bk \rVert}{\lVert k \rVert} \\
&= \frac{\lVert h - B\bigl(f(x+h) - f(x)\bigr) \rVert}{\lVert k \rVert} \\
&= \frac{\lVert B\bigl(f(x+h) - f(x) - f'(x)h\bigr) \rVert}{\lVert k \rVert} \\
&\leq \lVert B \rVert \frac{\lVert h \rVert}{\lVert k \rVert} \, \frac{\lVert f(x+h) - f(x) - f'(x)h \rVert}{\lVert h \rVert} \\
&\leq 2\lVert B \rVert \, \lVert A^{-1} \rVert \frac{\lVert f(x+h) - f(x) - f'(x)h \rVert}{\lVert h \rVert}.
\end{aligned}$$

As k goes to 0, so does h. So the right-hand side goes to 0 as f is differentiable, and hence the left-hand side also goes to 0. And B is precisely what we wanted $g'(y)$ to be.

We have g is differentiable, let us show it is $C^1(W)$. The function $g \colon W \to V$ is continuous (it is differentiable), f' is a continuous function from V to $L(\mathbb{R}^n)$, and $X \mapsto X^{-1}$ is a continuous function on the set of invertible operators. As $g'(y) = \Bigl(f'\bigl(g(y)\bigr)\Bigr)^{-1}$ is the composition of these three continuous functions, it is continuous. □

Corollary 8.5.2. *Suppose $U \subset \mathbb{R}^n$ is open and $f \colon U \to \mathbb{R}^n$ is a continuously differentiable mapping such that $f'(x)$ is invertible for all $x \in U$. Then for every open set $V \subset U$, the set $f(V)$ is open (f is said to be an* open mapping*).*

Proof. Without loss of generality, suppose $U = V$. For each point $y \in f(V)$, we pick $x \in f^{-1}(y)$ (there could be more than one such point), then by the inverse function theorem there is a neighborhood of x in V that maps onto a neighborhood of y. Hence $f(V)$ is open. □

Example 8.5.3: The theorem, and the corollary, is not true if $f'(x)$ is not invertible for some x. For example, the map $f(x,y) := (x,xy)$, maps $\mathbb{R}^2$ onto the set $\mathbb{R}^2 \setminus \{(0,y) : y \neq 0\}$, which is neither open nor closed. In fact $f^{-1}(0,0) = \{(0,y) : y \in \mathbb{R}\}$. This bad behavior only occurs on the y-axis, everywhere else the function is locally invertible. If we avoid the y-axis, f is even one-to-one.

Example 8.5.4: Just because $f'(x)$ is invertible everywhere does not mean that f is one-to-one globally. It is "locally" one-to-one but perhaps not "globally." For an example, take the map $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined by $f(x,y) := (x^2 - y^2, 2xy)$. It is left to student to show that f is differentiable and the derivative is invertible.

On the other hand, the mapping f is 2-to-1 globally. For every (a,b) that is not the origin, there are exactly two solutions to $x^2 - y^2 = a$ and $2xy = b$ (it is also onto). We leave it to the student to show that there is at least one solution, and then notice that replacing x and y with $-x$ and $-y$ we obtain another solution.

The invertibility of the derivative is not a necessary condition, just sufficient, for having a continuous inverse and being an open mapping. For example, the function $f(x) := x^3$ is an open mapping from $\mathbb{R}$ to $\mathbb{R}$ and is globally one-to-one with a continuous inverse, although the inverse is not differentiable at $x = 0$.

As a side note, there is a related famous, and as yet unsolved problem, called the *Jacobian conjecture*. If $F : \mathbb{R}^n \to \mathbb{R}^n$ is polynomial (each component is a polynomial) and J_F is a nonzero constant, does F have a polynomial inverse? The inverse function theorem gives a local C^1 inverse, but can one always find a global polynomial inverse is the question.

8.5.1 Implicit function theorem

The inverse function theorem is really a special case of the implicit function theorem, which we prove next. Although somewhat ironically we prove the implicit function theorem using the inverse function theorem. In the inverse function theorem we showed that the equation $x - f(y) = 0$ is solvable for y in terms of x if the derivative in terms of y is invertible, that is if $f'(y)$ is invertible. Then there is (locally) a function g such that $x - f\big(g(x)\big) = 0$.

OK, so how about the equation $f(x,y) = 0$. This equation is not solvable for y in terms of x in every case. For example, there is no solution when $f(x,y)$ does not actually depend on y. For a slightly more complicated example, notice that $x^2 + y^2 - 1 = 0$ defines the unit circle, and we can locally solve for y in terms of x when 1) we are near a point that lies on the unit circle and 2) when we are not at a point where the circle has a vertical tangency, or in other words where $\frac{\partial f}{\partial y} = 0$.

To make things simple, we fix some notation. We let $(x,y) \in \mathbb{R}^{n+m}$ denote the coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_m)$. A linear transformation $A \in L(\mathbb{R}^{n+m},\mathbb{R}^m)$ can then be written as $A = [A_x \, A_y]$ so that $A(x,y) = A_x x + A_y y$, where $A_x \in L(\mathbb{R}^n,\mathbb{R}^m)$ and $A_y \in L(\mathbb{R}^m)$.

Proposition 8.5.5. *Let $A = [A_x \, A_y] \in L(\mathbb{R}^{n+m},\mathbb{R}^m)$ and suppose A_y is invertible. If $B = -(A_y)^{-1}A_x$, then*

$$0 = A(x,Bx) = A_x x + A_y Bx.$$

Furthermore, $y = Bx$ is the unique $y \in \mathbb{R}^m$ such that $A(x,y) = 0$.

The proof is immediate: We solve and obtain $y = Bx$. Another way to solve is to "complete the basis," that is, add rows to the matrix until we have an invertible matrix. In this case, we construct a mapping $(x,y) \mapsto (x, A_x x + A_y y)$, and find that this operator in $L(\mathbb{R}^{n+m})$ is invertible, and the map B can be read off from the inverse. Let us show that the same can be done for C^1 functions.

Theorem 8.5.6 (Implicit function theorem)**.** *Let $U \subset \mathbb{R}^{n+m}$ be an open set and let $f : U \to \mathbb{R}^m$ be a $C^1(U)$ mapping. Let $(p,q) \in U$ be a point such that $f(p,q) = 0$ and such that*

$$\frac{\partial(f_1,\ldots,f_m)}{\partial(y_1,\ldots,y_m)}(p,q) \neq 0.$$

Then there exists an open set $W \subset \mathbb{R}^n$ with $p \in W$, an open set $W' \subset \mathbb{R}^m$ with $q \in W'$, with $W \times W' \subset U$, and a $C^1(W)$ mapping $g : W \to W'$, with $g(p) = q$, and for all $x \in W$, the point $g(x)$ is the unique point in W' such that

$$f\big(x, g(x)\big) = 0.$$

Furthermore, if $A = [A_x\ A_y] = f'(p,q)$, then

$$g'(p) = -(A_y)^{-1} A_x.$$

The condition $\frac{\partial(f_1,\ldots,f_m)}{\partial(y_1,\ldots,y_m)}(p,q) = \det(A_y) \neq 0$ simply means that A_y is invertible. If $n = m = 1$, the condition becomes $\frac{\partial f}{\partial y}(p,q) \neq 0$, W and W' are open intervals. See Figure 8.12.

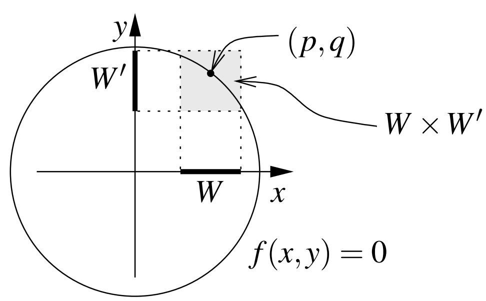

Figure 8.12: Implicit function theorem for $f(x,y) = x^2 + y^2 - 1$ in $U = \mathbb{R}^2$ and (p,q) in the first quadrant.

Proof. Define $F : U \to \mathbb{R}^{n+m}$ by $F(x,y) := \big(x, f(x,y)\big)$. It is clear that F is C^1, and we want to show that the derivative at (p,q) is invertible.

Let us compute the derivative. The quotient

$$\frac{\lVert f(p+h,q+k) - f(p,q) - A_x h - A_y k \rVert}{\lVert (h,k) \rVert}$$

goes to zero as $\lVert (h,k) \rVert = \sqrt{\lVert h \rVert^2 + \lVert k \rVert^2}$ goes to zero. But then so does

$$\begin{aligned} \frac{\lVert F(p+h,q+k) - F(p,q) - (h, A_x h + A_y k) \rVert}{\lVert (h,k) \rVert} &= \frac{\lVert \big(h, f(p+h,q+k) - f(p,q)\big) - (h, A_x h + A_y k) \rVert}{\lVert (h,k) \rVert} \\ &= \frac{\lVert f(p+h,q+k) - f(p,q) - A_x h - A_y k \rVert}{\lVert (h,k) \rVert}. \end{aligned}$$

So the derivative of F at (p,q) takes (h,k) to $(h,A_x h+A_y k)$. In block matrix form, it is $\left[\begin{smallmatrix} I & 0 \\ A_x & A_y \end{smallmatrix}\right]$. If $(h,A_x h+A_y k)=(0,0)$, then $h=0$, and so $A_y k=0$. As A_y is one-to-one, $k=0$. Thus $F'(p,q)$ is one-to-one or in other words invertible, and we apply the inverse function theorem.

That is, there exists an open set $V \subset \mathbb{R}^{n+m}$ with $F(p,q)=(p,0)\in V$, and a C^1 mapping $G\colon V\to\mathbb{R}^{n+m}$, such that $F\bigl(G(x,s)\bigr)=(x,s)$ for all $(x,s)\in V$, G is one-to-one, and $G(V)$ is open. Write $G=(G_1,G_2)$ (the first n and the second m components of G). Then

$$F\bigl(G_1(x,s),G_2(x,s)\bigr)=\Bigl(G_1(x,s),f\bigl(G_1(x,s),G_2(x,s)\bigr)\Bigr)=(x,s).$$

So $x=G_1(x,s)$ and $f\bigl(G_1(x,s),G_2(x,s)\bigr)=f\bigl(x,G_2(x,s)\bigr)=s$. Plugging in $s=0$, we obtain

$$f\bigl(x,G_2(x,0)\bigr)=0.$$

As the set $G(V)$ is open and $(p,q)\in G(V)$, there exist some open sets $\widetilde{W}$ and W' such that $\widetilde{W}\times W'\subset G(V)$ with $p\in\widetilde{W}$ and $q\in W'$. Take $W:=\bigl\{x\in\widetilde{W}:G_2(x,0)\in W'\bigr\}$. The function that takes x to $G_2(x,0)$ is continuous and therefore W is open. Define $g\colon W\to\mathbb{R}^m$ by $g(x):=G_2(x,0)$, which is the g in the theorem. The fact that $g(x)$ is the unique point in W' follows because $W\times W'\subset G(V)$ and G is one-to-one.

Next, differentiate

$$x\mapsto f\bigl(x,g(x)\bigr)$$

at p, which is the zero map, so its derivative is zero. Using the chain rule,

$$0=A\bigl(h,g'(p)h\bigr)=A_x h+A_y g'(p)h$$

for all $h\in\mathbb{R}^n$, and we obtain the desired derivative for g. □

In other words, in the context of the theorem, we have m equations in $n+m$ unknowns:

$$\begin{gathered}
f_1(x_1,\ldots,x_n,y_1,\ldots,y_m)=0,\\
f_2(x_1,\ldots,x_n,y_1,\ldots,y_m)=0,\\
\vdots\\
f_m(x_1,\ldots,x_n,y_1,\ldots,y_m)=0.
\end{gathered}$$

The condition guaranteeing a solution is that f is a C^1 mapping (all the components are C^1: partial derivatives in all variables exist and are continuous) and that the matrix

$$\begin{bmatrix}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \ldots & \frac{\partial f_1}{\partial y_m}\\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \ldots & \frac{\partial f_2}{\partial y_m}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial f_m}{\partial y_1} & \frac{\partial f_m}{\partial y_2} & \ldots & \frac{\partial f_m}{\partial y_m}
\end{bmatrix}$$

is invertible at (p,q).

Example 8.5.7: Consider the set given by $x^2+y^2-(z+1)^3=-1$ and $e^x+e^y+e^z=3$ near the point $(0,0,0)$. It is the zero set of the mapping

$$f(x,y,z)=\left(x^2+y^2-(z+1)^3+1, e^x+e^y+e^z-3\right),$$

whose derivative is

$$f'=\begin{bmatrix} 2x & 2y & -3(z+1)^2 \\ e^x & e^y & e^z \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 2(0) & -3(0+1)^2 \\ e^0 & e^0 \end{bmatrix}=\begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$$

is invertible. Hence near $(0,0,0)$ we can solve for y and z as C^1 functions of x such that for x near 0, we have

$$x^2+y(x)^2-\left(z(x)+1\right)^3=-1, \qquad e^x+e^{y(x)}+e^{z(x)}=3.$$

The theorem does not tell us how to find $y(x)$ and $z(x)$ explicitly, it just tells us they exist. In other words, near the origin the set of solutions is a smooth curve in $\mathbb{R}^3$ that goes through the origin.

An interesting observation from the proof is that we solved the equation $f\left(x,g(x)\right)=s$ for all s in some neighborhood of 0, not just $s=0$.

Remark 8.5.8. There are versions of the theorem for arbitrarily many derivatives. If f has k continuous derivatives, then the solution also has k continuous derivatives. See also the next section.

8.5.2 Exercises

***Exercise* 8.5.1:** *Let* $C:=\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\right\}$.

a) *Solve for y in terms of x near $(0,1)$ (that is, find the function g from the implicit function theorem for a neighborhood of the point $(p,q)=(0,1)$).*
b) *Solve for y in terms of x near $(0,-1)$.*
c) *Solve for x in terms of y near $(-1,0)$.*

***Exercise* 8.5.2:** *Define $f\colon \mathbb{R}^2\to\mathbb{R}^2$ by $f(x,y):=\left(x,y+h(x)\right)$ for some continuously differentiable function h of one variable.*

a) *Show that f is one-to-one and onto.*
b) *Compute f'.*
c) *Show that f' is invertible at all points, and compute its inverse.*

***Exercise* 8.5.3:** *Define $f\colon \mathbb{R}^2\to\mathbb{R}^2\setminus\left\{(0,0)\right\}$ by $f(x,y):=\left(e^x\cos(y), e^x\sin(y)\right)$.*

a) *Show that f is onto.*
b) *Show that f' is invertible at all points.*
c) *Show that f is not one-to-one, in fact for every $(a,b)\in\mathbb{R}^2\setminus\left\{(0,0)\right\}$, there exist infinitely many different points $(x,y)\in\mathbb{R}^2$ such that $f(x,y)=(a,b)$.*

Therefore, invertible derivative at every point does not mean that f is invertible globally.
Note: Feel free to use what you know about sine and cosine from calculus.

Exercise* 8.5.4*: *Find a map $f: \mathbb{R}^n \to \mathbb{R}^n$ that is one-to-one, onto, continuously differentiable, but $f'(0) = 0$. Hint: Generalize $f(x) = x^3$ from one to n dimensions.*

Exercise* 8.5.5*: *Consider $z^2 + xz + y = 0$ in $\mathbb{R}^3$. Find an equation $D(x,y) = 0$, such that if $D(x_0,y_0) \neq 0$ and $z^2 + x_0 z + y_0 = 0$ for some $z \in \mathbb{R}$, then for points near (x_0,y_0) there exist exactly two distinct continuously differentiable functions $r_1(x,y)$ and $r_2(x,y)$ such that $z = r_1(x,y)$ and $z = r_2(x,y)$ solve $z^2 + xz + y = 0$. Do you recognize the expression D from algebra?*

Exercise* 8.5.6*: *Suppose $f: (a,b) \to \mathbb{R}^2$ is continuously differentiable and the first component (the x component) of $\nabla f(t)$ is not equal to 0 for all $t \in (a,b)$. Prove that there exists an interval (c,d) and a continuously differentiable function $g: (c,d) \to \mathbb{R}$ such that $(x,y) \in f\big((a,b)\big)$ if and only if $x \in (c,d)$ and $y = g(x)$. In other words, the set $f\big((a,b)\big)$ is a graph of g.*

Exercise* 8.5.7*: *Define $f: \mathbb{R}^2 \to \mathbb{R}^2$*

$$f(x,y) := \begin{cases} \big(x^2 \sin(1/x) + x/2, y\big) & \text{if } x \neq 0, \\ (0,y) & \text{if } x = 0. \end{cases}$$

a) Show that f is differentiable everywhere.

b) Show that $f'(0,0)$ is invertible.

c) Show that f is not one-to-one in every neighborhood of the origin (it is not locally invertible, that is, the inverse function theorem does not work).

d) Show that f is not continuously differentiable.

Note: Feel free to use what you know about sine and cosine from calculus.

Exercise* 8.5.8** (Polar coordinates):*** *Define a mapping $F(r,\theta) := \big(r\cos(\theta), r\sin(\theta)\big)$.*

a) Show that F is continuously differentiable (for all $(r,\theta) \in \mathbb{R}^2$).

b) Compute $F'(0,\theta)$ for all θ.

c) Show that if $r \neq 0$, then $F'(r,\theta)$ is invertible, therefore an inverse of F exists locally as long as $r \neq 0$.

d) Show that $F: \mathbb{R}^2 \to \mathbb{R}^2$ is onto, and for each point $(x,y) \in \mathbb{R}^2$, the set $F^{-1}(x,y)$ is infinite.

e) Show that $F: \mathbb{R}^2 \to \mathbb{R}^2$ is an open map, despite not satisfying the condition of the inverse function theorem.

f) Show that $F|_{(0,\infty)\times[0,2\pi)}$ is one-to-one and onto $\mathbb{R}^2 \setminus \{(0,0)\}$.

Note: Feel free to use what you know about sine and cosine from calculus.

Exercise* 8.5.9*: *Let $H := \{(x,y) \in \mathbb{R}^2 : y > 0\}$, and for $(x,y) \in H$ define*

$$F(x,y) := \left(\frac{x^2 + y^2 - 1}{x^2 + 2y + y^2 + 1}, \frac{-2x}{x^2 + 2y + y^2 + 1} \right).$$

Prove that F is a bijective mapping from H to $B(0,1)$, it is continuously differentiable on H, and its inverse is also continuously differentiable.

Exercise* 8.5.10*: *Suppose $U \subset \mathbb{R}^2$ is open and $f: U \to \mathbb{R}$ is a C^1 function such that $\nabla f(x,y) \neq 0$ for all $(x,y) \in U$. Show that every level set is a C^1 smooth curve. That is, for every $(x,y) \in U$, there exists a C^1 function $\gamma: (-\delta,\delta) \to \mathbb{R}^2$ with $\gamma'(0) \neq 0$ such that $f\big(\gamma(t)\big)$ is constant for all $t \in (-\delta,\delta)$.*

Exercise* 8.5.11*: *Suppose $U \subset \mathbb{R}^2$ is open and $f \colon U \to \mathbb{R}$ is a C^1 function such that $\nabla f(x,y) \neq 0$ for all $(x,y) \in U$. Show that for every (x,y) there exists a neighborhood V of (x,y) an open set $W \subset \mathbb{R}^2$, a bijective C^1 function with a C^1 inverse $g \colon W \to V$ such that the level sets of $f \circ g$ are horizontal lines in W, that is, the set given by $(f \circ g)(s,t) = c$ for a constant c is a set of the form $\{(s,t_0) \in \mathbb{R}^2 : s \in \mathbb{R}, (s,t_0) \in W\}$, where t_0 is fixed. That is, the level curves can be locally "straightened."*

8.6 Higher order derivatives

Note: less than 1 lecture, partly depends on the optional §4.3 of volume I

Let $U \subset \mathbb{R}^n$ be an open set and $f: U \to \mathbb{R}$ a function. Denote by $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ our coordinates. Suppose $\frac{\partial f}{\partial x_j}$ exists everywhere in U, then it is also a function $\frac{\partial f}{\partial x_j}: U \to \mathbb{R}$. Therefore, it makes sense to talk about its partial derivatives. We denote the partial derivative of $\frac{\partial f}{\partial x_j}$ with respect to x_k by

$$\frac{\partial^2 f}{\partial x_k \partial x_j} := \frac{\partial \left(\frac{\partial f}{\partial x_j}\right)}{\partial x_k}.$$

If $k = j$, then we write $\frac{\partial^2 f}{\partial x_j^2}$ for simplicity.

We define higher order derivatives inductively. Suppose $j_1, j_2, \ldots, j_\ell$ are integers between 1 and n, and suppose

$$\frac{\partial^{\ell-1} f}{\partial x_{j_{\ell-1}} \partial x_{j_{\ell-2}} \cdots \partial x_{j_1}}$$

exists and is differentiable in the variable x_{j_ℓ}, then the partial derivative with respect to that variable is denoted by

$$\frac{\partial^\ell f}{\partial x_{j_\ell} \partial x_{j_{\ell-1}} \cdots \partial x_{j_1}} := \frac{\partial \left(\frac{\partial^{\ell-1} f}{\partial x_{j_{\ell-1}} \partial x_{j_{\ell-2}} \cdots \partial x_{j_1}}\right)}{\partial x_{j_\ell}}.$$

Such a derivative is called a *partial derivative of order ℓ*.

Sometimes the notation $f_{x_j x_k}$ is used for $\frac{\partial^2 f}{\partial x_k \partial x_j}$. This notation swaps the order in which we write the derivatives, which may be important.

Definition 8.6.1. Suppose $U \subset \mathbb{R}^n$ is an open set and $f: U \to \mathbb{R}$ is a function. We say f is *k-times continuously differentiable function*, or a C^k function, if all partial derivatives of all orders up to and including order k exist and are continuous.

So a continuously differentiable, or C^1, function is one where all partial derivatives exist and are continuous, which agrees with our previous definition due to Proposition 8.4.6. We could have required only that the kth order partial derivatives exist and are continuous, as the existence of lower order derivatives is clearly necessary to even define kth order partial derivatives, and these lower order derivatives are continuous as they are differentiable functions.

When the partial derivatives are continuous, we can swap their order.

Proposition 8.6.2. *Suppose $U \subset \mathbb{R}^n$ is open and $f: U \to \mathbb{R}$ is a C^2 function, and j and k are two integers from 1 to n. Then*

$$\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k}.$$

Proof. Fix a $p \in U$, and let e_j and e_k be the standard basis vectors. Pick two positive numbers s and t small enough so that $p + s_0 e_j + t_0 e_k \in U$ whenever $0 < s_0 \leq s$ and $0 < t_0 \leq t$. This can be done as U is open and so contains a small open ball (or a box if you wish) around p.

Use the mean value theorem on the function

$$\tau \mapsto f(p+se_j+\tau e_k) - f(x+\tau e_k),$$

on the interval $[0,t]$ to find a $t_0 \in (0,t)$ such that

$$\frac{f(p+se_j+te_k)-f(p+te_k)-f(p+se_j)+f(p)}{t} = \frac{\partial f}{\partial x_k}(p+se_j+t_0e_k) - \frac{\partial f}{\partial x_k}(p+t_0e_k).$$

Next there exists a number $s_0 \in (0,s)$

$$\frac{\frac{\partial f}{\partial x_k}(p+se_j+t_0e_k) - \frac{\partial f}{\partial x_k}(p+t_0e_k)}{s} = \frac{\partial^2 f}{\partial x_j \partial x_k}(p+s_0e_j+t_0e_k).$$

In other words,

$$g(s,t) := \frac{f(p+se_j+te_k)-f(p+te_k)-f(p+se_j)+f(p)}{st} = \frac{\partial^2 f}{\partial x_j \partial x_k}(p+s_0e_j+t_0e_k).$$

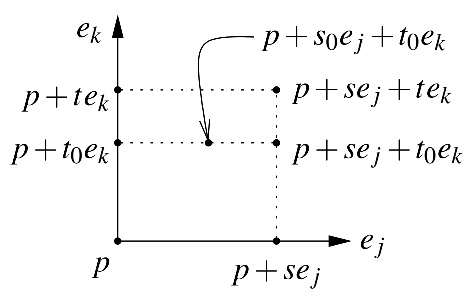

Figure 8.13: Using the mean value theorem to estimate a second order partial derivative by a certain difference quotient.

See Figure 8.13. The s_0 and t_0 depend on s and t, but $0 < s_0 < s$ and $0 < t_0 < t$. Denote by $\mathbb{R}^2_+$ the set of (s,t) where $s > 0$ and $t > 0$. The set $\mathbb{R}^2_+$ is the domain of g, and $(0,0)$ is a cluster point of $\mathbb{R}^2_+$. As $(s,t) \in \mathbb{R}^2_+$ goes to $(0,0)$, $(s_0,t_0) \in \mathbb{R}^2_+$ also goes to $(0,0)$. By continuity of the second partial derivatives,

$$\lim_{(s,t)\to(0,0)} g(s,t) = \frac{\partial^2 f}{\partial x_j \partial x_k}(p).$$

Now reverse the ordering. Start with the function $\sigma \mapsto f(p+\sigma e_j+te_k) - f(p+\sigma e_j)$ find an $s_1 \in (0,s)$ such that

$$\frac{f(p+te_k+se_j)-f(p+se_j)-f(p+te_k)+f(p)}{s} = \frac{\partial f}{\partial x_j}(p+te_k+s_1e_j) - \frac{\partial f}{\partial x_j}(p+s_1e_j).$$

Find a $t_1 \in (0,t)$ such that

$$\frac{\frac{\partial f}{\partial x_j}(p+te_k+s_1e_j) - \frac{\partial f}{\partial x_j}(p+s_1e_j)}{t} = \frac{\partial^2 f}{\partial x_k \partial x_j}(p+t_1e_k+s_1e_j).$$

So $g(s,t) = \frac{\partial^2 f}{\partial x_k \partial x_j}(p + t_1 e_k + s_1 e_j)$ for the same g as above. And as before

$$\lim_{(s,t)\to(0,0)} g(s,t) = \frac{\partial^2 f}{\partial x_k \partial x_j}(p).$$

Therefore the two partial derivatives are equal. $\square$

The proposition does not hold if the derivatives are not continuous. See the Exercise 8.6.2. Notice also that we did not really need a C^2 function, we only needed the two second order partial derivatives involved to be continuous functions.

8.6.1 Exercises

***Exercise* 8.6.1:** *Suppose $f: U \to \mathbb{R}$ is a C^2 function for some open $U \subset \mathbb{R}^n$ and $p \in U$. Use the proof of Proposition 8.6.2 to find an expression in terms of just the values of f (analogue of the difference quotient for the first derivative), whose limit is $\frac{\partial^2 f}{\partial x_j \partial x_k}(p)$.*

***Exercise* 8.6.2:** *Define*

$$f(x,y) := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that

a) The first order partial derivatives exist and are continuous.

b) The partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist, but are not continuous at the origin, and $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

***Exercise* 8.6.3:** *Suppose $f: U \to \mathbb{R}$ is a C^k function for some open $U \subset \mathbb{R}^n$ and $p \in U$. Suppose $j_1, j_2, \ldots, j_k$ are integers between 1 and n, and suppose $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ is a permutation of $(1,2,\ldots,k)$. Prove*

$$\frac{\partial^k f}{\partial x_{j_k} \partial x_{j_{k-1}} \cdots \partial x_{j_1}}(p) = \frac{\partial^k f}{\partial x_{j_{\sigma_k}} \partial x_{j_{\sigma_{k-1}}} \cdots \partial x_{j_{\sigma_1}}}(p).$$

***Exercise* 8.6.4:** *Suppose $\varphi: \mathbb{R}^2 \to \mathbb{R}$ is a C^k function such that $\varphi(0,\theta) = \varphi(0,\psi)$ for all $\theta, \psi \in \mathbb{R}$ and $\varphi(r,\theta) = \varphi(r,\theta + 2\pi)$ for all $r, \theta \in \mathbb{R}$. Let $F(r,\theta) := \big(r\cos(\theta), r\sin(\theta)\big)$ from Exercise 8.5.8. Show that a function $g: \mathbb{R}^2 \to \mathbb{R}$, given $g(x,y) := \varphi\big(F^{-1}(x,y)\big)$ is well-defined (notice that $F^{-1}(x,y)$ can only be defined locally), and when restricted to $\mathbb{R}^2 \setminus \{0\}$ it is a C^k function.*
Note: Feel free to use what you know about sine and cosine from calculus.

***Exercise* 8.6.5:** *Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function. For all $(x,y) \in \mathbb{R}^2$, compute*

$$\lim_{t\to 0} \frac{f(x+t,y) + f(x-t,y) + f(x,y+t) + f(x,y-t) - 4f(x,y)}{t^2}$$

in terms of the partial derivatives of f.

***Exercise* 8.6.6:** *Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is a function such that all first and second order partial derivatives exist. Furthermore, suppose that all second order partial derivatives are bounded functions. Prove that f is continuously differentiable.*

***Exercise* 8.6.7:** *Follow the strategy below to prove the following simple version of the second derivative test for functions defined on $\mathbb{R}^2$ (using (x,y) as coordinates):* Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a twice continuously differentiable function with a critical point at the origin, $f'(0,0) = 0$. If

$$\frac{\partial^2 f}{\partial x^2}(0,0) > 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}(0,0)\frac{\partial^2 f}{\partial y^2}(0,0) - \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right)^2 > 0,$$

then f has a (strict) local minimum at $(0,0)$. *Use the following technique: First suppose without loss of generality that $f(0,0) = 0$. Then prove:*

a) There exists an $A \in L(\mathbb{R}^2)$ such that $g = f \circ A$ is such that $\frac{\partial^2 g}{\partial x \partial y}(0,0) = 0$, and $\frac{\partial^2 g}{\partial x^2}(0,0) = \frac{\partial^2 g}{\partial y^2}(0,0) = 1$.

b) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left|g(x,y) - x^2 - y^2\right| < \varepsilon(x^2 + y^2)$ for all $(x,y) \in B\big((0,0),\delta\big)$. Hint: You can use Taylor's theorem in one variable.

c) This means that g, and therefore f, has a strict local minimum at $(0,0)$.

Note: You must avoid the temptation to just apply the one variable second derivative test along lines through the origin, see Exercise 8.3.11.

Chapter 9

One-dimensional Integrals in Several Variables

9.1 Differentiation under the integral

Note: less than 1 lecture

Let $f(x,y)$ be a function of two variables and define

$$g(y) := \int_a^b f(x,y)\,dx.$$

If f is continuous on the compact rectangle $[a,b] \times [c,d]$, then Proposition 7.5.12 from volume I says that g is continuous on $[c,d]$.

Suppose f is differentiable in y. The main question we want to ask is when can we "differentiate under the integral," that is, when is it true that g is differentiable and its derivative is

$$g'(y) \stackrel{?}{=} \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

Differentiation is a limit and therefore we are really asking when do the two limiting operations of integration and differentiation commute. This is not always possible and some extra hypothesis is necessary. The first question we would face is the integrability of $\frac{\partial f}{\partial y}$, but the formula above can fail even if $\frac{\partial f}{\partial y}$ is integrable as a function of x for every fixed y.

We prove a simple, but perhaps the most useful version of this kind of result.

Theorem 9.1.1 (Leibniz integral rule)**.** *Suppose $f\colon [a,b] \times [c,d] \to \mathbb{R}$ is a continuous function, such that $\frac{\partial f}{\partial y}$ exists for all $(x,y) \in [a,b] \times [c,d]$ and is continuous. Define*

$$g(y) := \int_a^b f(x,y)\,dx.$$

Then $g\colon [c,d] \to \mathbb{R}$ is continuously differentiable and

$$g'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

The hypotheses on f and $\frac{\partial f}{\partial y}$ can be weakened, see e.g. Exercise 9.1.8, but not dropped outright. The main point in the proof requires that $\frac{\partial f}{\partial y}$ exists and is continuous for all x up to the endpoints, but we only need a small interval in the y direction. In applications, we often make $[c,d]$ a small interval around the point where we need to differentiate.

Proof. Fix $y \in [c,d]$ and let $\varepsilon > 0$ be given. As $\frac{\partial f}{\partial y}$ is continuous on $[a,b] \times [c,d]$ it is uniformly continuous. In particular, there exists $\delta > 0$ such that whenever $y_1 \in [c,d]$ with $|y_1 - y| < \delta$ and all $x \in [a,b]$, we have

$$\left| \frac{\partial f}{\partial y}(x,y_1) - \frac{\partial f}{\partial y}(x,y) \right| < \varepsilon.$$

Suppose h is such that $y+h \in [c,d]$ and $|h| < \delta$. Fix x for a moment and apply the mean value theorem to find a y_1 between y and $y+h$ such that

$$\frac{f(x,y+h) - f(x,y)}{h} = \frac{\partial f}{\partial y}(x,y_1).$$

As $|y_1 - y| \leq |h| < \delta$,

$$\left| \frac{f(x,y+h) - f(x,y)}{h} - \frac{\partial f}{\partial y}(x,y) \right| = \left| \frac{\partial f}{\partial y}(x,y_1) - \frac{\partial f}{\partial y}(x,y) \right| < \varepsilon.$$

The argument worked for every $x \in [a,b]$ (different y_1 may have been used). Thus, as a function of x

$$x \mapsto \frac{f(x,y+h) - f(x,y)}{h} \qquad \text{converges uniformly to} \qquad x \mapsto \frac{\partial f}{\partial y}(x,y) \qquad \text{as } h \to 0.$$

We defined uniform convergence for sequences although the idea is the same. You may replace h with a sequence of nonzero numbers $\{h_n\}$ converging to 0 such that $y + h_n \in [c,d]$ and let $n \to \infty$.

Consider the difference quotient of g,

$$\frac{g(y+h) - g(y)}{h} = \frac{\int_a^b f(x,y+h)\,dx - \int_a^b f(x,y)\,dx}{h} = \int_a^b \frac{f(x,y+h) - f(x,y)}{h}\,dx.$$

Uniform convergence implies the limit can be taken underneath the integral. So

$$\lim_{h\to 0} \frac{g(y+h) - g(y)}{h} = \int_a^b \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

Then g' is continuous on $[c,d]$ by Proposition 7.5.12 from volume I mentioned above. □

Example 9.1.2: Let

$$f(y) = \int_0^1 \sin(x^2 - y^2)\,dx.$$

Then

$$f'(y) = \int_0^1 -2y\cos(x^2 - y^2)\,dx.$$

Example 9.1.3: Consider

$$\int_0^1 \frac{x-1}{\ln(x)}\,dx.$$

The function under the integral extends to be continuous on $[0,1]$, and hence the integral exists, see Exercise 9.1.1. Trouble is finding it. We introduce a parameter y and define a function:

$$g(y) := \int_0^1 \frac{x^y-1}{\ln(x)}\,dx.$$

The function $\frac{x^y-1}{\ln(x)}$ also extends to a continuous function of x and y for $(x,y) \in [0,1]\times[0,1]$ (also part of the exercise). See Figure 9.1.

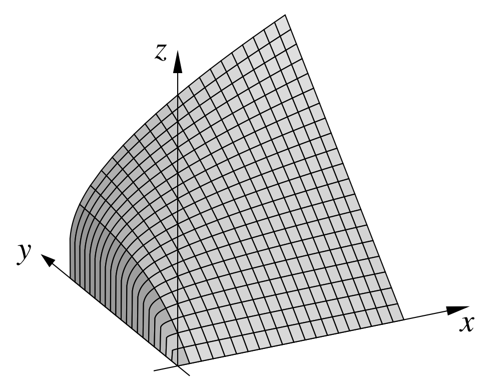

Figure 9.1: The graph $z = \frac{x^y-1}{\ln(x)}$ on $[0,1]\times[0,1]$.

Hence, g is a continuous function on $[0,1]$ and $g(0) = 0$. For every $\varepsilon > 0$, the y derivative of the integrand, x^y, is continuous on $[0,1]\times[\varepsilon,1]$. Therefore, for $y > 0$, we may differentiate under the integral sign,

$$g'(y) = \int_0^1 \frac{\ln(x)x^y}{\ln(x)}\,dx = \int_0^1 x^y\,dx = \frac{1}{y+1}.$$

We need to figure out $g(1)$ given that $g'(y) = \frac{1}{y+1}$ and $g(0) = 0$. Elementary calculus says that $g(1) = \int_0^1 g'(y)\,dy = \ln(2)$. Thus,

$$\int_0^1 \frac{x-1}{\ln(x)}\,dx = \ln(2).$$

9.1.1 Exercises

Exercise 9.1.1: *Prove the two statements that were asserted in Example 9.1.3:*

a) Prove $\frac{x-1}{\ln(x)}$ extends to a continuous function of $[0,1]$. That is, there exists a continuous function on $[0,1]$ that equals $\frac{x-1}{\ln(x)}$ on $(0,1)$.

b) Prove $\frac{x^y-1}{\ln(x)}$ extends to a continuous function on $[0,1]\times[0,1]$.

Exercise* 9.1.2*: *Suppose $h\colon \mathbb{R} \to \mathbb{R}$ is continuous and $g\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and compactly supported. That is, there exists some $M > 0$, such that $g(x) = 0$ whenever $|x| \geq M$. Define*

$$f(x) := \int_{-\infty}^{\infty} h(y)g(x-y)\, dy.$$

Show that f is differentiable.

Exercise* 9.1.3*: *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is infinitely differentiable (all derivatives exist) such that $f(0) = 0$. Then show that there exists an infinitely differentiable function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = x\,g(x)$. Show also that if $f'(0) \neq 0$, then $g(0) \neq 0$.*
Hint: First write $f(x) = \int_0^x f'(s)\, ds$ and then rewrite the integral to go from 0 to 1.

Exercise* 9.1.4*: *Compute $\int_0^1 e^{tx}\, dx$. Derive the formula for $\int_0^1 x^n e^x\, dx$ not using integration by parts, but by differentiation underneath the integral.*

Exercise* 9.1.5*: *Let $U \subset \mathbb{R}^n$ be an open set and suppose $f(x, y_1, y_2, \ldots, y_n)$ is a continuous function defined on $[0,1] \times U \subset \mathbb{R}^{n+1}$. Suppose $\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \ldots, \frac{\partial f}{\partial y_n}$ exist and are continuous on $[0,1] \times U$. Then prove that $F\colon U \to \mathbb{R}$ defined by*

$$F(y_1, y_2, \ldots, y_n) := \int_0^1 f(x, y_1, y_2, \ldots, y_n)\, dx$$

is continuously differentiable.

Exercise* 9.1.6*: *Work out the following counterexample: Let*

$$f(x,y) := \begin{cases} \frac{xy^3}{(x^2+y^2)^2} & \text{if } x \neq 0 \text{ or } y \neq 0, \\ 0 & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

a) Prove that for every fixed y, the function $x \mapsto f(x,y)$ is Riemann integrable on $[0,1]$, and

$$g(y) := \int_0^1 f(x,y)\, dx = \frac{y}{2y^2+2}.$$

Therefore, $g'(y)$ exists and its derivative is the continuous function

$$g'(y) = \frac{d}{dy} \int_0^1 f(x,y)\, dx = \frac{1-y^2}{2(y^2+1)^2}.$$

b) Prove $\frac{\partial f}{\partial y}$ exists at all x and y and compute it.

c) Show that for all y

$$\int_0^1 \frac{\partial f}{\partial y}(x,y)\, dx$$

exists, but

$$g'(0) \neq \int_0^1 \frac{\partial f}{\partial y}(x,0)\, dx.$$

Exercise* 9.1.7*: *Work out the following counterexample: Let*

$$f(x,y) := \begin{cases} x \sin\left(\frac{y}{x^2+y^2}\right) & \textit{if } (x,y) \neq (0,0), \\ 0 & \textit{if } (x,y) = (0,0). \end{cases}$$

a) Prove f is continuous on all of $\mathbb{R}^2$. Therefore the following function is well-defined for every $y \in \mathbb{R}$:

$$g(y) := \int_0^1 f(x,y)\,dx.$$

b) Prove $\frac{\partial f}{\partial y}$ exists for all (x,y), but is not continuous at $(0,0)$.

c) Show that $\int_0^1 \frac{\partial f}{\partial y}(x,0)\,dx$ does not exist even if we take improper integrals, that is, that the limit $\lim\limits_{h\to 0^+} \int_h^1 \frac{\partial f}{\partial y}(x,0)\,dx$ does not exist.

Note: Feel free to use what you know about sine and cosine from calculus.

Exercise* 9.1.8*: *Strengthen the Leibniz integral rule in the following way. Suppose $f\colon (a,b)\times(c,d) \to \mathbb{R}$ is a bounded continuous function, such that $\frac{\partial f}{\partial y}$ exists for all $(x,y) \in (a,b)\times(c,d)$ and is continuous and bounded. Define*

$$g(y) := \int_a^b f(x,y)\,dx.$$

Then $g\colon (c,d) \to \mathbb{R}$ is continuously differentiable and

$$g'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

Hint: See also Exercise 7.5.18 and Theorem 6.2.10 from volume I.

9.2 Path integrals

Note: 2–3 lectures

9.2.1 Piecewise smooth paths

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a function and write $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. Suppose γ is *continuously differentiable*, that is, it is differentiable and the derivative is continuous. In other words, there exists a continuous function $\gamma'\colon [a,b] \to \mathbb{R}^n$ such that for every $t \in [a,b]$, we have $\lim\limits_{h\to 0} \frac{\lVert \gamma(t+h)-\gamma(t)-\gamma'(t)\,h \rVert}{\lvert h \rvert} = 0$. We treat $\gamma'(t)$ either as a linear operator (an $n \times 1$ matrix) or a vector, $\gamma'(t) = \bigl(\gamma_1'(t), \gamma_2'(t), \ldots, \gamma_n'(t)\bigr)$. Equivalently, γ_j is a continuously differentiable function on $[a,b]$ for every $j = 1,2,\ldots,n$. By Exercise 8.2.6, the operator norm of the operator $\gamma'(t)$ is equal to the euclidean norm of the corresponding vector, so there is no confusion when writing $\lVert \gamma'(t) \rVert$.

Definition 9.2.1. A continuously differentiable function $\gamma\colon [a,b] \to \mathbb{R}^n$ is called a *smooth path* or a *continuously differentiable path** if γ is continuously differentiable and $\gamma'(t) \neq 0$ for all $t \in [a,b]$.

The function $\gamma\colon [a,b] \to \mathbb{R}^n$ is called a *piecewise smooth path* or a *piecewise continuously differentiable path* if there exist finitely many points $t_0 = a < t_1 < t_2 < \cdots < t_k = b$ such that the restriction $\gamma|_{[t_{j-1},t_j]}$ is smooth path for every $j = 1,2,\ldots,k$.

A path γ is a *closed path* if $\gamma(a) = \gamma(b)$, that is if the path starts and ends in the same point. A path γ is a *simple path* if either 1) γ is a one-to-one function, or 2) $\gamma|_{[a,b)}$ is one-to-one and $\gamma(a) = \gamma(b)$ (γ is a simple closed path).

Example 9.2.2: Let $\gamma\colon [0,4] \to \mathbb{R}^2$ be defined by

$$\gamma(t) := \begin{cases} (t,0) & \text{if } t \in [0,1], \\ (1,t-1) & \text{if } t \in (1,2], \\ (3-t,1) & \text{if } t \in (2,3], \\ (0,4-t) & \text{if } t \in (3,4]. \end{cases}$$

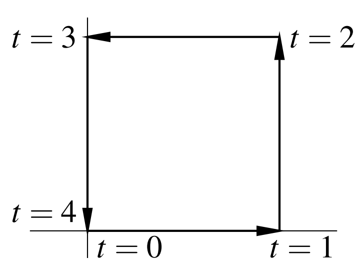

Figure 9.2: The path γ traversing the unit square.

The path γ is the unit square traversed counterclockwise. See Figure 9.2. It is a piecewise smooth path. For example, $\gamma|_{[1,2]}(t) = (1,t-1)$ and so $(\gamma|_{[1,2]})'(t) = (0,1) \neq 0$. Similarly for the

*The word “smooth” can sometimes mean “infinitely differentiable” in the literature.

other 3 sides. Notice that $(\gamma|_{[1,2]})'(1) = (0,1)$, $(\gamma|_{[0,1]})'(1) = (1,0)$, but $\gamma'(1)$ does not exist. At the corners γ is not differentiable. The path γ is a simple closed path, as $\gamma|_{[0,4)}$ is one-to-one and $\gamma(0) = \gamma(4)$.

The definition of a piecewise smooth path as we have given it implies continuity (exercise). For general functions, many authors also allow finitely many discontinuities, when they use the term *piecewise smooth*, and so one may say that we defined a piecewise smooth path to be a *continuous piecewise smooth* function. While one may get by with smooth paths, for computations, the simplest paths to write down are often piecewise smooth.

Generally, we are interested in the direct image $\gamma\big([a,b]\big)$, rather than the specific parametrization, although that is also important to some degree. When we informally talk about a path or a curve, we often mean the set $\gamma\big([a,b]\big)$, depending on context.

Example 9.2.3: The condition $\gamma'(t) \neq 0$ means that the image $\gamma\big([a,b]\big)$ has no "corners" where γ is smooth. Consider

$$\gamma(t) := \begin{cases} (t^2,0) & \text{if } t < 0, \\ (0,t^2) & \text{if } t \geq 0. \end{cases}$$

See Figure 9.3. It is left for the reader to check that γ is continuously differentiable, yet the image $\gamma(\mathbb{R}) = \big\{ (x,y) \in \mathbb{R}^2 : (x,y) = (s,0) \text{ or } (x,y) = (0,s) \text{ for some } s \geq 0 \big\}$ has a "corner" at the origin. And that is because $\gamma'(0) = (0,0)$. More complicated examples with, say, infinitely many corners exist, see the exercises.

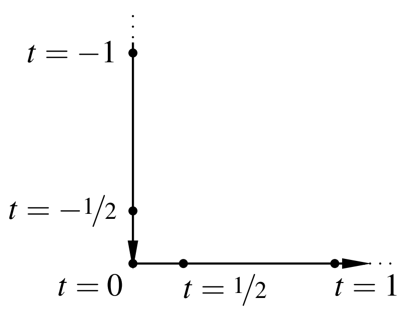

Figure 9.3: Smooth path with zero derivative with a corner. Several values of t are marked with dots.

The condition $\gamma'(t) \neq 0$ even at the endpoints guarantees not only no corners, but also that the path ends nicely, that is, it can extend a little bit past the endpoints. Again, see the exercises.

Example 9.2.4: A graph of a continuously differentiable function $f : [a,b] \to \mathbb{R}$ is a smooth path. Define $\gamma : [a,b] \to \mathbb{R}^2$ by

$$\gamma(t) := \big(t, f(t)\big).$$

Then $\gamma'(t) = \big(1, f'(t)\big)$, which is never zero, and $\gamma\big([a,b]\big)$ is the graph of f.

There are other ways of parametrizing the path. That is, there are different paths with the same image. The function $t \mapsto (1-t)a + tb$, takes the interval $[0,1]$ to $[a,b]$. Define $\alpha : [0,1] \to \mathbb{R}^2$ by

$$\alpha(t) := \big((1-t)a+tb, f((1-t)a+tb)\big).$$

Then $\alpha'(t) = \big(b-a,\ (b-a)f'((1-t)a+tb)\big)$, which is never zero. As sets, $\alpha\big([0,1]\big) = \gamma\big([a,b]\big) = \big\{(x,y) \in \mathbb{R}^2 : x \in [a,b] \text{ and } f(x) = y\big\}$, which is just the graph of f.

The last example leads us to a definition.

Definition 9.2.5. Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a smooth path and $h\colon [c,d] \to [a,b]$ a continuously differentiable bijective function such that $h'(t) \neq 0$ for all $t \in [c,d]$. Then the composition $\gamma \circ h$ is called a *smooth reparametrization* of γ.

Let γ be a piecewise smooth path, and h a piecewise smooth bijective function with nonzero one-sided limits of h'. The composition $\gamma \circ h$ is called a *piecewise smooth reparametrization* of γ.

If h is strictly increasing, then h is said to *preserve orientation*. If h does not preserve orientation, then h is said to *reverse orientation*.

A reparametrization is another path for the same set. That is, $(\gamma \circ h)\big([c,d]\big) = \gamma\big([a,b]\big)$.

The conditions on the piecewise smooth h mean that there is some partition $t_0 = c < t_1 < t_2 < \cdots < t_k = d$, such that $h|_{[t_{j-1},t_j]}$ is continuously differentiable and $(h|_{[t_{j-1},t_j]})'(t) \neq 0$ for all $t \in [t_{j-1},t_j]$. Since h is bijective, it is either strictly increasing or strictly decreasing. So either $(h|_{[t_{j-1},t_j]})'(t) > 0$ for all t or $(h|_{[t_{j-1},t_j]})'(t) < 0$ for all t.

Proposition 9.2.6. *If $\gamma\colon [a,b] \to \mathbb{R}^n$ is a piecewise smooth path, and $\gamma \circ h\colon [c,d] \to \mathbb{R}^n$ is a piecewise smooth reparametrization, then $\gamma \circ h$ is a piecewise smooth path.*

Proof. Assume that h preserves orientation, that is, h is strictly increasing. If $h\colon [c,d] \to [a,b]$ gives a piecewise smooth reparametrization, then for some partition $r_0 = c < r_1 < r_2 < \cdots < r_\ell = d$, the restriction $h|_{[r_{j-1},r_j]}$ is continuously differentiable with a positive derivative.

Let $t_0 = a < t_1 < t_2 < \cdots < t_k = b$ be the partition from the definition of piecewise smooth for γ together with the points $\{h(r_0), h(r_1), h(r_2), \ldots, h(r_\ell)\}$. Let $s_j := h^{-1}(t_j)$. Then $s_0 = c < s_1 < s_2 < \cdots < s_k = d$ is a partition that includes (is a refinement of) the $\{r_0, r_1, \ldots, r_\ell\}$. If $\tau \in [s_{j-1}, s_j]$, then $h(\tau) \in [t_{j-1}, t_j]$ since $h(s_{j-1}) = t_{j-1}$, $h(s_j) = t_j$, and h is strictly increasing. Also $h|_{[s_{j-1},s_j]}$ is continuously differentiable, and $\gamma|_{[t_{j-1},t_j]}$ is also continuously differentiable. Then

$$(\gamma \circ h)|_{[s_{j-1},s_j]}(\tau) = \gamma|_{[t_{j-1},t_j]}\big(h|_{[s_{j-1},s_j]}(\tau)\big).$$

The function $(\gamma \circ h)|_{[s_{j-1},s_j]}$ is therefore continuously differentiable and by the chain rule

$$\big((\gamma \circ h)|_{[s_{j-1},s_j]}\big)'(\tau) = \big(\gamma|_{[t_{j-1},t_j]}\big)'\big(h(\tau)\big)(h|_{[s_{j-1},s_j]})'(\tau) \neq 0.$$

Consequently, $\gamma \circ h$ is a piecewise smooth path. Orientation reversing h is left as an exercise. □

If two paths are simple and their images are the same, it is left as an exercise that there exists a reparametrization. Here is where our assumption that γ' is never zero is important.

9.2.2 Path integral of a one-form

Definition 9.2.7. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be our coordinates. Given n real-valued continuous functions $\omega_1, \omega_2, \ldots, \omega_n$ defined on a set $S \subset \mathbb{R}^n$, we define a *one-form* to be an object of the form

$$\omega = \omega_1\, dx_1 + \omega_2\, dx_2 + \cdots + \omega_n\, dx_n.$$

We could represent ω as a continuous function from S to $\mathbb{R}^n$, although it is better to think of it as a different object.

Example 9.2.8:

$$\omega(x,y) := \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

is a one-form defined on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Definition 9.2.9. Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a smooth path and let

$$\omega = \omega_1\,dx_1 + \omega_2\,dx_2 + \cdots + \omega_n\,dx_n,$$

be a one-form defined on the direct image $\gamma\bigl([a,b]\bigr)$. Write $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. Define:

$$\begin{aligned}\int_\gamma \omega &:= \int_a^b \Bigl(\omega_1\bigl(\gamma(t)\bigr)\gamma_1'(t) + \omega_2\bigl(\gamma(t)\bigr)\gamma_2'(t) + \cdots + \omega_n\bigl(\gamma(t)\bigr)\gamma_n'(t)\Bigr)\,dt \\ &= \int_a^b \left(\sum_{j=1}^n \omega_j\bigl(\gamma(t)\bigr)\gamma_j'(t)\right)\,dt.\end{aligned}$$

To remember the definition note that x_j is $\gamma_j(t)$, so dx_j becomes $\gamma_j'(t)\,dt$.

If γ is piecewise smooth, take the corresponding partition $t_0 = a < t_1 < t_2 < \ldots < t_k = b$, and assume the partition is minimal in the sense that γ is not differentiable at $t_1, t_2, \ldots, t_{k-1}$. As each $\gamma|_{[t_{j-1},t_j]}$ is a smooth path, define

$$\int_\gamma \omega := \int_{\gamma|_{[t_0,t_1]}} \omega + \int_{\gamma|_{[t_1,t_2]}} \omega + \cdots + \int_{\gamma|_{[t_{k-1},t_k]}} \omega.$$

The notation makes sense from the formula you remember from calculus, let us state it somewhat informally: If $x_j(t) = \gamma_j(t)$, then $dx_j = \gamma_j'(t)\,dt$.

Paths can be cut up or concatenated. The proof is a direct application of the additivity of the Riemann integral, and is left as an exercise. The proposition justifies why we defined the integral over a piecewise smooth path in the way we did, and it justifies that we may as well have taken any partition not just the minimal one in the definition.

Proposition 9.2.10. *Let $\gamma\colon [a,c] \to \mathbb{R}^n$ be a piecewise smooth path, and $b \in (a,c)$. Define the piecewise smooth paths $\alpha := \gamma|_{[a,b]}$ and $\beta := \gamma|_{[b,c]}$. Let ω be a one-form defined on $\gamma\bigl([a,c]\bigr)$. Then*

$$\int_\gamma \omega = \int_\alpha \omega + \int_\beta \omega.$$

Example 9.2.11: Let the one-form ω and the path $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ be defined by

$$\omega(x,y) := \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy, \qquad \gamma(t) := \bigl(\cos(t), \sin(t)\bigr).$$

Then

$$\begin{aligned}\int_\gamma \omega &= \int_0^{2\pi} \left(\frac{-\sin(t)}{\bigl(\cos(t)\bigr)^2 + \bigl(\sin(t)\bigr)^2}\bigl(-\sin(t)\bigr) + \frac{\cos(t)}{\bigl(\cos(t)\bigr)^2 + \bigl(\sin(t)\bigr)^2}\bigl(\cos(t)\bigr)\right)\,dt \\ &= \int_0^{2\pi} 1\,dt = 2\pi.\end{aligned}$$

Next, let us parametrize the same curve as $\alpha\colon [0,1] \to \mathbb{R}^2$ defined by $\alpha(t) := \big(\cos(2\pi t), \sin(2\pi t)\big)$, that is α is a smooth reparametrization of γ. Then

$$\begin{aligned}\int_\alpha \omega = \int_0^1 \Bigg(& \frac{-\sin(2\pi t)}{\big(\cos(2\pi t)\big)^2 + \big(\sin(2\pi t)\big)^2}\big(-2\pi\sin(2\pi t)\big) \\ & + \frac{\cos(2\pi t)}{\big(\cos(2\pi t)\big)^2 + \big(\sin(2\pi t)\big)^2}\big(2\pi\cos(2\pi t)\big)\Bigg)\,dt \\ = \int_0^1 & 2\pi\,dt = 2\pi.\end{aligned}$$

Now let us reparametrize with $\beta\colon [0,2\pi] \to \mathbb{R}^2$ as $\beta(t) := \big(\cos(-t), \sin(-t)\big)$. Then

$$\begin{aligned}\int_\beta \omega &= \int_0^{2\pi} \Bigg(\frac{-\sin(-t)}{\big(\cos(-t)\big)^2 + \big(\sin(-t)\big)^2}\big(\sin(-t)\big) + \frac{\cos(-t)}{\big(\cos(-t)\big)^2 + \big(\sin(-t)\big)^2}\big(-\cos(-t)\big)\Bigg)\,dt \\ &= \int_0^{2\pi} (-1)\,dt = -2\pi.\end{aligned}$$

The path α is an orientation preserving reparametrization of γ, and the integrals are the same. The path β is an orientation reversing reparametrization of γ and the integral is minus the original. See Figure 9.4.

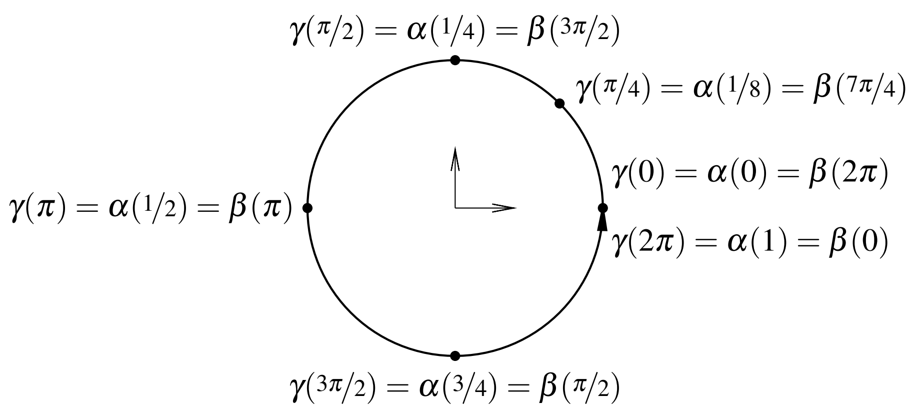

Figure 9.4: A circular path reparametrized in two different ways. The arrow indicates the orientation of γ and α. The path β traverses the circle in the opposite direction.

The previous example is not a fluke. The path integral does not depend on the parametrization of the curve, the only thing that matters is the direction in which the curve is traversed.

Proposition 9.2.12. *Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a piecewise smooth path and $\gamma \circ h\colon [c,d] \to \mathbb{R}^n$ a piecewise smooth reparametrization. Suppose ω is a one-form defined on the set $\gamma\big([a,b]\big)$. Then*

$$\int_{\gamma\circ h} \omega = \begin{cases} \int_\gamma \omega & \textit{if } h \textit{ preserves orientation,} \\ -\int_\gamma \omega & \textit{if } h \textit{ reverses orientation.}\end{cases}$$

Proof. Assume first that γ and h are both smooth. Write $\omega = \omega_1 \, dx_1 + \omega_2 \, dx_2 + \cdots + \omega_n \, dx_n$. Suppose that h is orientation preserving. Use the change of variables formula for the Riemann integral:

$$\begin{aligned}
\int_\gamma \omega &= \int_a^b \left(\sum_{j=1}^n \omega_j\bigl(\gamma(t)\bigr) \gamma_j'(t) \right) dt \\
&= \int_c^d \left(\sum_{j=1}^n \omega_j\Bigl(\gamma\bigl(h(\tau)\bigr)\Bigr) \gamma_j'\bigl(h(\tau)\bigr) \right) h'(\tau) \, d\tau \\
&= \int_c^d \left(\sum_{j=1}^n \omega_j\Bigl(\gamma\bigl(h(\tau)\bigr)\Bigr) (\gamma_j \circ h)'(\tau) \right) d\tau = \int_{\gamma \circ h} \omega.
\end{aligned}$$

If h is orientation reversing, it swaps the order of the limits on the integral and introduces a minus sign. The details, along with finishing the proof for piecewise smooth paths, is left as Exercise 9.2.4. □

Due to this proposition (and the exercises), if $\Gamma \subset \mathbb{R}^n$ is the image of a simple piecewise smooth path $\gamma\bigl([a,b]\bigr)$, then as long as we somehow indicate the orientation, that is, the direction in which we traverse the curve, we can write

$$\int_\Gamma \omega,$$

without mentioning the specific γ. Furthermore, for a simple closed path, it does not even matter where we start the parametrization. See the exercises.

Recall that *simple* means that γ is one-to-one except perhaps at the endpoints, in particular it is one-to-one when restricted to $[a,b)$. We may relax the condition that the path is simple a little bit. For example, it is enough to suppose that $\gamma\colon [a,b] \to \mathbb{R}^n$ is one-to-one except at finitely many points. See Exercise 9.2.14. But we cannot remove the condition completely as is illustrated by the following example.

Example 9.2.13: Suppose $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ is given by $\gamma(t) := \bigl(\cos(t), \sin(t)\bigr)$, and $\beta\colon [0,2\pi] \to \mathbb{R}^2$ is given by $\beta(t) := \bigl(\cos(2t), \sin(2t)\bigr)$. Notice that $\gamma\bigl([0,2\pi]\bigr) = \beta\bigl([0,2\pi]\bigr)$, and we travel around the same curve, the unit circle. But γ goes around the unit circle once in the counter clockwise direction, and β goes around the unit circle twice (in the same direction). See Figure 9.5.

Compute

$$\begin{aligned}
\int_\gamma -y \, dx + x \, dy &= \int_0^{2\pi} \Bigl(\bigl(-\sin(t)\bigr)\bigl(-\sin(t)\bigr) + \cos(t)\cos(t) \Bigr) dt = 2\pi, \\
\int_\beta -y \, dx + x \, dy &= \int_0^{2\pi} \Bigl(\bigl(-\sin(2t)\bigr)\bigl(-2\sin(2t)\bigr) + \cos(t)\bigl(2\cos(t)\bigr) \Bigr) dt = 4\pi.
\end{aligned}$$

It is sometimes convenient to define a path integral over $\gamma\colon [a,b] \to \mathbb{R}^n$ that is not a path. Define

$$\int_\gamma \omega := \int_a^b \left(\sum_{j=1}^n \omega_j\bigl(\gamma(t)\bigr) \gamma_j'(t) \right) dt$$

for every continuously differentiable γ. A case that comes up naturally is when γ is constant. Then $\gamma'(t) = 0$ for all t, and $\gamma\bigl([a,b]\bigr)$ is a single point, which we regard as a "curve" of length zero. Then, $\int_\gamma \omega = 0$ for every ω.

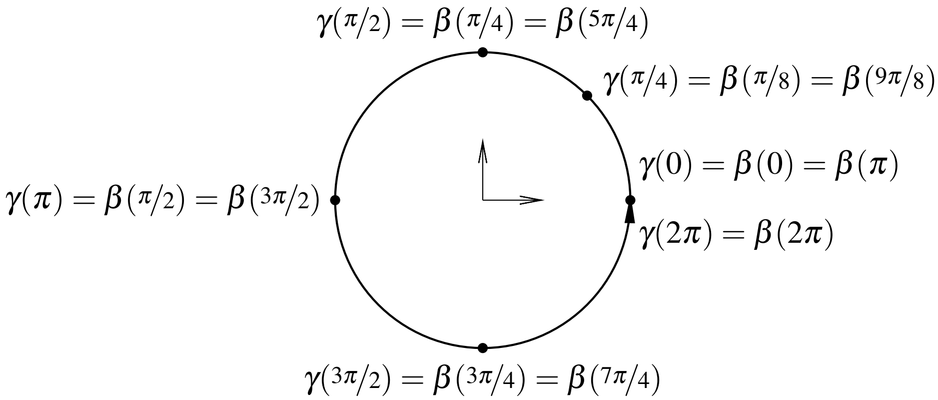

Figure 9.5: Circular path traversed once by $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ and twice by $\beta\colon [0,2\pi] \to \mathbb{R}^2$.

9.2.3 Path integral of a function

Next we integrate a function against the so-called *arc-length measure* ds. The geometric picture we have in mind is the area under the graph of the function over a path. Imagine a fence erected over γ with height given by the function and the integral is the area of the fence. See Figure 9.6.

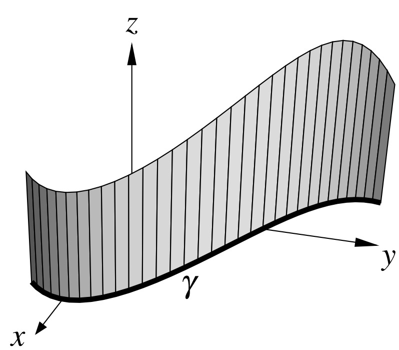

Figure 9.6: A path $\gamma\colon [a,b] \to \mathbb{R}^2$ in the xy-plane (bold curve), and a function $z = f(x,y)$ graphed above it in the z direction. The integral is the shaded area depicted.

Definition 9.2.14. Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ is a smooth path, and f is a continuous function defined on the image $\gamma\bigl([a,b]\bigr)$. Then define

$$\int_\gamma f\,ds := \int_a^b f\bigl(\gamma(t)\bigr)\,\lVert \gamma'(t) \rVert\,dt.$$

To emphasize the variables we may use

$$\int_\gamma f(x)\,ds(x) := \int_\gamma f\,ds.$$

The definition for a piecewise smooth path is similar as before and is left to the reader.

The path integral of a function is also independent of the parametrization, and in this case, the orientation does not matter.

Proposition 9.2.15. *Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a piecewise smooth path and $\gamma \circ h\colon [c,d] \to \mathbb{R}^n$ a piecewise smooth reparametrization. Suppose f is a continuous function defined on the set $\gamma\bigl([a,b]\bigr)$. Then*

$$\int_{\gamma \circ h} f\, ds = \int_{\gamma} f\, ds.$$

Proof. Suppose first that h is orientation preserving and that γ and h are both smooth. Then

$$\begin{aligned}
\int_{\gamma} f\, ds &= \int_a^b f\bigl(\gamma(t)\bigr) \|\gamma'(t)\|\, dt \\
&= \int_c^d f\Bigl(\gamma\bigl(h(\tau)\bigr)\Bigr) \|\gamma'\bigl(h(\tau)\bigr)\| h'(\tau)\, d\tau \\
&= \int_c^d f\Bigl(\gamma\bigl(h(\tau)\bigr)\Bigr) \|\gamma'\bigl(h(\tau)\bigr) h'(\tau)\|\, d\tau \\
&= \int_c^d f\bigl((\gamma \circ h)(\tau)\bigr) \|(\gamma \circ h)'(\tau)\|\, d\tau \\
&= \int_{\gamma \circ h} f\, ds.
\end{aligned}$$

If h is orientation reversing it swaps the order of the limits on the integral, but you also have to introduce a minus sign in order to take h' inside the norm. The details, along with finishing the proof for piecewise smooth paths is left to the reader as Exercise 9.2.5. □

As before, due to this proposition (and the exercises), if γ is simple, it does not matter which parametrization we use. Therefore, if $\Gamma = \gamma\bigl([a,b]\bigr)$, we can simply write

$$\int_{\Gamma} f\, ds.$$

In this case we also do not need to worry about orientation, either way we get the same integral.

Example 9.2.16: Let $f(x,y) := x$. Let $C \subset \mathbb{R}^2$ be half of the unit circle for $x \geq 0$. We wish to compute

$$\int_C f\, ds.$$

Parametrize the curve C via $\gamma\colon [-\pi/2, \pi/2] \to \mathbb{R}^2$ defined as $\gamma(t) := \bigl(\cos(t), \sin(t)\bigr)$. Then $\gamma'(t) = \bigl(-\sin(t), \cos(t)\bigr)$, and

$$\int_C f\, ds = \int_{\gamma} f\, ds = \int_{-\pi/2}^{\pi/2} \cos(t) \sqrt{\bigl(-\sin(t)\bigr)^2 + \bigl(\cos(t)\bigr)^2}\, dt = \int_{-\pi/2}^{\pi/2} \cos(t)\, dt = 2.$$

Definition 9.2.17. Suppose $\Gamma \subset \mathbb{R}^n$ is parametrized by a simple piecewise smooth path $\gamma\colon [a,b] \to \mathbb{R}^n$, that is $\gamma\bigl([a,b]\bigr) = \Gamma$. We define the *length* by

$$\ell(\Gamma) := \int_{\Gamma} ds = \int_{\gamma} ds.$$

If γ is smooth,

$$\ell(\Gamma) = \int_a^b \|\gamma'(t)\|\,dt.$$

This may be a good time to mention that it is common to write $\int_a^b \|\gamma'(t)\|\,dt$ even if the path is only piecewise smooth. That is because $\|\gamma'(t)\|$ is defined and continuous at all but finitely many points and is bounded, and so the integral exists.

Example 9.2.18: Let $x, y \in \mathbb{R}^n$ be two points and write $[x, y]$ as the straight line segment between the two points x and y. Parametrize $[x, y]$ by $\gamma(t) := (1-t)x + ty$ for t running between 0 and 1. See Figure 9.7. Then $\gamma'(t) = y - x$, and therefore

$$\ell\big([x,y]\big) = \int_{[x,y]} ds = \int_0^1 \|y-x\|\,dt = \|y-x\|.$$

So the length of $[x, y]$ is the standard euclidean distance between x and y, justifying the name.

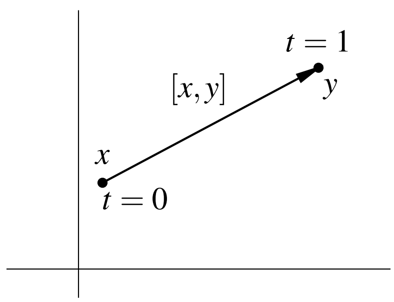

Figure 9.7: Straight path between x and y parametrized by $(1-t)x + ty$.

A simple piecewise smooth path $\gamma\colon [0, r] \to \mathbb{R}^n$ is said to be an *arc-length parametrization* if for all $t \in [0, r]$, we have

$$\ell\big(\gamma([0,t])\big) = t.$$

If γ is smooth, then

$$\int_0^t d\tau = t = \ell\big(\gamma([0,t])\big) = \int_0^t \|\gamma'(\tau)\|\,d\tau$$

for all t, which means that $\|\gamma'(t)\| = 1$ for all t. Similarly for piecewise smooth γ, we get $\|\gamma'(t)\| = 1$ for all t where the derivative exists. So you can think of such a parametrization as moving around your curve at speed 1. If $\gamma\colon [0, r] \to \mathbb{R}^n$ is an arclength parametrization, it is common to use s as the variable as $\int_\gamma f\,ds = \int_0^r f\big(\gamma(s)\big)\,ds$.

9.2.4 Exercises

***Exercise* 9.2.1:** *Show that if $\varphi\colon [a, b] \to \mathbb{R}^n$ is a piecewise smooth path as we defined it, then φ is a continuous function.*

***Exercise* 9.2.2:** *Finish the proof of Proposition 9.2.6 for orientation reversing reparametrizations.*

Exercise* 9.2.3*: *Prove Proposition 9.2.10.*

Exercise* 9.2.4*: *Finish the proof of Proposition 9.2.12 for*

a) *orientation reversing reparametrizations, and*
b) *piecewise smooth paths and reparametrizations.*

Exercise* 9.2.5*: *Finish the proof of Proposition 9.2.15 for*

a) *orientation reversing reparametrizations, and*
b) *piecewise smooth paths and reparametrizations.*

Exercise* 9.2.6*: *Suppose* $\gamma\colon [a,b] \to \mathbb{R}^n$ *is a piecewise smooth path, and* f *is a continuous function defined on the image* $\gamma\big([a,b]\big)$. *Provide a definition of* $\int_\gamma f\, ds$.

Exercise* 9.2.7*: *Directly using the definitions compute:*

a) *The arc-length of the unit square from Example 9.2.2 using the given parametrization.*
b) *The arc-length of the unit circle using the parametrization* $\gamma\colon [0,1] \to \mathbb{R}^2$, $\gamma(t) := \big(\cos(2\pi t), \sin(2\pi t)\big)$.
c) *The arc-length of the unit circle using the parametrization* $\beta\colon [0,2\pi] \to \mathbb{R}^2$, $\beta(t) := \big(\cos(t), \sin(t)\big)$.

Note: Feel free to use what you know about sine and cosine from calculus.

Exercise* 9.2.8*: *Suppose* $\gamma\colon [0,1] \to \mathbb{R}^n$ *is a smooth path, and* ω *is a one-form defined on the image* $\gamma\big([a,b]\big)$. *For* $r \in [0,1]$, *let* $\gamma_r\colon [0,r] \to \mathbb{R}^n$ *be defined as simply the restriction of* γ *to* $[0,r]$. *Show that the function* $h(r) := \int_{\gamma_r} \omega$ *is a continuously differentiable function on* $[0,1]$.

Exercise* 9.2.9*: *Suppose* $\gamma\colon [a,b] \to \mathbb{R}^n$ *is a smooth path. Show that there exists an* $\varepsilon > 0$ *and a smooth function* $\widetilde{\gamma}\colon (a-\varepsilon, b+\varepsilon) \to \mathbb{R}^n$ *with* $\widetilde{\gamma}(t) = \gamma(t)$ *for all* $t \in [a,b]$ *and* $\widetilde{\gamma}'(t) \neq 0$ *for all* $t \in (a-\varepsilon, b+\varepsilon)$. *That is, prove that a smooth path extends some small distance past the end points.*

Exercise* 9.2.10*: *Suppose* $\alpha\colon [a,b] \to \mathbb{R}^n$ *and* $\beta\colon [c,d] \to \mathbb{R}^n$ *are piecewise smooth paths such that* $\Gamma := \alpha\big([a,b]\big) = \beta\big([c,d]\big)$. *Show that there exist finitely many points* $\{p_1, p_2, \ldots, p_k\} \in \Gamma$, *such that the sets* $\alpha^{-1}\big(\{p_1, p_2, \ldots, p_k\}\big)$ *and* $\beta^{-1}\big(\{p_1, p_2, \ldots, p_k\}\big)$ *are partitions of* $[a,b]$ *and* $[c,d]$ *such that on every subinterval the paths are smooth (that is, they are partitions as in the definition of piecewise smooth path).*

Exercise* 9.2.11*:

a) *Suppose* $\gamma\colon [a,b] \to \mathbb{R}^n$ *and* $\alpha\colon [c,d] \to \mathbb{R}^n$ *are two smooth paths that are one-to-one and* $\gamma\big([a,b]\big) = \alpha\big([c,d]\big)$. *Then there exists a smooth reparametrization* $h\colon [a,b] \to [c,d]$ *such that* $\gamma = \alpha \circ h$.
Hint 1: It is not hard to show h *exists. The trick is to prove it is continuously differentiable with a nonzero derivative. Apply the implicit function theorem though it may at first seem the dimensions are wrong.*
Hint 2: Worry about derivative of h *in* (a,b) *first.*
b) *Prove the same thing as part a, but now for simple closed paths with the further assumption that* $\gamma(a) = \gamma(b) = \alpha(c) = \alpha(d)$.
c) *Prove parts a) and b) but for piecewise smooth paths, obtaining piecewise smooth reparametrizations.*
Hint: The trick is to find two partitions such that when restricted to a subinterval of the partition both paths have the same image and are smooth, see the exercise above.

Exercise* 9.2.12*: *Suppose $\alpha\colon [a,b] \to \mathbb{R}^n$ and $\beta\colon [b,c] \to \mathbb{R}^n$ are piecewise smooth paths with $\alpha(b) = \beta(b)$. Let $\gamma\colon [a,c] \to \mathbb{R}^n$ be defined by*

$$\gamma(t) := \begin{cases} \alpha(t) & \text{if } t \in [a,b], \\ \beta(t) & \text{if } t \in (b,c]. \end{cases}$$

Show that γ is a piecewise smooth path, and that if ω is a one-form defined on the curve given by γ, then

$$\int_\gamma \omega = \int_\alpha \omega + \int_\beta \omega.$$

Exercise* 9.2.13*: *Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ and $\beta\colon [c,d] \to \mathbb{R}^n$ are two simple closed piecewise smooth paths. That is, $\gamma(a) = \gamma(b)$ and $\beta(c) = \beta(d)$ and the restrictions $\gamma|_{[a,b)}$ and $\beta|_{[c,d)}$ are one-to-one. Suppose $\Gamma = \gamma\big([a,b]\big) = \beta\big([c,d]\big)$ and ω is a one-form defined on $\Gamma \subset \mathbb{R}^n$. Show that either*

$$\int_\gamma \omega = \int_\beta \omega, \qquad \textit{or} \qquad \int_\gamma \omega = -\int_\beta \omega.$$

In particular, the notation $\int_\Gamma \omega$ makes sense if we indicate the direction in which the integral is evaluated. Hint: See previous three exercises.

Exercise* 9.2.14*: *Suppose $\gamma\colon [a,b] \to \mathbb{R}^n$ and $\beta\colon [c,d] \to \mathbb{R}^n$ are two piecewise smooth paths which are one-to-one except at finitely many points. That is, there exist finite sets $S \subset [a,b]$ and $T \subset [c,d]$ such that $\gamma|_{[a,b]\setminus S}$ and $\beta|_{[c,d]\setminus T}$ are one-to-one. Suppose $\Gamma = \gamma\big([a,b]\big) = \beta\big([c,d]\big)$ and ω is a one-form defined on $\Gamma \subset \mathbb{R}^n$. Show that either*

$$\int_\gamma \omega = \int_\beta \omega, \qquad \textit{or} \qquad \int_\gamma \omega = -\int_\beta \omega.$$

In particular, the notation $\int_\Gamma \omega$ makes sense if we indicate the direction in which the integral is evaluated. Hint: Same hint as the last exercise.

Exercise* 9.2.15*: *Define $\gamma\colon [0,1] \to \mathbb{R}^2$ by $\gamma(t) := \Big(t^3 \sin(1/t),\, t\big(3t^2\sin(1/t) - t\cos(1/t)\big)^2\Big)$ for $t \neq 0$ and $\gamma(0) = (0,0)$. Show that*

a) γ is continuously differentiable on $[0,1]$.

b) Show that there exists an infinite sequence $\{t_n\}$ in $[0,1]$ converging to 0, such that $\gamma'(t_n) = (0,0)$.

c) Show that the points $\gamma(t_n)$ lie on the line $y = 0$ and such that the x-coordinate of $\gamma(t_n)$ alternates between positive and negative (if they do not alternate you only found a subsequence, you need to find them all).

d) Show that there is no piecewise smooth α whose image equals $\gamma\big([0,1]\big)$. Hint: Look at part c) and show that α' must be zero where it reaches the origin.

e) (Computer) If you know a plotting software that allows you to plot parametric curves, make a plot of the curve, but only for t in the range $[0,0.1]$ otherwise you will not see the behavior. In particular, you should notice that $\gamma\big([0,1]\big)$ has infinitely many "corners" near the origin.

Note: Feel free to use what you know about sine and cosine from calculus.

9.3 Path independence

Note: 2 lectures

9.3.1 Path independent integrals

Let $U \subset \mathbb{R}^n$ be a set and ω a one-form defined on U. The integral of ω is said to be *path independent* if for every pair of points $x, y \in U$ and every pair of piecewise smooth paths $\gamma\colon [a,b] \to U$ and $\beta\colon [c,d] \to U$ such that $\gamma(a) = \beta(c) = x$ and $\gamma(b) = \beta(d) = y$, we have

$$\int_\gamma \omega = \int_\beta \omega.$$

In this case, we simply write

$$\int_x^y \omega := \int_\gamma \omega = \int_\beta \omega.$$

Not every one-form gives a path independent integral. Most do not.

Example 9.3.1: Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be the path $\gamma(t) := (t,0)$ going from $(0,0)$ to $(1,0)$. Let $\beta\colon [0,1] \to \mathbb{R}^2$ be the path $\beta(t) := \bigl(t,(1-t)t\bigr)$ also going between the same points. Then

$$\begin{aligned}
\int_\gamma y\,dx &= \int_0^1 \gamma_2(t)\gamma_1'(t)\,dt = \int_0^1 0(1)\,dt = 0, \\
\int_\beta y\,dx &= \int_0^1 \beta_2(t)\beta_1'(t)\,dt = \int_0^1 (1-t)t(1)\,dt = \frac{1}{6}.
\end{aligned}$$

The integral of $y\,dx$ is not path independent. In particular, $\int_{(0,0)}^{(1,0)} y\,dx$ does not make sense.

Definition 9.3.2. Let $U \subset \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}$ a continuously differentiable function. The one-form

$$df := \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n$$

is called the *total derivative* of f.

An open set $U \subset \mathbb{R}^n$ is said to be *path connected** if for every two points x and y in U, there exists a piecewise smooth path starting at x and ending at y.

We leave as an exercise that every connected open set is path connected.

Proposition 9.3.3. *Let $U \subset \mathbb{R}^n$ be a path connected open set and ω a one-form defined on U. Then $\int_x^y \omega$ is path independent (for all $x, y \in U$) if and only if there exists a continuously differentiable $f\colon U \to \mathbb{R}$ such that $\omega = df$.*

In fact, if such an f exists, then for every pair of points $x, y \in U$

$$\int_x^y \omega = f(y) - f(x).$$

*Normally only a continuous path is used in this definition, but for open sets the two definitions are equivalent. See the exercises.

In other words, if we fix $p \in U$, then $f(x) = C + \int_p^x \omega$ for some constant C.

Proof. First suppose that the integral is path independent. Pick $p \in U$. Since U is path connected, there exists a path from p to every $x \in U$. Define

$$f(x) := \int_p^x \omega.$$

Write $\omega = \omega_1 \, dx_1 + \omega_2 \, dx_2 + \cdots + \omega_n \, dx_n$. We wish to show that for every $j = 1, 2, \ldots, n$, the partial derivative $\frac{\partial f}{\partial x_j}$ exists and is equal to ω_j.

Let e_j be an arbitrary standard basis vector, and h a nonzero real number. Compute

$$\frac{f(x+he_j) - f(x)}{h} = \frac{1}{h} \left(\int_p^{x+he_j} \omega - \int_p^x \omega \right) = \frac{1}{h} \int_x^{x+he_j} \omega,$$

which follows by Proposition 9.2.10 and path independence as $\int_p^{x+he_j} \omega = \int_p^x \omega + \int_x^{x+he_j} \omega$, because we pick a path from p to $x+he_j$ that also happens to pass through x, and then we cut this path in two, see Figure 9.8.

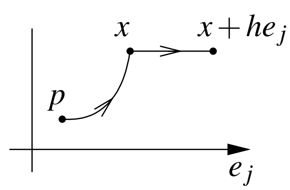

Figure 9.8: Using path independence in computing the partial derivative.

Since U is open, suppose h is so small so that all points of distance $|h|$ or less from x are in U. As the integral is path independent, pick the simplest path possible from x to $x+he_j$, that is $\gamma(t) := x + the_j$ for $t \in [0,1]$. The path is in U. Notice $\gamma'(t) = he_j$ has only one nonzero component and that is the jth component, which is h. Therefore,

$$\frac{1}{h} \int_x^{x+he_j} \omega = \frac{1}{h} \int_\gamma \omega = \frac{1}{h} \int_0^1 \omega_j(x+the_j) h \, dt = \int_0^1 \omega_j(x+the_j) \, dt.$$

We wish to take the limit as $h \to 0$. The function ω_j is continuous at x. Given $\varepsilon > 0$, suppose h is small enough so that $\left|\omega_j(x) - \omega_j(y)\right| < \varepsilon$ whenever $\|x-y\| \leq |h|$. Thus, $\left|\omega_j(x+the_j) - \omega_j(x)\right| < \varepsilon$ for all $t \in [0,1]$, and we estimate

$$\left| \int_0^1 \omega_j(x+the_j) \, dt - \omega_j(x) \right| = \left| \int_0^1 \bigl(\omega_j(x+the_j) - \omega_j(x) \bigr) \, dt \right| \leq \varepsilon.$$

That is,

$$\lim_{h \to 0} \frac{f(x+he_j) - f(x)}{h} = \omega_j(x).$$

All partials of f exist and are equal to ω_j, which are continuous functions. Thus, f is continuously differentiable, and furthermore $df = \omega$.

For the other direction, suppose a continuously differentiable f exists such that $df = \omega$. Take a smooth path $\gamma\colon [a,b] \to U$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then

$$\begin{aligned}\int_\gamma df &= \int_a^b \left(\frac{\partial f}{\partial x_1}\big(\gamma(t)\big)\gamma_1'(t) + \frac{\partial f}{\partial x_2}\big(\gamma(t)\big)\gamma_2'(t) + \cdots + \frac{\partial f}{\partial x_n}\big(\gamma(t)\big)\gamma_n'(t) \right) dt \\ &= \int_a^b \frac{d}{dt}\Big[f\big(\gamma(t)\big)\Big]\, dt \\ &= f(y) - f(x).\end{aligned}$$

The value of the integral only depends on x and y, not the path taken. Therefore the integral is path independent. We leave checking this fact for a piecewise smooth path as an exercise. □

Path independence can be stated more neatly in terms of integrals over closed paths.

Proposition 9.3.4. *Let $U \subset \mathbb{R}^n$ be a path connected open set and ω a one-form defined on U. Then $\omega = df$ for some continuously differentiable $f\colon U \to \mathbb{R}$ if and only if*

$$\int_\gamma \omega = 0 \qquad \text{for every piecewise smooth closed path } \gamma\colon [a,b] \to U.$$

Proof. Suppose $\omega = df$ and let γ be a piecewise smooth closed path. Since $\gamma(a) = \gamma(b)$ for a closed path, the previous proposition says

$$\int_\gamma \omega = f\big(\gamma(b)\big) - f\big(\gamma(a)\big) = 0.$$

Now suppose that for every piecewise smooth closed path γ, $\int_\gamma \omega = 0$. Let x, y be two points in U and let $\alpha\colon [0,1] \to U$ and $\beta\colon [0,1] \to U$ be two piecewise smooth paths with $\alpha(0) = \beta(0) = x$ and $\alpha(1) = \beta(1) = y$. See Figure 9.9.

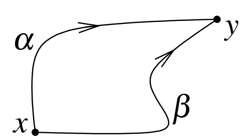

Figure 9.9: Two paths from x to y.

Define $\gamma\colon [0,2] \to U$ by

$$\gamma(t) := \begin{cases} \alpha(t) & \text{if } t \in [0,1], \\ \beta(2-t) & \text{if } t \in (1,2]. \end{cases}$$

This path is piecewise smooth. This is due to the fact that $\gamma|_{[0,1]}(t) = \alpha(t)$ and $\gamma|_{[1,2]}(t) = \beta(2-t)$ (note especially $\gamma(1) = \alpha(1) = \beta(2-1)$). It is also closed as $\gamma(0) = \alpha(0) = \beta(0) = \gamma(2)$. So

$$0 = \int_\gamma \omega = \int_\alpha \omega - \int_\beta \omega.$$

This follows first by Proposition 9.2.10, and then noticing that the second part is β traveled backwards so that we get minus the β integral. Thus the integral of ω on U is path independent. □

However one states path independence, it is often a difficult criterion to check, you have to check something "for all paths." There is a local criterion, a differential equation, that guarantees path independence, or in other words it guarantees an *antiderivative* f whose total derivative is the given one-form ω. Since the criterion is local, we generally only find the function f locally. We can find the antiderivative in every so-called *simply connected* domain, which informally is a domain where every path between two points can be "continuously deformed" into any other path between those two points. But to make matters simple, we prove the result for so-called star-shaped domains, which is often good enough. As a bonus the proof in the star-shaped case constructs the antiderivative explicitly. As balls are star-shaped we then have the result locally.

Definition 9.3.5. Let $U \subset \mathbb{R}^n$ be an open set and $p \in U$. We say U is a *star-shaped domain* with respect to p if for every other point $x \in U$, the line segment $[p,x]$ is in U, that is, if $(1-t)p+tx \in U$ for all $t \in [0,1]$. If we say simply *star-shaped*, then U is star-shaped with respect to some $p \in U$. See Figure 9.10.

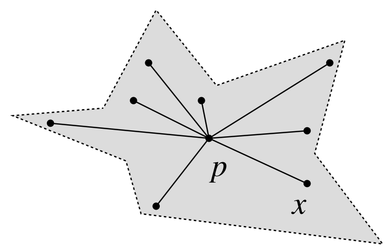

Figure 9.10: A star-shaped domain with respect to p.

Notice the difference between star-shaped and convex. A convex domain is star-shaped, but a star-shaped domain need not be convex.

Theorem 9.3.6 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^n$ be a star-shaped domain and ω a continuously differentiable one-form defined on U. That is, if*

$$\omega = \omega_1\, dx_1 + \omega_2\, dx_2 + \cdots + \omega_n\, dx_n,$$

then $\omega_1, \omega_2, \ldots, \omega_n$ are continuously differentiable functions. Suppose that for every j and k

$$\frac{\partial \omega_j}{\partial x_k} = \frac{\partial \omega_k}{\partial x_j},$$

then there exists a twice continuously differentiable function $f \colon U \to \mathbb{R}$ such that $df = \omega$.

The condition on the derivatives of ω is precisely the condition that the second partial derivatives commute. That is, if $df = \omega$, and f is twice continuously differentiable, then

$$\frac{\partial \omega_j}{\partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial \omega_k}{\partial x_j}.$$

The condition is clearly necessary. The Poincaré lemma says that it is sufficient for a star-shaped U.

Proof. Suppose U is a star-shaped domain with respect to $p = (p_1, p_2, \ldots, p_n) \in U$. Given $x = (x_1, x_2, \ldots, x_n) \in U$, define the path $\gamma\colon [0,1] \to U$ as $\gamma(t) := (1-t)p + tx$, so $\gamma'(t) = x - p$. Let

$$f(x) := \int_\gamma \omega = \int_0^1 \left(\sum_{k=1}^n \omega_k\bigl((1-t)p+tx\bigr)\,(x_k - p_k) \right) dt.$$

We differentiate in x_j under the integral, which is allowed as everything, including the partials, is continuous:

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(x) &= \int_0^1 \left(\left(\sum_{k=1}^n \frac{\partial \omega_k}{\partial x_j}\bigl((1-t)p+tx\bigr)\,t(x_k - p_k) \right) + \omega_j\bigl((1-t)p+tx\bigr) \right) dt \\
&= \int_0^1 \left(\left(\sum_{k=1}^n \frac{\partial \omega_j}{\partial x_k}\bigl((1-t)p+tx\bigr)\,t(x_k - p_k) \right) + \omega_j\bigl((1-t)p+tx\bigr) \right) dt \\
&= \int_0^1 \frac{d}{dt}\Bigl[t\omega_j\bigl((1-t)p+tx\bigr) \Bigr]\, dt \\
&= \omega_j(x).
\end{aligned}$$

And this is precisely what we wanted. $\square$

Example 9.3.7: Without some hypothesis on U the theorem is not true. Let

$$\omega(x,y) := \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

be defined on $\mathbb{R}^2 \setminus \{0\}$. Then

$$\frac{\partial}{\partial y}\left[\frac{-y}{x^2+y^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right].$$

However, there is no $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $df = \omega$. In Example 9.2.11 we integrated from $(1,0)$ to $(1,0)$ along the unit circle counterclockwise, that is $\gamma(t) = \bigl(\cos(t), \sin(t)\bigr)$ for $t \in [0, 2\pi]$, and we found the integral to be 2π. We would have gotten 0 if the integral was path independent, or in other words if there would exist an f such that $df = \omega$.

9.3.2 Vector fields

A common object to integrate is a so-called vector field.

Definition 9.3.8. Let $U \subset \mathbb{R}^n$ be a set. A continuous function $v\colon U \to \mathbb{R}^n$ is called a *vector field*. Write $v = (v_1, v_2, \ldots, v_n)$.

Given a smooth path $\gamma\colon [a,b] \to \mathbb{R}^n$ with $\gamma\bigl([a,b]\bigr) \subset U$ we define the path integral of the vectorfield v as

$$\int_\gamma v \cdot d\gamma := \int_a^b v\bigl(\gamma(t)\bigr) \cdot \gamma'(t)\, dt,$$

where the dot in the definition is the standard dot product. The definition for a piecewise smooth path is, again, done by integrating over each smooth interval and adding the results.

Unraveling the definition, we find that

$$\int_\gamma v \cdot d\gamma = \int_\gamma v_1 \, dx_1 + v_2 \, dx_2 + \cdots + v_n \, dx_n.$$

What we know about integration of one-forms carries over to the integration of vector fields. For example, path independence for integration of vector fields is simply that

$$\int_x^y v \cdot d\gamma$$

is path independent if and only if $v = \nabla f$, that is, v is the gradient of a function. The function f is then called a *potential* for v.

A vector field v whose path integrals are path independent is called a *conservative vector field*. The rationale for the naming is that such vector fields arise in physical systems where a certain quantity, the energy, is conserved.

9.3.3 Exercises

Exercise* 9.3.1*: *Find an $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that $df = xe^{x^2+y^2} \, dx + ye^{x^2+y^2} \, dy$.*

Exercise* 9.3.2*: *Find an $\omega_2 \colon \mathbb{R}^2 \to \mathbb{R}$ such that there exists a continuously differentiable $f \colon \mathbb{R}^2 \to \mathbb{R}$ for which $df = e^{xy} \, dx + \omega_2 \, dy$.*

Exercise* 9.3.3*: *Finish the proof of Proposition 9.3.3, that is, we only proved the second direction for a smooth path, not a piecewise smooth path.*

Exercise* 9.3.4*: *Show that a star-shaped domain $U \subset \mathbb{R}^n$ is path connected.*

Exercise* 9.3.5*: *Show that $U := \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : x \leq 0, y = 0\}$ is star-shaped and find all points $(x_0,y_0) \in U$ such that U is star-shaped with respect to (x_0,y_0).*

Exercise* 9.3.6*: *Suppose U_1 and U_2 are two open sets in $\mathbb{R}^n$ with $U_1 \cap U_2$ nonempty and path connected. Suppose there exists an $f_1 \colon U_1 \to \mathbb{R}$ and $f_2 \colon U_2 \to \mathbb{R}$, both twice continuously differentiable such that $df_1 = df_2$ on $U_1 \cap U_2$. Then there exists a twice differentiable function $F \colon U_1 \cup U_2 \to \mathbb{R}$ such that $dF = df_1$ on U_1 and $dF = df_2$ on U_2.*

Exercise* 9.3.7** (Hard):*** *Let $\gamma \colon [a,b] \to \mathbb{R}^n$ be a simple nonclosed piecewise smooth path (so γ is one-to-one). Suppose ω is a continuously differentiable one-form defined on some open set V with $\gamma\big([a,b]\big) \subset V$ and $\frac{\partial \omega_j}{\partial x_k} = \frac{\partial \omega_k}{\partial x_j}$ for all j and k. Prove that there exists an open set U with $\gamma\big([a,b]\big) \subset U \subset V$ and a twice continuously differentiable function $f \colon U \to \mathbb{R}$ such that $df = \omega$.*
Hint 1: $\gamma\big([a,b]\big)$ is compact.
Hint 2: Show that you can cover the curve by finitely many balls in sequence so that the kth ball only intersects the $(k-1)$th ball.
Hint 3: See previous exercise.

Exercise* 9.3.8*:

a) Show that a connected open set $U \subset \mathbb{R}^n$ is path connected. Hint: Start with a point $x \in U$, and let $U_x \subset U$ is the set of points that are reachable by a path from x. Show that U_x and $U \setminus U_x$ are both open, and since U_x is nonempty ($x \in U_x$) it must be that $U_x = U$.

b) Prove the converse, that is, an open path connected set $U \subset \mathbb{R}^n$ is connected. Hint: For contradiction assume there exist two open and disjoint nonempty open sets and then assume there is a piecewise smooth (and therefore continuous) path between a point in one to a point in the other.*

Exercise* 9.3.9*: *Usually path connectedness is defined using continuous paths rather than piecewise smooth paths. Prove that for open subsets of $\mathbb{R}^n$ the definitions are equivalent, in other words prove:*
Suppose $U \subset \mathbb{R}^n$ is open and for every $x, y \in U$, there exists a continuous function $\gamma \colon [a,b] \to U$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then U is path connected, that is, there is a piecewise smooth path in U from x to y.

Exercise* 9.3.10** (Hard):*** *Take*

$$\omega(x,y) = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Let $\gamma \colon [a,b] \to \mathbb{R}^2 \setminus \{(0,0)\}$ be a closed piecewise smooth path. Let $R := \{(x,y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = 0\}$. Suppose $R \cap \gamma\big([a,b]\big)$ is a finite set of k points. Prove that

$$\int_\gamma \omega = 2\pi\ell$$

for some integer ℓ with $|\ell| \leq k$.
Hint 1: First prove that for a path β that starts and end on R but does not intersect it otherwise, you find that $\int_\beta \omega$ is -2π, 0, or 2π.
Hint 2: You proved above that $\mathbb{R}^2 \setminus R$ is star-shaped.
Note: The number ℓ is called the winding number *it measures how many times does γ wind around the origin in the clockwise direction.*

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*If the definition of "path connected" is as in the next exercise, "open" would not be needed for this part.

Chapter 10

Multivariable Integral

10.1 Riemann integral over rectangles

Note: 2–3 lectures

As in chapter 5, we define the Riemann integral using the Darboux upper and lower integrals. The ideas in this section are very similar to integration in one dimension. The complication is mostly notational. The differences between one and several dimensions will grow more pronounced in the sections following.

10.1.1 Rectangles and partitions

Definition 10.1.1. Let $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ be such that $a_k \leq b_k$ for all k. A set of the form $[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$ is called a *closed rectangle*. In this setting it is sometimes useful to allow $a_k = b_k$, in which case we think of $[a_k,b_k] = \{a_k\}$ as usual. If $a_k < b_k$ for all k, then a set of the form $(a_1,b_1)\times(a_2,b_2)\times\cdots\times(a_n,b_n)$ is called an *open rectangle*.

For an open or closed rectangle $R := [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n] \subset \mathbb{R}^n$ or $R := (a_1,b_1)\times(a_2,b_2)\times\cdots\times(a_n,b_n) \subset \mathbb{R}^n$, we define the *n-dimensional volume* by

$$V(R) := (b_1-a_1)(b_2-a_2)\cdots(b_n-a_n).$$

A *partition* P of the closed rectangle $R = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$ is given by partitions $P_1,P_2,\ldots,P_n$ of the intervals $[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]$. We write $P = (P_1,P_2,\ldots,P_n)$. That is, for every $k = 1,2,\ldots,n$ there is an integer ℓ_k and a finite set of numbers $P_k = \{x_{k,0},x_{k,1},x_{k,2},\ldots,x_{k,\ell_k}\}$ such that

$$a_k = x_{k,0} < x_{k,1} < x_{k,2} < \cdots < x_{k,\ell_k-1} < x_{k,\ell_k} = b_k.$$

Picking a set of n integers $j_1,j_2,\ldots,j_n$ where $j_k \in \{1,2,\ldots,\ell_k\}$ we get the *subrectangle*

$$[x_{1,j_1-1}\ ,\ x_{1,j_1}]\times[x_{2,j_2-1}\ ,\ x_{2,j_2}]\times\cdots\times[x_{n,j_n-1}\ ,\ x_{n,j_n}].$$

We order the subrectangles somehow and we say $\{R_1,R_2,\ldots,R_N\}$ are the subrectangles corresponding to the partition P of R, or more simply, subrectangles of P. In other words, we subdivided the original rectangle into many smaller subrectangles. See Figure 10.1.

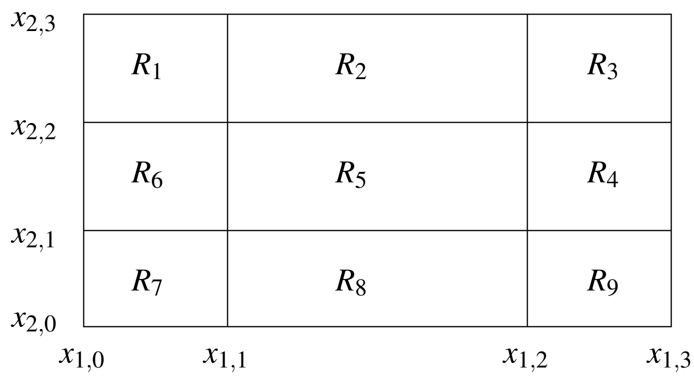

Figure 10.1: Example partition of a rectangle in $\mathbb{R}^2$. The order of the subrectangles is not important.

Let $R \subset \mathbb{R}^n$ be a closed rectangle and let $f : R \to \mathbb{R}$ be a bounded function. Let P be a partition of R with N subrectangles $R_1, R_2, \ldots, R_N$. Define

$$m_i := \inf\{f(x) : x \in R_i\}, \qquad M_i := \sup\{f(x) : x \in R_i\},$$
$$L(P,f) := \sum_{i=1}^{N} m_i V(R_i), \qquad U(P,f) := \sum_{i=1}^{N} M_i V(R_i).$$

We call $L(P,f)$ the *lower Darboux sum* and $U(P,f)$ the *upper Darboux sum*.

To see the relationship to the Δ notation from the one-variable definition, note that when

$$R_i = [x_{1,j_1-1}\,,\, x_{1,j_1}] \times [x_{2,j_2-1}\,,\, x_{2,j_2}] \times \cdots \times [x_{n,j_n-1}\,,\, x_{n,j_n}],$$

then

$$V(R_i) = (x_{1,j_1} - x_{1,j_1-1})(x_{2,j_2} - x_{2,j_2-1}) \cdots (x_{n,j_n} - x_{n,j_n-1}) = \Delta x_{1,j_1} \Delta x_{2,j_2} \cdots \Delta x_{n,j_n}.$$

It is not difficult to see that the subrectangles of P cover our original R, and their volumes sum to that of R. That is,

$$R = \bigcup_{k=1}^{N} R_k, \qquad \text{and} \qquad V(R) = \sum_{k=1}^{N} V(R_k).$$

The indexing in the definition may be complicated, but fortunately we do not need to go back directly to the definition often. We start by proving facts about the Darboux sums analogous to the one-variable results.

Proposition 10.1.2. *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f : R \to \mathbb{R}$ is a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in R$, we have $m \leq f(x) \leq M$. Then for every partition P of R,*

$$mV(R) \leq L(P,f) \leq U(P,f) \leq MV(R).$$

Proof. Let P be a partition of R. For all i, we have $m \leq m_i \leq M_i \leq M$. Also $\sum_{i=1}^{N} V(R_i) = V(R)$. Therefore,

$$mV(R) = m\left(\sum_{i=1}^{N} V(R_i)\right) = \sum_{i=1}^{N} mV(R_i) \leq \sum_{i=1}^{N} m_i V(R_i) \leq$$
$$\leq \sum_{i=1}^{N} M_i V(R_i) \leq \sum_{i=1}^{N} M V(R_i) = M\left(\sum_{i=1}^{N} V(R_i)\right) = MV(R). \quad \square$$

10.1.2 Upper and lower integrals

By Proposition 10.1.2, the set of upper and lower Darboux sums are bounded sets and we can take their infima and suprema. As in one variable, we make the following definition.

Definition 10.1.3. Let $f\colon R \to \mathbb{R}$ be a bounded function on a closed rectangle $R \subset \mathbb{R}^n$. Define

$$\underline{\int_R} f := \sup\{L(P,f) : P \text{ a partition of } R\}, \qquad \overline{\int_R} f := \inf\{U(P,f) : P \text{ a partition of } R\}.$$

We call $\underline{\int}$ the *lower Darboux integral* and $\overline{\int}$ the *upper Darboux integral*.

And as in one dimension, we define refinements of partitions.

Definition 10.1.4. Let $R \subset \mathbb{R}^n$ be a closed rectangle. Let $P = (P_1, P_2, \ldots, P_n)$ and $\widetilde{P} = (\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_n)$ be partitions of R. We say $\widetilde{P}$ a *refinement* of P if, as sets, $P_k \subset \widetilde{P}_k$ for all $k = 1, 2, \ldots, n$.

If $\widetilde{P}$ is a refinement of P, then subrectangles of P are unions of subrectangles of $\widetilde{P}$. Simply put, in a refinement, we take the subrectangles of P, and we cut them into smaller subrectangles and call that $\widetilde{P}$. See Figure 10.2.

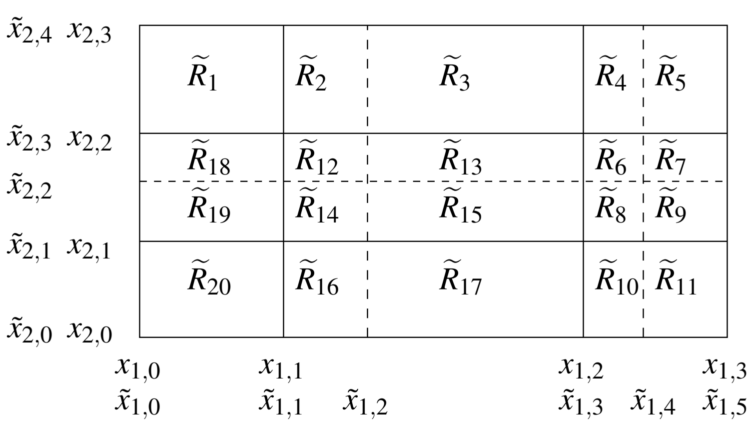

Figure 10.2: Example refinement of the partition from Figure 10.1. New "cuts" are marked in dashed lines. The exact order of the new subrectangles does not matter.

Proposition 10.1.5. *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle, P is a partition of R, and $\widetilde{P}$ is a refinement of P. If $f \colon R \to \mathbb{R}$ is bounded, then*

$$L(P,f) \leq L(\widetilde{P},f) \qquad \text{and} \qquad U(\widetilde{P},f) \leq U(P,f).$$

Proof. We prove the first inequality, and the second follows similarly. Let $R_1, R_2, \ldots, R_N$ be the subrectangles of P and $\widetilde{R}_1, \widetilde{R}_2, \ldots, \widetilde{R}_{\widetilde{N}}$ be the subrectangles of $\widetilde{R}$. Let I_k be the set of all indices j such that $\widetilde{R}_j \subset R_k$. For example, in figures 10.1 and 10.2, $I_4 = \{6,7,8,9\}$ as $R_4 = \widetilde{R}_6 \cup \widetilde{R}_7 \cup \widetilde{R}_8 \cup \widetilde{R}_9$. Then,

$$R_k = \bigcup_{j \in I_k} \widetilde{R}_j, \qquad V(R_k) = \sum_{j \in I_k} V(\widetilde{R}_j).$$

Let $m_j := \inf\{f(x) : x \in R_j\}$, and $\widetilde{m}_j := \inf\{f(x) : \in \widetilde{R}_j\}$ as usual. If $j \in I_k$, then $m_k \leq \widetilde{m}_j$. Then

$$L(P,f) = \sum_{k=1}^{N} m_k V(R_k) = \sum_{k=1}^{N} \sum_{j \in I_k} m_k V(\widetilde{R}_j) \leq \sum_{k=1}^{N} \sum_{j \in I_k} \widetilde{m}_j V(\widetilde{R}_j) = \sum_{j=1}^{\widetilde{N}} \widetilde{m}_j V(\widetilde{R}_j) = L(\widetilde{P},f). \qquad \square$$

The key point of this next proposition is that the lower Darboux integral is less than or equal to the upper Darboux integral.

Proposition 10.1.6. *Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f \colon R \to \mathbb{R}$ a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in R$, we have $m \leq f(x) \leq M$. Then*

$$m V(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq M V(R). \tag{10.1}$$

Proof. For every partition P, via Proposition 10.1.2,

$$m V(R) \leq L(P,f) \leq U(P,f) \leq M V(R).$$

Taking supremum of $L(P,f)$ and infimum of $U(P,f)$ over all partitions P, we obtain the first and the last inequality in (10.1).

The key inequality in (10.1) is the middle one. Let $P = (P_1, P_2, \ldots, P_n)$ and $Q = (Q_1, Q_2, \ldots, Q_n)$ be partitions of R. Define $\widetilde{P} = (\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_n)$ by letting $\widetilde{P}_k := P_k \cup Q_k$. Then $\widetilde{P}$ is a partition of R as can easily be checked, and $\widetilde{P}$ is a refinement of P and a refinement of Q. By Proposition 10.1.5, $L(P,f) \leq L(\widetilde{P},f)$ and $U(\widetilde{P},f) \leq U(Q,f)$. Therefore,

$$L(P,f) \leq L(\widetilde{P},f) \leq U(\widetilde{P},f) \leq U(Q,f).$$

In other words, for two arbitrary partitions P and Q, we have $L(P,f) \leq U(Q,f)$. Via Proposition 1.2.7 from volume I, we obtain

$$\sup \bigl\{ L(P,f) : P \text{ a partition of } R \bigr\} \leq \inf \bigl\{ U(P,f) : P \text{ a partition of } R \bigr\}.$$

In other words, $\underline{\int_R} f \leq \overline{\int_R} f$. $\square$

10.1.3 The Riemann integral

We have all we need to define the Riemann integral in n-dimensions over rectangles. As in one dimension, the Riemann integral is only defined on a certain class of functions, called the Riemann integrable functions.

Definition 10.1.7. Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f\colon R \to \mathbb{R}$ a bounded function such that

$$\underline{\int_R} f(x)\,dx = \overline{\int_R} f(x)\,dx.$$

Then f is said to be *Riemann integrable*, and we sometimes say simply *integrable*. The set of Riemann integrable functions on R is denoted by $\mathscr{R}(R)$. For $f \in \mathscr{R}(R)$ define the *Riemann integral*

$$\int_R f := \underline{\int_R} f = \overline{\int_R} f.$$

When the variable $x \in \mathbb{R}^n$ needs to be emphasized, we write

$$\int_R f(x)\,dx, \qquad \int_R f(x_1,\ldots,x_n)\,dx_1\cdots dx_n, \qquad \text{or} \qquad \int_R f(x)\,dV.$$

If $R \subset \mathbb{R}^2$, then we often say area instead of volume, and we write

$$\int_R f(x)\,dA.$$

Proposition 10.1.6 immediately implies the following proposition.

Proposition 10.1.8. *Let $f\colon R \to \mathbb{R}$ be a Riemann integrable function on a closed rectangle $R \subset \mathbb{R}^n$. Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in R$. Then*

$$m V(R) \leq \int_R f \leq M V(R).$$

Example 10.1.9: A constant function is Riemann integrable. Suppose $f(x) = c$ for all x on R. Then

$$c V(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq c V(R).$$

So f is integrable, and furthermore $\int_R f = c V(R)$.

The proofs of linearity and monotonicity are almost completely identical as the proofs from one variable. We therefore leave it as an exercise to prove the next two propositions.

Proposition 10.1.10 (Linearity)**.** *Let $R \subset \mathbb{R}^n$ be a closed rectangle and let f and g be in $\mathscr{R}(R)$ and $\alpha \in \mathbb{R}$.*

(i) αf is in $\mathscr{R}(R)$ and

$$\int_R \alpha f = \alpha \int_R f.$$

(ii) $f+g$ is in $\mathscr{R}(R)$ and

$$\int_R (f+g) = \int_R f + \int_R g.$$

Proposition 10.1.11 (Monotonicity)**.** *Let $R \subset \mathbb{R}^n$ be a closed rectangle, let f and g be in $\mathscr{R}(R)$, and suppose $f(x) \leq g(x)$ for all $x \in R$. Then*

$$\int_R f \leq \int_R g.$$

Checking for integrability using the definition often involves the following technique, as in the single variable case.

Proposition 10.1.12. *Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f \colon R \to \mathbb{R}$ a bounded function. Then $f \in \mathscr{R}(R)$ if and only if for every $\varepsilon > 0$, there exists a partition P of R such that*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof. First, if f is integrable, then the supremum of $L(P, f)$ and infimum of $U(P, f)$ are equal and hence the infimum of $U(P, f) - L(P, f)$ is zero. Therefore, for every $\varepsilon > 0$ there must be some partition P such that $U(P, f) - L(P, f) < \varepsilon$.

For the other direction, given an $\varepsilon > 0$ find P such that $U(P, f) - L(P, f) < \varepsilon$.

$$\overline{\int_R} f - \underline{\int_R} f \leq U(P, f) - L(P, f) < \varepsilon.$$

As $\overline{\int_R} f \geq \underline{\int_R} f$ and the above holds for every $\varepsilon > 0$, we conclude $\overline{\int_R} f = \underline{\int_R} f$ and $f \in \mathscr{R}(R)$. $\square$

Suppose $f \colon S \to \mathbb{R}$ is a function and $R \subset S$ is a closed rectangle. If the restriction $f|_R$ is integrable, then for simplicity we say f is *integrable on R*, or $f \in \mathscr{R}(R)$ and we write

$$\int_R f := \int_R f|_R.$$

Proposition 10.1.13. *Let $S \subset \mathbb{R}^n$ be a closed rectangle. If $f \colon S \to \mathbb{R}$ is integrable and $R \subset S$ is a closed rectangle, then f is integrable on R.*

Proof. Given $\varepsilon > 0$, we find a partition P of S such that $U(P, f) - L(P, f) < \varepsilon$. By making a refinement of P if necessary, we assume that the endpoints of R are in P. In other words, R is a union of subrectangles of P. The subrectangles of P divide into two collections, ones that are subsets of R and ones whose intersection with the interior of R is empty. Suppose $R_1, R_2 \ldots, R_K$ are the subrectangles that are subsets of R and let $R_{K+1}, \ldots, R_N$ be the rest. Let $\widetilde{P}$ be the partition of R composed of those subrectangles of P contained in R. Using the same notation as before,

$$\begin{aligned} \varepsilon > U(P, f) - L(P, f) &= \sum_{k=1}^{K} (M_k - m_k) V(R_k) + \sum_{k=K+1}^{N} (M_k - m_k) V(R_k) \\ &\geq \sum_{k=1}^{K} (M_k - m_k) V(R_k) = U(\widetilde{P}, f|_R) - L(\widetilde{P}, f|_R). \end{aligned}$$

Therefore, $f|_R$ is integrable. $\square$

10.1.4 Integrals of continuous functions

Although we will prove a more general result later, it is useful to start with integrability of continuous functions. First we wish to measure the fineness of partitions. In one variable, we measured the length of a subinterval, in several variables, we similarly measure the sides of a subrectangle. We say a rectangle $R = [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_n,b_n]$ has *longest side at most* α if $b_k - a_k \leq \alpha$ for all $k = 1,2,\ldots,n$.

Proposition 10.1.14. *If a rectangle $R \subset \mathbb{R}^n$ has longest side at most α, then for all $x,y \in R$,*

$$\|x-y\| \leq \sqrt{n}\,\alpha.$$

Proof.

$$\begin{aligned} \|x-y\| &= \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} \\ &\leq \sqrt{(b_1-a_1)^2+(b_2-a_2)^2+\cdots+(b_n-a_n)^2} \\ &\leq \sqrt{\alpha^2+\alpha^2+\cdots+\alpha^2} = \sqrt{n}\,\alpha. \end{aligned}$$

□

Theorem 10.1.15. *Let $R \subset \mathbb{R}^n$ be a closed rectangle. If $f \colon R \to \mathbb{R}$ is continuous, then $f \in \mathscr{R}(R)$.*

Proof. The proof is analogous to the one-variable proof with some complications. The set R is a closed and bounded subset of $\mathbb{R}^n$, and hence compact. So f is not just continuous, but in fact uniformly continuous by Theorem 7.5.11 from volume I. Let $\varepsilon > 0$ be given. Find a $\delta > 0$ such that $\|x-y\| < \delta$ implies $|f(x)-f(y)| < \frac{\varepsilon}{V(R)}$.

Let P be a partition of R, such that longest side of every subrectangle is strictly less than $\frac{\delta}{\sqrt{n}}$. If $x,y \in R_k$ for some subrectangle R_k of P, then, by the proposition above, $\|x-y\| < \sqrt{n}\frac{\delta}{\sqrt{n}} = \delta$. Therefore,

$$f(x)-f(y) \leq |f(x)-f(y)| < \frac{\varepsilon}{V(R)}.$$

As f is continuous on R_k, which is compact, f attains a maximum and a minimum on this subrectangle. Let x be a point where f attains the maximum and y be a point where f attains the minimum. Then $f(x) = M_k$ and $f(y) = m_k$ in the notation from the definition of the integral. Therefore,

$$M_k - m_k = f(x)-f(y) < \frac{\varepsilon}{V(R)}.$$

And so

$$\begin{aligned} U(P,f)-L(P,f) &= \left(\sum_{k=1}^{N} M_k V(R_k)\right) - \left(\sum_{k=1}^{N} m_k V(R_k)\right) \\ &= \sum_{k=1}^{N} (M_k-m_k) V(R_k) \\ &< \frac{\varepsilon}{V(R)} \sum_{k=1}^{N} V(R_k) = \varepsilon. \end{aligned}$$

Via application of Proposition 10.1.12, we find that $f \in \mathscr{R}(R)$. □

10.1.5 Integration of functions with compact support

Let $U \subset \mathbb{R}^n$ be an open set and $f: U \to \mathbb{R}$ be a function. The *support* of f is the set

$$\operatorname{supp}(f) := \overline{\{x \in U : f(x) \neq 0\}},$$

where the closure is with respect to the subspace topology on U. Taking the closure with respect to the subspace topology is the same as $\overline{\{x \in U : f(x) \neq 0\}} \cap U$, where the closure is with respect to the ambient euclidean space $\mathbb{R}^n$. In particular, $\operatorname{supp}(f) \subset U$. The support is the closure (in U) of the set of points where the function is nonzero. Its complement in U is open. If $x \in U$ and x is not in the support of f, then f is constantly zero in a whole neighborhood of x.

A function f is said to have *compact support* if $\operatorname{supp}(f)$ is a compact set.

Example 10.1.16: The function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} -x(x^2+y^2-1)^2 & \text{if } \sqrt{x^2+y^2} \leq 1, \\ 0 & \text{else,} \end{cases}$$

is continuous and its support is the closed unit disc $C(0,1) = \{(x,y) : \sqrt{x^2+y^2} \leq 1\}$, which is a compact set, so f has compact support. Do note that the function is zero on the entire y-axis and on the unit circle, but all points that lie in the closed unit disc are still within the support as they are in the closure of points where f is nonzero. See Figure 10.3.

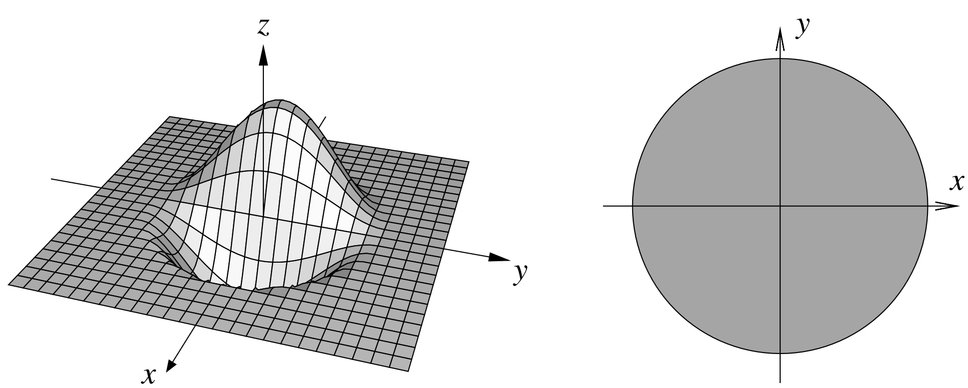

Figure 10.3: Function with compact support (left), the support is the closed unit disc (right).

If $U \neq \mathbb{R}^n$, then you must be careful to take the closure in U. Consider the following two examples.

Example 10.1.17: Let $B(0,1) \subset \mathbb{R}^2$ be the unit disc. The function $f: B(0,1) \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} 0 & \text{if } \sqrt{x^2+y^2} > 1/2, \\ 1/2 - \sqrt{x^2+y^2} & \text{if } \sqrt{x^2+y^2} \leq 1/2, \end{cases}$$

is continuous on $B(0,1)$ and its support is the smaller closed ball $C(0,1/2)$. As that is a compact set, f has compact support.

The function $g: B(0,1) \to \mathbb{R}$ defined by

$$g(x,y) := \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x > 0, \end{cases}$$

is continuous on $B(0,1)$, but its support is the set $\{(x,y) \in B(0,1) : x \geq 0\}$. In particular, g is not compactly supported.

We really only need to consider the case when $U = \mathbb{R}^n$. In light of Exercise 10.1.1, which says every continuous function on an open $U \subset \mathbb{R}^n$ with compact support can be extended to a continuous function with compact support on $\mathbb{R}^n$, considering $U = \mathbb{R}^n$ is not an oversimplification.

Example 10.1.18: The continuous function $f: B(0,1) \to \mathbb{R}$ defined by $f(x,y) := \sin\left(\frac{1}{1-x^2-y^2}\right)$, does not have compact support; as f is not constantly zero on any neighborhood of every point in $B(0,1)$, the support is the entire disc $B(0,1)$. The function does not extend as above to a continuous function on $\mathbb{R}^2$. In fact, it is not difficult to show that f cannot be extended in any way whatsoever to be continuous on all of $\mathbb{R}^2$ (the boundary of the disc is the problem).

Proposition 10.1.19. *Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is a continuous function with compact support. If R and S are closed rectangles such that* $\operatorname{supp}(f) \subset R$ *and* $\operatorname{supp}(f) \subset S$*, then*

$$\int_S f = \int_R f.$$

Proof. As f is continuous, it is automatically integrable on the rectangles R, S, and $R \cap S$. Then Exercise 10.1.7 says $\int_S f = \int_{S \cap R} f = \int_R f$. □

Because of this proposition, when $f: \mathbb{R}^n \to \mathbb{R}$ has compact support and is integrable on a rectangle R containing the support we write

$$\int f := \int_R f \qquad \text{or} \qquad \int_{\mathbb{R}^n} f := \int_R f.$$

For example, if f is continuous and of compact support, then $\int_{\mathbb{R}^n} f$ exists.

10.1.6 Exercises

***Exercise* 10.1.1:** *Suppose $U \subset \mathbb{R}^n$ is open and $f: U \to \mathbb{R}$ is continuous and of compact support. Show that the function $\widetilde{f}: \mathbb{R}^n \to \mathbb{R}$*

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{otherwise,} \end{cases}$$

is continuous.

***Exercise* 10.1.2:** *Prove Proposition 10.1.10.*

***Exercise* 10.1.3:** *Suppose R is a rectangle with the length of one of the sides equal to 0. For every bounded function f, show that $f \in \mathscr{R}(R)$ and $\int_R f = 0$.*

Exercise* 10.1.4*: *Suppose R is a rectangle with the length of one of the sides equal to 0, and suppose S is a rectangle with $R \subset S$. If f is a bounded function such that $f(x) = 0$ for $x \in R \setminus S$, show that $f \in \mathscr{R}(R)$ and $\int_R f = 0$.*

Exercise* 10.1.5*: *Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is such that $f(x) := 0$ if $x \neq 0$ and $f(0) := 1$. Show that f is integrable on $R := [-1,1] \times [-1,1] \times \cdots \times [-1,1]$ directly using the definition, and find $\int_R f$.*

Exercise* 10.1.6*: *Suppose R is a closed rectangle and $h: R \to \mathbb{R}$ is a bounded function such that $h(x) = 0$ if $x \notin \partial R$ (the boundary of R). Let S be a closed rectangle. Show that $h \in \mathscr{R}(S)$ and*

$$\int_S h = 0.$$

Hint: Write h as a sum of functions as in Exercise 10.1.4.

Exercise* 10.1.7*: *Suppose R and R' are two closed rectangles with $R' \subset R$. Suppose $f: R \to \mathbb{R}$ is in $\mathscr{R}(R')$ and $f(x) = 0$ for $x \in R \setminus R'$. Show that $f \in \mathscr{R}(R)$ and*

$$\int_{R'} f = \int_R f.$$

Do this in the following steps.

a) First do the proof assuming that furthermore $f(x) = 0$ whenever $x \in \overline{R \setminus R'}$.

b) Write $f(x) = g(x) + h(x)$ where $g(x) = 0$ whenever $x \in \overline{R \setminus R'}$, and $h(x)$ is zero except perhaps on $\partial R'$. Then show $\int_R h = \int_{R'} h = 0$ (see Exercise 10.1.6).

c) Show $\int_{R'} f = \int_R f$.

Exercise* 10.1.8*: *Suppose $R' \subset \mathbb{R}^n$ and $R'' \subset \mathbb{R}^n$ are two rectangles such that $R = R' \cup R''$ is a rectangle, and $R' \cap R''$ is rectangle with one of the sides having length 0 (that is $V(R' \cap R'') = 0$). Let $f: R \to \mathbb{R}$ be a function such that $f \in \mathscr{R}(R')$ and $f \in \mathscr{R}(R'')$. Show that $f \in \mathscr{R}(R)$ and*

$$\int_R f = \int_{R'} f + \int_{R''} f.$$

Hint: See previous exercise.

Exercise* 10.1.9*: *Prove a stronger version of Proposition 10.1.19. Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is a function with compact support but not necessarily continuous. Prove that if R is a closed rectangle such that $\operatorname{supp}(f) \subset R$ and f is integrable on R, then for every other closed rectangle S with $\operatorname{supp}(f) \subset S$, the function f is integrable on S and $\int_S f = \int_R f$. Hint: See Exercise 10.1.7.*

Exercise* 10.1.10*: *Suppose R and S are closed rectangles of $\mathbb{R}^n$. Define $f: \mathbb{R}^n \to \mathbb{R}$ as $f(x) := 1$ if $x \in R$, and $f(x) := 0$ otherwise. Prove f is integrable on S and compute $\int_S f$. Hint: Consider $S \cap R$.*

Exercise* 10.1.11*: *Let $R := [0,1] \times [0,1] \subset \mathbb{R}^2$.*

a) Suppose $f: R \to \mathbb{R}$ is defined by

$$f(x,y) := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{else.} \end{cases}$$

Show that $f \in \mathscr{R}(R)$ and compute $\int_R f$.

b) Suppose $f: R \to \mathbb{R}$ is defined by

$$f(x,y) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}, \\ 0 & \text{else.} \end{cases}$$

Show that $f \notin \mathscr{R}(R)$.

Exercise* 10.1.12*: *Suppose R is a closed rectangle, and suppose S_j are closed rectangles such that $S_j \subset R$ and $S_j \subset S_{j+1}$ for all j. Suppose $f \colon R \to \mathbb{R}$ is bounded and $f \in \mathscr{R}(S_j)$ for all j. Show that $f \in \mathscr{R}(R)$ and*

$$\lim_{j\to\infty} \int_{S_j} f = \int_R f.$$

Exercise* 10.1.13*: *Suppose $f \colon [-1,1] \times [-1,1] \to \mathbb{R}$ is a Riemann integrable function such $f(x) = -f(-x)$. Using the definition prove*

$$\int_{[-1,1]\times[-1,1]} f = 0.$$

10.2 Iterated integrals and Fubini theorem

Note: 1–2 lectures

The Riemann integral in several variables is hard to compute by the definition. For one-dimensional Riemann integral, we have the fundamental theorem of calculus, and we can compute many integrals without having to appeal to the definition of the integral. We will rewrite a Riemann integral in several variables into several one-dimensional Riemann integrals by iterating. However, if $f\colon [0,1]^2 \to \mathbb{R}$ is a Riemann integrable function, it is not immediately clear if the three expressions

$$\int_{[0,1]^2} f, \qquad \int_0^1 \int_0^1 f(x,y)\,dx\,dy, \qquad \text{and} \qquad \int_0^1 \int_0^1 f(x,y)\,dy\,dx$$

are equal, or if the last two are even well-defined.

Example 10.2.1: Define

$$f(x,y) := \begin{cases} 1 & \text{if } x = 1/2 \text{ and } y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is Riemann integrable on $R := [0,1]^2$ and $\int_R f = 0$. Furthermore, $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0$. However,

$$\int_0^1 f(1/2,y)\,dy$$

does not exist, so we cannot even write $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$.

Proof: Let us start with integrability of f. Consider the partition of $[0,1]^2$ where the partition in the x direction is $\{0, 1/2-\varepsilon, 1/2+\varepsilon, 1\}$ and in the y direction $\{0,1\}$. The subrectangles of the partition are

$$R_1 := [0, 1/2-\varepsilon] \times [0,1], \qquad R_2 := [1/2-\varepsilon, 1/2+\varepsilon] \times [0,1], \qquad R_3 := [1/2+\varepsilon, 1] \times [0,1].$$

We have $m_1 = M_1 = 0$, $m_2 = 0$, $M_2 = 1$, and $m_3 = M_3 = 0$. Therefore,

$$L(P,f) = m_1 V(R_1) + m_2 V(R_2) + m_3 V(R_3) = 0(1/2-\varepsilon) + 0(2\varepsilon) + 0(1/2-\varepsilon) = 0,$$

and

$$U(P,f) = M_1 V(R_1) + M_2 V(R_2) + M_3 V(R_3) = 0(1/2-\varepsilon) + 1(2\varepsilon) + 0(1/2-\varepsilon) = 2\varepsilon.$$

The upper and lower sums are arbitrarily close and the lower sum is always zero, so the function is integrable and $\int_R f = 0$.

For every fixed y, the function that takes x to $f(x,y)$ is zero except perhaps at a single point $x = 1/2$. Such a function is integrable and $\int_0^1 f(x,y)\,dx = 0$. Therefore, $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0$.

However, if $x = 1/2$, the function that takes y to $f(1/2,y)$ is the nonintegrable function that is 1 on the rationals and 0 on the irrationals. See Example 5.1.4 from volume I.

We solve this problem of undefined inside integrals by using the upper and lower integrals, which are always defined.

Split the coordinates of $\mathbb{R}^{n+m}$ into two parts: Write the coordinates on $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ as (x,y) where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For a function $f(x,y)$, write

$$f_x(y) := f(x,y)$$

when x is fixed and we wish to speak of the function in terms of y. Write

$$f^y(x) := f(x,y)$$

when y is fixed and we wish to speak of the function in terms of x.

Theorem 10.2.2 (Fubini version A*). *Let $R \times S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed rectangle and $f \colon R \times S \to \mathbb{R}$ be integrable. The functions $g \colon R \to \mathbb{R}$ and $h \colon R \to \mathbb{R}$ defined by*

$$g(x) := \underline{\int_S} f_x \qquad \textit{and} \qquad h(x) := \overline{\int_S} f_x$$

are integrable on R and

$$\int_R g = \int_R h = \int_{R \times S} f.$$

In other words,

$$\int_{R \times S} f = \int_R \left(\underline{\int_S} f(x,y)\, dy \right) dx = \int_R \left(\overline{\int_S} f(x,y)\, dy \right) dx.$$

If it turns out that f_x is integrable for all x, for example when f is continuous, then we obtain the more familiar

$$\int_{R \times S} f = \int_R \int_S f(x,y)\, dy\, dx.$$

Proof. Any partition of $R \times S$ is a concatenation of a partition of R and a partition of S. That is, write a partition of $R \times S$ as $(P,P') = (P_1, P_2, \ldots, P_n, P'_1, P'_2, \ldots, P'_m)$, where $P = (P_1, P_2, \ldots, P_n)$ and $P' = (P'_1, P'_2, \ldots, P'_m)$ are partitions of R and S respectively. Let $R_1, R_2, \ldots, R_N$ be the subrectangles of P and $R'_1, R'_2, \ldots, R'_K$ be the subrectangles of P'. Then the subrectangles of (P,P') are $R_j \times R'_k$ where $1 \leq j \leq N$ and $1 \leq k \leq K$.

Let

$$m_{j,k} := \inf_{(x,y) \in R_j \times R'_k} f(x,y).$$

We notice that $V(R_j \times R'_k) = V(R_j) V(R'_k)$ and hence

$$L\big((P,P'),f\big) = \sum_{j=1}^{N} \sum_{k=1}^{K} m_{j,k} V(R_j \times R'_k) = \sum_{j=1}^{N} \left(\sum_{k=1}^{K} m_{j,k} V(R'_k) \right) V(R_j).$$

If we let

$$m_k(x) := \inf_{y \in R'_k} f(x,y) = \inf_{y \in R'_k} f_x(y),$$

*Named after the Italian mathematician Guido Fubini (1879–1943).

then for $x \in R_j$, we have $m_{j,k} \leq m_k(x)$. Therefore,

$$\sum_{k=1}^{K} m_{j,k} V(R'_k) \leq \sum_{k=1}^{K} m_k(x) V(R'_k) = L(P', f_x) \leq \underline{\int_S} f_x = g(x).$$

As the inequality holds for all $x \in R_j$, we have

$$\sum_{k=1}^{K} m_{j,k} V(R'_k) \leq \inf_{x \in R_j} g(x).$$

We thus obtain

$$L\big((P,P'), f\big) \leq \sum_{j=1}^{N} \left(\inf_{x \in R_j} g(x) \right) V(R_j) = L(P,g).$$

Similarly, $U\big((P,P'), f\big) \geq U(P,h)$, and the proof of this inequality is left as an exercise. Putting the two inequalities together with the fact that $g(x) \leq h(x)$ for all x, we have

$$L\big((P,P'), f\big) \leq L(P,g) \leq U(P,g) \leq U(P,h) \leq U\big((P,P'), f\big).$$

And since f is integrable, it must be that g is integrable as

$$U(P,g) - L(P,g) \leq U\big((P,P'), f\big) - L\big((P,P'), f\big),$$

and we can make the right-hand side arbitrarily small. As for any partition we have $L\big((P,P'), f\big) \leq L(P,g) \leq U\big((P,P'), f\big)$, we must have $\int_R g = \int_{R \times S} f$.

Similarly,

$$L\big((P,P'), f\big) \leq L(P,g) \leq L(P,h) \leq U(P,h) \leq U\big((P,P'), f\big),$$

and hence

$$U(P,h) - L(P,h) \leq U\big((P,P'), f\big) - L\big((P,P'), f\big).$$

If f is integrable, so is h. As $L\big((P,P'), f\big) \leq L(P,h) \leq U\big((P,P'), f\big)$ we must have that $\int_R h = \int_{R \times S} f$. □

We can also do the iterated integration in the opposite order. The proof of this version is almost identical to version A (or follows quickly from version A). We leave it as an exercise to the reader.

Theorem 10.2.3 (Fubini version B)**.** *Let $R \times S \subset \mathbb{R}^n \times \mathbb{R}^m$ be a closed rectangle and $f \colon R \times S \to \mathbb{R}$ be integrable. The functions $g \colon S \to \mathbb{R}$ and $h \colon S \to \mathbb{R}$ defined by*

$$g(y) := \underline{\int_R} f^y \qquad \text{and} \qquad h(y) := \overline{\int_R} f^y$$

are integrable on S and

$$\int_S g = \int_S h = \int_{R \times S} f.$$

That is,

$$\int_{R\times S} f = \int_S \left(\underline{\int_R} f(x,y)\,dx \right) dy = \int_S \left(\overline{\int_R} f(x,y)\,dx \right) dy.$$

Next suppose f_x and f^y are integrable. For example, suppose f is continuous. By putting the two versions together we obtain the familiar

$$\int_{R\times S} f = \int_R \int_S f(x,y)\,dy\,dx = \int_S \int_R f(x,y)\,dx\,dy.$$

Often the Fubini theorem is stated in two dimensions for a continuous function $f\colon R \to \mathbb{R}$ on a rectangle $R = [a,b] \times [c,d]$. Then the Fubini theorem states that

$$\int_R f = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

The Fubini theorem is commonly thought of as *the* theorem that allows us to swap the order of iterated integrals, although there are many variations on Fubini, and we have seen but two of them.

Repeatedly applying Fubini theorem gets us the following corollary: Let $R := [a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_n,b_n] \subset \mathbb{R}^n$ be a closed rectangle and let $f\colon R \to \mathbb{R}$ be continuous. Then

$$\int_R f = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1,x_2,\ldots,x_n)\,dx_n\,dx_{n-1}\cdots dx_1.$$

Clearly we may switch the order of integration to any order we please. We may also relax the continuity requirement by making sure that all the intermediate functions are integrable, or by using upper or lower integrals appropriately.

10.2.1 Exercises

Exercise* 10.2.1*: *Compute* $\int_0^1 \int_{-1}^1 xe^{xy}\,dx\,dy$ *in a simple way.*

Exercise* 10.2.2*: *Prove the assertion* $U\big((P,P'),f\big) \geq U(P,h)$ *from the proof of Theorem 10.2.2.*

Exercise* 10.2.3** (Easy):*** *Prove Theorem 10.2.3.*

Exercise* 10.2.4*: *Let* $R := [a,b] \times [c,d]$ *and* $f(x,y)$ *is an integrable function on* R *such that for every fixed* y*, the function that takes* x *to* $f(x,y)$ *is zero except at finitely many points. Show*

$$\int_R f = 0.$$

Exercise* 10.2.5*: *Let* $R := [a,b] \times [c,d]$ *and* $f(x,y) := g(x)h(y)$ *for two continuous functions* $g\colon [a,b] \to \mathbb{R}$ *and* $h\colon [c,d] \to \mathbb{R}$*. Prove*

$$\int_R f = \left(\int_a^b g\right)\left(\int_c^d h\right).$$

Exercise* 10.2.6*: *Compute (using calculus)*

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy \qquad \textit{and} \qquad \int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx.$$

You will need to interpret the integrals as improper, that is, the limit of $\int_\varepsilon^1$ *as* $\varepsilon \to 0^+$*.*

***Exercise* 10.2.7:** *Suppose $f(x,y) := g(x)$ where $g\colon [a,b] \to \mathbb{R}$ is Riemann integrable. Show that f is Riemann integrable for every $R = [a,b] \times [c,d]$ and*

$$\int_R f = (d-c) \int_a^b g.$$

***Exercise* 10.2.8:** *Define $f\colon [-1,1] \times [0,1] \to \mathbb{R}$ by*

$$f(x,y) := \begin{cases} x & \text{if } y \in \mathbb{Q}, \\ 0 & \text{else.} \end{cases}$$

a) Show $\int_0^1 \int_{-1}^1 f(x,y)\,dx\,dy$ exists, but $\int_{-1}^1 \int_0^1 f(x,y)\,dy\,dx$ does not.

b) Compute $\int_{-1}^1 \overline{\int_0^1} f(x,y)\,dy\,dx$ and $\int_{-1}^1 \underline{\int_0^1} f(x,y)\,dy\,dx$.

c) Show f is not Riemann integrable on $[-1,1] \times [0,1]$ (use Fubini).

***Exercise* 10.2.9:** *Define $f\colon [0,1] \times [0,1] \to \mathbb{R}$ by*

$$f(x,y) := \begin{cases} 1/q & \text{if } x \in \mathbb{Q}, y \in \mathbb{Q}, \text{ and } y = p/q \text{ in lowest terms,} \\ 0 & \text{else.} \end{cases}$$

a) Show f is Riemann integrable on $[0,1] \times [0,1]$.

b) Find $\overline{\int_0^1} f(x,y)\,dx$ and $\underline{\int_0^1} f(x,y)\,dx$ for all $y \in [0,1]$, and show they are unequal for all $y \in \mathbb{Q}$.

c) Show $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ exists, but $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ does not.

Note: By Fubini, $\int_0^1 \overline{\int_0^1} f(x,y)\,dy\,dx$ and $\int_0^1 \underline{\int_0^1} f(x,y)\,dy\,dx$ do exist and equal the integral of f on R.

10.3 Outer measure and null sets

Note: 2 lectures

10.3.1 Outer measure and null sets

Before we characterize all Riemann integrable functions, we need to make a slight detour. We introduce a way of measuring the size of sets in $\mathbb{R}^n$.

Definition 10.3.1. Let $S \subset \mathbb{R}^n$ be a subset. Define the *outer measure* of S as

$$m^*(S) := \inf \sum_{j=1}^{\infty} V(R_j),$$

where the infimum is taken over all sequences $\{R_j\}$ of open rectangles such that $S \subset \bigcup_{j=1}^{\infty} R_j$, and we are allowing both the sum and the infimum to be ∞. See Figure 10.4. In particular, S is of *measure zero* or a *null set* if $m^*(S) = 0$.

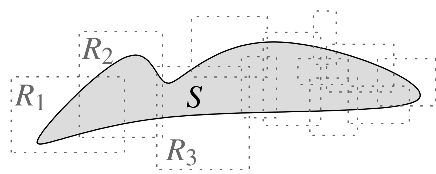

Figure 10.4: Outer measure construction, in this case $S \subset R_1 \cup R_2 \cup R_3 \cup \cdots$, so $m^*(S) \leq V(R_1) + V(R_2) + V(R_3) + \cdots$.

An immediate consequence (Exercise 10.3.2) of the definition is that if $A \subset B$, then $m^*(A) \leq m^*(B)$. It is also not difficult to show (Exercise 10.3.13) that we obtain the same number $m^*(S)$ if we also allow both finite and infinite sequences of rectangles in the definition. It is not enough, however, to allow only finite sequences.

The theory of measures on $\mathbb{R}^n$ is a very complicated subject. We will only require measure-zero sets and so we focus on these. A set S is of measure zero if for every $\varepsilon > 0$ there exists a sequence of open rectangles $\{R_j\}$ such that

$$S \subset \bigcup_{j=1}^{\infty} R_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} V(R_j) < \varepsilon. \tag{10.2}$$

If S is of measure zero and $S' \subset S$, then S' is of measure zero. We can use the same exact rectangles.

It is sometimes more convenient to use balls instead of rectangles. Furthermore, we can choose balls no bigger than a fixed radius.

Proposition 10.3.2. *Let $\delta > 0$ be given. A set $S \subset \mathbb{R}^n$ is of measure zero if and only if for every $\varepsilon > 0$, there exists a sequence of open balls $\{B_j\}$, where the radius of B_j is $r_j < \delta$, and such that*

$$S \subset \bigcup_{j=1}^{\infty} B_j \qquad \textit{and} \qquad \sum_{j=1}^{\infty} r_j^n < \varepsilon.$$

Note that the "volume" of B_j is proportional to r_j^n.

Proof. If C is a closed cube (rectangle with all sides equal) of side s, then C is contained in a closed ball of radius $\sqrt{n}\,s$ by Proposition 10.1.14, and therefore in an open ball of radius $2\sqrt{n}\,s$.

Suppose R is a rectangle of positive volume. Let $s > 0$ be a number that is less than the smallest side of R and also so that $2\sqrt{n}\,s < \delta$. We claim R is contained in a union of closed cubes $C_1, C_2, \ldots, C_k$ of sides s such that

$$\sum_{j=1}^{k} V(C_j) \leq 2^n V(R).$$

It is clearly true (without the 2^n) if R has sides that are integer multiples of s. So if a side is of length $(\ell+\alpha)s$, for $\ell \in \mathbb{N}$ and $0 \leq \alpha < 1$, then $(\ell+\alpha)s \leq 2\ell s$. Increasing the side to $2\ell s$, and then doing the same for every side, we obtain a new larger rectangle of volume at most 2^n times larger, but whose sides are multiples of s.

So suppose that S is a null set and there exist $\{R_j\}$ whose union contains S and such that (10.2) is true. As we have seen above, we can choose closed cubes $\{C_k\}$ with C_k of side s_k as above that cover all the rectangles $\{R_j\}$ and so that

$$\sum_{k=1}^{\infty} s_k^n = \sum_{k=1}^{\infty} V(C_k) \leq 2^n \sum_{j=1}^{\infty} V(R_k) < 2^n \varepsilon.$$

Covering C_k with balls B_k of radius $r_k = 2\sqrt{n}\,s_k < \delta$ we obtain

$$\sum_{k=1}^{\infty} r_k^n = \sum_{k=1}^{\infty} (2\sqrt{n})^n s_k^n < (4\sqrt{n})^n \varepsilon.$$

And as $S \subset \bigcup_j R_j \subset \bigcup_k C_k \subset \bigcup_k B_k$, we are finished.

For the other direction, suppose S is covered by balls B_j of radii r_j, such that $\sum r_j^n < \varepsilon$, as in the statement of the proposition. Each B_j is contained in an open cube R_j of side $2r_j$. So $V(R_j) = (2r_j)^n = 2^n r_j^n$. Therefore,

$$S \subset \bigcup_{j=1}^{\infty} R_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} V(R_j) \leq \sum_{j=1}^{\infty} 2^n r_j^n < 2^n \varepsilon. \qquad \square$$

The definition of outer measure (not just null sets) could have been done with open balls as well. We leave this generalization to the reader.

10.3.2 Examples and basic properties

Example 10.3.3: The set $\mathbb{Q}^n \subset \mathbb{R}^n$ of points with rational coordinates is a set of measure zero.

Proof: The set $\mathbb{Q}^n$ is countable and therefore let us write it as a sequence $q_1, q_2, \ldots$. For each q_j find an open rectangle R_j with $q_j \in R_j$ and $V(R_j) < \varepsilon 2^{-j}$. Then

$$\mathbb{Q}^n \subset \bigcup_{j=1}^{\infty} R_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} V(R_j) < \sum_{j=1}^{\infty} \varepsilon 2^{-j} = \varepsilon.$$

The example points to a more general result.

Proposition 10.3.4. *A countable union of measure zero sets is of measure zero.*

Proof. Suppose

$$S = \bigcup_{j=1}^{\infty} S_j,$$

where S_j are all measure zero sets. Let $\varepsilon > 0$ be given. For each j there exists a sequence of open rectangles $\{R_{j,k}\}_{k=1}^{\infty}$ such that

$$S_j \subset \bigcup_{k=1}^{\infty} R_{j,k}$$

and

$$\sum_{k=1}^{\infty} V(R_{j,k}) < 2^{-j}\varepsilon.$$

Then

$$S \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} R_{j,k}.$$

As $V(R_{j,k})$ is always positive, the sum over all j and k can be done in any manner. In particular, it can be done as

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} V(R_{j,k}) < \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon. \qquad \square$$

The next example is not just interesting, it will be useful later.

Example 10.3.5: Let $P := \{x \in \mathbb{R}^n : x_k = c\}$ for a fixed $k = 1, 2, \ldots, n$ and a fixed constant $c \in \mathbb{R}$. Then P is of measure zero.

Proof: First fix s and let us prove that

$$P_s := \left\{x \in \mathbb{R}^n : x_k = c, |x_j| \leq s \text{ for all } j \neq k\right\}$$

is of measure zero. Given any $\varepsilon > 0$ define the open rectangle

$$R := \left\{x \in \mathbb{R}^n : c - \varepsilon < x_k < c + \varepsilon, |x_j| < s + 1 \text{ for all } j \neq k\right\}.$$

It is clear that $P_s \subset R$. Furthermore,

$$V(R) = 2\varepsilon\big(2(s+1)\big)^{n-1}.$$

As s is fixed, we make $V(R)$ arbitrarily small by picking ε small enough. So P_s is measure zero.

Next

$$P = \bigcup_{j=1}^{\infty} P_j$$

and a countable union of measure zero sets is measure zero.

Example 10.3.6: If $a < b$, then $m^*\big([a,b]\big) = b - a$.

Proof: In $\mathbb{R}$, open rectangles are open intervals. Since $[a,b] \subset (a-\varepsilon, b+\varepsilon)$ for all $\varepsilon > 0$. Hence, $m^*\big([a,b]\big) \leq b-a$.

Let us prove the other inequality. Suppose $\big\{(a_j,b_j)\big\}$ are open intervals such that

$$[a,b] \subset \bigcup_{j=1}^{\infty} (a_j,b_j).$$

We wish to bound $\sum (b_j - a_j)$ from below. Since $[a,b]$ is compact, then finitely many of the open intervals still cover $[a,b]$. As throwing out some of the intervals only makes the sum smaller, we only need to consider the finite number of intervals still covering $[a,b]$. If $(a_i,b_i) \subset (a_j,b_j)$, then we can throw out (a_i,b_i) as well, in other words the intervals that are left have distinct left endpoints, and whenever $a_j < a_i < b_j$, then $b_j < b_i$. Therefore, $[a,b] \subset \bigcup_{j=1}^{k} (a_j,b_j)$ for some k, and we assume that the intervals are sorted such that $a_1 < a_2 < \cdots < a_k$. Since (a_2,b_2) is not contained in (a_1,b_1), since $a_j > a_2$ for all $j > 2$, and since the intervals must contain every point in $[a,b]$, we find that $a_2 < b_1$, or in other words $a_1 < a_2 < b_1 < b_2$. Similarly $a_j < a_{j+1} < b_j < b_{j+1}$ for all j. Furthermore, $a_1 < a$ and $b_k > b$. Thus,

$$m^*\big([a,b]\big) \geq \sum_{j=1}^{k} (b_j - a_j) \geq \sum_{j=1}^{k-1} (a_{j+1} - a_j) + (b_k - a_k) = b_k - a_1 > b - a.$$

Proposition 10.3.7. *Suppose $E \subset \mathbb{R}^n$ is a compact set of measure zero. Then for every $\varepsilon > 0$, there exist finitely many open rectangles $R_1, R_2, \ldots, R_k$ such that*

$$E \subset R_1 \cup R_2 \cup \cdots \cup R_k \qquad \text{and} \qquad \sum_{j=1}^{k} V(R_j) < \varepsilon.$$

Moreover, for every $\varepsilon > 0$ and every $\delta > 0$, there exist finitely many open balls $B_1, B_2, \ldots, B_\ell$ of radii $r_1, r_2, \ldots, r_\ell < \delta$ such that

$$E \subset B_1 \cup B_2 \cup \cdots \cup B_\ell \qquad \text{and} \qquad \sum_{j=1}^{\ell} r_j^n < \varepsilon.$$

Proof. As E is of measure zero, there exists a sequence of open rectangles $\{R_j\}$ such that

$$E \subset \bigcup_{j=1}^{\infty} R_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} V(R_j) < \varepsilon.$$

By compactness, there are finitely many of these rectangles that still contain E. That is, there is some k such that $E \subset R_1 \cup R_2 \cup \cdots \cup R_k$. Hence

$$\sum_{j=1}^{k} V(R_j) \leq \sum_{j=1}^{\infty} V(R_j) < \varepsilon.$$

The proof that we can choose balls instead of rectangles is left as an exercise. □

Example 10.3.8: So that the reader is not under the impression that there are only few measure zero sets and that these sets are uncomplicated, let us give an uncountable, compact, measure zero subset of $[0,1]$. For every $x \in [0,1]$, write its representation in ternary notation

$$x = \sum_{n=1}^{\infty} d_n 3^{-n}, \qquad \text{where } d_n = 0, 1, \text{ or } 2.$$

See §1.5 in volume I, in particular Exercise 1.5.4. Define the *Cantor set* C as

$$C := \left\{ x \in [0,1] : x = \sum_{n=1}^{\infty} d_n 3^{-n}, \text{ where } d_n = 0 \text{ or } d_n = 2 \text{ for all } n \right\}.$$

That is, x is in C if it has a ternary expansion in only 0s and 2s. If x has two expansions, as long as one of them does not have any 1s, then x is in C. Define $C_0 := [0,1]$ and

$$C_k := \left\{ x \in [0,1] : x = \sum_{n=1}^{\infty} d_n 3^{-n}, \text{ where } d_n = 0 \text{ or } d_n = 2 \text{ for all } n = 1, 2, \ldots, k \right\}.$$

Clearly,

$$C = \bigcap_{k=1}^{\infty} C_k.$$

See Figure 10.5.

We leave as an exercise to prove that

(i) Each C_k is a finite union of closed intervals. It is obtained by taking C_{k-1}, and from each closed interval removing the "middle third."

(ii) Each C_k is closed, and so C is closed.

(iii) $m^*(C_k) = 1 - \sum_{n=1}^{k} \frac{2^n}{3^{n+1}}$.

(iv) Hence, $m^*(C) = 0$.

(v) The set C is in one-to-one correspondence with $[0,1]$, in other words, C is uncountable.

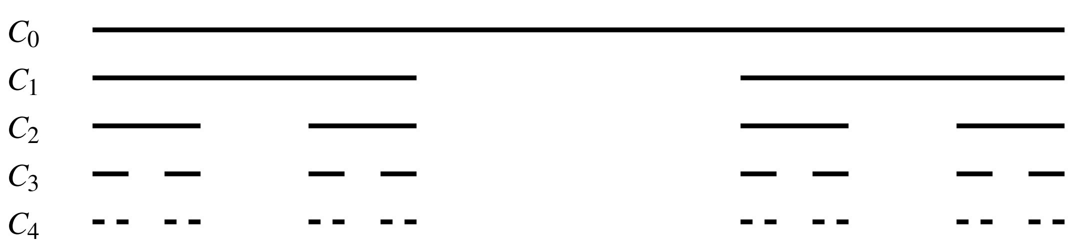

Figure 10.5: Cantor set construction.

10.3.3 Images of null sets under differentiable functions

Before we look at images of measure zero sets, let us see what a continuously differentiable function does to a ball.

Lemma 10.3.9. *Suppose $U \subset \mathbb{R}^n$ is an open set, $B \subset U$ is an open (resp. closed) ball of radius at most r, $f : B \to \mathbb{R}^n$ is continuously differentiable and suppose $\|f'(x)\| \leq M$ for all $x \in B$. Then $f(B) \subset B'$, where B' is an open (resp. closed) ball of radius at most Mr.*

Proof. Suppose that B is open. The ball B is convex, and so via Proposition 8.4.2, $\|f(x) - f(y)\| \leq M\|x - y\|$. So if $\|x - y\| < r$, then $\|f(x) - f(y)\| < Mr$, or in other words, if $B = B(y,r)$, then $f(B) \subset B\big(f(y), Mr\big)$. If B is closed, then $\overline{B(y,r)} = B$. As f is continuous, $f(B) = f\big(\overline{B(y,r)}\big) \subset \overline{B\big(f(y), Mr\big)}$, as $f(\overline{A}) \subset \overline{f(A)}$ for any set A. $\square$

The image of a measure zero set using a continuous map is not necessarily a measure zero set, although this is not easy to show (see the exercises). However, if the mapping is continuously differentiable, then the mapping cannot "stretch" the set that much.

Proposition 10.3.10. *Suppose $U \subset \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^n$ is a continuously differentiable mapping. If $E \subset U$ is a measure zero set, then $f(E)$ is measure zero.*

Proof. We prove the proposition for a compact E and leave the general case as an exercise.

Suppose E is compact and of measure zero. First, we will replace U by a smaller open set to make $\|f'(x)\|$ bounded. At each point $x \in E$ pick an open ball $B(x, r_x)$ such that the closed ball $C(x, r_x) \subset U$. By compactness we only need to take finitely many points $x_1, x_2, \ldots, x_q$ to cover E we the balls $B(x_j, r_{x_j})$. Define

$$U' := \bigcup_{j=1}^{q} B(x_j, r_{x_j}), \qquad K := \bigcup_{j=1}^{q} C(x_j, r_{x_j}).$$

We have $E \subset U' \subset K \subset U$. The set K, being a finite union of compact sets, is compact. The function that takes x to $\|f'(x)\|$ is continuous, and therefore there exists an $M > 0$ such that $\|f'(x)\| \leq M$ for all $x \in K$. So without loss of generality we may replace U by U' and from now on suppose that $\|f'(x)\| \leq M$ for all $x \in U$.

At each $x \in E$, take the maximum radius δ_x such that $B(x, \delta_x) \subset U$ (we may assume $U \neq \mathbb{R}^n$). Let $\delta := \inf_{x \in E} \delta_x$. We want to show that $\delta > 0$. Take a sequence $\{x_j\} \subset E$ so that $\delta_{x_j} \to \delta$. As E is compact, we can pick the sequence to be convergent to some $y \in E$. Once $\|x_j - y\| < \frac{\delta_y}{2}$, then $\delta_{x_j} > \frac{\delta_y}{2}$ by the triangle inequality. Thus, $\delta > 0$.

Given $\varepsilon > 0$, there exist balls $B_1, B_2, \ldots, B_k$ of radii $r_1, r_2, \ldots, r_k < \frac{\delta}{2}$ such that

$$E \subset B_1 \cup B_2 \cup \cdots \cup B_k \qquad \text{and} \qquad \sum_{j=1}^{k} r_j^n < \varepsilon.$$

We can assume that each ball contains a point of E and so the balls are contained in U. Suppose $B_1', B_2', \ldots, B_k'$ are the balls of radius $Mr_1, Mr_2, \ldots, Mr_k$ from Lemma 10.3.9, such that $f(B_j) \subset B_j'$ for all j. Then,

$$f(E) \subset f(B_1) \cup f(B_2) \cup \cdots \cup f(B_k) \subset B_1' \cup B_2' \cup \cdots \cup B_k' \qquad \text{and} \qquad \sum_{j=1}^{k} (Mr_j)^n < M^n \varepsilon. \quad \square$$

10.3.4 Exercises

Exercise* 10.3.1*: *Finish the proof of Proposition 10.3.7, that is, show that you can use balls instead of rectangles.*

Exercise* 10.3.2*: *If $A \subset B$, then $m^*(A) \leq m^*(B)$.*

Exercise* 10.3.3*: *Suppose $X \subset \mathbb{R}^n$ is a set such that for every $\varepsilon > 0$, there exists a set Y such that $X \subset Y$ and $m^*(Y) \leq \varepsilon$. Prove that X is a measure zero set.*

Exercise* 10.3.4*: *Show that if $R \subset \mathbb{R}^n$ is a closed rectangle, then $m^*(R) = V(R)$.*

Exercise* 10.3.5*: *The closure of a measure zero set can be quite large. Find an example set $S \subset \mathbb{R}^n$ that is of measure zero, but whose closure $\bar{S} = \mathbb{R}^n$.*

Exercise* 10.3.6*: *Prove the general case of Proposition 10.3.10 without using compactness:*

a) *Mimic the proof to first prove that the proposition holds if E is* relatively compact*; a set $E \subset U$ is relatively compact if the closure of E in the subspace topology on U is compact, or in other words if there exists a compact set K with $K \subset U$ and $E \subset K$.*
   Hint: The bound on the size of the derivative still holds, but you need to use countably many balls in the second part of the proof. Be careful as the closure of E need no longer be measure zero.

b) *Now prove it for every null set E.*
   Hint: First show that $\{x \in U : \|x-y\| \geq {}^1/_m \text{ for all } y \notin U \text{ and } \|x\| \leq m\}$ is compact for every $m > 0$.

Exercise* 10.3.7*: *Let $U \subset \mathbb{R}^n$ be an open set and let $f \colon U \to \mathbb{R}$ be a continuously differentiable function. Let $G := \{(x,y) \in U \times \mathbb{R} : y = f(x)\}$ be the graph of f. Show that G is of measure zero.*

Exercise* 10.3.8*: *Given a closed rectangle $R \subset \mathbb{R}^n$, show that for every $\varepsilon > 0$, there exists a number $s > 0$ and finitely many open cubes $C_1, C_2, \ldots, C_k$ of side s such that $R \subset C_1 \cup C_2 \cup \cdots \cup C_k$ and*

$$\sum_{j=1}^{k} V(C_j) \leq V(R) + \varepsilon.$$

Exercise* 10.3.9*: *Show that there exists a number $k = k(n,r,\delta)$ depending only on n, r and δ such the following holds. Given $B(x,r) \subset \mathbb{R}^n$ and $\delta > 0$, there exist k open balls $B_1, B_2, \ldots, B_k$ of radius at most δ such that $B(x,r) \subset B_1 \cup B_2 \cup \cdots \cup B_k$. Note that you can find k that really only depends on n and the ratio ${}^\delta/_r$.*

Exercise* 10.3.10** (Challenging):*** *Prove the statements of Example 10.3.8. That is, prove:*

a) *Each C_k is a finite union of closed intervals, and so C is closed.*
b) *$m^*(C_k) = 1 - \sum_{n=1}^{k} \frac{2^n}{3^{n+1}}$.*
c) *$m^*(C) = 0$.*
d) *The set C is in one-to-one correspondence with $[0,1]$.*

Exercise* 10.3.11*: *Prove that the Cantor set of Example 10.3.8 contains no interval. That is, whenever $a < b$, there exists a point $x \notin C$ such that $a < x < b$.*
Note a consequence of this statement. While every open set in $\mathbb{R}$ is a countable disjoint union of intervals, a closed set (even though it is just the complement of an open set) need not be a union of intervals.

Exercise* 10.3.12** (Challenging):*** *Let us construct the so-called* Cantor function *or the* Devil's staircase. *Let C be the Cantor set and let C_k be as in Example 10.3.8. Write $x \in [0,1]$ in ternary representation $x = \sum_{n=1} d_n 3^{-n}$. If $d_n \neq 1$ for all n, then let $c_n := \frac{d_n}{2}$ for all n. Otherwise, let k be the smallest integer such that $d_k = 1$. Then let $c_n := \frac{d_n}{2}$ if $n < k$, $c_k := 1$, and $c_n := 0$ if $n > k$. Then define*

$$\varphi(x) := \sum_{n=1}^{\infty} c_n 2^{-n}.$$

a) Prove that φ is continuous and increasing (see Figure 10.5).

b) Prove that for $x \notin C$, φ is differentiable at x and $\varphi'(x) = 0$. (Notice that φ' exists and is zero except for a set of measure zero, yet the function manages to climb from 0 *to* 1*.)*

c) Define $\psi \colon [0,1] \to [0,2]$ by $\psi(x) := \varphi(x) + x$. Show that ψ is continuous, strictly increasing, and bijective.

d) Prove that while $m^*(C) = 0$, $m^*\big(\psi(C)\big) \neq 0$. That is, continuous functions need take measure zero sets to measure zero sets. Hint: $m^*\big(\psi([0,1] \setminus C)\big) = 1$, but $m^*\big([0,2]\big) = 2$.

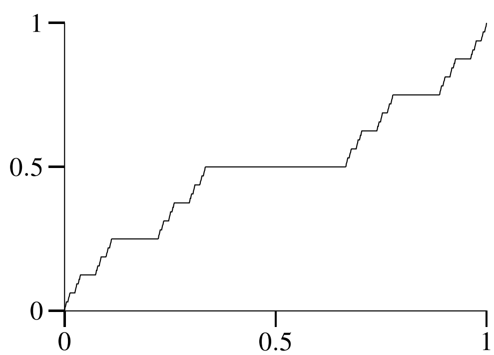

Figure 10.6: Cantor function or Devil's staircase (the function φ from the exercise).

Exercise* 10.3.13*: *Prove that we obtain the same outer measure if we allow both finite and infinite sequences in the definition. That is, define $\mu^*(S) := \inf \sum_{j \in I} V(R_j)$ where the infimum is taken over all countable (finite or infinite) sets of open rectangles $\{R_j\}_{j \in I}$ such that $S \subset \bigcup_{j \in I} R_j$. Prove that for every $S \subset \mathbb{R}^n$, $\mu^*(S) = m^*(S)$.*

Exercise* 10.3.14*: *Prove that for any two subsets $A, B \subset \mathbb{R}^n$, we have $m^*(A \cup B) \leq m^*(A) + m^*(B)$.*

Exercise* 10.3.15*: *Suppose $A, B \subset \mathbb{R}^n$ are such that $m^*(B) = 0$. Prove that $m^*(A \cup B) = m^*(A)$.*

Exercise* 10.3.16*: *Suppose $R_1, R_2, \ldots, R_n$ are pairwise disjoint open rectangles. Prove that $m^*(R_1 \cup R_2 \cup \cdots \cup R_n) = m^*(R_1) + m^*(R_2) + \cdots + m^*(R_n)$.*

10.4 The set of Riemann integrable functions

Note: 1 lecture

10.4.1 Oscillation and continuity

Consider $D \subset \mathbb{R}^n$ and $f \colon D \to \mathbb{R}$. Instead of just saying that f is or is not continuous at a point $x \in D$, we want to quantify how discontinuous f is at x. For every $\delta > 0$, define the *oscillation* of f on the δ-ball in subspace topology, $B_D(x,\delta) = B_{\mathbb{R}^n}(x,\delta) \cap D$, as

$$o(f,x,\delta) := \sup_{y \in B_D(x,\delta)} f(y) - \inf_{y \in B_D(x,\delta)} f(y) = \sup_{y_1,y_2 \in B_D(x,\delta)} \bigl(f(y_1) - f(y_2)\bigr).$$

That is, $o(f,x,\delta)$ is the length of the smallest interval that contains the image $f\bigl(B_D(x,\delta)\bigr)$. For unbounded functions, the oscillation could be ∞, although we only need to worry about bounded functions. Clearly $o(f,x,\delta) \geq 0$ and $o(f,x,\delta) \leq o(f,x,\delta')$ whenever $\delta < \delta'$. Therefore, the limit as $\delta \to 0$ from the right exists, and we define the *oscillation* of f at x as

$$o(f,x) := \lim_{\delta \to 0^+} o(f,x,\delta) = \inf_{\delta > 0} o(f,x,\delta).$$

Proposition 10.4.1. *A function $f \colon D \to \mathbb{R}$ is continuous at $x \in D$ if and only if $o(f,x) = 0$.*

Proof. First suppose that f is continuous at $x \in D$. Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for $y \in B_D(x,\delta)$, we have $|f(x) - f(y)| < \varepsilon$. Therefore, if $y_1, y_2 \in B_D(x,\delta)$, then

$$f(y_1) - f(y_2) = \bigl(f(y_1) - f(x)\bigr) - \bigl(f(y_2) - f(x)\bigr) < \varepsilon + \varepsilon = 2\varepsilon.$$

We take the supremum over y_1 and y_2

$$o(f,x,\delta) = \sup_{y_1,y_2 \in B_D(x,\delta)} \bigl(f(y_1) - f(y_2)\bigr) \leq 2\varepsilon.$$

As $o(x,f) \leq o(f,x,\delta) \leq 2\varepsilon$, and $\varepsilon > 0$ was arbitrary, $o(x,f) = 0$.

On the other hand suppose $o(x,f) = 0$. Given any $\varepsilon > 0$, find a $\delta > 0$ such that $o(f,x,\delta) < \varepsilon$. If $y \in B_D(x,\delta)$, then

$$|f(x) - f(y)| \leq \sup_{y_1,y_2 \in B_D(x,\delta)} \bigl(f(y_1) - f(y_2)\bigr) = o(f,x,\delta) < \varepsilon. \qquad \square$$

Proposition 10.4.2. *Let $D \subset \mathbb{R}^n$ be closed, $f \colon D \to \mathbb{R}$, and $\varepsilon > 0$. The set $\bigl\{x \in D : o(f,x) \geq \varepsilon\bigr\}$ is closed.*

Proof. Equivalently, we want to show that $G := \bigl\{x \in D : o(f,x) < \varepsilon\bigr\}$ is open in the subspace topology. Consider $x \in G$. As $\inf_{\delta>0} o(f,x,\delta) < \varepsilon$, find a $\delta > 0$ such that

$$o(f,x,\delta) < \varepsilon$$

Take any $\xi \in B_D(x, \delta/2)$. Notice that $B_D(\xi, \delta/2) \subset B_D(x,\delta)$. Therefore,

$$o(f,\xi,\delta/2) = \sup_{y_1,y_2 \in B_D(\xi,\delta/2)} \bigl(f(y_1) - f(y_2)\bigr) \leq \sup_{y_1,y_2 \in B_D(x,\delta)} \bigl(f(y_1) - f(y_2)\bigr) = o(f,x,\delta) < \varepsilon.$$

So $o(f,\xi) < \varepsilon$ as well. As this is true for all $\xi \in B_D(x,\delta/2)$, we get that G is open in the subspace topology and $D \setminus G$ is closed as claimed. $\square$

10.4.2 The set of Riemann integrable functions

We have seen that continuous functions are Riemann integrable, but we also know that certain kinds of discontinuities are allowed. It turns out that as long as the discontinuities happen on a set of measure zero, the function is integrable, and vice versa.

Theorem 10.4.3 (Riemann–Lebesgue)**.** *Let $R \subset \mathbb{R}^n$ be a closed rectangle and $f\colon R \to \mathbb{R}$ bounded. Then f is Riemann integrable if and only if the set of discontinuities of f is of measure zero.*

Proof. Let $S \subset R$ be the set of discontinuities of f, that is, $S = \{x \in R : o(f,x) > 0\}$. Suppose S is a measure zero set: $m^*(S) = 0$. The trick to proving that f is integrable is to isolate the bad set into a small set of subrectangles of a partition. A partition has finitely many subrectangles, so we need compactness. If S were closed, then it would be compact and we could cover it by finitely many small rectangles. Unfortunately, S itself is not closed in general, but the following set is.

Given $\varepsilon > 0$, define
$$S_\varepsilon := \{x \in R : o(f,x) \geq \varepsilon\}.$$
By Proposition 10.4.2, S_ε is closed and as it is a subset of R, which is bounded, S_ε is compact. Furthermore, $S_\varepsilon \subset S$ and S is of measure zero, so S_ε is of measure zero. Via Proposition 10.3.7, finitely many open rectangles $O_1, O_2, \ldots, O_k$ cover S_ε and $\sum V(O_j) < \varepsilon$.

The set $T := R \setminus (O_1 \cup \cdots \cup O_k)$ is closed, bounded, and so compact. As $o(f,x) < \varepsilon$ for all $x \in T$, for each $x \in T$, there is a $\delta > 0$ such that $o(f,x,\delta) < \varepsilon$, so there exists a small closed rectangle $T_x \subset B(x,\delta)$ with x in the interior of T_x, such that
$$\sup_{y \in T_x} f(y) - \inf_{y \in T_x} f(y) < \varepsilon.$$
The interiors of the rectangles T_x cover T. As T is compact, finitely many such rectangles $T_1, T_2, \ldots, T_m$ cover T. Take the rectangles $T_1, T_2, \ldots, T_m$ and $O_1, O_2, \ldots, O_k$ and construct a partition out of their endpoints. That is, construct a partition P of R with subrectangles $R_1, R_2, \ldots, R_p$ such that every R_j is contained in T_ℓ for some ℓ or the closure of O_ℓ for some ℓ. Order the rectangles so that $R_1, R_2, \ldots, R_q$ are those that are contained in some T_ℓ, and $R_{q+1}, R_{q+2}, \ldots, R_p$ are the rest. So
$$\sum_{j=1}^{q} V(R_j) \leq V(R) \qquad \text{and} \qquad \sum_{j=q+1}^{p} V(R_j) \leq \sum_{\ell=1}^{k} V(O_\ell) < \varepsilon.$$
Let m_j and M_j be the inf and sup of f over R_j as before. If $R_j \subset T_\ell$ for some ℓ, then $M_j - m_j < 2\varepsilon$. Let $B \in \mathbb{R}$ be such that $|f(x)| \leq B$ for all $x \in R$, so $M_j - m_j \leq 2B$ over all rectangles. Then
$$\begin{aligned}
U(P,f) - L(P,f) &= \sum_{j=1}^{p} (M_j - m_j) V(R_j) \\
&= \left(\sum_{j=1}^{q} (M_j - m_j) V(R_j) \right) + \left(\sum_{j=q+1}^{p} (M_j - m_j) V(R_j) \right) \\
&< \left(\sum_{j=1}^{q} 2\varepsilon V(R_j) \right) + \left(\sum_{j=q+1}^{p} 2B V(R_j) \right) \\
&< 2\varepsilon V(R) + 2B\varepsilon = \varepsilon\big(2V(R) + 2B\big).
\end{aligned}$$
We can make the right-hand side as small as we want, and hence f is integrable.

For the other direction, suppose f is Riemann integrable on R. Let S be the set of discontinuities again. Consider the sequence of sets

$$S_{1/k} = \{x \in R : o(f,x) \geq 1/k\}.$$

Fix a $k \in \mathbb{N}$. Given an $\varepsilon > 0$, find a partition P with subrectangles $R_1, R_2, \ldots, R_p$ such that

$$U(P,f) - L(P,f) = \sum_{j=1}^{p} (M_j - m_j) V(R_j) < \varepsilon$$

Suppose $R_1, R_2, \ldots, R_p$ are ordered so that the interiors of $R_1, R_2, \ldots, R_q$ intersect $S_{1/k}$, while the interiors of $R_{q+1}, R_{q+2}, \ldots, R_p$ are disjoint from $S_{1/k}$. If $x \in R_j \cap S_{1/k}$ and x is in the interior of R_j, then $o(f,x) \geq 1/k$. As sufficiently small δ-balls are completely inside R_j and $o(f,x,\delta) \geq o(f,x) \geq 1/k$, we get $M_j - m_j \geq 1/k$. Then

$$\varepsilon > \sum_{j=1}^{p} (M_j - m_j) V(R_j) \geq \sum_{j=1}^{q} (M_j - m_j) V(R_j) \geq \frac{1}{k} \sum_{j=1}^{q} V(R_j)$$

In other words, $\sum_{j=1}^{q} V(R_j) < k\varepsilon$. Let G be the set of all boundaries of all the subrectangles of P. The set G is of measure zero (as it can be covered by finitely many sets from Example 10.3.5). Let R_j° denote the interior of R_j, then

$$S_{1/k} \subset R_1^\circ \cup R_2^\circ \cup \cdots \cup R_q^\circ \cup G.$$

As G can be covered by open rectangles arbitrarily small volume, $S_{1/k}$ must be of measure zero. As

$$S = \bigcup_{k=1}^{\infty} S_{1/k}$$

and a countable union of measure zero sets is of measure zero, S is of measure zero. $\square$

Corollary 10.4.4. *Let $R \subset \mathbb{R}^n$ be a closed rectangle. Let $\mathscr{R}(R)$ be the set of Riemann integrable functions on R. Then*

(i) $\mathscr{R}(R)$ is a real algebra: If $f, g \in \mathscr{R}(R)$ and $a \in \mathbb{R}$, then $af \in \mathscr{R}(R)$, $f+g \in \mathscr{R}(R)$ and $fg \in \mathscr{R}(R)$.

(ii) If $f, g \in \mathscr{R}(R)$ and

$$\varphi(x) := \max\{f(x), g(x)\}, \qquad \psi(x) := \min\{f(x), g(x)\},$$

then $\varphi, \psi \in \mathscr{R}(R)$.

(iii) If $f \in \mathscr{R}(R)$, then $|f| \in \mathscr{R}(R)$, where $|f|(x) := |f(x)|$.

(iv) If $R' \subset \mathbb{R}^n$ is another closed rectangle, $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^n$ are open sets such that $R \subset U$ and $R' \subset U'$, $g \colon U \to U'$ is continuously differentiable, bijective, g^{-1} is continuously differentiable, $g(R) \subset R'$, and $f \in \mathscr{R}(R')$, then the composition $f \circ g$ is Riemann integrable on R.

The proof is contained in the exercises.

10.4.3 Exercises

Exercise* 10.4.1*: *Suppose $f : (a,b) \times (c,d) \to \mathbb{R}$ is a bounded continuous function. Show that the integral of f over $R = [a,b] \times [c,d]$ makes sense and is uniquely defined. That is, set f to be anything on the boundary of R and compute the integral, showing that the values on the boundary are irrelevant.*

Exercise* 10.4.2*: *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle. Show that $\mathscr{R}(R)$, the set of Riemann integrable functions, is an algebra. That is, show that if $f, g \in \mathscr{R}(R)$ and $a \in \mathbb{R}$, then $af \in \mathscr{R}(R)$, $f + g \in \mathscr{R}(R)$, and $fg \in \mathscr{R}(R)$.*

Exercise* 10.4.3*: *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f : R \to \mathbb{R}$ is a bounded function which is zero except on a closed set $E \subset R$ of measure zero. Show that $\int_R f$ exists and compute it.*

Exercise* 10.4.4*: *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$ are two Riemann integrable functions. Suppose $f = g$ except for a closed set $E \subset R$ of measure zero. Show that $\int_R f = \int_R g$.*

Exercise* 10.4.5*: *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f : R \to \mathbb{R}$ is a bounded function.*

a) Suppose there exists a closed set $E \subset R$ of measure zero such that $f|_{R \setminus E}$ is continuous. Then $f \in \mathscr{R}(R)$.

b) Find an example where $E \subset R$ is a set of measure zero (not closed) such that $f|_{R \setminus E}$ is continuous and $f \notin \mathscr{R}(R)$.

Exercise* 10.4.6*: *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$ are Riemann integrable. Show that*

$$\varphi(x) := \max\bigl\{f(x), g(x)\bigr\}, \qquad \psi(x) := \min\bigl\{f(x), g(x)\bigr\},$$

are Riemann integrable.

Exercise* 10.4.7*: *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle and $f : R \to \mathbb{R}$ is Riemann integrable. Show that $\lvert f \rvert$ is Riemann integrable. Hint: Define $f_+(x) := \max\bigl\{f(x), 0\bigr\}$ and $f_-(x) := \max\bigl\{-f(x), 0\bigr\}$, and then write $\lvert f \rvert$ in terms of f_+ and f_-.*

Exercise* 10.4.8*:

a) Suppose $R \subset \mathbb{R}^n$ and $R' \subset \mathbb{R}^n$ are closed rectangles, $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^n$ are open sets such that $R \subset U$ and $R' \subset U'$, $g : U \to U'$ is continuously differentiable, bijective, g^{-1} is continuously differentiable, $g(R) \subset R'$, and $f \in \mathscr{R}(R')$, then the composition $f \circ g$ is Riemann integrable on R.

b) Find a counterexample when g is not one-to-one. Hint: Try $g(x,y) := (x,0)$ and $R = R' = [0,1] \times [0,1]$.

Exercise* 10.4.9*: *Suppose $f : [0,1]^2 \to \mathbb{R}$ is defined by*

$$f(x,y) := \begin{cases} \frac{1}{kq} & \text{if } x, y \in \mathbb{Q} \text{ and } x = \frac{\ell}{k} \text{ and } y = \frac{p}{q} \text{ in lowest terms,} \\ 0 & \text{else.} \end{cases}$$

Show that $f \in \mathscr{R}\bigl([0,1]^2\bigr)$.

Exercise* 10.4.10*: *Compute the oscillation $o\bigl(f,(x,y)\bigr)$ for all $(x,y) \in \mathbb{R}^2$ for the function*

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Exercise* 10.4.11*: *Consider the popcorn function $f : [0,1] \to \mathbb{R}$,*

$$f(x) := \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms,} \\ 0 & \text{else.} \end{cases}$$

Compute $o(f,x)$ for all $x \in [0,1]$.

10.5 Jordan measurable sets

Note: 1 lecture

10.5.1 Volume and Jordan measurable sets

Given a set $S \subset \mathbb{R}^n$, its *characteristic function* or *indicator function* $\chi_S : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\chi_S(x) := \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

A bounded set S is *Jordan measurable** if for some closed rectangle R such that $S \subset R$, the function χ_S is Riemann integrable, that is, $\chi_S \in \mathscr{R}(R)$. Take two closed rectangles R and R' with $S \subset R$ and $S \subset R'$, then $R \cap R'$ is a closed rectangle also containing S. By Proposition 10.1.13 and Exercise 10.1.7, $\chi_S \in \mathscr{R}(R \cap R')$ and so $\chi_S \in \mathscr{R}(R')$. Thus

$$\int_R \chi_S = \int_{R'} \chi_S = \int_{R \cap R'} \chi_S.$$

We define the *n-dimensional volume* of the bounded Jordan measurable set S as

$$V(S) := \int_R \chi_S,$$

where R is any closed rectangle containing S.

Proposition 10.5.1. *A bounded set $S \subset \mathbb{R}^n$ is Jordan measurable if and only if the boundary ∂S is a measure zero set.*

Proof. Suppose R is a closed rectangle such that S is contained in the interior of R. If $x \in \partial S$, then for every $\delta > 0$, the sets $S \cap B(x,\delta)$ (where χ_S is 1) and the sets $(R \setminus S) \cap B(x,\delta)$ (where χ_S is 0) are both nonempty. So χ_S is not continuous at x. If x is either in the interior of S or in the complement of the closure $\bar{S}$, then χ_S is either identically 1 or identically 0 in a whole neighborhood of x and hence χ_S is continuous at x. Therefore, the set of discontinuities of χ_S is precisely the boundary ∂S. The proposition follows. □

Proposition 10.5.2. *Suppose S and T are bounded Jordan measurable sets. Then*

(i) The closure $\bar{S}$ is Jordan measurable.

(ii) The interior S° is Jordan measurable.

(iii) $S \cup T$ is Jordan measurable.

(iv) $S \cap T$ is Jordan measurable.

(v) $S \setminus T$ is Jordan measurable.

The proof of the proposition is left as an exercise. Next, we find that the volume that we defined above coincides with the outer measure we defined above.

*Named after the French mathematician Marie Ennemond Camille Jordan (1838–1922).

Proposition 10.5.3. *If $S \subset \mathbb{R}^n$ is Jordan measurable, then $V(S) = m^*(S)$.*

Proof. Given $\varepsilon > 0$, let R be a closed rectangle that contains S. Let P be a partition of R such that

$$U(P, \chi_S) \leq \left(\int_R \chi_S \right) + \varepsilon = V(S) + \varepsilon \qquad \text{and} \qquad L(P, \chi_S) \geq \left(\int_R \chi_S \right) - \varepsilon = V(S) - \varepsilon.$$

Let $R_1, R_2, \ldots, R_k$ be all the subrectangles of P such that χ_S is not identically zero on each R_j. That is, there is some point $x \in R_j$ such that $x \in S$ (i.e., $\chi_S(x) = 1$). Let O_j be an open rectangle such that $R_j \subset O_j$ and $V(O_j) < V(R_j) + \varepsilon/k$. Notice that $S \subset \bigcup_j O_j$. Then

$$U(P, \chi_S) = \sum_{j=1}^{k} V(R_j) > \left(\sum_{j=1}^{k} V(O_j) \right) - \varepsilon \geq m^*(S) - \varepsilon.$$

As $U(P, \chi_S) \leq V(S) + \varepsilon$, then $m^*(S) - \varepsilon \leq V(S) + \varepsilon$, or in other words $m^*(S) \leq V(S)$.

Let $R'_1, R'_2, \ldots, R'_\ell$ be all the subrectangles of P such that χ_S is identically one on each R'_j. In other words, these are the subrectangles contained in S. The interiors of the subrectangles R'°_j are disjoint and $V(R'^{\circ}_j) = V(R'_j)$. Via Exercise 10.3.16,

$$m^* \left(\bigcup_{j=1}^{\ell} R'^{\circ}_j \right) = \sum_{j=1}^{\ell} V(R'^{\circ}_j).$$

Hence

$$m^*(S) \geq m^* \left(\bigcup_{j=1}^{\ell} R'_j \right) \geq m^* \left(\bigcup_{j=1}^{\ell} R'^{\circ}_j \right) = \sum_{j=1}^{\ell} V(R'^{\circ}_j) = \sum_{j=1}^{\ell} V(R'_j) = L(P, f) \geq V(S) - \varepsilon.$$

Therefore $m^*(S) \geq V(S)$ as well. □

10.5.2 Integration over Jordan measurable sets

In $\mathbb{R}$ there is only one reasonable type of set to integrate over: an interval. In $\mathbb{R}^n$ there are many kinds of sets. The ones that work with the Riemann integral are the Jordan measurable sets.

Definition 10.5.4. Let $S \subset \mathbb{R}^n$ be a bounded Jordan measurable set. A bounded function $f : S \to \mathbb{R}$ is said to be *Riemann integrable on* S, or $f \in \mathscr{R}(S)$, if for a closed rectangle R such that $S \subset R$, the function $\widetilde{f} : R \to \mathbb{R}$ defined by

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x \in S, \\ 0 & \text{otherwise,} \end{cases}$$

is in $\mathscr{R}(R)$. In this case we write

$$\int_S f := \int_R \widetilde{f}.$$

When f is defined on a larger set and we wish to integrate over S, then we apply the definition to the restriction $f|_S$. As the restriction can be defined by the product $f\xi_S$, and the product of Riemann integrable sets is Riemann integrable, $f|_S$ is automatically Riemann integrable. In particular, if $f : R \to \mathbb{R}$ for a closed rectangle R, and $S \subset R$ is a Jordan measurable subset, then

$$\int_S f = \int_R f\chi_S.$$

Proposition 10.5.5. *If $S \subset \mathbb{R}^n$ is a bounded Jordan measurable set and $f : S \to \mathbb{R}$ is a bounded continuous function, then f is integrable on S.*

Proof. Define the function $\widetilde{f}$ as above for some closed rectangle R with $S \subset R$. If $x \in R \setminus \bar{S}$, then $\widetilde{f}$ is identically zero in a neighborhood of x. Similarly if x is in the interior of S, then $\widetilde{f} = f$ on a neighborhood of x and f is continuous at x. Therefore, $\widetilde{f}$ is only ever possibly discontinuous at ∂S, which is a set of measure zero, and we are finished. □

10.5.3 Images of Jordan measurable subsets

Finally, images of Jordan measurable sets are Jordan measurable under nice enough mappings. For simplicity, we assume that the Jacobian determinant never vanishes.

Proposition 10.5.6. *Suppose $U \subset \mathbb{R}^n$ is open and $S \subset U$ is a compact Jordan measurable set. Suppose $g : U \to \mathbb{R}^n$ is a one-to-one continuously differentiable mapping such that the Jacobian determinant J_g is never zero on S. Then $g(S)$ is bounded and Jordan measurable.*

Proof. Let $T := g(S)$. By Lemma 7.5.5 from volume I, the set T is also compact and so closed and bounded. We claim $\partial T \subset g(\partial S)$. Suppose the claim is proved. As S is Jordan measurable, then ∂S is measure zero. Then $g(\partial S)$ is measure zero by Proposition 10.3.10. As $\partial T \subset g(\partial S)$, then T is Jordan measurable.

It is therefore left to prove the claim. As T is closed, $\partial T \subset T$. Suppose $y \in \partial T$, then there must exist an $x \in S$ such that $g(x) = y$, and by hypothesis $J_g(x) \neq 0$. We use the inverse function theorem (Theorem 8.5.1). We find a neighborhood $V \subset U$ of x and an open set W such that the restriction $f|_V$ is a one-to-one and onto function from V to W with a continuously differentiable inverse. In particular, $g(x) = y \in W$. As $y \in \partial T$, there exists a sequence $\{y_k\}$ in W with $\lim y_k = y$ and $y_k \notin T$. As $g|_V$ is invertible and in particular has a continuous inverse, there exists a sequence $\{x_k\}$ in V such that $g(x_k) = y_k$ and $\lim x_k = x$. Since $y_k \notin T = g(S)$, clearly $x_k \notin S$. Since $x \in S$, we conclude that $x \in \partial S$. The claim is proved, $\partial T \subset g(\partial S)$. □

10.5.4 Exercises

***Exercise* 10.5.1:** *Prove Proposition 10.5.2.*

***Exercise* 10.5.2:** *Prove that a bounded convex set is Jordan measurable. Hint: Induction on dimension.*

Exercise* 10.5.3*: *Let $f \colon [a,b] \to \mathbb{R}$ and $g \colon [a,b] \to \mathbb{R}$ be continuous functions and such that for all $x \in (a,b)$, $f(x) < g(x)$. Let*

$$U := \bigl\{ (x,y) \in \mathbb{R}^2 : a < x < b \text{ and } f(x) < y < g(x) \bigr\}.$$

a) Show that U is Jordan measurable.

b) If $\varphi \colon U \to \mathbb{R}$ is Riemann integrable on U, then

$$\int_U \varphi = \int_a^b \int_{g(x)}^{f(x)} \varphi(x,y)\, dy\, dx.$$

Exercise* 10.5.4*: *Let us construct an example of a non-Jordan measurable open set. Start in one dimension. Let $\{r_j\}$ be an enumeration of all rational numbers in $(0,1)$. Let (a_j,b_j) be open intervals such that $(a_j,b_j) \subset (0,1)$ for all j, $r_j \in (a_j,b_j)$, and $\sum_{j=1}^{\infty}(b_j - a_j) < 1/2$. Now let $U := \bigcup_{j=1}^{\infty}(a_j,b_j)$.*

a) Show the open intervals (a_j,b_j) as above actually exist.

b) Prove $\partial U = [0,1] \setminus U$.

c) Prove ∂U is not of measure zero, and therefore U is not Jordan measurable.

d) Show that $W := \bigl(U \times (0,2)\bigr) \cup \bigl((0,1) \times (1,2)\bigr)$ is a connected bounded open set in $\mathbb{R}^2$ that is not Jordan measurable.

Exercise* 10.5.5*: *Suppose $K \subset \mathbb{R}^n$ is a closed measure zero set.*

a) If K is bounded, prove that K is Jordan measurable.

b) If $S \subset \mathbb{R}^n$ is bounded and Jordan measurable, prove that $S \setminus K$ is Jordan measurable.

c) Construct a bounded Jordan measurable $S \subset \mathbb{R}^n$ and a bounded $T \subset \mathbb{R}^n$ of measure zero, such that neither T nor $S \setminus T$ is Jordan measurable.

Exercise* 10.5.6*: *Suppose $U \subset \mathbb{R}^n$ is open and $K \subset U$ is compact. Find a compact Jordan measurable set S such that $S \subset U$ and $K \subset S^{\circ}$ (K is in the interior of S).*

Exercise* 10.5.7*: *Prove a version of Corollary 10.4.4, replacing all closed rectangles with closed and bounded Jordan measurable sets.*

10.6 Green's theorem

Note: 1 lecture, requires chapter 9

One of the most important theorems of analysis in several variables is the so-called generalized Stokes' theorem, a generalization of the fundamental theorem of calculus. The two-dimensional version is called Green's theorem*. We will state the theorem in general, but we will only prove a special, but important, case.

Definition 10.6.1. Let $U \subset \mathbb{R}^2$ be a bounded connected open set. Suppose the boundary ∂U is a disjoint union of (the images of) finitely many simple closed piecewise smooth paths such that every $p \in \partial U$ is in the closure of $\mathbb{R}^2 \setminus \overline{U}$. Then U is called a *bounded domain with piecewise smooth boundary* in $\mathbb{R}^2$.

The condition about points outside the closure says that locally ∂U separates $\mathbb{R}^2$ into an "inside" and an "outside." The condition prevents ∂U from being just a "cut" inside U. As we travel along the path in a certain orientation, there is a well-defined left and a right, and either U is on the left and the complement of U is on the right, or vice versa. The orientation on U is the direction in which we travel along the paths. We can switch orientation if needed by reparametrizing the path.

Definition 10.6.2. Let $U \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary, let ∂U be oriented , and let $\gamma \colon [a,b] \to \mathbb{R}^2$ be a parametrization of ∂U giving the orientation. Write $\gamma(t) = \big(x(t), y(t)\big)$. If the vector $n(t) := \big(-y'(t), x'(t)\big)$ points into the domain, that is, $\varepsilon n(t) + \gamma(t)$ is in U for all small enough $\varepsilon > 0$, then ∂U is *positively oriented*. See Figure 10.7. Otherwise it is *negatively oriented*.

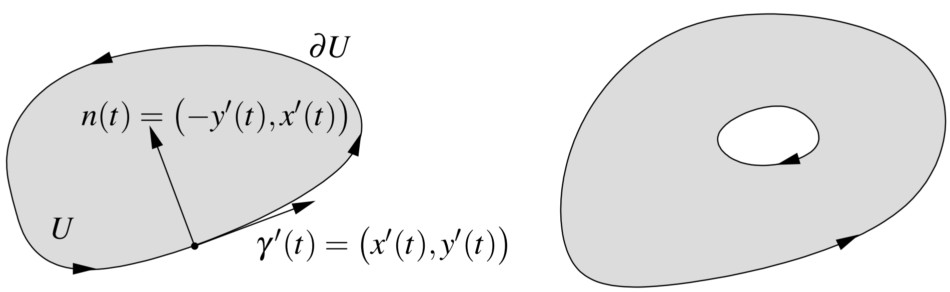

Figure 10.7: Positively oriented domain (left), and a positively oriented domain with a hole (right).

The vector $n(t)$ turns $\gamma'(t)$ counterclockwise by 90°, that is to the left. When we travel along a positively oriented boundary in the direction of its orientation, the domain is "on our left." For example, if U is a bounded domain with "no holes," that is ∂U is connected, then the positive orientation means we are traveling counterclockwise around ∂U. If we do have "holes," then we travel around them clockwise.

Proposition 10.6.3. *Let $U \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary. Then U is Jordan measurable.*

*Named after the British mathematical physicist George Green (1793–1841).

Proof. We must show that ∂U is a null set. As ∂U is a finite union of piecewise smooth paths, which are finite unions of smooth paths, we need only show that a smooth path in $\mathbb{R}^2$ is a null set. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a smooth path. It is enough to show that $\gamma\big((a,b)\big)$ is a null set, as adding the points $\gamma(a)$ and $\gamma(b)$, to a null set still results in a null set. Define

$$f\colon (a,b) \times (-1,1) \to \mathbb{R}^2, \qquad \text{as} \qquad f(x,y) := \gamma(x).$$

The set $(a,b) \times \{0\}$ is a null set in $\mathbb{R}^2$ and $\gamma\big((a,b)\big) = f\big((a,b) \times \{0\}\big)$. By Proposition 10.3.10, $\gamma\big((a,b)\big)$ is a null set in $\mathbb{R}^2$ and so $\gamma\big([a,b]\big)$ is a null set, and so finally ∂U is a null set. □

Theorem 10.6.4 (Green)**.** *Suppose $U \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary with the boundary positively oriented. Suppose P and Q are continuously differentiable functions defined on some open set that contains the closure $\overline{U}$. Then*

$$\int_{\partial U} P\,dx + Q\,dy = \int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

We stated Green's theorem in general, although we will only prove a special version of it. That is, we will only prove it for a special kind of domain. The general version follows from the special case by application of further geometry, and cutting up the general domain into smaller domains on which to apply the special case. We will not prove the general case.

Let $U \subset \mathbb{R}^2$ be a domain with piecewise smooth boundary. We say U is of *type I* if there exist numbers $a < b$, and continuous functions $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$, such that

$$U := \{(x,y) \in \mathbb{R}^2 : a < x < b \text{ and } f(x) < y < g(x)\}.$$

Similarly, U is of *type II* if there exist numbers $c < d$, and continuous functions $h\colon [c,d] \to \mathbb{R}$ and $k\colon [c,d] \to \mathbb{R}$, such that

$$U := \{(x,y) \in \mathbb{R}^2 : c < y < d \text{ and } h(y) < x < k(y)\}.$$

Finally, $U \subset \mathbb{R}^2$ is of *type III* if it is both of type I and type II. See Figure 10.8.

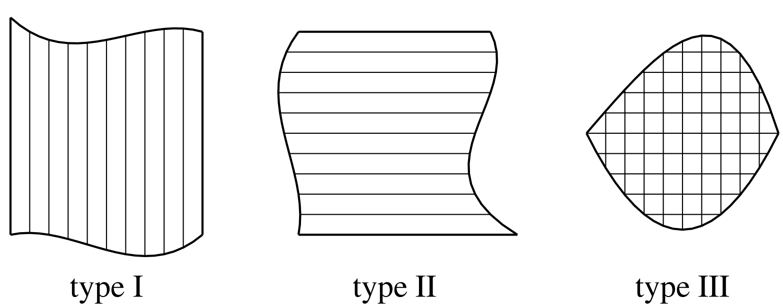

Figure 10.8: Domain types for Green's theorem.

Common domains to apply Green's theorem to are rectangles and discs, and these are type III domains. We will only prove Green's theorem for type III domains.

Proof of Green's theorem for U of type III. Let f,g,h,k be the functions defined above. Using Exercise 10.5.3, U is Jordan measurable and as U is of type I, then

$$\begin{aligned}\int_U \left(-\frac{\partial P}{\partial y}\right) &= \int_a^b \int_{g(x)}^{f(x)} \left(-\frac{\partial P}{\partial y}(x,y)\right) dy\,dx \\ &= \int_a^b \Big(-P\big(x,f(x)\big) + P\big(x,g(x)\big)\Big)\,dx \\ &= \int_a^b P\big(x,g(x)\big)\,dx - \int_a^b P\big(x,f(x)\big)\,dx.\end{aligned}$$

We integrate $P\,dx$ along the boundary. The one-form $P\,dx$ integrates to zero along the straight vertical lines in the boundary. Therefore it is only integrated along the top and along the bottom. As a parameter, x runs from left to right. If we use the parametrizations that take x to $\big(x,f(x)\big)$ and to $\big(x,g(x)\big)$ we recognize path integrals above. However the second path integral is in the wrong direction; the top should be going right to left, and so we must switch orientation.

$$\int_{\partial U} P\,dx = \int_a^b P\big(x,g(x)\big)\,dx + \int_b^a P\big(x,f(x)\big)\,dx = \int_U \left(-\frac{\partial P}{\partial y}\right).$$

Similarly, U is also of type II. The form $Q\,dy$ integrates to zero along horizontal lines. So

$$\int_U \frac{\partial Q}{\partial x} = \int_c^d \int_{k(y)}^{h(y)} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy = \int_a^b \Big(Q\big(y,h(y)\big) - Q\big(y,k(y)\big)\Big)\,dx = \int_{\partial U} Q\,dy.$$

Putting the two together we obtain

$$\int_{\partial U} P\,dx + Q\,dy = \int_{\partial U} P\,dx + \int_{\partial U} Q\,dy = \int_U \left(-\frac{\partial P}{\partial y}\right) + \int_U \frac{\partial Q}{\partial x} = \int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right). \qquad \square$$

We illustrate the usefulness of Green's theorem on a fundamental result about harmonic functions.

Example 10.6.5: Suppose $U \subset \mathbb{R}^2$ is open and $f\colon U \to \mathbb{R}$ is harmonic, that is, f is twice continuously differentiable and $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. We will prove one of the most fundamental properties of harmonic functions.

Let $D_r := B(p,r)$ be closed disc such that its closure $C(p,r) \subset U$. Write $p = (x_0, y_0)$. We orient ∂D_r positively. See Exercise 10.6.1. Then

$$\begin{aligned}0 &= \frac{1}{2\pi r}\int_{D_r} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \\ &= \frac{1}{2\pi r}\int_{\partial D_r} -\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial x}\,dy \\ &= \frac{1}{2\pi r}\int_0^{2\pi} \Bigg(-\frac{\partial f}{\partial y}\big(x_0 + r\cos(t), y_0 + r\sin(t)\big)\big(-r\sin(t)\big) \\ &\qquad\qquad + \frac{\partial f}{\partial x}\big(x_0 + r\cos(t), y_0 + r\sin(t)\big) r\cos(t)\Bigg)\,dt \\ &= \frac{d}{dr}\left[\frac{1}{2\pi}\int_0^{2\pi} f\big(x_0 + r\cos(t), y_0 + r\sin(t)\big)\,dt\right].\end{aligned}$$

Let $g(r) := \frac{1}{2\pi}\int_0^{2\pi} f\big(x_0 + r\cos(t), y_0 + r\sin(t)\big)\,dt$. Then $g'(r) = 0$ for all $r > 0$. The function is constant for $r > 0$ and continuous at $r = 0$ (exercise). Therefore, $g(0) = g(r)$ for all $r > 0$, and

$$g(r) = g(0) = \frac{1}{2\pi}\int_0^{2\pi} f\big(x_0 + 0\cos(t), y_0 + 0\sin(t)\big)\,dt = f(x_0, y_0).$$

We proved the *mean value property* of harmonic functions:

$$f(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} f\big(x_0 + r\cos(t), y_0 + r\sin(t)\big)\,dt = \frac{1}{2\pi r}\int_{\partial D_r} f\,ds.$$

That is, the value at $p = (x_0, y_0)$ is the average over a circle of any radius r centered at (x_0, y_0).

10.6.1 Exercises

Exercise* 10.6.1*: *Prove that a disc $B(p,r) \subset \mathbb{R}^2$ is a type III domain, and prove that the orientation given by the parametrization $\gamma(t) = \big(x_0 + r\cos(t), y_0 + r\sin(t)\big)$ where $p = (x_0, y_0)$ is the positive orientation of the boundary $\partial B(p,r)$.*
Note: Feel free to use what you know about sine and cosine from calculus.

Exercise* 10.6.2*: *Prove that a convex bounded domain with piecewise smooth boundary is a type III domain.*

Exercise* 10.6.3*: *Suppose $V \subset \mathbb{R}^2$ is a domain with piecewise smooth boundary that is a type III domain and suppose that $U \subset \mathbb{R}^2$ is a domain such that $\overline{V} \subset U$. Suppose $f \colon U \to \mathbb{R}$ is a twice continuously differentiable function. Prove that $\int_{\partial V} \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$.*

Exercise* 10.6.4*: *For a disc $B(p,r) \subset \mathbb{R}^2$, orient the boundary $\partial B(p,r)$ positively.*

a) Compute $\displaystyle\int_{\partial B(p,r)} -y\,dx$.

b) Compute $\displaystyle\int_{\partial B(p,r)} x\,dy$.

c) Compute $\displaystyle\int_{\partial B(p,r)} \frac{-y}{2}\,dx + \frac{x}{2}\,dy$.

Exercise* 10.6.5*: *Using Green's theorem show that the area of a triangle with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3) is $\frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1|$. Hint: See previous exercise.*

Exercise* 10.6.6*: *Using the mean value property prove the* maximum principle *for harmonic functions: Suppose $U \subset \mathbb{R}^2$ is a connected open set and $f \colon U \to \mathbb{R}$ is harmonic. Prove that if f attains a maximum at $p \in U$, then f is constant.*

Exercise* 10.6.7*: *Let $f(x,y) := \ln\sqrt{x^2+y^2}$.*

a) Show f is harmonic where defined.

b) Show $\lim_{(x,y)\to 0} f(x,y) = -\infty$.

c) Using a circle C_r of radius r around the origin, compute $\frac{1}{2\pi r}\int_{\partial C_r} f\,ds$. What happens as $r \to 0$?

d) Why can't you use Green's theorem?

10.7 Change of variables

Note: 1 lecture

In one variable, we have the familiar change of variables

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx.$$

The analogue in higher dimensions is quite a bit more complicated. The first complication is orientation. If we use the definition of integral from this chapter, then we do not have the notion of $\int_a^b$ versus $\int_b^a$. We are simply integrating over an interval $[a,b]$. With this notation, the change of variables becomes

$$\int_{[a,b]} f\big(g(x)\big)\,|g'(x)|\,dx = \int_{g([a,b])} f(x)\,dx.$$

In this section we will obtain the several-variable analogue of this form.

Let us remark the role of $|g'(x)|$ in the formula. The integral measures volumes in general, so in one dimension it measures length. Notice that $|g'(x)|$ scales the dx and so it scales the lengths. If our g is linear, that is, $g(x) = Lx$, then $g'(x) = L$ and the length of the interval $g([a,b])$ is simply $|L|(b-a)$. That is because $g([a,b])$ is either $[La,Lb]$ or $[Lb,La]$. This property holds in higher dimension with $|L|$ replaced by the absolute value of the determinant.

Proposition 10.7.1. *Suppose $R \subset \mathbb{R}^n$ is a rectangle and $A \colon \mathbb{R}^n \to \mathbb{R}^n$ is linear. Then $A(R)$ is Jordan measurable and $V\big(A(R)\big) = |\det(A)|V(R)$.*

Proof. It is enough to prove for elementary matrices. The proof is left as an exercise. $\square$

Let us prove that absolute value of the Jacobian determinant $|J_g(x)| = \big|\det\big(g'(x)\big)\big|$ is the replacement of $|g'(x)|$ for multiple dimensions in the change of variables formula. The following theorem holds in more generality, but this statement is sufficient for many uses.

Theorem 10.7.2. *Suppose $U \subset \mathbb{R}^n$ is open, $S \subset U$ is a compact Jordan measurable set, and $g \colon U \to \mathbb{R}^n$ is a one-to-one continuously differentiable mapping, such that J_g is never zero on S. Suppose $f \colon g(S) \to \mathbb{R}$ is Riemann integrable. Then $f \circ g$ is Riemann integrable on S and*

$$\int_{g(S)} f(x)\,dx = \int_S f\big(g(x)\big)\,|J_g(x)|\,dx.$$

The set $g(S)$ is Jordan measurable by Proposition 10.5.6, so the left-hand side does make sense. That the right-hand side makes sense follows by Corollary 10.4.4 (actually Exercise 10.5.7).

Proof. The set S can be covered by finitely many closed rectangles $P_1, P_2, \ldots, P_k$, whose interiors do not overlap such that each $P_j \subset U$ (Exercise 10.7.2). Proving the theorem for $P_j \cap S$ instead of S is enough. Define $f(y) := 0$ for all $y \notin g(S)$. The new f is still Riemann integrable since $g(S)$ is Jordan measurable. We can now replace the integrals over S with integrals over the whole rectangle. We therefore assume that S is equal to a rectangle R.

Let $\varepsilon > 0$ be given. For every $x \in R$, let

$$W_x := \big\{ y \in U : \|g'(x) - g'(y)\| < \varepsilon/2 \big\}.$$

By Exercise 10.7.3, W_x is open. As $x \in W_x$ for every x, it is an open cover. By the Lebesgue covering lemma (Lemma 7.4.10 from volume I), there exists a $\delta > 0$ such that for every $y \in R$, there is an x such that $B(y,\delta) \subset W_x$. In other words, if P is a rectangle of maximum side length less than $\frac{\delta}{\sqrt{n}}$ and $y \in P$, then $P \subset B(y,\delta) \subset W_x$. By triangle inequality, $\|g'(\xi) - g'(\eta)\| < \varepsilon$ for all $\xi, \eta \in P$.

Let $R_1, R_2, \ldots, R_N$ be subrectangles partitioning R such that the maximum side of every R_j is less than $\frac{\delta}{\sqrt{n}}$. We also make sure that the minimum side length is at least $\frac{\delta}{2\sqrt{n}}$, which we can do if δ is sufficiently small relative to the sides of R (Exercise 10.7.4).

Consider some R_j and some fixed $x_j \in R_j$. First suppose $x_j = 0$, $g(0) = 0$, and $g'(0) = I$. For any given $y \in R_j$, apply the fundamental theorem of calculus to the function $t \mapsto g(ty)$ to find $g(y) = \int_0^1 g'(ty)y\,dt$. As the side of R_j is at most $\frac{\delta}{\sqrt{n}}$, then $\|y\| \leq \delta$. So

$$\|g(y) - y\| = \left\| \int_0^1 \bigl(g'(ty)y - y\bigr)\,dt \right\| \leq \int_0^1 \|g'(ty)y - y\|\,dt \leq \|y\| \int_0^1 \|g'(ty) - I\|\,dt \leq \delta\varepsilon.$$

Therefore, $g(R_j) \subset \widetilde{R}_j$, where $\widetilde{R}_j$ is a rectangle obtained from R_j by extending by $\delta\varepsilon$ on all sides. See Figure 10.9.

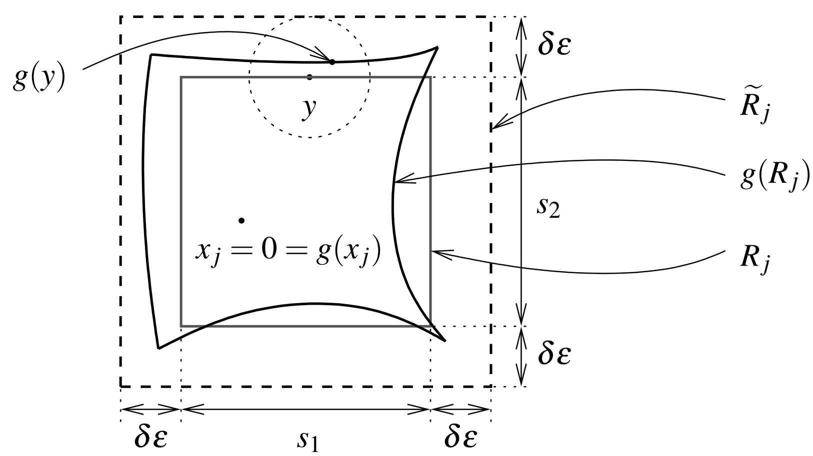

Figure 10.9: Image of R_j under g lies inside $\widetilde{R}_j$. A sample point $y \in R_j$ (on the boundary of R_j in fact) is marked and $g(y)$ must lie within with a radius of $\delta\varepsilon$ (also marked).

If the sides of R_j are $s_1, s_2, \ldots, s_n$, then $V(R_j) = s_1 s_2 \cdots s_n$. Recall $\delta \leq 2\sqrt{n}\,s_j$. Thus,

$$\begin{aligned} V(\widetilde{R}_j) &= (s_1 + 2\delta\varepsilon)(s_2 + 2\delta\varepsilon) \cdots (s_n + 2\delta\varepsilon) \\ &\leq (s_1 + 4\sqrt{n}\,s_1\varepsilon)(s_2 + 4\sqrt{n}\,s_2\varepsilon) \cdots (s_n + 4\sqrt{n}\,s_n\varepsilon) \\ &= s_1(1 + 4\sqrt{n}\,\varepsilon)\,s_2(1 + 4\sqrt{n}\,\varepsilon) \cdots s_n(1 + 4\sqrt{n}\,\varepsilon) = V(R_j)(1 + 4\sqrt{n}\,\varepsilon)^n. \end{aligned}$$

In other words,

$$V\bigl(g(R_j)\bigr) \leq V(\widetilde{R}_j) \leq V(R_j)(1 + 4\sqrt{n}\,\varepsilon)^n.$$

Next, suppose $A := g'(0)$ is not necessarily the identity. Write $g = A \circ \widetilde{g}$ where $\widetilde{g}'(0) = I$. By Proposition 10.7.1, $V\big(A(R_j)\big) = |\det(A)| V(R_j)$, and hence

$$\begin{aligned} V\big(g(R_j)\big) &\leq |\det(A)| V(R_j) (1 + 4\sqrt{n}\,\varepsilon)^n \\ &= |J_g(0)| V(R_j) (1 + 4\sqrt{n}\,\varepsilon)^n. \end{aligned}$$

Translation does not change volume, and therefore for every R_j, and $x_j \in R_j$, including when $x_j \neq 0$ and $g(x_j) \neq 0$, we find

$$V\big(g(R_j)\big) \leq |J_g(x_j)| V(R_j) (1 + 4\sqrt{n}\,\varepsilon)^n.$$

Write f as $f = f_+ - f_-$ for two nonnegative Riemann integrable functions f_+ and f_-:

$$f_+(x) := \max\{f(x), 0\}, \qquad f_-(x) := \max\{-f(x), 0\}.$$

So, if we prove the theorem for a nonnegative f, we obtain the theorem for arbitrary f. Therefore, suppose that $f(y) \geq 0$ for all $y \in R$.

For a small enough $\delta > 0$, we have

$$\begin{aligned} \varepsilon + \int_R f\big(g(x)\big) |J_g(x)| \, dx &\geq \sum_{j=1}^N \left(\sup_{x \in R_j} f\big(g(x)\big) |J_g(x)| \right) V(R_j) \\ &\geq \sum_{j=1}^N \left(\sup_{x \in R_j} f\big(g(x)\big) \right) |J_g(x_j)| V(R_j) \\ &\geq \sum_{j=1}^N \left(\sup_{y \in g(R_j)} f(y) \right) V\big(g(R_j)\big) \frac{1}{(1 + 4\sqrt{n}\,\varepsilon)^n} \\ &\geq \sum_{j=1}^N \left(\int_{g(R_j)} f(y) \, dy \right) \frac{1}{(1 + 4\sqrt{n}\,\varepsilon)^n} \\ &= \frac{1}{(1 + 4\sqrt{n}\,\varepsilon)^n} \int_{g(R)} f(y) \, dy. \end{aligned}$$

The last equality follows because the overlaps of the rectangles are their boundaries, which are of measure zero, and hence the image of their boundaries is also measure zero. Let ε go to zero to find

$$\int_R f\big(g(x)\big) |J_g(x)| \, dx \geq \int_{g(R)} f(y) \, dy.$$

By adding this result for several rectangles covering an S we obtain the result for an arbitrary bounded Jordan measurable $S \subset U$, and nonnegative integrable function f:

$$\int_S f\big(g(x)\big) |J_g(x)| \, dx \geq \int_{g(S)} f(y) \, dy.$$

Recall that g^{-1} exists and $g^{-1}\big(g(S)\big) = S$. Also $1 = J_{g \circ g^{-1}} = J_g(g^{-1}(y)) J_{g^{-1}}(y)$ for $y \in g(S)$. So

$$\begin{aligned} \int_{g(S)} f(y) \, dy &= \int_{g(S)} f\big(g\big(g^{-1}(y)\big)\big) |J_g\big(g^{-1}(y)\big)| \, |J_{g^{-1}}(y)| \, dy \\ &\geq \int_{g^{-1}(g(S))} f\big(g(x)\big) |J_g(x)| \, dx = \int_S f\big(g(x)\big) |J_g(x)| \, dx. \end{aligned}$$

The conclusion of the theorem holds for all nonnegative f and as we mentioned above, it thus holds for all Riemann integrable f. □

10.7.1 Exercises

***Exercise* 10.7.1:** *Prove Proposition 10.7.1.*

***Exercise* 10.7.2:** *Suppose $U \subset \mathbb{R}^n$ is open and $S \subset U$ is a compact Jordan measurable set. Show that there exist finitely many closed rectangles $P_1, P_2, \ldots, P_k$ such that $P_j \subset U$, $S \subset P_1 \cup P_2 \cup \cdots \cup P_k$, and the interiors are mutually disjoint, that is $P_j^\circ \cap P_\ell^\circ = \emptyset$ whenever $j \neq \ell$.*

***Exercise* 10.7.3:** *Suppose $U \subset \mathbb{R}^n$ is open, $x \in U$, and $g \colon U \to \mathbb{R}^n$ is a continuously differentiable mapping. For every $\varepsilon > 0$, show that*

$$W_x := \left\{ y \in U : \|g'(x) - g'(y)\| < \varepsilon/2 \right\}$$

is an open set.

***Exercise* 10.7.4:** *Suppose $R \subset \mathbb{R}^n$ is a closed rectangle. Show that if $\delta' > 0$ is sufficiently small relative to the sides of R, then R can be partitioned into subrectangles where each side of every subrectangle is between $\frac{\delta'}{2}$ and δ'.*

***Exercise* 10.7.5:** *Prove the following version of the theorem:* Suppose $f \colon \mathbb{R}^n \to \mathbb{R}$ is a Riemann integrable compactly supported function. Suppose $K \subset \mathbb{R}^n$ is the support of f, S is a compact set, and $g \colon \mathbb{R}^n \to \mathbb{R}^n$ is a function that when restricted to a neighborhood U of S is one-to-one and continuously differentiable, $g(S) = K$ and J_g is never zero on S (in the formula assume $J_g(x) = 0$ if g not differentiable at x, that is when $x \notin U$). Then

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_{\mathbb{R}^n} f\big(g(x)\big) \left| J_g(x) \right| dx.$$

***Exercise* 10.7.6:** *Prove the following version of the theorem:* Suppose $S \subset \mathbb{R}^n$ is an open bounded Jordan measurable set, $g \colon S \to \mathbb{R}^n$ is a one-to-one continuously differentiable mapping such that J_g is never zero on S, and such that $g(S)$ is bounded and Jordan measurable (it is also open). Suppose $f \colon g(S) \to \mathbb{R}$ is Riemann integrable. Then $f \circ g$ is Riemann integrable on S and

$$\int_{g(S)} f(x)\, dx = \int_S f\big(g(x)\big) \left| J_g(x) \right| dx.$$

Hint: Write S as an increasing union of compact Jordan measurable sets, then apply the theorem of the section to those. Then prove that you can take the limit.

Chapter 11

Functions as Limits

11.1 Complex numbers

Note: half a lecture

11.1.1 The complex plane

In this chapter we consider approximation of functions, or in other words functions as limits of sequences and series. We will extend some results we already saw to a somewhat more general setting, and we will look at some completely new results. In particular, we consider complex-valued functions. We gave complex numbers as examples before, but let us start from scratch and properly define the complex number field.

A complex number is just a pair $(x,y) \in \mathbb{R}^2$ on which we define multiplication (see below). We call the set the *complex numbers* and denote it by $\mathbb{C}$. We identify $x \in \mathbb{R}$ with $(x,0) \in \mathbb{C}$. The x-axis is then called the *real axis* and the y-axis is called the *imaginary axis*. The set $\mathbb{C}$ is sometimes called the *complex plane*.

Define:

$$\begin{aligned}(x,y)+(s,t) &:= (x+s,y+t),\\ (x,y)(s,t) &:= (xs-yt,xt+ys).\end{aligned}$$

Under the identification above, we have $0 = (0,0)$ and $1 = (1,0)$. These two operations make the plane into a field (exercise).

We write a complex number (x,y) as $x+iy$, where we define*

$$i := (0,1).$$

Notice that $i^2 = (0,1)(0,1) = (0-1,0+0) = -1$. That is, i is a solution to the polynomial equation

$$z^2+1 = 0.$$

From now on, we will not use the notation (x,y) and use only $x+iy$. See Figure 11.1.

We generally use x,y,r,s,t for real values and z,w,ξ,ζ for complex values, although that is not a hard and fast rule. In particular, z is often used as a third real variable in $\mathbb{R}^3$.

*Note that engineers use j instead of i.

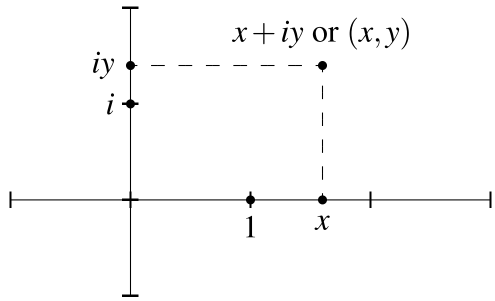

Figure 11.1: The points 1, i, x, iy, and $x+iy$ in the complex plane.

Definition 11.1.1. Suppose $z = x + iy$. We call x the *real part* of z, and we call y the *imaginary part* of z. We write

$$\operatorname{Re} z := x, \qquad \operatorname{Im} z := y.$$

Define *complex conjugate* as

$$\bar{z} := x - iy,$$

and define *modulus* as

$$|z| := \sqrt{x^2 + y^2}.$$

Modulus is the complex analogue of the absolute value and has similar properties. For example, $|zw| = |z|\,|w|$ (exercise). The complex conjugate is a reflection of the plane across the real axis. The real numbers are precisely those numbers for which the imaginary part $y = 0$. In particular, they are precisely those numbers which satisfy the equation

$$z = \bar{z}.$$

As $\mathbb{C}$ is really $\mathbb{R}^2$, we let the metric on $\mathbb{C}$ be the standard euclidean metric on $\mathbb{R}^2$. In particular,

$$|z| = d(z,0), \qquad \text{and also} \qquad |z-w| = d(z,w).$$

So the topology on $\mathbb{C}$ is the same exact topology as the standard topology on $\mathbb{R}^2$ with the euclidean metric, and $|z|$ is equal to the euclidean norm on $\mathbb{R}^2$. Importantly, since $\mathbb{R}^2$ is a complete metric space, then so is $\mathbb{C}$. As $|z|$ is the euclidean norm on $\mathbb{R}^2$, we have the *triangle inequality* of both flavors:

$$|z+w| \leq |z| + |w| \qquad \text{and} \qquad \big||z| - |w|\big| \leq |z-w|.$$

The complex conjugate and the modulus are even more intimately related:

$$|z|^2 = x^2 + y^2 = (x+iy)(x-iy) = z\bar{z}.$$

Remark 11.1.2. There is no natural ordering on the complex numbers. In particular, no ordering that makes the complex numbers into an ordered field. Ordering is one of the things we lose when we go from real to complex numbers.

11.1.2 Complex numbers and limits

It is not hard to show that the algebraic operations are continuous. This is because convergence in $\mathbb{R}^2$ is the same as convergence for each component and we already know that the real algebraic operations are continuous. For example, write $z_n = x_n + i\,y_n$ and $w_n = s_n + i\,t_n$, and suppose that $\lim z_n = z = x + i\,y$ and $\lim w_n = w = s + i\,t$. Let us show

$$\lim_{n\to\infty} z_n w_n = zw.$$

First,

$$z_n w_n = (x_n s_n - y_n t_n) + i(x_n t_n + y_n s_n).$$

The topology on $\mathbb{C}$ is the same as on $\mathbb{R}^2$, and so $x_n \to x$, $y_n \to y$, $s_n \to s$, and $t_n \to t$. Hence,

$$\lim_{n\to\infty} (x_n s_n - y_n t_n) = xs - yt \qquad \text{and} \qquad \lim_{n\to\infty} (x_n t_n + y_n s_n) = xt + ys.$$

As $(xs - yt) + i(xt + ys) = zw$, then

$$\lim_{n\to\infty} z_n w_n = zw.$$

Similarly the modulus and the complex conjugate are continuous functions. We leave the proof of the following proposition as an exercise.

Proposition 11.1.3. *Suppose $\{z_n\}$, $\{w_n\}$ are sequences of complex numbers converging to z and w respectively. Then*

(i) $\lim\limits_{n\to\infty} z_n + w_n = z + w.$

(ii) $\lim\limits_{n\to\infty} z_n w_n = zw.$

(iii) Assuming $w_n \neq 0$ for all n and $w \neq 0$, $\lim\limits_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}.$

(iv) $\lim\limits_{n\to\infty} \lvert z_n \rvert = \lvert z \rvert.$

(v) $\lim\limits_{n\to\infty} \bar{z}_n = \bar{z}.$

As we have seen above, convergence in $\mathbb{C}$ is the same as convergence in $\mathbb{R}^2$. In particular, a sequence in $\mathbb{C}$ converges if and only if the real and imaginary parts converge. Therefore, feel free to apply everything you have learned about convergence in $\mathbb{R}^2$, as well as applying results about real numbers to the real and imaginary parts.

We also need convergence of complex series. Let $\{z_n\}$ be a sequence of complex numbers. The series

$$\sum_{n=1}^{\infty} z_n$$

converges if the limit of partial sums converges, that is, if

$$\lim_{k\to\infty} \sum_{n=1}^{k} z_n \qquad \text{exists.}$$

As before, we sometimes write $\sum z_n$ for the series. A series *converges absolutely* if $\sum \lvert z_n \rvert$ converges.

We say a series is *Cauchy* if the sequence of partial sums is Cauchy. The following two propositions have essentially the same proofs as for real series and we leave them as exercises.

Proposition 11.1.4. *The complex series $\sum z_n$ is Cauchy if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for every $n \geq M$ and every $k > n$, we have*

$$\left| \sum_{j=n+1}^{k} z_j \right| < \varepsilon.$$

Proposition 11.1.5. *If a complex series $\sum z_n$ converges absolutely, then it converges.*

The series $\sum |z_n|$ is a real series. All the convergence tests (ratio test, root test, etc.) that talk about absolute convergence work with the numbers $|z_n|$, that is, they are really talking about convergence of series of nonnegative real numbers. You can directly apply these tests them without needing to reprove anything for complex series.

11.1.3 Complex-valued functions

When we deal with complex-valued functions $f : X \to \mathbb{C}$, what we often do is to write $f = u + iv$ for real-valued functions $u : X \to \mathbb{R}$ and $v : X \to \mathbb{R}$.

Suppose we wish to integrate $f : [a,b] \to \mathbb{C}$. We write $f = u + iv$ for real-valued u and v. We say that f is *Riemann integrable* if u and v are Riemann integrable, and in this case we define

$$\int_a^b f := \int_a^b u + i \int_a^b v.$$

We make the same definition for every other type of integral (improper, multivariable, etc.).

Similarly when we differentiate, write $f : [a,b] \to \mathbb{C}$ as $f = u + iv$. Thinking of $\mathbb{C}$ as $\mathbb{R}^2$ we say that f is differentiable if u and v are differentiable. For a function valued in $\mathbb{R}^2$, the derivative was represented by a vector in $\mathbb{R}^2$. Now a vector in $\mathbb{R}^2$ is a complex number. In other words, we write the *derivative* as

$$f'(t) := u'(t) + i v'(t).$$

The linear operator representing the derivative is the multiplication by the complex number $f'(t)$, so nothing is lost in this identification.

11.1.4 Exercises

***Exercise* 11.1.1:** *Check that $\mathbb{C}$ is a field.*

***Exercise* 11.1.2:** *Prove that for $z, w \in \mathbb{C}$, we have $|zw| = |z|\,|w|$.*

***Exercise* 11.1.3:** *Finish the proof of Proposition 11.1.3.*

***Exercise* 11.1.4:** *Prove Proposition 11.1.4.*

***Exercise* 11.1.5:** *Prove Proposition 11.1.5.*

Exercise* 11.1.6*: *Considering the definition of complex multiplication, given $x+iy$ define the matrix $\left[\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right]$. Prove that*

a) *The action of this matrix on a vector (s,t) is the same as the action of multiplying $(x+iy)(s+it)$.*

b) *Multiplying two such matrices is the same multiplying the underlying complex numbers and then finding the corresponding matrix for the product. In other words, we can think of the field $\mathbb{C}$ as also a subset of the 2-by-2 matrices.*

c) *Show that $\left[\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right]$ has eigenvalues $x+iy$ and $x-iy$. Recall that λ is an eigenvalue of a matrix A if $A-\lambda I$ (a complex matrix in our case) is not invertible, or in other words if it has linearly dependent rows: That is, one row is a (complex) multiple of the other*

Exercise* 11.1.7*: *Prove the Bolzano–Weierstrass theorem for complex sequences. Suppose $\{z_n\}$ is a bounded sequence of complex numbers, that is, there exists an M such that $|z_n| \leq M$ for all n. Prove that there exists a subsequence $\{z_{n_k}\}$ that converges to some $z \in \mathbb{C}$.*

Exercise* 11.1.8*:

a) *Prove that there is no simple mean value theorem for complex-valued functions: Find a differentiable function $f\colon [0,1] \to \mathbb{C}$ such that $f(0) = f(1) = 0$, but $f'(t) \neq 0$ for all $t \in [0,1]$.*

b) *However, there is a weaker form of the mean value theorem as there is for vector-valued functions. Prove: If $f\colon [a,b] \to \mathbb{C}$ is continuous and differentiable in (a,b), and for some M, $|f'(x)| \leq M$ for all $x \in (a,b)$, then $|f(b) - f(a)| \leq M|b-a|$.*

Exercise* 11.1.9*: *Prove that there is no simple mean value theorem for integrals for complex-valued functions: Find a continuous function $f\colon [0,1] \to \mathbb{C}$ such that $\int_0^1 f = 0$ but $f(t) \neq 0$ for all $t \in [0,1]$.*

11.2 Swapping limits

Note: 2 lectures

11.2.1 Continuity

Let us get back to swapping limits and expand on chapter 6 of volume I. Let $\{f_n\}$ be a sequence of functions $f_n \colon X \to Y$ for a set X and a metric space Y. Let $f \colon X \to Y$ be a function and for every $x \in X$ suppose that

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We say the sequence $\{f_n\}$ *converges pointwise* to f.

For $Y = \mathbb{C}$, a series *converges pointwise* if for every $x \in X$, we have

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{\infty} f_k(x).$$

The question is: If f_n are all continuous, is f continuous? Differentiable? Integrable? What are the derivatives or integrals of f?

For example, for continuity of the pointwise limit of a sequence $\{f_n\}$, we are asking if

$$\lim_{x\to x_0} \lim_{n\to\infty} f_n(x) \overset{?}{=} \lim_{n\to\infty} \lim_{x\to x_0} f_n(x).$$

We don't even a priory know if both sides exist, let alone if they are equal each other.

Example 11.2.1: The functions $f_n \colon \mathbb{R} \to \mathbb{R}$,

$$f_n(x) := \frac{1}{1+nx^2},$$

are continuous and converge pointwise to the discontinuous function

$$f(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{else.} \end{cases}$$

So pointwise convergence is not enough to preserve continuity (nor even boundedness). For that, we need uniform convergence.

Let $f_n \colon X \to Y$ be functions. Then $\{f_n\}$ *converges uniformly* to f if for every $\varepsilon > 0$, there exists an M such that for all $n \geq M$ and all $x \in X$, we have

$$d\bigl(f_n(x), f(x)\bigr) < \varepsilon.$$

A series $\sum f_n$ of complex-valued functions converges uniformly if the sequence of partial sums converges uniformly, that is for every $\varepsilon > 0$ there exists an M such that for all $n \geq M$ and all $x \in X$

$$\left| \left(\sum_{k=1}^{n} f_k(x) \right) - f(x) \right| < \varepsilon.$$

The simplest property preserved by uniform convergence is boundedness. We leave the proof of the following proposition as an exercise. It is almost identical to the proof for real-valued functions.

Proposition 11.2.2. *Let X be a set and (Y,d) a metric space. If $f_n \colon X \to Y$ are bounded functions and converge uniformly to $f \colon X \to Y$, then f is bounded.*

If X is a set and (Y,d) is a metric space, then a sequence $f_n \colon X \to Y$ is *uniformly Cauchy* if for every $\varepsilon > 0$, there is an M such that for all $n,m \geq M$ and all $x \in X$, we have

$$d\bigl(f_n(x), f_m(x)\bigr) < \varepsilon.$$

The notion is the same as for real-valued functions. The proof of the following proposition is again essentially the same as in that setting and is left as an exercise.

Proposition 11.2.3. *Let X be a set, (Y,d) be a metric space, and $f_n \colon X \to Y$ be functions. If $\{f_n\}$ converges uniformly, then $\{f_n\}$ is uniformly Cauchy. Conversely, if $\{f_n\}$ is uniformly Cauchy and (Y,d) is Cauchy-complete, then $\{f_n\}$ converges uniformly.*

For $f \colon X \to \mathbb{C}$, we write

$$\|f\|_u := \sup_{x\in X} |f(x)|.$$

We call $\|\cdot\|_u$ the *supremum norm* or *uniform norm*. Then a sequence of functions $f_n \colon X \to \mathbb{C}$ converges uniformly to $f \colon X \to \mathbb{C}$ if and only if

$$\lim_{n\to\infty} \|f_n - f\|_u = 0.$$

The supremum norm satisfies the triangle inequality: For every $x \in X$,

$$|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_u + \|g\|_u.$$

Take a supremum on the left to get

$$\|f+g\|_u \leq \|f\|_u + \|g\|_u.$$

For a compact metric space X, the uniform norm is a norm on the vector space $C(X,\mathbb{C})$. We leave it as an exercise. While we will not need it, $C(X,\mathbb{C})$ is in fact a complex vector space, that is, in the definition of a vector space we can replace $\mathbb{R}$ with $\mathbb{C}$. Convergence in the metric space $C(X,\mathbb{C})$ is uniform convergence.

We will study a couple of types of series of functions, and a useful test for uniform convergence of a series is the so-called *Weierstrass M-test*.

Theorem 11.2.4 (Weierstrass M-test)**.** *Let X be a set. Suppose $f_n \colon X \to \mathbb{C}$ are functions and $M_n > 0$ numbers such that*

$$|f_n(x)| \leq M_n \quad \text{for all } x \in X, \qquad \text{and} \qquad \sum_{n=1}^{\infty} M_n \quad \text{converges}.$$

Then

$$\sum_{n=1}^{\infty} f_n(x) \quad \text{converges uniformly}.$$

Another way to state the theorem is to say that if $\sum \lVert f_n \rVert_u$ converges, then $\sum f_n$ converges uniformly. Note that the converse of this theorem is not true. Also note that applying the theorem to $\sum \lvert f_n(x) \rvert$ gives that a series satisfying the M-test also converges uniformly, so the series converges both absolutely and uniformly.

Proof. Suppose $\sum M_n$ converges. Given $\varepsilon > 0$, we have that the partial sums of $\sum M_n$ are Cauchy so there is an N such that for all $m,n \geq N$ with $m \geq n$, we have

$$\sum_{k=n+1}^{m} M_k < \varepsilon.$$

We estimate a Cauchy difference of the partial sums of the functions

$$\left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} \lvert f_k(x) \rvert \leq \sum_{k=n+1}^{m} M_k < \varepsilon.$$

We are done by Proposition 11.1.4. □

Example 11.2.5: The series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

converges uniformly on $\mathbb{R}$. See Figure 11.2. This is a Fourier series, we will see more of these in a later section. Proof: The series converges uniformly because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and

$$\left| \frac{\sin(nx)}{n^2} \right| \leq \frac{1}{n^2}.$$

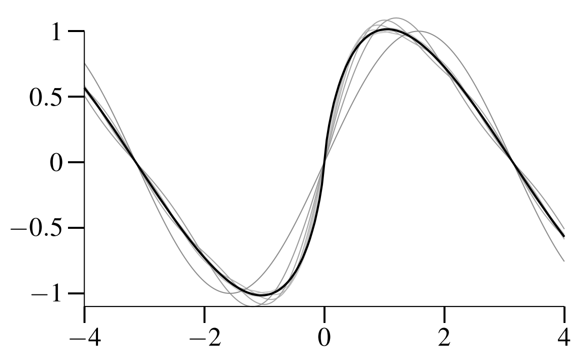

Figure 11.2: Plot of $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ including the first 8 partial sums in various shades of gray.

Example 11.2.6: The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges uniformly on every bounded interval. This series is a power series that we will study shortly. Proof: Take the interval $[-r,r] \subset \mathbb{R}$ (every bounded interval is contained in some $[-r,r]$). The series $\sum_{n=0}^{\infty} \frac{r^n}{n!}$ converges by the ratio test, so $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on $[-r,r]$ as

$$\left| \frac{x^n}{n!} \right| \leq \frac{r^n}{n!}.$$

Now we would love to say something about the limit. For example, is it continuous?

Proposition 11.2.7. *Let (X,d_X) and (Y,d_Y) be metric spaces. Suppose $f_n \colon X \to Y$ converge uniformly to $f \colon X \to Y$. Let $\{x_k\}$ be a sequence in X and $x := \lim x_k$. Suppose that*

$$a_n := \lim_{k\to\infty} f_n(x_k)$$

exists for all n. Then $\{a_n\}$ converges and

$$\lim_{k\to\infty} f(x_k) = \lim_{n\to\infty} a_n.$$

In other words,

$$\lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k) = \lim_{n\to\infty} \lim_{k\to\infty} f_n(x_k).$$

Proof. First we show that $\{a_n\}$ converges. As $\{f_n\}$ converges uniformly it is uniformly Cauchy. Let $\varepsilon > 0$ be given. There is an M such that for all $m,n \geq M$, we have

$$d_Y\big(f_n(x_k), f_m(x_k)\big) < \varepsilon \qquad \text{for all } k.$$

Note that $d_Y(a_n,a_m) \leq d_Y\big(a_n, f_n(x_k)\big) + d_Y\big(f_n(x_k), f_m(x_k)\big) + d_Y\big(f_m(x_k), a_m\big)$ and take the limit as $k \to \infty$ to find

$$d_Y(a_n,a_m) \leq \varepsilon.$$

Hence $\{a_n\}$ is Cauchy and converges since Y is complete. Write $a := \lim a_n$.

Find a $k \in \mathbb{N}$ such that

$$d_Y\big(f_k(p), f(p)\big) < \varepsilon/3$$

for all $p \in X$. Assume k is large enough so that

$$d_Y(a_k,a) < \varepsilon/3.$$

Find an $N \in \mathbb{N}$ such that for $m \geq N$,

$$d_Y\big(f_k(x_m), a_k\big) < \varepsilon/3.$$

Then for $m \geq N$,

$$d_Y\big(f(x_m), a\big) \leq d_Y\big(f(x_m), f_k(x_m)\big) + d_Y\big(f_k(x_m), a_k\big) + d_Y\big(a_k, a\big) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \qquad \square$$

We obtain an immediate corollary about continuity.

Corollary 11.2.8. *Let X and Y be metric spaces. If $f_n \colon X \to Y$ are continuous functions such that $\{f_n\}$ converges uniformly to $f \colon X \to Y$, then f is continuous.*

The converse is not true. Just because the limit is continuous doesn't mean that the convergence is uniform. For example: $f_n\colon (0,1) \to \mathbb{R}$ defined by $f_n(x) := x^n$ converge to the zero function, but not uniformly. However, if we add extra conditions on the sequence, we can obtain a partial converse such as Dini's theorem, see Exercise 6.2.10 from volume I.

In Exercise 11.2.3 the reader is asked to prove that for a compact X, $C(X,\mathbb{C})$ is a normed vector space with the uniform norm, and hence a metric space. We have just shown that $C(X,\mathbb{C})$ is Cauchy-complete: Proposition 11.2.3 says that a Cauchy sequence in $C(X,\mathbb{C})$ converges uniformly to some function, and Corollary 11.2.8 shows that the limit is continuous and hence in $C(X,\mathbb{C})$.

Corollary 11.2.9. *Let (X,d) be a compact metric space. Then $C(X,\mathbb{C})$ is a Cauchy-complete metric space.*

Example 11.2.10: By Example 11.2.5 the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

converges uniformly and hence is continuous by Corollary 11.2.8 (as is visible in Figure 11.2).

11.2.2 Integration

Proposition 11.2.11. *Suppose $f_n\colon [a,b] \to \mathbb{C}$ are Riemann integrable and suppose that $\{f_n\}$ converges uniformly to $f\colon [a,b] \to \mathbb{C}$. Then f is Riemann integrable and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n .$$

Since the integral of a complex-valued function is just the integral of the real and imaginary parts separately, the proof follows directly by the results of chapter 6 of volume I. We leave the details as an exercise.

Corollary 11.2.12. *Suppose $f_n\colon [a,b] \to \mathbb{C}$ are Riemann integrable and suppose that*

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly. Then the series is Riemann integrable on $[a,b]$ and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

Example 11.2.13: Let us show how to integrate a Fourier series.

$$\int_0^x \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2}\,dt = \sum_{n=1}^{\infty} \int_0^x \frac{\cos(nt)}{n^2}\,dt = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$$

The swapping of integral and sum is possible because of uniform convergence, which we have proved before using the Weierstrass M-test (Theorem 11.2.4).

We remark that we can swap integrals and limits under far less stringent hypotheses, but for that we would need a stronger integral than the Riemann integral. E.g. the Lebesgue integral.

11.2.3 Differentiation

Recall that a complex-valued function $f\colon [a,b] \to \mathbb{C}$, where $f(x) = u(x) + i\,v(x)$, is differentiable, if u and v are differentiable and the derivative is

$$f'(x) = u'(x) + i\,v'(x).$$

The proof of the following theorem is to apply the corresponding theorem for real functions to u and v, and is left as an exercise.

Theorem 11.2.14. *Let $I \subset \mathbb{R}$ be a bounded interval and let $f_n\colon I \to \mathbb{C}$ be continuously differentiable functions. Suppose $\{f_n'\}$ converges uniformly to $g\colon I \to \mathbb{C}$, and suppose $\{f_n(c)\}_{n=1}^\infty$ is a convergent sequence for some $c \in I$. Then $\{f_n\}$ converges uniformly to a continuously differentiable function $f\colon I \to \mathbb{C}$, and $f' = g$.*

Uniform limits of the functions themselves are not enough, and can make matters even worse. In §11.7 we will prove that continuous functions are uniform limits of polynomials, yet as the following example demonstrates, a continuous function need not be differentiable anywhere.

Example 11.2.15: There exist continuous nowhere differentiable functions. Such functions are often called *Weierstrass functions*, although this particular one, essentially due to Takagi*, is a different example than what Weierstrass gave.

Define

$$\varphi(x) := |x| \qquad \text{for } x \in [-1,1].$$

Extend the definition of φ to all of $\mathbb{R}$ by making it 2-periodic: Decree that $\varphi(x) = \varphi(x+2)$. The function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is continuous, in fact $|\varphi(x) - \varphi(y)| \le |x-y|$ (why?). See Figure 11.3.

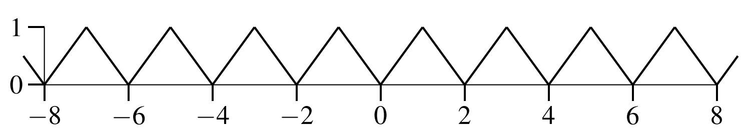

Figure 11.3: The 2-periodic function φ.

As $\sum \left(\frac{3}{4}\right)^n$ converges and $|\varphi(x)| \le 1$ for all x, we have by the M-test (Theorem 11.2.4) that

$$f(x) := \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$

converges uniformly and hence is continuous. See Figure 11.4.

We claim $f\colon \mathbb{R} \to \mathbb{R}$ is nowhere differentiable. Fix x, and we will show f is not differentiable at x. Define

$$\delta_m := \pm \frac{1}{2} 4^{-m},$$

where the sign is chosen so that there is no integer between $4^m x$ and $4^m(x+\delta_m) = 4^m x \pm \frac{1}{2}$.

*Teiji Takagi (1875–1960) was a Japanese mathematician.

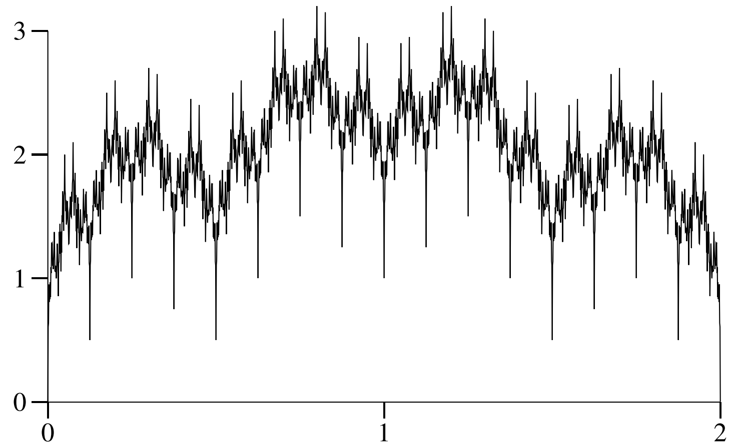

Figure 11.4: Plot of the nowhere differentiable function f.

We want to look at the difference quotient

$$\frac{f(x+\delta_m)-f(x)}{\delta_m}=\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\frac{\varphi\bigl(4^n(x+\delta_m)\bigr)-\varphi(4^nx)}{\delta_m}.$$

Fix m for a moment. Consider the expression inside the series:

$$\gamma_n:=\frac{\varphi\bigl(4^n(x+\delta_m)\bigr)-\varphi(4^nx)}{\delta_m}.$$

If $n>m$, then $4^n\delta_m$ is an even integer. As φ is 2-periodic we get that $\gamma_n=0$.

As there is no integer between $4^m(x+\delta_m)=4^mx\pm 1/2$ and 4^mx, then on this interval $\varphi(t)=\pm t+\ell$ for some integer ℓ. In particular, $\bigl|\varphi\bigl(4^m(x+\delta_m)\bigr)-\varphi(4^mx)\bigr|=|4^mx\pm 1/2-4^mx|=1/2$. Therefore,

$$|\gamma_m|=\left|\frac{\varphi\bigl(4^m(x+\delta_m)\bigr)-\varphi(4^mx)}{\pm(1/2)4^{-m}}\right|=4^m.$$

Similarly, suppose $n<m$. Since $|\varphi(s)-\varphi(t)|\leq|s-t|$,

$$|\gamma_n|=\left|\frac{\varphi\bigl(4^nx\pm(1/2)4^{n-m}\bigr)-\varphi(4^nx)}{\pm(1/2)4^{-m}}\right|\leq\left|\frac{\pm(1/2)4^{n-m}}{\pm(1/2)4^{-m}}\right|=4^n.$$

And so

$$\begin{aligned}\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right|=\left|\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\gamma_n\right|&=\left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n\gamma_n\right|\\&\geq\left|\left(\frac{3}{4}\right)^m\gamma_m\right|-\left|\sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n\gamma_n\right|\\&\geq 3^m-\sum_{n=0}^{m-1}3^n=3^m-\frac{3^m-1}{3-1}=\frac{3^m+1}{2}.\end{aligned}$$

As $m\to\infty$, we have $\delta_m\to 0$, but $\frac{3^m+1}{2}$ goes to infinity. Hence f cannot be differentiable at x.

11.2.4 Exercises

Exercise* 11.2.1*: *Prove Proposition 11.2.2.*

Exercise* 11.2.2*: *Prove Proposition 11.2.3.*

Exercise* 11.2.3*: *Suppose (X,d) is a compact metric space. Prove that $\|\cdot\|_u$ is a norm on the vector space of continuous complex-valued functions $C(X,\mathbb{C})$.*

Exercise* 11.2.4*:

a) Prove that $f_n(x) := 2^{-n}\sin(2^n x)$ converge uniformly to zero, but there exists a dense set $D \subset \mathbb{R}$ such that $\lim_{n\to\infty} f_n'(x) = 1$ for all $x \in D$.

b) Prove that $\sum_{n=1}^{\infty} 2^{-n}\sin(2^n x)$ converges uniformly to a continuous function, and there exists a dense set $D \subset \mathbb{R}$ where the derivatives of the partial sums do not converge.

Exercise* 11.2.5*: *Suppose (X,d) is a compact metric space. Prove that $\|f\|_{C^1} := \|f\|_u + \|f'\|_u$ is a norm on the vector space of continuously differentiable complex-valued functions $C^1(X,\mathbb{C})$.*

Exercise* 11.2.6*: *Prove Theorem 11.2.14.*

Exercise* 11.2.7*: *Prove Proposition 11.2.11 by reducing to the real result.*

Exercise* 11.2.8*: *Work through the following counterexample to the converse of the Weierstrass M-test (Theorem 11.2.4). Define $f_n \colon [0,1] \to \mathbb{R}$ by*

$$f_n(x) := \begin{cases} \frac{1}{n} & \text{if } \frac{1}{n+1} < x < \frac{1}{n}, \\ 0 & \text{else.} \end{cases}$$

Prove that $\sum f_n$ converges uniformly, but $\sum \|f_n\|_u$ does not converge.

Exercise* 11.2.9*: *Suppose $f_n \colon [0,1] \to \mathbb{R}$ are monotone increasing functions and suppose that $\sum f_n$ converges pointwise. Prove that $\sum f_n$ converges uniformly.*

Exercise* 11.2.10*: *Prove that*

$$\sum_{n=1}^{\infty} e^{-nx}$$

converges for all $x > 0$ to a differentiable function.

11.3 Power series and analytic functions

Note: 2–3 lectures

11.3.1 Analytic functions

A (complex) power series is a series of the form

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

for $c_n, z, a \in \mathbb{C}$. We say the series *converges* if the series converges for some $z \neq a$.

Let $U \subset \mathbb{C}$ be an open set and $f \colon U \to \mathbb{C}$ a function. Suppose that for every $a \in U$ there exists a $\rho > 0$ and a power series convergent to the function

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

for all $z \in B(a,\rho)$. Then we say f is an *analytic* function.

Similarly if we have an interval $(a,b) \subset \mathbb{R}$, we say that $f \colon (a,b) \to \mathbb{C}$ is analytic or perhaps *real-analytic* if for each point $c \in (a,b)$ there is a power series around c that converges in some $(c-\rho, c+\rho)$ for some $\rho > 0$.

As we will sometimes talk about real and sometimes about complex power series we will use z to denote a complex number and x a real number, but we will always mention which case we are working with.

An analytic function has different expansions around different points. Also the convergence does not automatically happen on the entire domain of the function. For example, if $|z| < 1$, then

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n .$$

While the left-hand side exists on all of $z \neq 1$, the right-hand side happens to converge only if $|z| < 1$. See a graph of a small piece of $\frac{1}{1-z}$ in Figure 11.5. Notice that we can't graph the function itself, we can only graph its real or imaginary parts for lack of dimensions in our universe.

11.3.2 Convergence of power series

We proved several results for power series of a real variable in §2.6 of volume I. For the most part the convergence properties of power series deal with the series $\sum |c_k| \, |z-a|^k$ and so we have already proved many results about complex power series. In particular, we computed the so-called *radius of convergence* of a power series.

Proposition 11.3.1. *Let $\sum_{n=0}^{\infty} c_n (z-a)^n$ be a power series. There exists a $\rho \in [0,\infty]$ such that*

(i) If $\rho = 0$, then the series diverges.

(ii) If $\rho = \infty$, then the series converges for all $z \in \mathbb{C}$.

(iii) If $0 < \rho < \infty$, then the series converges absolutely on $B(a,\rho)$, and diverges when $|z-a| > \rho$.

Furthermore, if $0 < r < \rho$, then the series converges uniformly on the closed ball $C(a,r)$.

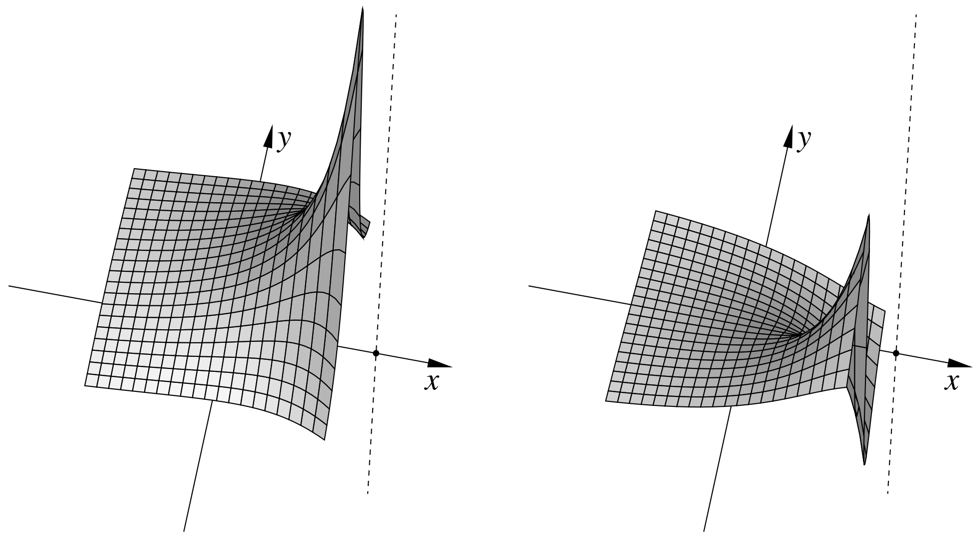

Figure 11.5: Graphs of the real and imaginary parts of $z = x + iy \mapsto \frac{1}{1-z}$ in the square $[-0.8, 0.8]^2$. The singularity at $z = 1$ is marked with a vertical dashed line.

Proof. We use the real version of this proposition, Proposition 2.6.10 in volume I. Let

$$R := \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$

If $R = 0$, then $\sum_{n=0}^{\infty} |c_n| \, |z-a|^n$ converges for all z. If $R = \infty$, then $\sum_{n=0}^{\infty} |c_n| \, |z-a|^n$ converges only at $z = a$. Otherwise, let $\rho := {}^1\!/_R$ and $\sum_{n=0}^{\infty} |c_n| \, |z-a|^n$ converges when $|z-a| < \rho$, and diverges (in fact the terms of the series do not go to zero) when $|z-a| > \rho$.

To prove the furthermore suppose $0 < r < \rho$ and $z \in C(a,r)$. Then consider the partial sums

$$\left| \sum_{n=0}^{k} c_n (z-a)^n \right| \leq \sum_{n=0}^{k} |c_n| |z-a|^n \leq \sum_{n=0}^{k} |c_n| r^n. \qquad \square$$

The number ρ is called the *radius of convergence*. See Figure 11.6. The radius of convergence gives us a disk around a where the series converges. A power series is convergent if $\rho > 0$.

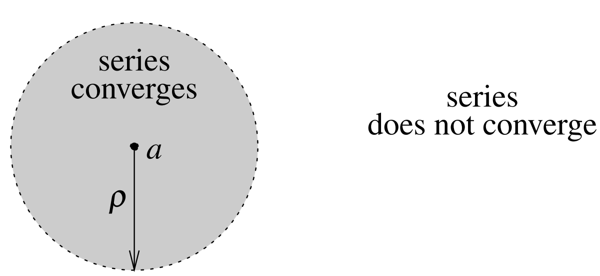

Figure 11.6: Radius of convergence.

If $\sum c_n(z-a)^n$ converges for some z, then

$$\sum c_n(w-a)^n$$

converges absolutely whenever $|w-a| < |z-a|$. Conversely if the series diverges at z, then it must diverge at w whenever $|w-a| > |z-a|$. This means that to show that the radius of convergence is at least some number, we simply need to show convergence at some point by any method we know.

Example 11.3.2: Let us list some series we already know:

$$\sum_{n=0}^{\infty} z^n \qquad \text{has radius of convergence } 1.$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n \qquad \text{has radius of convergence } \infty.$$

$$\sum_{n=0}^{\infty} n^n z^n \qquad \text{has radius of convergence } 0.$$

Example 11.3.3: Note the difference between $\frac{1}{1-z}$ and its power series. Let us expand $\frac{1}{1-z}$ as power series around a point $a \neq 1$. Let $c := \frac{1}{1-a}$, then

$$\frac{1}{1-z} = \frac{c}{1-c(z-a)} = c\sum_{n=0}^{\infty} c^n(z-a)^n = \sum_{n=0}^{\infty} \left(\frac{1}{(1-a)^{n+1}} \right) (z-a)^n.$$

The series $\sum c^n(z-a)^n$ converges if and only if the series on the right-hand side converges and

$$\limsup_{n\to\infty} \sqrt[n]{|c^n|} = |c| = \frac{1}{|1-a|}.$$

The radius of convergence of the power series is $|1-a|$, that is the distance from 1 to a. The function $\frac{1}{1-z}$ has a power series representation around every $a \neq 1$ and so is analytic in $\mathbb{C} \setminus \{1\}$. The domain of the function is bigger than the region of convergence of the power series representing the function at any point.

It turns out that if a function has a power series representation converging to the function on some ball, then it has a power series representation at every point in the ball. We will prove this result later.

11.3.3 Properties of analytic functions

Proposition 11.3.4. *If*

$$f(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$$

is convergent in $B(a,\rho)$ for some $\rho > 0$, then $f\colon B(a,\rho) \to \mathbb{C}$ is continuous. In particular, analytic functions are continuous.

Proof. For $z_0 \in B(a,\rho)$, pick $r < \rho$ such that $z_0 \in B(a,r)$. On $B(a,r)$ the partial sums (which are continuous) converge uniformly, and so the limit $f|_{B(a,r)}$ is continuous. Any sequence converging to z_0 has some tail that is completely in the open ball $B(a,r)$, hence f is continuous at z_0. □

In Corollary 6.2.13 of volume I we proved that we can differentiate real power series term by term. That is we proved that if

$$f(x) := \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for real x in an interval around $a \in \mathbb{R}$, then we can differentiate term by term and obtain a series

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} (x-a)^n$$

with the same radius of convergence. We only proved this theorem when c_n is real, however, for complex c_n, we write $c_n = s_n + i t_n$, and as x and a are real

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} s_n (x-a)^n + i \sum_{n=0}^{\infty} t_n (x-a)^n.$$

We apply the theorem to the real and imaginary part.

By iterating this theorem we find that an analytic function is infinitely differentiable:

$$f^{(\ell)}(x) = \sum_{n=\ell}^{\infty} n(n-1)\cdots(n-\ell+1) c_k (x-a)^{n-\ell} = \sum_{n=0}^{\infty} (n+\ell)(n+\ell-1)\cdots(n+1) c_{n+\ell} (x-a)^n.$$

In particular,

$$f^{(\ell)}(a) = \ell!\, c_\ell. \tag{11.1}$$

So the coefficients are uniquely determined by the derivatives of the function, and vice versa.

On the other hand, just because we have an infinitely differentiable function doesn't mean that the numbers c_n obtained by $c_n = \frac{f^{(n)}(0)}{n!}$ give a convergent power series. There is a theorem, which we will not prove, that given an arbitrary sequence $\{c_n\}$, there exists an infinitely differentiable function f such that $c_n = \frac{f^{(n)}(0)}{n!}$. Moreover, even if the obtained series converges it may not converge to the function we started with. For an example, see Exercise 5.4.11 in volume I: The function

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

is infinitely differentiable, and all derivatives at the origin are zero. So its series at the origin would be just the zero series, and while that series converges, it does not converge to f for $x > 0$.

Note that we can always apply an affine transformation $z \mapsto z + a$ that converts a power series to a series at the origin. That is, if

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad \text{we consider} \qquad f(z+a) = \sum_{n=0}^{\infty} c_n z^n.$$

Therefore it is usually sufficient to prove results about power series at the origin. From now on, we often assume $a = 0$ for simplicity.

11.3.4 Power series as analytic functions

We need a theorem on swapping limits of series, that is, Fubini's theorem for sums.

Theorem 11.3.5 (Fubini for sums)**.** *Let $\{a_{kj}\}_{k=1,j=1}^{\infty}$ be a double sequence of complex numbers and suppose that for every k the series*

$$\sum_{j=1}^{\infty} |a_{kj}| \qquad \textit{converges}$$

and furthermore that

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{kj}| \right) \qquad \textit{converges}.$$

Then

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj} \right) = \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{kj} \right),$$

where all the series involved converge.

Proof. Let E be the set $\{1/n : n \in \mathbb{N}\} \cup \{0\}$, and treat it as a metric space with the metric inherited from $\mathbb{R}$. Define the sequence of functions $f_k \colon E \to \mathbb{C}$ by

$$f_k(1/n) := \sum_{j=1}^{n} a_{kj} \qquad \text{and} \qquad f_k(0) := \sum_{j=1}^{\infty} a_{kj}.$$

As the series converges, each f_k is continuous at 0 (since 0 is the only cluster point, they are continuous at every point of E, but we don't need that). For all $x \in E$, we have

$$|f_k(x)| \leq \sum_{j=1}^{\infty} |a_{kj}|.$$

By knowing that $\sum_k \sum_j |a_{kj}|$ converges (and does not depend on x), we know that

$$\sum_{k=1}^{n} f_k(x)$$

converges uniformly on E. Define

$$g(x) := \sum_{k=1}^{\infty} f_k(x),$$

which is therefore a continuous function at 0. So

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{kj} \right) &= \sum_{k=1}^{\infty} f_k(0) = g(0) = \lim_{n\to\infty} g(1/n) \\
&= \lim_{n\to\infty} \sum_{k=1}^{\infty} f_k(1/n) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \sum_{j=1}^{n} a_{kj} \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{k=1}^{\infty} a_{kj} = \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{kj} \right).
\end{aligned}$$

□

Now we prove that once we have a series converging to a function in some interval, we can expand the function around every point.

Theorem 11.3.6 (Taylor's theorem for real-analytic functions)**.** *Let*

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

be a power series converging in $(-\rho,\rho)$ *for some* $\rho > 0$*. Given any* $a \in (-\rho,\rho)$*, and* x *such that* $|x-a| < \rho - |a|$*, we obtain*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The power series at a could of course converge in a larger interval, but the one above is guaranteed. It is the largest symmetric interval about a that fits in $(-\rho,\rho)$.

Proof. Given a and x as in the theorem, write

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k\bigl((x-a)+a\bigr)^k \\ &= \sum_{k=0}^{\infty} a_k \sum_{m=0}^{k} \binom{k}{m} a^{k-m}(x-a)^m. \end{aligned}$$

Define $c_{k,m} := a_k\binom{k}{m}a^{k-m}$ if $m \leq k$ and 0 if $m > k$. Then

$$f(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{k,m}(x-a)^m. \tag{11.2}$$

Let us show that the double sum converges absolutely.

$$\begin{aligned} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \left|c_{k,m}(x-a)^m\right| &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} \left|a_k\binom{k}{m}a^{k-m}(x-a)^m\right| \\ &= \sum_{k=0}^{\infty}|a_k|\sum_{m=0}^{k}\binom{k}{m}|a|^{k-m}|x-a|^m \\ &= \sum_{k=0}^{\infty}|a_k|\bigl(|x-a|+|a|\bigr)^k, \end{aligned}$$

and this series converges as long as $(|x-a|+|a|) < \rho$ or in other words if $|x-a| < \rho - |a|$.

Using Theorem 11.3.5, swap the order of summation in (11.2), and the following series converges when $|x-a| < \rho - |a|$:

$$f(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} c_{k,m}(x-a)^m = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty} c_{k,m}\right)(x-a)^m.$$

The formula in terms of derivatives at a follows by differentiating the series to obtain (11.1). $\square$

Note that if a series converges for real $x \in (a-\rho, a+\rho)$ it also converges for all complex numbers in $B(a,\rho)$. We have the following corollary, which says that functions defined by power series are analytic.

Corollary 11.3.7. *For every $a \in \mathbb{C}$, if $\sum c_k (z-a)^k$ converges to $f(z)$ in $B(a,\rho)$ and $b \in B(a,\rho)$, then there exists a power series $\sum d_k (z-b)^k$ that converges to $f(z)$ in $B(b, \rho - |b-a|)$.*

Proof. Without loss of generality assume that $a = 0$. We can rotate to assume that b is real, but since that is harder to picture, let us do it explicitly. Let $\alpha := \frac{\bar{b}}{|b|}$. Notice that

$$|1/\alpha| = |\alpha| = 1.$$

Therefore the series $\sum c_k (z/\alpha)^k = \sum c_k \alpha^{-k} z^k$ converges to $f(z/\alpha)$ in $B(0,\rho)$. When $z = x$ is real we apply Theorem 11.3.6 at $|b|$ and get a series that converges to $f(z/\alpha)$ on $B(|b|, \rho - |b|)$. That is, there is a convergent series

$$f(z/\alpha) = \sum_{k=0}^{\infty} a_k \big(z - |b|\big)^k.$$

Using $\alpha b = |b|$, we find

$$f(z) = f(\alpha z/\alpha) = \sum_{k=0}^{\infty} a_k (\alpha z - |b|)^k = \sum_{k=0}^{\infty} a_k \alpha^k \big(z - |b|/\alpha\big)^k = \sum_{k=0}^{\infty} a_k \alpha^k (z-b)^k,$$

and this series converges for all z such that $\big|\alpha z - |b|\big| < \rho - |b|$ or $|z-b| < \rho - |b|$. $\square$

We proved above that a convergent power series is an analytic function where it converges. We have also shown before that $\frac{1}{1-z}$ is analytic outside of $z = 1$.

Note that just because a real analytic function is analytic on the entire real line it does not necessarily mean that it has a power series representation that converges everywhere. For example, the function

$$f(x) = \frac{1}{1+x^2}$$

happens to be real analytic function on $\mathbb{R}$ (exercise). A power series around the origin converging to f has a radius of convergence of exactly 1. Can you see why? (exercise)

11.3.5 Identity theorem for analytic functions

Lemma 11.3.8. *Suppose $f(z) = \sum a_k z^k$ is a convergent power series and $\{z_n\}$ is a sequence of nonzero complex numbers converging to 0, such that $f(z_n) = 0$ for all n. Then $a_k = 0$ for every k.*

Proof. By continuity we know $f(0) = 0$ so $a_0 = 0$. Suppose there exists some nonzero a_k. Let m be the smallest m such that $a_m \neq 0$. Then

$$f(z) = \sum_{k=m}^{\infty} a_k z^k = z^m \sum_{k=m}^{\infty} a_k z^{k-m} = z^m \sum_{k=0}^{\infty} a_{k+m} z^k.$$

Write $g(z) = \sum_{k=0}^{\infty} a_{k+m} z^k$ (this series converges in on the same set as f). g is continuous and $g(0) = a_m \neq 0$. Thus there exists some $\delta > 0$ such that $g(z) \neq 0$ for all $z \in B(0,\delta)$. As $f(z) = z^m g(z)$, the only point in $B(0,\delta)$ where $f(z) = 0$ is when $z = 0$, but this contradicts the assumption that $f(z_n) = 0$ for all n. $\square$

Recall that in a metric space X, a *cluster point* (or sometimes *limit point*) of a set E is a point $p \in X$ such that $B(p,\varepsilon) \setminus \{p\}$ contains points of E for all $\varepsilon > 0$.

Theorem 11.3.9 (Identity theorem)**.** *Let $U \subset \mathbb{C}$ be open and connected. If $f \colon U \to \mathbb{C}$ and $g \colon U \to \mathbb{C}$ are analytic functions that are equal on a set $E \subset U$, and E has a cluster point in U, then $f(z) = g(z)$ for all $z \in U$.*

In most common applications of this theorem E is an open set or perhaps a curve.

Proof. Without loss of generality suppose E is the set of all points $z \in U$ such that $g(z) = f(z)$. Note that E must be closed as f and g are continuous.

Suppose E has a cluster point. Without loss of generality assume that 0 is this cluster point. Near 0, we have the expansions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \qquad \text{and} \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

which converge in some ball $B(0,\rho)$. Therefore the series

$$0 = f(z) - g(z) = \sum_{k=0}^{\infty} (a_k - b_k) z^k$$

converges in $B(0,\rho)$. As 0 is a cluster point of E, there is a sequence of nonzero points $\{z_n\}$ such that $f(z_n) - g(z_n) = 0$. Hence, by the lemma above $a_k = b_k$ for all k. Therefore, $B(0,\rho) \subset E$.

Thus the set of cluster points of E is open. The set of cluster points of E is also closed: A limit of cluster points of E is in E as it is closed, and it is clearly a cluster point of E. As U is connected, the set of cluster points of E is equal to U, or in other words $E = U$. □

By restricting our attention to real x, we obtain the same theorem for connected open subsets of $\mathbb{R}$, which are just open intervals.

11.3.6 Exercises

***Exercise* 11.3.1:** *Let*

$$a_{kj} := \begin{cases} 1 & \text{if } k = j, \\ -2^{k-j} & \text{if } k < j, \\ 0 & \text{if } k > j. \end{cases}$$

Compute (or show the limit doesn't exist):

a) $\displaystyle\sum_{j=1}^{\infty} |a_{kj}|$ *for every k,* *b)* $\displaystyle\sum_{k=1}^{\infty} |a_{kj}|$ *for every j,* *c)* $\displaystyle\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |a_{kj}|$, *d)* $\displaystyle\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{kj}$, *e)* $\displaystyle\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{kj}$.

Hint: Fubini for sums does not apply, in fact, answers to d) and e) are different.

***Exercise* 11.3.2:** *Let $f(x) := \frac{1}{1+x^2}$. Prove that*

a) f is analytic function on all of $\mathbb{R}$ by finding a power series for f at every $a \in \mathbb{R}$,

b) the radius of convergence of the power series for f at the origin is 1.

Exercise* 11.3.3*: *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is analytic. Show that for each n, there are at most finitely many zeros of f in $B(0,n)$, that is, $f^{-1}(0) \cap B(0,n)$ is finite for each n.*

Exercise* 11.3.4*: *Suppose $U \subset \mathbb{C}$ is open and connected, $0 \in U$, and $f\colon U \to \mathbb{C}$ is analytic. Treating f as a function of a real x at the origin, suppose $f^{(n)}(0) = 0$ for all n. Show that $f(z) = 0$ for all $z \in U$.*

Exercise* 11.3.5*: *Suppose $U \subset \mathbb{C}$ is open and connected, $0 \in U$, and $f\colon U \to \mathbb{C}$ is analytic. For real x and y, let $h(x) := f(x)$ and $g(y) := -i\,f(iy)$. Show that h and g are infinitely differentiable at the origin and $h'(0) = g'(0)$.*

Exercise* 11.3.6*: *Suppose a function f is analytic in some neighborhood of the origin, and that there exists an M such that $|f^{(n)}(0)| \leq M$ for all n. Prove that the series of f at the origin converges for all $z \in \mathbb{C}$.*

Exercise* 11.3.7*: *Suppose $f(z) := \sum c_n z^n$ with a radius of convergence 1. Suppose $f(0) = 0$, but f is not the zero function. Show that there exists a $k \in \mathbb{N}$ and a convergent power series $g(z) := \sum d_n z^n$ with radius of convergence 1 such that $f(z) = z^k g(z)$ for all $z \in B(0,1)$, and $g(0) \neq 0$.*

Exercise* 11.3.8*: *Suppose $U \subset \mathbb{C}$ is open and connected. Suppose that $f\colon U \to \mathbb{C}$ is analytic, $U \cap \mathbb{R} \neq \emptyset$ and $f(x) = 0$ for all $x \in U \cap \mathbb{R}$. Show that $f(z) = 0$ for all $z \in U$.*

Exercise* 11.3.9*: *For $\alpha \in \mathbb{C}$ and $k = 0,1,2,3\ldots$, define*

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k)}{k!}.$$

a) Show that the series

$$f(z) := \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$$

converges whenever $|z| < 1$. In fact, prove that for $\alpha = 0,1,2,3,\ldots$ the radius of convergence is ∞, and for all other α the radius of convergence is 1.

b) Show that for $x \in \mathbb{R}$, $|x| < 1$, we have

$$(1+x)f'(x) = \alpha f(x),$$

meaning that $f(x) = (1+x)^{\alpha}$.

Exercise* 11.3.10*: *Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is analytic and suppose that for some open interval $(a,b) \subset \mathbb{R}$, f is real valued on (a,b). Show that f is real-valued on $\mathbb{R}$.*

Exercise* 11.3.11*: *Let $\mathbb{D} := B(0,1)$ be the unit disc. Suppose $f\colon \mathbb{D} \to \mathbb{C}$ is analytic with power series $\sum c_n z^n$. Suppose $|c_n| \leq 1$ for all n. Prove that for all $z \in \mathbb{D}$, we have $|f(z)| \leq \frac{1}{1-|z|}$.*

11.4 The complex exponential and the trigonometric functions

Note: 1 lecture

11.4.1 The complex exponential

Define

$$E(z) := \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

This series converges for all $z \in \mathbb{C}$ and so by Corollary 11.3.7, E is analytic on $\mathbb{C}$. We notice that $E(0) = 1$, and that for $z = x \in \mathbb{R}$, $E(x) \in \mathbb{R}$. Keeping x real, we find

$$\frac{d}{dx}\big(E(x)\big) = E(x)$$

by direct calculation. In §5.4 of volume I (or by Picard's theorem), we proved that the unique function satisfying $E' = E$ and $E(0) = 1$ is the exponential. In other words, for $x \in \mathbb{R}$, $e^x = E(x)$.

For complex numbers z, we define

$$e^z := E(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

On the real line this new definition agrees with our previous one. See Figure 11.7. Notice that in the x direction (the real direction) the graph behaves like the real exponential, and in the y direction (the imaginary direction) the graph oscillates.

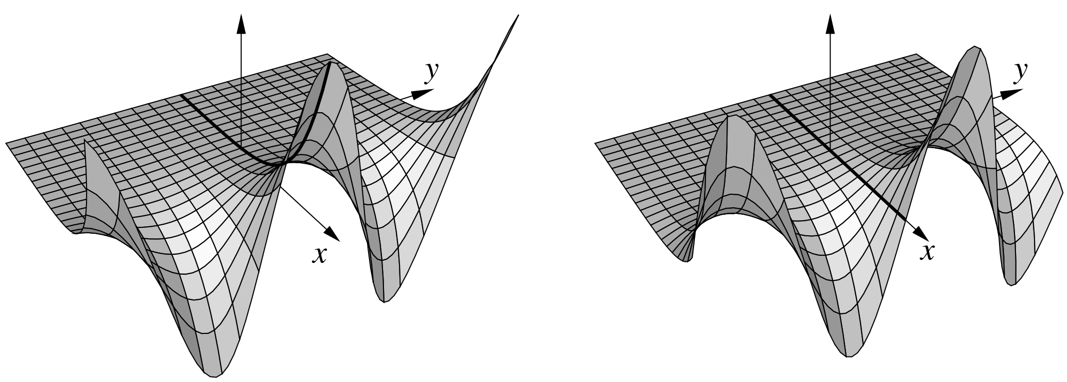

Figure 11.7: Graphs of the real part (left) and imaginary part (right) of the complex exponential $e^z = e^{x+iy}$. The x-axis goes from -4 to 4, the y-axis goes from -6 to 6, and the vertical axis goes from $-e^4 \approx -54.6$ to $e^4 \approx 54.6$. The plot of the real exponential ($y = 0$) is marked in a bold line.

Proposition 11.4.1. *Let $z, w \in \mathbb{C}$ be complex numbers. Then*

$$e^{z+w} = e^z e^w.$$

Proof. We already know that the equality $e^{x+y} = e^x e^y$ holds for all real numbers x and y. For every fixed $y \in \mathbb{R}$, consider the expressions as functions of x and apply the identity theorem (Theorem 11.3.9) to get that $e^{z+y} = e^z e^y$ for all $z \in \mathbb{C}$. Fixing an arbitrary $z \in \mathbb{C}$, we get $e^{z+y} = e^z e^y$ for all $y \in \mathbb{R}$. Again by the identity theorem $e^{z+w} = e^z e^w$ for all $w \in \mathbb{C}$. □

A simple consequence is that $e^z \neq 0$ for all $z \in \mathbb{C}$, as $e^z e^{-z} = e^{z-z} = 1$. A more complicated consequence is that we can easily compute the power series for the exponential at a point $a \in \mathbb{C}$: $e^z = e^a e^{z-a} = \sum \frac{e^a}{k!}(z-a)^k$.

11.4.2 Trigonometric functions and π

We can now finally define *sine* and *cosine* by the equation

$$e^{x+iy} = e^x \bigl(\cos(y) + i\sin(y)\bigr).$$

In fact, we define sine and cosine for all complex z:

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin(z) := \frac{e^{iz} - e^{-iz}}{2i}.$$

Let us use our definition to prove the common properties we usually associate with sine and cosine. In the process we also define the number π.

Proposition 11.4.2. *The sine and cosine functions have the following properties:*

(i) *For all* $z \in \mathbb{C}$,
$$e^{iz} = \cos(z) + i\sin(z) \qquad \textit{(Euler's formula)}.$$

(ii) $\cos(0) = 1$, $\sin(0) = 0$.

(iii) *For all* $z \in \mathbb{C}$,
$$\cos(-z) = \cos(z), \qquad \sin(-z) = -\sin(z).$$

(iv) *For all* $z \in \mathbb{C}$,
$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}, \qquad \sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.$$

(v) *For all* $x \in \mathbb{R}$
$$\cos(x) = \operatorname{Re}(e^{ix}) \qquad \textit{and} \qquad \sin(x) = \operatorname{Im}(e^{ix}).$$

(vi) *For all* $x \in \mathbb{R}$,
$$\bigl(\cos(x)\bigr)^2 + \bigl(\sin(x)\bigr)^2 = 1.$$

(vii) *For all* $x \in \mathbb{R}$,
$$|\sin(x)| \leq 1, \qquad |\cos(x)| \leq 1.$$

(viii) *For all* $x \in \mathbb{R}$
$$\frac{d}{dx}\bigl[\cos(x)\bigr] = -\sin(x) \qquad \textit{and} \qquad \frac{d}{dx}\bigl[\sin(x)\bigr] = \cos(x).$$

(ix) *For all* $x \geq 0$,
$$\sin(x) \leq x.$$

(x) *There exists an* $x > 0$ *such that* $\cos(x) = 0$. *We define*
$$\pi := 2 \inf\{x > 0 : \cos(x) = 0\}.$$

(xi) *For all* $z \in \mathbb{C}$,
$$e^{2\pi i} = 1, \qquad \textit{and} \qquad e^{z+i2\pi} = e^z.$$

(xii) *Sine and cosine are* 2π*-periodic and not periodic with any smaller period. That is,* 2π *is the smallest number such that for all* $z \in \mathbb{C}$,
$$\sin(z+2\pi) = \sin(z) \qquad \textit{and} \qquad \cos(z+2\pi) = \cos(z).$$

(xiii) *The function* $x \mapsto e^{ix}$ *is a bijective map from* $[0,2\pi)$ *onto the set of* $z \in \mathbb{C}$ *such that* $|z| = 1$.

The proposition immediately implies that $\sin(x)$ and $\cos(x)$ are real whenever x is real.

Proof. The first three items follow directly from the definition. The computation of the power series for both is left as an exercise.

As complex conjugate is a continuous function, the definition of e^z implies $\overline{(e^z)} = e^{\bar{z}}$. If x is real,
$$\overline{(e^{ix})} = e^{-ix}.$$
Thus for real x, $\cos(x) = \operatorname{Re}(e^{ix})$ and $\sin(x) = \operatorname{Im}(e^{ix})$.

For real x we compute
$$1 = e^{ix}e^{-ix} = |e^{ix}|^2 = \big(\cos(x)\big)^2 + \big(\sin(x)\big)^2.$$
In particular, is e^{ix} is unimodular, the values lie on the unit circle. A square is always nonnegative:
$$\big(\sin(x)\big)^2 = 1 - \big(\cos(x)\big)^2 \leq 1.$$
So $|\sin(x)| \leq 1$ and similarly $|\cos(x)| \leq 1$.

We leave the computation of the derivatives to the reader as exercises.

Let us now prove that $\sin(x) \leq x$ for $x \geq 0$. Consider $f(x) := x - \sin(x)$ and differentiate:
$$f'(x) = \frac{d}{dx}\big[x - \sin(x)\big] = 1 - \cos(x) \geq 0,$$
for all x as $|\cos(x)| \leq 1$. In other words, f is increasing and $f(0) = 0$. So f must be nonnegative when $x \geq 0$.

We claim there exists a positive x such that $\cos(x) = 0$. As $\cos(0) = 1 > 0$, $\cos(x) > 0$ for x near 0. Namely, there is some $y > 0$, such that $\cos(x) > 0$ on $[0,y)$. Then $\sin(x)$ is strictly increasing on $[0,y)$. As $\sin(0) = 0$, then $\sin(x) > 0$ for $x \in (0,y)$. Take $a \in (0,y)$. By the mean value theorem there is a $c \in (a,y)$ such that
$$2 \geq \cos(a) - \cos(y) = \sin(c)(y-a) \geq \sin(a)(y-a).$$

As $a \in (0,y)$, then $\sin(a) > 0$ and so

$$y \leq \frac{2}{\sin(a)} + a.$$

Hence there is some largest y such that $\cos(x) > 0$ in $[0,y)$, and let y be the largest such number. By continuity, $\cos(y) = 0$. In fact, y is the smallest positive y such that $\cos(y) = 0$. As mentioned π is defined to be $2y$.

As $\cos(\pi/2) = 0$, then $\bigl(\sin(\pi/2)\bigr)^2 = 1$. As sin is positive on $(0,y)$, we have $\sin(\pi/2) = 1$. Hence,

$$e^{i\pi/2} = i,$$

and by the addition formula

$$e^{i\pi} = -1, \qquad e^{i2\pi} = 1.$$

So $e^{i2\pi} = 1 = e^0$. The addition formula says

$$e^{z+i2\pi} = e^z$$

for all $z \in \mathbb{C}$. Immediately we also obtain $\cos(z+2\pi) = \cos(z)$ and $\sin(z+2\pi) = \sin(z)$. So sin and cos are 2π-periodic.

We claim that sin and cos are not periodic with a smaller period. It would suffice to show that if $e^{ix} = 1$ for the smallest positive x, then $x = 2\pi$. So let x be the smallest positive x such that $e^{ix} = 1$. Of course, $x \leq 2\pi$. By the addition formula,

$$\bigl(e^{ix/4}\bigr)^4 = 1.$$

If $e^{ix/4} = a + ib$, then

$$(a+ib)^4 = a^4 - 6a^2b^2 + b^4 + i\bigl(4ab(a^2-b^2)\bigr) = 1.$$

As $x/4 \leq \pi/2$, then $a = \cos(x/4) \geq 0$ and $0 < b = \sin(x/4)$. Then either $a = 0$ or $a^2 = b^2$. If $a^2 = b^2$, then $a^4 - 6a^2b^2 + b^4 = -4a^4 < 0$ and in particular not equal to 1. Therefore $a = 0$ in which case $x/4 = \pi/2$. Hence 2π is the smallest period we could choose for e^{ix} and so also for cos and sin.

Finally, we also wish to show that e^{ix} is one-to-one and onto from the set $[0,2\pi)$ to the set of $z \in \mathbb{C}$ such that $|z| = 1$. Suppose $e^{ix} = e^{iy}$ and $x > y$. Then $e^{i(x-y)} = 1$, meaning $x - y$ is a multiple of 2π and hence only one of them can live in $[0,2\pi)$. To show onto, pick $(a,b) \in \mathbb{R}^2$ such that $a^2 + b^2 = 1$. Suppose first that $a, b \geq 0$. By the intermediate value theorem there must exist an $x \in [0,\pi/2]$ such that $\cos(x) = a$, and hence $b^2 = \bigl(\sin(x)\bigr)^2$. As b and $\sin(x)$ are nonnegative, we have $b = \sin(x)$. Since $-\sin(x)$ is the derivative of $\cos(x)$ and $\cos(-x) = \cos(x)$, then $\sin(x) < 0$ for $x \in [-\pi/2, 0)$. Using the same reasoning we obtain that if $a > 0$ and $b \leq 0$, we can find an x in $[-\pi/2, 0)$, and by periodicity, $x \in [3\pi/2, 2\pi)$ such that $\cos(x) = a$ and $\sin(x) = b$. Multiplying by -1 is the same as multiplying by $e^{i\pi}$ or $e^{-i\pi}$. So we can always assume that $a \geq 0$ (details are left as exercise). $\square$

11.4.3 The unit circle and polar coordinates

The arclength of a curve parametrized by $\gamma\colon [a,b] \to \mathbb{C}$ is given by

$$\int_a^b |\gamma'(t)|\, dt.$$

We have that e^{it} parametrizes the circle for t in $[0,2\pi)$. As $\frac{d}{dt}\left(e^{it}\right) = ie^{it}$, the circumference of the circle (the arclength) is

$$\int_0^{2\pi} |ie^{it}|\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

More generally we notice that e^{it} parametrizes the circle by arclength. That is, t measures the arclength, and hence a circle of radius 1 by the angle in radians. So the definitions of sin and cos we have used above agree with the standard geometric definitions.

All the points on the unit circle can be achieved by e^{it} for some t. Therefore, we can write a complex number $z \in \mathbb{C}$ (in so-called *polar coordinates*) as

$$z = re^{i\theta}$$

for some $r \geq 0$ and $\theta \in \mathbb{R}$. The θ is, of course, not unique as θ or $\theta + 2\pi$ gives the same number. The formula $e^{a+b} = e^a e^b$ leads to a useful formula for powers and products of complex numbers in polar coordinates:

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta}, \qquad (re^{i\theta})(se^{i\gamma}) = rse^{i(\theta+\gamma)}.$$

11.4.4 Exercises

Exercise* 11.4.1*: *Derive the power series for* $\sin(z)$ *and* $\cos(z)$ *at the origin.*

Exercise* 11.4.2*: *Using the power series, show that for x real, we have* $\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$ *and* $\frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$.

Exercise* 11.4.3*: *Finish the proof of the argument that $x \mapsto e^{ix}$ from $[0,2\pi)$ is onto the unit circle. In particular, assume that we get all points of the form (a,b) where $a^2+b^2 = 1$ for $a \geq 0$. By multiplying by $e^{i\pi}$ or $e^{-i\pi}$ show that we get everything.*

Exercise* 11.4.4*: *Prove that there is no $z \in \mathbb{C}$ such that $e^z = 0$.*

Exercise* 11.4.5*: *Prove that for every $w \neq 0$ and every $\varepsilon > 0$, there exists a $z \in \mathbb{C}$, $|z| < \varepsilon$ such that $e^{1/z} = w$.*

Exercise* 11.4.6*: *We showed $\left(\cos(x)\right)^2 + \left(\sin(x)\right)^2 = 1$ for all $x \in \mathbb{R}$. Prove that $\left(\cos(z)\right)^2 + \left(\sin(z)\right)^2 = 1$ for all $z \in \mathbb{C}$.*

Exercise* 11.4.7*: *Prove the trigonometric identities* $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$ *and* $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$ *for all $z, w \in \mathbb{C}$.*

Exercise* 11.4.8*: *Define* $\operatorname{sinc}(z) := \frac{\sin(z)}{z}$ *for $z \neq 0$ and* $\operatorname{sinc}(0) := 1$. *Show that sinc is analytic and compute its power series at zero.*

Define the *hyperbolic sine* and *hyperbolic cosine* by

$$\sinh(z) := \frac{e^z - e^{-z}}{2}, \qquad \cosh(z) := \frac{e^z + e^{-z}}{2}.$$

Exercise* 11.4.9*: *Derive the power series at the origin for the hyperbolic sine and cosine.*

Exercise* 11.4.10*: *Show*

a) $\sinh(0) = 0$, $\cosh(0) = 1$.

b) $\frac{d}{dx}\big[\sinh(x)\big] = \cosh(x)$ *and* $\frac{d}{dx}\big[\cosh(x)\big] = \sinh(x)$.

c) $\cosh(x) > 0$ *for all* $x \in \mathbb{R}$ *and show that* $\sinh(x)$ *is strictly increasing and bijective from* $\mathbb{R}$ *to* $\mathbb{R}$.

d) $\big(\cosh(x)\big)^2 = 1 + \big(\sinh(x)\big)^2$ *for all* x.

Exercise* 11.4.11*: *Define* $\tan(x) := \frac{\sin(x)}{\cos(x)}$ *as usual.*

a) Show that for $x \in (-\pi/2, \pi/2)$ *both* $\sin$ *and* $\tan$ *are strictly increasing, and hence* $\sin^{-1}$ *and* $\tan^{-1}$ *exist when we restrict to that interval.*

b) Show that $\sin^{-1}$ *and* $\tan^{-1}$ *are differentiable and that* $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ *and* $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.

c) Using the finite geometric sum formula show

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^s}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

converges for all $-1 \leq x \leq 1$ *(including the end points). Hint: Integrate the finite sum, not the series.*

d) Use this to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

11.5 Fundamental theorem of algebra

Note: half a lecture, optional

In this section we study the local behavior of polynomials and the growth of polynomials as z goes to infinity. As an application we prove the fundamental theorem of algebra: Any nonconstant polynomial has a complex root.

Lemma 11.5.1. *Let $p(z)$ be a nonconstant complex polynomial. If $p(z_0) \neq 0$, then there exist $w \in \mathbb{C}$ such that $|p(w)| < |p(z_0)|$. In fact, we can pick w to be arbitrarily close to z_0.*

Proof. Without loss of generality assume that $z_0 = 0$ and $p(0) = 1$. Write

$$p(z) = 1 + a_k z^k + a_{k+1} z^{k+1} + \cdots + a_d z^d,$$

where $a_k \neq 0$. Pick t such that $a_k e^{ikt} = -|a_k|$, which we can do by the discussion on trigonometric functions. Suppose $r > 0$ is small enough such that $1 - r^k|a_k| > 0$. We have

$$p(re^{it}) = 1 - r^k|a_k| + r^{k+1} a_{k+1} e^{i(k+1)t} + \cdots + r^d a_d e^{idt}.$$

So

$$\begin{aligned} \left|p(re^{it})\right| - \left|r^{k+1} a_{k+1} e^{i(k+1)t} + \cdots + r^d a_d e^{idt}\right| &\leq \left|p(re^{it}) - r^{k+1} a_{k+1} e^{i(k+1)t} - \cdots - r^d a_d e^{idt}\right| \\ &= \left|1 - r^k|a_k|\right| = 1 - r^k|a_k|. \end{aligned}$$

In other words,

$$\left|p(re^{it})\right| \leq 1 - r^k \left(|a_k| - r\left|a_{k+1} e^{i(k+1)t} + \cdots + r^{d-k-1} a_d e^{idt}\right|\right).$$

For small enough r, the expression in the parentheses is positive as $|a_k| > 0$. Hence, $\left|p(re^{it})\right| < 1 = p(0)$. $\square$

Remark 11.5.2. The lemma above holds essentially with an unchanged proof for (complex) analytic functions. A proof of this generalization is left as an exercise to the reader. What the lemma says is that the only minima the modulus of analytic functions (polynomials) has are precisely at the zeros.

Remark 11.5.3. The lemma does not hold if we restrict to real numbers. For example, $x^2 + 1$ has a minimum at $x = 0$, but no zero there. The thing is that there is a w arbitrarily close to 0 such that $|w^2 + 1| < 1$, but this w is necessarily not real. Letting $w = i\varepsilon$ for small $\varepsilon > 0$ works.

The moral of the story is that if $p(0) = 1$, then very close to 0, the polynomial looks like $1 + az^k$, and $1 + az^k$ has no minimum at the origin. All the higher powers of z are too small to make a difference. We find similar behavior at infinity.

Lemma 11.5.4. *Let $p(z)$ be a nonconstant complex polynomial. Then for an $M > 0$, there exists an $R > 0$ such that $|p(z)| \geq M$ whenever $|z| \geq R$.*

Proof. Write $p(z) = a_0 + a_1 z + \cdots + a_d z^d$ and suppose that $d \geq 1$ and $a_d \neq 0$. Suppose $|z| \geq R$ (so also $|z|^{-1} \leq R^{-1}$). We estimate:

$$\begin{aligned} |p(z)| &\geq |a_d z^d| - |a_0| - |a_1 z| - \cdots - |a_{d-1} z^{d-1}| \\ &= |z|^d \left(|a_d| - |a_0|\,|z|^{-d} - |a_1|\,|z|^{-d+1} - \cdots - |a_{d-1}|\,|z|^{-1} \right) \\ &\geq R^d \left(|a_d| - |a_0| R^{-d} - |a_1| R^{1-d} - \cdots - |a_{d-1}| R^{-1} \right). \end{aligned}$$

Then the expression in parentheses is eventually positive for large enough R. In particular, for large enough R we get that this expression is greater than $\frac{|a_d|}{2}$, and so

$$|p(z)| \geq R^d \frac{|a_d|}{2}.$$

Therefore, we can pick R large enough to be bigger than a given M. □

The lemma above does *not* generalize to analytic functions, even those defined in all of $\mathbb{C}$. The function $\cos(z)$ is a counterexample. Note that we had to look at the term with the largest degree, and we only have such a term for a polynomial. In fact, something that we will not prove is that an analytic function defined on all of $\mathbb{C}$ satisfying the conclusion of the lemma must be a polynomial.

The moral of the story here is that for very large $|z|$ (far away from the origin) a polynomial of degree d really looks like a constant multiple of z^d.

Theorem 11.5.5 (Fundamental theorem of algebra)**.** *Let $p(z)$ be a nonconstant complex polynomial, then there exists a $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. Let $\mu := \inf\{ |p(z)| : z \in \mathbb{C} \}$. Find an R such that for all z with $|z| \geq R$, we have $|p(z)| \geq \mu + 1$. Therefore, every z with $|p(z)|$ close to μ must be in the closed ball $C(0,R) = \{ z \in \mathbb{C} : |z| \leq R \}$. As $|p(z)|$ is a continuous real-valued function, it achieves its minimum on the compact set $C(0,R)$ (closed and bounded) and this minimum must be μ. So there is a $z_0 \in C(0,R)$ such that $|p(z_0)| = \mu$. As that is a minimum of $|p(z)|$ on $\mathbb{C}$, then by the first lemma above, we have $|p(z_0)| = 0$. □

The fundamental theorem also does not generalize to analytic functions. For example, e^z is an analytic function on $\mathbb{C}$ with no zeros.

11.5.1 Exercises

Exercise* 11.5.1*: *Prove Lemma 11.5.1 for an analytic function. That is, suppose that $p(z)$ is a power series around z_0.*

Exercise* 11.5.2*: *Use Exercise 11.5.1 to prove the* maximum principle for analytic functions*:* If $U \subset \mathbb{C}$ is open and connected, $f : U \to \mathbb{C}$ is analytic, and $|f(z)|$ attains a relative maximum at $z_0 \in U$, then f is constant.

Exercise* 11.5.3*: *Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Suppose $f : U \to \mathbb{C}$ is analytic and $f(z_0) = 0$. Show that there exists an $\varepsilon > 0$ such that either $f(z) \neq 0$ for all z with $0 < |z| < \varepsilon$ or $f(z) = 0$ for all $z \in B(z_0, \varepsilon)$. In other words, zeros of analytic functions are isolated. Of course, same holds for polynomials.*

A *rational function* is a function $f(z) := \frac{p(z)}{q(z)}$ where p and q are polynomials and q is not identically zero. A point $z_0 \in \mathbb{C}$ where $f(z_0) = 0$ (and therefore $p(z_0) = 0$) is called a *zero*. A point $z_0 \in \mathbb{C}$ is called an *singularity* of f if $q(z_0) = 0$. As all zeros are isolated and so all singularities of rational functions are isolated and so are called an *isolated singularity*. An isolated singularity is called *removable* if $\lim_{z \mapsto z_0} f(z)$ exists. An isolated singularity is called a *pole* if $\lim_{z \mapsto z_0} |f(z)| = \infty$. We say f has pole at ∞ if

$$\lim_{z \to \infty} |f(z)| = \infty,$$

that is, if for every $M > 0$ there exists an $R > 0$ such that $|f(z)| > M$ for all z with $|z| > R$.

Exercise* 11.5.4*: *Show that a rational function which is not identically zero has at most finitely many zeros and singularities. In fact, show that if p is a polynomial of degree $n > 0$ it has at most n zeros.*
Hint: If z_0 is a zero of p, without loss of generality assume $z_0 = 0$. Then use induction.

Exercise* 11.5.5*: *Prove that if z_0 is a removable singularity of a rational function $f(z) := \frac{p(z)}{q(z)}$, then there exist polynomials $\widetilde{p}$ and $\widetilde{q}$ such that $\widetilde{q}(z_0) \neq 0$ and $f(z) = \frac{\widetilde{p}(z)}{\widetilde{q}(z)}$.*
Hint: Without loss of generality assume $z_0 = 0$.

Exercise* 11.5.6*: *Given a rational function f with an isolated singularity at z_0, show that z_0 is either removable or a pole.*
Hint: See the previous exercise.

Exercise* 11.5.7*: *Let f be a rational function and $S \subset \mathbb{C}$ is the set of the singularities of f. Prove that f is equal to a polynomial on $\mathbb{C} \setminus S$ if and only if f has a pole at infinity and all the singularities are removable.*
Hint: See previous exercises.

11.6 Equicontinuity and the Arzelà–Ascoli theorem

Note: 2 lectures

We would like an analogue of Bolzano–Weierstrass. Something to the tune of "every bounded sequence of functions (with some property) has a convergent subsequence." Matters are not as simple even for continuous functions. Not every bounded sequence in the metric space $C([0,1],\mathbb{R})$ has a convergent subsequence.

Definition 11.6.1. Let X be a set. Let $f_n \colon X \to \mathbb{C}$ be functions in a sequence. We say that $\{f_n\}$ is *pointwise bounded* if for every $x \in X$, there is an $M_x \in \mathbb{R}$ such that

$$|f_n(x)| \leq M_x \qquad \text{for all } n \in \mathbb{N}.$$

We say that $\{f_n\}$ is *uniformly bounded* if there is an $M \in \mathbb{R}$ such that

$$|f_n(x)| \leq M \qquad \text{for all } n \in \mathbb{N} \text{ and all } x \in X.$$

If X is a compact metric space, then a sequence in $C(X,\mathbb{C})$ is uniformly bounded if it is bounded as a set in the metric space $C(X,\mathbb{C})$ using the uniform norm.

Example 11.6.2: There exist sequences of continuous functions on $[0,1]$ that are uniformly bounded but contain no subsequence converging even pointwise. Let us state without proof that $f_n(x) := \sin(2\pi n x)$ is one such sequence. Below we will show that there must always exist a subsequence converging at countably many points, but $[0,1]$ is uncountable.

Example 11.6.3: The sequence $f_n(x) := x^n$ of continuous functions on $[0,1]$ is uniformly bounded, but contains no subsequence that converges uniformly, although the sequence converges pointwise (to a discontinuous function).

Example 11.6.4: The sequence $\{f_n\}$ of functions in $C([0,1],\mathbb{R})$ given by $f_n(x) := \frac{n^3 x}{1+n^4x^2}$ converges pointwise to the zero function (obvious at $x=0$, and for $x>0$, we have $\frac{n^3x}{1+n^4x^2} \leq \frac{1}{nx}$). As for each x, $\{f_n(x)\}$ converges to 0, it is bounded so $\{f_n\}$ is pointwise bounded.

By calculus, we maximize f_n on $[0,1]$, and we find the maximum occurs at the critical point $x = 1/n^2$:

$$\|f_n\|_u = f_n\left(1/n^2\right) = n/2.$$

So $\lim \|f_n\|_u = \infty$, and this sequence is not uniformly bounded.

When the domain is countable, we can locate a subsequence converging at least pointwise. The proof uses a very common and useful diagonal argument.

Proposition 11.6.5. *Let X be a countable set and $f_n \colon X \to \mathbb{C}$ give a pointwise bounded sequence of functions. Then $\{f_n\}$ has a subsequence that converges pointwise.*

Proof. Let $x_1, x_2, x_3, \ldots$ be an enumeration of the elements of X. The sequence $\{f_n(x_1)\}_{n=1}^\infty$ is bounded and hence we have a subsequence of $\{f_n\}_{n=1}^\infty$, which we denote by $\{f_{1,k}\}_{k=1}^\infty$, such that $\{f_{1,k}(x_1)\}_{k=1}^\infty$ converges. Next $\{f_{1,k}(x_2)\}_{k=1}^\infty$ is bounded and so $\{f_{1,k}\}_{k=1}^\infty$ has a subsequence $\{f_{2,k}\}_{k=1}^\infty$ such that $\{f_{2,k}(x_2)\}_{k=1}^\infty$ converges. Note that $\{f_{2,k}(x_1)\}_{k=1}^\infty$ is still convergent.

In general, we have a sequence $\{f_{m,k}\}_{k=1}^{\infty}$, which is a subsequence of $\{f_{m-1,k}\}_{k=1}^{\infty}$, such that $\{f_{m,k}(x_j)\}_{k=1}^{\infty}$ converges for $j = 1,2,\ldots,m$. We let $\{f_{m+1,k}\}_{k=1}^{\infty}$ be a subsequence of $\{f_{m,k}\}_{k=1}^{\infty}$ such that $\{f_{m+1,k}(x_{m+1})\}_{k=1}^{\infty}$ converges (and hence it converges for all x_j for $j = 1,2,\ldots,m+1$). Rinse and repeat.

If X is finite, we are done as the process stops at some point. If X is countably infinite, we pick the sequence $\{f_{k,k}\}_{k=1}^{\infty}$. This is a subsequence of the original sequence $\{f_n\}_{n=1}^{\infty}$. For every m, the tail $\{f_{k,k}\}_{k=m}^{\infty}$ is a subsequence of $\{f_{m,k}\}_{k=1}^{\infty}$ and hence for any m the sequence $\{f_{k,k}(x_m)\}_{k=1}^{\infty}$ converges. $\square$

For larger than countable sets, we need the functions of the sequence to be related. When we look at continuous functions, the concept we need is equicontinuity.

Definition 11.6.6. Let (X,d) be a metric space. A set S of functions $f\colon X \to \mathbb{C}$ is *uniformly equicontinuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $x,y \in X$ with $d(x,y) < \delta$, we have

$$|f(x) - f(y)| < \varepsilon \qquad \text{for all } f \in S.$$

Notice that functions in a uniformly equicontinuous sequence are all uniformly continuous. It is not hard to show that a finite set of uniformly continuous functions is uniformly equicontinuous. The definition is really interesting if S is infinite.

Just as for continuity, one can define equicontinuity at a point. That is, S is *equicontinuous* at $x \in X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for $y \in X$ with $d(x,y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$ for all $f \in S$. We will only deal with compact X here, and one can prove (exercise) that for a compact metric space X, if S is equicontinuous at every $x \in X$, then it is uniformly equicontinuous. For simplicity we stick to uniform equicontinuity.

Proposition 11.6.7. *Suppose (X,d) is a compact metric space, $f_n \in C(X,\mathbb{C})$, and $\{f_n\}$ converges uniformly, then $\{f_n\}$ is uniformly equicontinuous.*

Proof. Let $\varepsilon > 0$ be given. As $\{f_n\}$ converges uniformly, there is an $N \in \mathbb{N}$ such that for all $n \geq N$

$$|f_n(x) - f_N(x)| < \varepsilon/3 \qquad \text{for all } x \in X.$$

As X is compact, every continuous function is uniformly continuous. So $\{f_1, f_2, \ldots, f_N\}$ is a finite set of uniformly continuous functions. And so, as we mentioned above, the set is uniformly equicontinuous. Hence there is a $\delta > 0$ such that

$$|f_j(x) - f_j(y)| < \varepsilon/3 < \varepsilon$$

whenever $d(x,y) < \delta$ and $1 \leq j \leq N$.

Take $n > N$. For $d(x,y) < \delta$, we have

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \quad \square$$

Proposition 11.6.8. *A compact metric space (X,d) contains a countable dense subset, that is, there exists a countable $D \subset X$ such that $\overline{D} = X$.*

Proof. For each $n \in \mathbb{N}$ there are finitely many balls of radius $1/n$ that cover X (as X is compact). That is, for every n, there exists a finite set of points $x_{n,1}, x_{n,2}, \ldots, x_{n,k_n}$ such that

$$X = \bigcup_{j=1}^{k_n} B(x_{n,j}, 1/n).$$

Let $D := \bigcup_{n=1}^{\infty} \{x_{n,1}, x_{n,2}, \ldots, x_{n,k_n}\}$. The set D is countable as it is a countable union of finite sets. For every $x \in X$ and every $\varepsilon > 0$, there exists an n such that $1/n < \varepsilon$ and an $x_{n,j} \in D$ such that

$$x \in B(x_{n,j}, 1/n) \subset B(x_{n,j}, \varepsilon).$$

Hence $x \in \overline{D}$, so $\overline{D} = X$, and D is dense. □

We are now ready for the main result of this section, the Arzelà–Ascoli theorem* about existence of convergent subsequences.

Theorem 11.6.9 (Arzelà–Ascoli)**.** *Let (X,d) be a compact metric space, and let $\{f_n\}$ be pointwise bounded and uniformly equicontinuous sequence of functions $f_n \in C(X,\mathbb{C})$. Then $\{f_n\}$ is uniformly bounded and $\{f_n\}$ contains a uniformly convergent subsequence.*

Basically, a uniformly equicontinuous sequence in the metric space $C(X,\mathbb{C})$ that is pointwise bounded is bounded (in $C(X,\mathbb{C})$) and furthermore contains a convergent subsequence in $C(X,\mathbb{C})$.

As we mentioned before, as X is compact, it is enough to just assume that $\{f_n\}$ is equicontinuous as uniform equicontinuity is automatic via an exercise.

Proof. Let us first show that the sequence is uniformly bounded.

By uniform equicontinuity, there is a $\delta > 0$ such that for all $x \in X$ and all $n \in \mathbb{N}$,

$$B(x,\delta) \subset f_n^{-1}\big(B(f_n(x),1)\big).$$

The space X is compact, so there exist $x_1, x_2, \ldots, x_k$ such that

$$X = \bigcup_{j=1}^{k} B(x_j, \delta).$$

As $\{f_n\}$ is pointwise bounded there exist $M_1, M_2, \ldots, M_k$ such that for $j = 1, 2, \ldots, k$, we have

$$|f_n(x_j)| \leq M_j \qquad \text{for all } n.$$

Let $M := 1 + \max\{M_1, M_2, \ldots, M_k\}$. Given any $x \in X$, there is a j such that $x \in B(x_j, \delta)$. Therefore, for all n, we have $x \in f_n^{-1}\big(B(f_n(x_j),1)\big)$, or in other words

$$|f_n(x) - f_n(x_j)| < 1.$$

By reverse triangle inequality,

$$|f_n(x)| < 1 + |f_n(x_j)| \leq 1 + M_j \leq M.$$

*Named after the Italian mathematicians Cesare Arzelà (1847–1912), and Giulio Ascoli (1843–1896).

And as x was arbitrary, $\{f_n\}$ is uniformly bounded.

Next, pick a countable dense subset $D \subset X$. By Proposition 11.6.5, we find a subsequence $\{f_{n_j}\}$ that converges pointwise on D. Write $g_j := f_{n_j}$ for simplicity. The sequence $\{g_n\}$ is uniformly equicontinuous. Let $\varepsilon > 0$ be given, then there exists a $\delta > 0$ such that for all $x \in X$ and all $n \in \mathbb{N}$

$$B(x,\delta) \subset g_n^{-1}\bigl(B(g_n(x), \varepsilon/3)\bigr).$$

By density of D and because δ is fixed, every $x \in X$ is in some $B(y,\delta)$ for some $y \in D$. By compactness of X, there is a finite subset $\{x_1, x_2, \ldots, x_k\} \subset D$ such that

$$X = \bigcup_{j=1}^{k} B(x_j, \delta).$$

As there are finitely many points and $\{g_n\}$ converges pointwise on D, there exists a single N such that for all $n,m \geq N$, we have

$$|g_n(x_j) - g_m(x_j)| < \varepsilon/3 \qquad \text{for all } j = 1,2,\ldots,k.$$

Let $x \in X$ be arbitrary. There is some j such that $x \in B(x_j, \delta)$ and so for all $\ell \in \mathbb{N}$,

$$|g_\ell(x) - g_\ell(x_j)| < \varepsilon/3.$$

So for $n,m \geq N$,

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_j)| + |g_n(x_j) - g_m(x_j)| + |g_m(x_j) - g_m(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence, the sequence is uniformly Cauchy. By completeness of $\mathbb{C}$, it is uniformly convergent. □

Corollary 11.6.10. *Let (X,d) be a compact metric space. Let $S \subset C(X,\mathbb{C})$ be a closed, bounded and uniformly equicontinuous set. Then S is compact.*

The theorem says that S is sequentially compact and that means compact in a metric space. Recall that the closed unit ball in $C([0,1],\mathbb{R})$ (and therefore also in $C([0,1],\mathbb{C})$) is not compact. Hence it cannot be a uniformly equicontinuous set.

Corollary 11.6.11. *Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a,b]$, $\{f_n'\}$ is uniformly bounded, and there is an $x_0 \in [a,b]$ such that $\{f_n(x_0)\}$ is bounded. Then there exists a uniformly convergent subsequence $\{f_{n_j}\}$.*

Proof. The trick is to use the mean value theorem. If M is the uniform bound on $\{f_n'\}$, then by the mean value theorem for every n

$$|f_n(x) - f_n(y)| \leq M|x-y| \qquad \text{for all } x,y \in X.$$

All the f_n are Lipschitz with the same constant and hence the sequence is uniformly equicontinuous.

Suppose $|f_n(x_0)| \leq M_0$ for all n. For all $x \in [a,b]$,

$$|f_n(x)| \leq |f_n(x_0)| + |f_n(x) - f_n(x_0)| \leq M_0 + M|x-x_0| \leq M_0 + M(b-a).$$

So $\{f_n\}$ is uniformly bounded. We apply Arzelà–Ascoli to find the subsequence. □

A classic application of the corollary above to Arzelà–Ascoli in the theory of differential equations is to prove the Peano existence theorem, that is, the existence of solutions to ordinary differential equations. See Exercise 11.6.11 below.

Another application of Arzelà–Ascoli using the same idea as the corollary above is the following. Take a continuous $k \colon [0,1] \times [0,1] \to \mathbb{C}$. For every $f \in C([0,1],\mathbb{C})$ define

$$T\big(f\big)(x) := \int_0^1 f(t)\, k(x,t)\, dt.$$

In exercises to earlier sections you have shown that T is a linear operator on $C([0,1],\mathbb{C})$. Via Arzelà–Ascoli, we also find (exercise) that the image of the unit ball of functions

$$T\big(B(0,1)\big) = \big\{ Tf \in C([0,1],\mathbb{C}) : \|f\|_u < 1 \big\}$$

has compact closure, usually called *relatively compact*. Such an operator is called a *compact operator*. And they are very useful. Generally operators defined by integration tend to be compact.

11.6.1 Exercises

***Exercise* 11.6.1:** *Let $f_n \colon [-1,1] \to \mathbb{R}$ be given by $f_n(x) := \frac{nx}{1+(nx)^2}$. Prove that the sequence is uniformly bounded, converges pointwise to 0, yet there is no subsequence that converges uniformly. Which hypothesis of Arzelà–Ascoli is not satisfied? Prove your assertion.*

***Exercise* 11.6.2:** *Define $f_n \colon \mathbb{R} \to \mathbb{R}$ by $f_n(x) := \frac{1}{(x-n)^2+1}$. Prove that this sequence is uniformly bounded, uniformly equicontinuous, the sequence converges pointwise to zero, yet there is no subsequence that converges uniformly. Which hypothesis of Arzelà–Ascoli is not satisfied? Prove your assertion.*

***Exercise* 11.6.3:** *Let (X,d) be a compact metric space, $C > 0$, $0 < \alpha \leq 1$, and suppose $f_n \colon X \to \mathbb{C}$ are functions such as $|f_n(x) - f_n(y)| \leq C d(x,y)^\alpha$ for all $x, y \in X$ and $n \in \mathbb{N}$. Suppose also that there is a point $p \in X$ such that $f_n(p) = 0$ for all n. Show that there exists a uniformly convergent subsequence converging to an $f \colon X \to \mathbb{C}$ that also satisfies $f(p) = 0$ and $|f(x) - f(y)| \leq C d(x,y)^\alpha$.*

***Exercise* 11.6.4:** *Let $T \colon C([0,1],\mathbb{C}) \to C([0,1],\mathbb{C})$ be the operator given by*

$$T\big(f\big)(x) := \int_0^x f(t)\, dt.$$

(That T is linear and that Tf is continuous follows from linearity of the integral and the fundamental theorem of calculus.)

a) Show that T takes the unit ball centered at 0 in $C([0,1],\mathbb{C})$ into a relatively compact set (a set with compact closure). That is, T is a compact operator.
Hint: See Exercise 7.4.20 in volume I.

b) Let $C \subset C([0,1],\mathbb{C})$ the closed unit ball, prove that the image $T(C)$ is not closed (though it is relatively compact).

***Exercise* 11.6.5:** *Given $k \in C([0,1] \times [0,1],\mathbb{C})$, let $T \colon C([0,1],\mathbb{C}) \to C([0,1],\mathbb{C})$ be the operator defined by*

$$T\big(f\big)(x) := \int_0^1 f(t)\, k(x,t)\, dt.$$

Show that T takes the unit ball centered at 0 in $C([0,1],\mathbb{C})$ into a relatively compact set (a set with compact closure). That is, T is a compact operator.
Hint: See Exercise 7.4.20 in volume I.
Note: That T is a well-defined linear operator was proved in Exercise 8.1.6.

***Exercise* 11.6.6:** *Suppose $S^1 \subset \mathbb{C}$ is the unit circle, that is the set where $|z| = 1$. Suppose the continuous functions $f_n : S^1 \to \mathbb{C}$ are uniformly bounded. Let $\gamma : [0,1] \to S^1$ be a parametrization of S^1, and $g(z,w)$ a continuous function on $C(0,1) \times S^1$ (here $C(0,1) \subset \mathbb{C}$ is the closed unit ball). Define the functions $F_n : C(0,1) \to \mathbb{C}$ by the path integral (see §9.2)*

$$F_n(z) := \int_\gamma f_n(w)\, g(z,w)\, ds(w).$$

Show that $\{F_n\}$ has a uniformly convergent subsequence.

***Exercise* 11.6.7:** *Suppose (X,d) is a compact metric space, $\{f_n\}$ a uniformly equicontinuous sequence of functions in $C(X,\mathbb{C})$. Suppose $\{f_n\}$ converges pointwise. Show that it converges uniformly.*

***Exercise* 11.6.8:** *Suppose that $\{f_n\}$ is a uniformly equicontinuous uniformly bounded sequence of 2π-periodic functions $f_n : \mathbb{R} \to \mathbb{R}$. Show that there is a uniformly convergent subsequence.*

***Exercise* 11.6.9:** *Show that for a compact metric space X, a sequence $\{f_n\}$ that is equicontinuous at every $x \in X$ is uniformly equicontinuous.*

***Exercise* 11.6.10:** *Define $f_n : [0,1] \to \mathbb{C}$ by $f_n(t) := e^{i(2\pi t + n)}$. This is a uniformly equicontinuous uniformly bounded sequence. Prove more than just the conclusion of Arzelà–Ascoli for this sequence. Let $\gamma \in \mathbb{R}$ be given, and define $g(t) := e^{i(2\pi t + \gamma)}$. Show that there exists a subsequence of $\{f_n\}$ converging uniformly to g. Hint: Feel free to use the* Kronecker density theorem[*]*: The sequence $\{e^{in}\}_{n=1}^\infty$ is dense in the unit circle.*

***Exercise* 11.6.11:** *Prove the* Peano existence theorem *(note the weaker hypotheses than Picard, but also the lack of uniqueness in this theorem):*

Theorem: Suppose $F : I \times J \to \mathbb{R}$ is a continuous function where $I, J \subset \mathbb{R}$ are closed bounded intervals, let I° and J° be their interiors, and let $(x_0, y_0) \in I^\circ \times J^\circ$. Then there exists an $h > 0$ and a differentiable function $f : [x_0 - h, x_0 + h] \to J \subset \mathbb{R}$, such that

$$f'(x) = F\big(x, f(x)\big) \qquad \textit{and} \qquad f(x_0) = y_0.$$

Use the following outline:

a) *We wish to define the Picard iterates, that is, set $f_0(x) := y_0$, and*

$$f_{n+1}(x) := y_0 + \int_{x_0}^x F\big(t, f_n(t)\big)\, dt.$$

Prove that there exists an $h > 0$ such that $f_n : [x_0 - h, x_0 + h] \to \mathbb{C}$ is well-defined for all n. Hint: F is bounded (why?).

b) *Show that $\{f_n\}$ is equicontinuous and bounded, in fact it is Lipschitz with a uniform Lipschitz constant. Arzelà–Ascoli then says that there exists a uniformly convergent subsequence $\{f_{n_k}\}$.*

c) *Prove $\big\{F\big(x, f_{n_k}(x)\big)\big\}_{k=1}^\infty$ converges uniformly on $[x_0 - h, x_0 + h]$. Hint: F is uniformly continuous (why?).*

d) *Finish the proof of the theorem by taking the limit under the integral and applying the fundamental theorem of calculus.*

[*]Named after the German mathematician Leopold Kronecker (1823–1891).

11.7 The Stone–Weierstrass theorem

Note: 3 lectures

11.7.1 Weierstrass approximation

Perhaps surprisingly, even a very badly behaved continuous function is a uniform limit of polynomials. And we cannot really get any "nicer" functions than polynomials. The idea of the proof is a very common approximation or "smoothing" idea (convolution with an approximate delta function) that has applications far beyond pure mathematics.

Theorem 11.7.1 (Weierstrass approximation theorem)**.** *If $f\colon [a,b] \to \mathbb{C}$ is continuous, then there exists a sequence $\{p_n\}$ of polynomials converging to f uniformly on $[a,b]$. Furthermore, if f is real-valued, we can find p_n with real coefficients.*

Proof. For $x \in [0,1]$ define

$$g(x) := f\bigl((b-a)x+a\bigr) - f(a) - x\bigl(f(b)-f(a)\bigr).$$

If we prove the theorem for g and find the sequence $\{p_n\}$ for g, it is proved for f as we simply composed with an invertible affine function and added an affine function to f: We reverse the process and apply that to our p_n, to obtain polynomials approximating f.

The function g is defined on $[0,1]$ and $g(0) = g(1) = 0$. For simplicity, assume that g is defined on the whole real line by letting $g(x) := 0$ if $x < 0$ or $x > 1$. This extended g is continuous.

Define

$$c_n := \left(\int_{-1}^{1} (1-x^2)^n \, dx \right)^{-1}, \qquad q_n(x) := c_n (1-x^2)^n.$$

The choice of c_n is so that $\int_{-1}^{1} q_n(x)\,dx = 1$. See Figure 11.8.

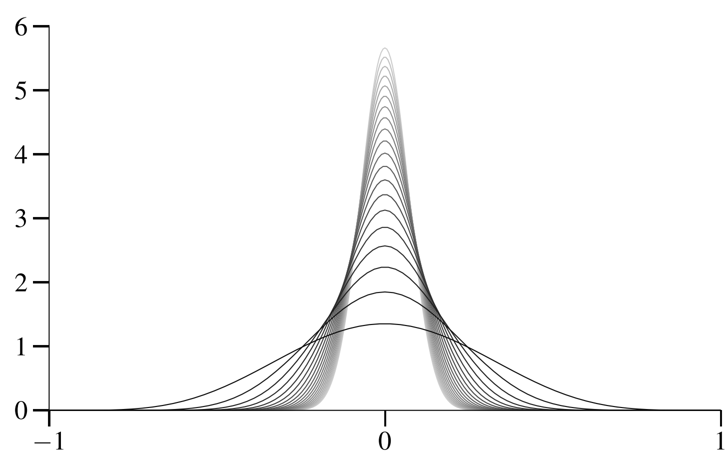

Figure 11.8: Plot of the approximate delta functions q_n on $[-1,1]$ for $n = 5, 10, 15, 20, \ldots, 100$ with higher n in lighter shade.

The functions q_n are peaks around 0 (ignoring what happens outside of $[-1,1]$) that get narrower and taller as n increases, while the area underneath is always 1. A classic approximation idea is to do a *convolution* integral with peaks like this: For for $x \in [0,1]$, let

$$p_n(x) := \int_0^1 g(t) q_n(x-t)\,dt \qquad \left(= \int_{-\infty}^{\infty} g(t) q_n(x-t)\,dt \right).$$

The idea of this convolution is that we do a “weighted average” of the function g around the point x using q_n as the weight. See Figure 11.9.

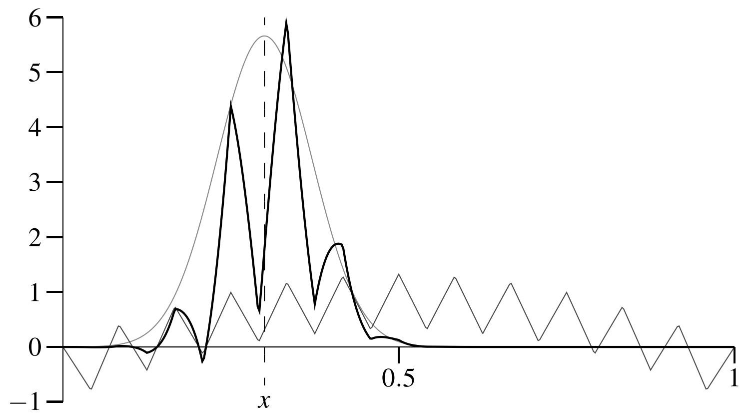

Figure 11.9: For $x = 0.3$, the plot of $q_{100}(x-t)$ (light gray peak centered at x), some continuous function $g(t)$ (the jagged line) and the product $g(t)q_{100}(x-t)$ (the bold line).

As q_n is a narrow peak, the integral mostly sees the values of g that are close to x and it does the weighted average of them. When the peak gets narrower, we compute this average closer to x and we expect the result to get closer to the value of $g(x)$. Really we are approximating what is called a delta function* (don’t worry if you have not heard of this concept), and functions like q_n are often called *approximate delta functions*. We could do this with any set of polynomials that look like narrower and narrower peaks near zero. These just happen to be the simplest ones. We only need this behavior on $[-1,1]$ as the convolution sees nothing further than this as g is zero outside $[0,1]$.

Because q_n is a polynomial we write

$$q_n(x-t) = a_0(t) + a_1(t)\,x + \cdots + a_{2n}(t)\,x^{2n},$$

where $a_k(t)$ are polynomials in t, in particular continuous and hence integrable functions. So

$$\begin{aligned} p_n(x) &= \int_0^1 g(t) q_n(x-t)\,dt \\ &= \left(\int_0^1 g(t) a_0(t)\,dt \right) + \left(\int_0^1 g(t) a_1(t)\,dt \right) x + \cdots + \left(\int_0^1 g(t) a_{2n}(t)\,dt \right) x^{2n}. \end{aligned}$$

In other words, p_n is a polynomial† in x. If $g(t)$ is real-valued, then the functions $g(t)a_j(t)$ are real-valued and hence p_n has real coefficients, proving the “furthermore” part of the theorem.

*The delta function is not actually a function, it is a “thing” that supposed to give “$\int_{-\infty}^{\infty} g(x)\delta(x-t)\,dt = g(x)$.”

†Do note that the functions a_j depend on n, so the coefficients of p_n change as n changes.

We still need to prove that $\{p_n\}$ converges to g. First let us get some handle on the size of c_n. For $x \in [0,1]$, we have that $1-x \leq 1-x^2$. We estimate

$$\begin{aligned} c_n^{-1} = \int_{-1}^{1} (1-x^2)^n \, dx = 2 \int_0^1 (1-x^2)^n \, dx \\ \geq 2 \int_0^1 (1-x)^n \, dx = \frac{2}{n+1}. \end{aligned}$$

So $c_n \leq \frac{n+1}{2} \leq n$.

Let us see how small q_n is, if we ignore some small interval around the origin, which is where the peak is. Given any $\delta > 0$, $\delta < 1$, for x such that $\delta \leq |x| \leq 1$, we have

$$q_n(x) \leq c_n (1-\delta^2)^n \leq n(1-\delta^2)^n,$$

because q_n is increasing on $[-1,0]$ and decreasing on $[0,1]$. By the ratio test, $n(1-\delta^2)^n$ goes to 0 as n goes to infinity.

The function q_n is even, $q_n(t) = q_n(-t)$, and g is zero outside of $[0,1]$. So for $x \in [0,1]$,

$$p_n(x) = \int_0^1 g(t) q_n(x-t) \, dt = \int_{-x}^{1-x} g(x+t) q_n(-t) \, dt = \int_{-1}^{1} g(x+t) q_n(t) \, dt.$$

Let $\varepsilon > 0$ be given. As $[-1,2]$ is compact and g is continuous on $[-1,2]$, we have that g is uniformly continuous. Pick $0 < \delta < 1$ such that if $|x-y| < \delta$ (and $x, y \in [-1,2]$), then

$$|g(x) - g(y)| < \frac{\varepsilon}{2}.$$

Let M be such that $|g(x)| \leq M$ for all x. Let N be such that for all $n \geq N$,

$$4Mn(1-\delta^2)^n < \frac{\varepsilon}{2}.$$

Note that $\int_{-1}^{1} q_n(t) \, dt = 1$ and $q_n(t) \geq 0$ on $[-1,1]$. So for $n \geq N$ and every $x \in [0,1]$,

$$\begin{aligned} |p_n(x) - g(x)| &= \left| \int_{-1}^{1} g(x+t) q_n(t) \, dt - g(x) \int_{-1}^{1} q_n(t) \, dt \right| \\ &= \left| \int_{-1}^{1} \big(g(x+t) - g(x)\big) q_n(t) \, dt \right| \\ &\leq \int_{-1}^{1} |g(x+t) - g(x)| q_n(t) \, dt \\ &= \int_{-1}^{-\delta} |g(x+t) - g(x)| q_n(t) \, dt \quad + \int_{-\delta}^{\delta} |g(x+t) - g(x)| q_n(t) \, dt \\ &\quad + \int_{\delta}^{1} |g(x+t) - g(x)| q_n(t) \, dt \\ &\leq 2M \int_{-1}^{-\delta} q_n(t) \, dt \quad + \quad \frac{\varepsilon}{2} \int_{-\delta}^{\delta} q_n(t) \, dt \quad + \quad 2M \int_{\delta}^{1} q_n(t) \, dt \\ &\leq 2Mn(1-\delta^2)^n (1-\delta) \quad + \quad \frac{\varepsilon}{2} \quad + \quad 2Mn(1-\delta^2)^n (1-\delta) \\ &< 4Mn(1-\delta^2)^n + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

□

A convolution often inherits some property of the functions we are convolving. In our case the convolution p_n inherited the property of being a polynomial from q_n. The same idea of the proof is often used to get other properties. If q_n or g is infinitely differentiable, so is p_n. If q_n or g is a solution to a linear differential equation, so is p_n. Etc.

Let us note an immediate application of the Weierstrass theorem. We have already seen that countable dense subsets can be very useful.

Corollary 11.7.2. *The metric space $C([a,b],\mathbb{C})$ contains a countable dense subset.*

Proof. Without loss of generality suppose that we are dealing with $C([a,b],\mathbb{R})$ (why?). The real polynomials are dense in $C([a,b],\mathbb{R})$ by Weierstrass. If we show that every real polynomial can be approximated by polynomials with rational coefficients, we are done. This is because there are only countably many rational numbers and so there are only countably many polynomials with rational coefficients (a countable union of countable sets is still countable).

Further without loss of generality, suppose $[a,b] = [0,1]$. Let

$$p(x) := \sum_{k=0}^{n} a_k x^k$$

be a polynomial of degree n where $a_k \in \mathbb{R}$. Given $\varepsilon > 0$, pick $b_k \in \mathbb{Q}$ such that $|a_k - b_k| < \frac{\varepsilon}{n+1}$. Then if we let

$$q(x) := \sum_{k=0}^{n} b_k x^k,$$

we have

$$|p(x) - q(x)| = \left| \sum_{k=0}^{n} (a_k - b_k) x^k \right| \leq \sum_{k=0}^{n} |a_k - b_k| x^k \leq \sum_{k=0}^{n} |a_k - b_k| < \sum_{k=0}^{n} \frac{\varepsilon}{n+1} = \varepsilon. \qquad \square$$

Remark 11.7.3. While we will not prove this, the corollary above implies that $C([a,b],\mathbb{C})$ has the same cardinality as $\mathbb{R}$, which may be a bit surprising. The set of all functions $[a,b] \to \mathbb{C}$ has cardinality that is strictly greater than the cardinality of $\mathbb{R}$, it has the cardinality of the power set of $\mathbb{R}$. So the set of continuous functions is a very tiny subset of the set of all functions.

Warning! The fact that every continuous function $f\colon [-1,1] \to \mathbb{C}$ (or any interval $[a,b]$) can be uniformly approximated by polynomials

$$\sum_{k=0}^{n} a_k x^k$$

does not mean that every continuous f is analytic, that is, equal to a power series

$$\sum_{k=0}^{\infty} c_k x^k.$$

An analytic function is infinitely differentiable, so the function $|x|$ provides a counterexample.

The key distinction is that the polynomials coming from the Weierstrass theorem are not the partial sums of a power series. For each one, the coefficients a_k above can be completely different—they do not need to come from a single sequence $\{c_k\}$.

11.7.2 Stone–Weierstrass approximation

We want to abstract away what is not really necessary and prove a general version of the Weierstrass theorem. The polynomials are dense in the space of continuous functions on a compact interval. What other kind of families of functions are also dense? And if the domain is an arbitrary metric space, then we no longer have polynomials to begin with.

The theorem we will prove is the Stone–Weierstrass theorem*. First, we need a very special case of the Weierstrass theorem though.

Corollary 11.7.4. *Let $[-a,a]$ be an interval. Then there is a sequence of real polynomials $\{p_n\}$ that converges uniformly to $|x|$ on $[-a,a]$ and such that $p_n(0) = 0$ for all n.*

Proof. As $f(x) := |x|$ is continuous and real-valued on $[-a,a]$, the Weierstrass theorem gives a sequence of real polynomials $\{\widetilde{p}_n\}$ that converges to f uniformly on $[-a,a]$. Let

$$p_n(x) := \widetilde{p}_n(x) - \widetilde{p}_n(0).$$

Obviously $p_n(0) = 0$.

Given $\varepsilon > 0$, let N be such that for $n \geq N$, we have $\bigl|\widetilde{p}_n(x) - |x|\bigr| < \varepsilon/2$ for all $x \in [-a,a]$. In particular, $|\widetilde{p}_n(0)| < \varepsilon/2$. Then for $n \geq N$,

$$\bigl|p_n(x) - |x|\bigr| = \bigl|\widetilde{p}_n(x) - \widetilde{p}_n(0) - |x|\bigr| \leq \bigl|\widetilde{p}_n(x) - |x|\bigr| + |\widetilde{p}_n(0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \square$$

Generalizing the corollary, we can always make the polynomials from the Weierstrass theorem be equal to our target function at one point, not just for $|x|$, but that's the one we will need.

Definition 11.7.5. A set $\mathscr{A}$ of complex-valued functions $f\colon X \to \mathbb{C}$ is said to be an *algebra* (sometimes *complex algebra* or *algebra over* $\mathbb{C}$) if for all $f, g \in \mathscr{A}$ and $c \in \mathbb{C}$, we have

(i) $f + g \in \mathscr{A}$.

(ii) $fg \in \mathscr{A}$.

(iii) $cg \in \mathscr{A}$.

A *real algebra* or an *algebra over* $\mathbb{R}$ is a set of real-valued functions that satisfies the three properties above for $c \in \mathbb{R}$.

We are interested in the case when X is a compact metric space. Then $C(X,\mathbb{C})$ and $C(X,\mathbb{R})$ are metric spaces. Given a set $\mathscr{A} \subset C(X,\mathbb{C})$, the set of all uniform limits is the metric space closure $\overline{\mathscr{A}}$. When we talk about closure of an algebra from now on we mean the closure in $C(X,\mathbb{C})$ as a metric space. Same for $C(X,\mathbb{R})$.

The set $\mathscr{P}$ of all polynomials is an algebra in $C([a,b],\mathbb{C})$, and we have shown that its closure $\overline{\mathscr{P}} = C([a,b],\mathbb{C})$. That is, it is dense. That is the sort of result that we wish to prove.

We leave the following proposition as an exercise.

Proposition 11.7.6. *Suppose X is a compact metric space. If $\mathscr{A} \subset C(X,\mathbb{C})$ is an algebra, then the closure $\overline{\mathscr{A}}$ is also an algebra. Similarly for a real algebra in $C(X,\mathbb{R})$.*

*Named after the American mathematician Marshall Harvey Stone (1903–1989), and the German mathematician Karl Theodor Wilhelm Weierstrass (1815–1897).

Let us distill the properties of polynomials that are sufficient for an approximation theorem.

Definition 11.7.7. Let $\mathscr{A}$ be a set of complex-valued functions defined on a set X.

(i) $\mathscr{A}$ *separates points* if for every $x, y \in X$, with $x \neq y$ there is a function $f \in \mathscr{A}$ such that $f(x) \neq f(y)$.

(ii) $\mathscr{A}$ *vanishes at no point* if for every $x \in X$ there is an $f \in \mathscr{A}$ such that $f(x) \neq 0$.

Example 11.7.8: The set $\mathscr{P}$ of polynomials separates points and vanishes at no point on $\mathbb{R}$. That is, $1 \in \mathscr{P}$ so it vanishes at no point. And for $x, y \in \mathbb{R}$, $x \neq y$, take $f(t) := t$. Then $f(x) = x \neq y = f(y)$. So $\mathscr{P}$ separates points.

Example 11.7.9: The set of functions of the form

$$f(t) = a_0 + \sum_{n=1}^{k} a_n \cos(nt)$$

is an algebra, which follows by the identity $\cos(mt)\cos(nt) = \frac{\cos((n+m)t)}{2} + \frac{\cos((n-m)t)}{2}$. The algebra does not separate points if the domain is an interval of the form $[-a, a]$, because $f(-t) = f(t)$ for all t. It does separate points if the domain is $[0, \pi]$, as $\cos(t)$ is one-to-one on that set.

Example 11.7.10: The set of polynomials with no constant term vanishes at the origin.

Proposition 11.7.11. *Suppose $\mathscr{A}$ is an algebra of complex-valued functions on a set X, that separates points and vanishes at no point. Suppose x, y are distinct points of X, and $c, d \in \mathbb{C}$. Then there is an $f \in \mathscr{A}$ such that*

$$f(x) = c, \qquad f(y) = d.$$

If $\mathscr{A}$ is a real algebra, the conclusion holds for $c, d \in \mathbb{R}$.

Proof. There must exist an $g, h, k \in \mathscr{A}$ such that

$$g(x) \neq g(y), \quad h(x) \neq 0, \quad k(y) \neq 0.$$

Let

$$f := c\frac{\big(g - g(y)\big)h}{\big(g(x) - g(y)\big)h(x)} + d\frac{\big(g - g(x)\big)k}{\big(g(y) - g(x)\big)k(y)} = c\frac{gh - g(y)h}{g(x)h(x) - g(y)h(x)} + d\frac{gk - g(x)k}{g(y)k(y) - g(x)k(y)}.$$

Do note that we are not dividing by zero (clear from the first formula). Also from the first formula we see that $f(x) = c$ and $f(y) = d$. By the second formula we see that $f \in \mathscr{A}$ (as $\mathscr{A}$ is an algebra). $\square$

Theorem 11.7.12 (Stone–Weierstrass, real version)**.** *Let X be a compact metric space and $\mathscr{A}$ an algebra of real-valued continuous functions on X, such that $\mathscr{A}$ separates points and vanishes at no point. Then the closure $\overline{\mathscr{A}} = C(X, \mathbb{R})$.*

The proof is divided into several claims.

Claim 1: *If $f \in \overline{\mathscr{A}}$, then $|f| \in \overline{\mathscr{A}}$.*

Proof. The function f is bounded (continuous on a compact set), so there is an M such that $|f(x)| \leq M$ for all $x \in X$.

Let $\varepsilon > 0$ be given. By the corollary to the Weierstrass theorem there exists a real polynomial $c_1 y + c_2 y^2 + \cdots + c_n y^n$ (vanishing at $y = 0$) such that

$$\left| |y| - \sum_{j=1}^{N} c_j y^j \right| < \varepsilon$$

for all $y \in [-M, M]$. Because $\overline{\mathscr{A}}$ is an algebra and because there is no constant term in the polynomial,

$$\sum_{j=1}^{N} c_j f^j \in \overline{\mathscr{A}}.$$

As $|f(x)| \leq M$, then for all $x \in X$

$$\left| |f(x)| - \sum_{j=1}^{N} c_j \big(f(x)\big)^j \right| < \varepsilon.$$

So $|f|$ is in the closure of $\overline{\mathscr{A}}$, which is closed. In other words, $|f| \in \overline{\mathscr{A}}$. □

Claim 2: *If $f \in \overline{\mathscr{A}}$ and $g \in \overline{\mathscr{A}}$, then $\max(f,g) \in \overline{\mathscr{A}}$ and $\min(f,g) \in \overline{\mathscr{A}}$, where*

$$\big(\max(f,g)\big)(x) := \max\big\{f(x), g(x)\big\}, \qquad \text{and} \qquad \big(\min(f,g)\big)(x) := \min\big\{f(x), g(x)\big\}.$$

Proof. Write:

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$

and

$$\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

As $\overline{\mathscr{A}}$ is an algebra we are done. □

The claim is true for the minimum or maximum of every finite collection of functions as well.

Claim 3: *Given $f \in C(X, \mathbb{R})$, $x \in X$ and $\varepsilon > 0$ there exists a $g_x \in \overline{\mathscr{A}}$ with $g_x(x) = f(x)$ and*

$$g_x(t) > f(t) - \varepsilon \qquad \text{for all } t \in X.$$

Proof. Fix f, x, and ε. By Proposition 11.7.11, for every $y \in X$ we find an $h_y \in \mathscr{A}$ such that

$$h_y(x) = f(x), \qquad h_y(y) = f(y).$$

As h_y and f are continuous, the function $h_y - f$ is continuous, and the set

$$U_y := \big\{t \in X : h_y(t) > f(t) - \varepsilon\big\} = (h_y - f)^{-1}\big((-\varepsilon, \infty)\big)$$

is open (it is the inverse image of an open set by a continuous function). Furthermore $y \in U_y$. So the sets U_y cover X.

The space X is compact so there exist finitely many points $y_1, y_2, \ldots, y_n$ in X such that

$$X = \bigcup_{j=1}^{n} U_{y_j}.$$

Let

$$g_x := \max(h_{y_1}, h_{y_2}, \ldots, h_{y_n}).$$

By Claim 2, $g_x \in \overline{\mathscr{A}}$. Furthermore,

$$g_x(t) > f(t) - \varepsilon$$

for all $t \in X$, since for every t, there is a y_j such that $t \in U_{y_j}$, and so $h_{y_j}(t) > f(t) - \varepsilon$.

Finally $h_y(x) = f(x)$ for all $y \in X$, so $g_x(x) = f(x)$. □

Claim 4: *If $f \in C(X,\mathbb{R})$ and $\varepsilon > 0$ is given, then there exists an $\varphi \in \overline{\mathscr{A}}$ such that*

$$|f(x) - \varphi(x)| < \varepsilon.$$

Proof. For every x find the function g_x as in Claim 3.

Let

$$V_x := \{ t \in X : g_x(t) < f(t) + \varepsilon \}.$$

The sets V_x are open as g_x and f are continuous. As $g_x(x) = f(x)$, then $x \in V_x$. So the sets V_x cover X. By compactness of X, there are finitely many points $x_1, x_2, \ldots, x_k$ such that

$$X = \bigcup_{j=1}^{k} V_{x_j}.$$

Let

$$\varphi := \min(g_{x_1}, g_{x_2}, \ldots, g_{x_k}).$$

By Claim 2, $\varphi \in \overline{\mathscr{A}}$. Similarly as before (same argument as in Claim 3), for all $t \in X$,

$$\varphi(t) < f(t) + \varepsilon.$$

Since all the g_x satisfy $g_x(t) > f(t) - \varepsilon$ for all $t \in X$, $\varphi(t) > f(t) - \varepsilon$ as well. Therefore, for all t,

$$-\varepsilon < \varphi(t) - f(t) < \varepsilon,$$

which is the desired conclusion. □

The proof of the theorem follows from Claim 4. The claim states that an arbitrary continuous function is in the closure of $\overline{\mathscr{A}}$, which itself is closed. So the theorem is proved.

Example 11.7.13: The functions of the form

$$f(t) = \sum_{j=1}^{n} c_j e^{jt},$$

for $c_j \in \mathbb{R}$, are dense in $C([a,b],\mathbb{R})$. We need to note that such functions are a real algebra, which follows from $e^{jt} e^{kt} = e^{(j+k)t}$. They separate points as e^t is one-to-one, and $e^t > 0$ for all t so the algebra does not vanish at any point.

In general if we have a set of functions that separates points and does not vanish at any point, we can let these functions *generate* an algebra by considering all the linear combinations of arbitrary multiples of such functions. That is, we consider all real polynomials without constant term of such functions. In the example above, the algebra is generated by e^t. We consider polynomials in e^t without constant term.

Example 11.7.14: We mentioned that the set of all functions of the form

$$a_0 + \sum_{n=1}^{N} a_n \cos(nt)$$

is an algebra. When considered on $[0,\pi]$, it separates points and vanishes nowhere so Stone–Weierstrass applies. As for polynomials, you *do not* want to conclude that every continuous function on $[0,\pi]$ has a uniformly convergent Fourier cosine series, that is, that every continuous function can be written as

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nt).$$

That is *not true*! There exist continuous functions whose Fourier series does not converge even pointwise let alone uniformly.

To obtain Stone–Weierstrass for complex algebras, we must make an extra assumption.

Definition 11.7.15. An algebra $\mathscr{A}$ is *self-adjoint* if for all $f \in \mathscr{A}$, the function $\bar{f}$ defined by $\bar{f}(x) := \overline{f(x)}$ is in $\mathscr{A}$, where by the bar we mean the complex conjugate.

Theorem 11.7.16 (Stone–Weierstrass, complex version)**.** *Let X be a compact metric space and $\mathscr{A}$ an algebra of complex-valued continuous functions on X, such that $\mathscr{A}$ separates points, vanishes at no point, and is self-adjoint. Then the closure $\overline{\mathscr{A}} = C(X,\mathbb{C})$.*

Proof. Suppose $\mathscr{A}_{\mathbb{R}} \subset \mathscr{A}$ is the set of the real-valued elements of $\mathscr{A}$. For $f \in \mathscr{A}$, write $f = u + iv$ where u and v are real-valued. Then

$$u = \frac{f+\bar{f}}{2}, \qquad v = \frac{f-\bar{f}}{2i}.$$

So $u,v \in \mathscr{A}$ as $\mathscr{A}$ is a self-adjoint algebra, and since they are real-valued $u,v \in \mathscr{A}_{\mathbb{R}}$.

If $x \neq y$, then find an $f \in \mathscr{A}$ such that $f(x) \neq f(y)$. If $f = u+iv$, then it is obvious that either $u(x) \neq u(y)$ or $v(x) \neq v(y)$. So $\mathscr{A}_{\mathbb{R}}$ separates points.

Similarly, for every x find $f \in \mathscr{A}$ such that $f(x) \neq 0$. If $f = u+iv$, then either $u(x) \neq 0$ or $v(x) \neq 0$. So $\mathscr{A}_{\mathbb{R}}$ vanishes at no point.

The set $\mathscr{A}_{\mathbb{R}}$ is a real algebra, and satisfies the hypotheses of the real Stone–Weierstrass theorem. Given any $f = u+iv \in C(X,\mathbb{C})$, we find $g,h \in \mathscr{A}_{\mathbb{R}}$ such that $|u(t)-g(t)| < \varepsilon/2$ and $|v(t)-h(t)| < \varepsilon/2$ for all $t \in X$. Next, $g+ih \in \mathscr{A}$, and

$$\begin{aligned} \bigl|f(t) - \bigl(g(t)+ih(t)\bigr)\bigr| &= \bigl|u(t)+iv(t) - \bigl(g(t)+ih(t)\bigr)\bigr| \\ &\leq |u(t)-g(t)| + |v(t)-h(t)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $t \in X$. So $\overline{\mathscr{A}} = C(X,\mathbb{C})$. $\square$

The self-adjoint requirement is necessary although it is not so obvious to see it. For an example see Exercise 11.7.9.

Here is an interesting application. When working with functions of two variables, it may be useful to work with functions of the form $f(x)g(y)$ rather than $F(x,y)$. For example, they are easier to integrate. We have the following.

Example 11.7.17: Any continuous function $F\colon [0,1]\times[0,1]\to\mathbb{C}$ can be approximated uniformly by functions of the form

$$\sum_{j=1}^{n} f_j(x)g_j(y),$$

where $f_j\colon [0,1]\to\mathbb{C}$ and $g_j\colon [0,1]\to\mathbb{C}$ are continuous.

Proof: It is not hard to see that the functions of the above form are a complex algebra. It is equally easy to show that they vanish nowhere, separate points, and the algebra is self-adjoint. As $[0,1]\times[0,1]$ is compact we apply Stone–Weierstrass to obtain the result.

11.7.3 Exercises

***Exercise* 11.7.1:** *Prove Proposition 11.7.6. Hint: If $\{f_n\}$ is a sequence in $C(X,\mathbb{R})$ converging to f, then as f is bounded, you can show that f_n is uniformly bounded, that is, there exists a single bound for all f_n (and f).*

***Exercise* 11.7.2:** *Suppose $X := \mathbb{R}$ (not compact in particular). Show that $f(t) := e^t$ is not possible to uniformly approximate by polynomials on X. Hint: Consider $\left|\frac{e^t}{t^n}\right|$ as $t\to\infty$.*

***Exercise* 11.7.3:** *Suppose $f\colon [0,1]\to\mathbb{C}$ is a uniform limit of a sequence of polynomials of degree at most d, then the limit is a polynomial of degree at most d. Conclude that to approximate a function which is not a polynomial, we need the degree of the approximations to go to infinity.*
Hint: First prove that if a sequence of polynomials of degree d converges uniformly to the zero function, then the coefficients converge to zero. One way to do this is linear algebra: Consider a polynomial p evaluated at $d+1$ points to be a linear operator taking the coefficients of p to the values of p (an operator in $L(\mathbb{R}^{d+1})$).

***Exercise* 11.7.4:** *Suppose $f\colon [0,1]\to\mathbb{R}$ is continuous and $\int_0^1 f(x)x^n\,dx = 0$ for all $n=0,1,2,\ldots$. Show that $f(x)=0$ for all $x\in[0,1]$. Hint: Approximate by polynomials to show that $\int_0^1 \bigl(f(x)\bigr)^2\,dx = 0$.*

***Exercise* 11.7.5:** *Suppose $I\colon C([0,1],\mathbb{R})\to\mathbb{R}$ is a linear continuous function such that $I(x^n)=\frac{1}{n+1}$ for all $n=0,1,2,3,\ldots$. Prove that $I(f)=\int_0^1 f$ for all $f\in C([0,1],\mathbb{R})$.*

***Exercise* 11.7.6:** *Let $\mathscr{A}$ be the collection of real polynomials in x^2, that is polynomials of the form $c_0+c_1x^2+c_2x^4+\cdots+c_dx^{2d}$.*

a) Show that every $f\in C([0,1],\mathbb{R})$ is a uniform limit of polynomials from $\mathscr{A}$.

b) Find an $f\in C([-1,1],\mathbb{R})$ that is not a uniform limit of polynomials from $\mathscr{A}$.

c) Which hypothesis of the real Stone–Weierstrass is not satisfied for the domain $[-1,1]$?

Exercise* 11.7.7*: *Let $|z| = 1$ define the unit circle $S^1 \subset \mathbb{C}$.*

a) *Show that functions of the form*
$$\sum_{k=-n}^{n} c_k z^k$$
are dense in $C(S^1,\mathbb{C})$. Notice the negative powers.

b) *Show that functions of the form*
$$c_0 + \sum_{k=1}^{n} c_k z^k + \sum_{k=1}^{n} c_{-k} \bar{z}^k$$
are dense in $C(S^1,\mathbb{C})$. These are so-called harmonic polynomials, and this approximation leads to, for example, the solution of the steady state heat problem.

Hint: A good way to write the equation for S^1 is $z\bar{z} = 1$.

Exercise* 11.7.8*: *Show that for complex numbers c_j, the set of functions of x on $[-\pi,\pi]$ of the form*
$$\sum_{k=-n}^{n} c_k e^{ikx}$$
satisfies the hypotheses of the complex Stone–Weierstrass theorem and therefore such functions are dense in the $C([-\pi,\pi],\mathbb{C})$.

Exercise* 11.7.9*: *Let $S^1 \subset \mathbb{C}$ be the unit circle, that is the set where $|z| = 1$. Orient this set counterclockwise. Let $\gamma(t) := e^{it}$. For the one-form $f(z)\,dz$ we write*[*]
$$\int_{S^1} f(z)\,dz := \int_0^{2\pi} f(e^{it})\,ie^{it}\,dt.$$

a) *Prove that for all nonnegative integers $k = 0,1,2,3,\ldots$, we have $\int_{S^1} z^k\,dz = 0$.*

b) *Prove that if $P(z) = \sum_{k=0}^{n} c_k z^k$ is a polynomial in z, then $\int_{S^1} P(z)\,dz = 0$.*

c) *Prove $\int_{S^1} \bar{z}\,dz \neq 0$.*

d) *Conclude that polynomials in z (this algebra of functions is not self-adjoint) are not dense in $C(S^1,\mathbb{C})$.*

Exercise* 11.7.10*: *Let (X,d) be a compact metric space and suppose $\mathscr{A} \subset C(X,\mathbb{R})$ is a real algebra that separates points, but such that for some x_0, $f(x_0) = 0$ for all $f \in \mathscr{A}$. Prove that every function $g \in C(X,\mathbb{R})$ such that $g(x_0) = 0$ is a uniform limit of functions from $\mathscr{A}$.*

Exercise* 11.7.11*: *Let (X,d) be a compact metric space and suppose $\mathscr{A} \subset C(X,\mathbb{R})$ is a real algebra. Suppose that for each $y \in X$ the closure $\overline{\mathscr{A}}$ contains the function $\varphi_y(x) := d(y,x)$. Then $\overline{\mathscr{A}} = C(X,\mathbb{R})$.*

Exercise* 11.7.12*:

a) *Suppose $f\colon [a,b] \to \mathbb{C}$ is continuously differentiable. Show that there exists a sequence of polynomials $\{p_n\}$ that converges in the C^1 norm to f, that is $\|f - p_n\|_u + \|f' - p_n'\|_u \to 0$ as $n \to \infty$.*

b) *Suppose $f\colon [a,b] \to \mathbb{C}$ is k times continuously differentiable. Show that there exists a sequence of polynomials $\{p_n\}$ that converges in the C^k norm to f, that is,*
$$\sum_{j=0}^{k} \|f^{(j)} - p_n^{(j)}\|_u \to 0 \qquad \text{as} \qquad n \to \infty.$$

[*]One could also define $dz := dx + i\,dy$ and then extend the path integral from chapter 9 to complex-valued one-forms.

Exercise* 11.7.13*:

a) *Show that an even function* $f: [-1,1] \to \mathbb{R}$ *is a uniform limit of polynomials with even powers only, that is, polynomials of the form* $a_0 + a_1x^2 + a_2x^4 + \cdots + a_kx^{2k}$.

b) *Show that an odd function* $f: [-1,1] \to \mathbb{R}$ *is a uniform limit of polynomials with odd powers only, that is, polynomials of the form* $b_1x + b_2x^3 + b_3x^5 + \cdots + b_kx^{2k-1}$.

11.8 Fourier series

Note: 3–4 lectures

Fourier series* is perhaps the most important (and most difficult to understand) of the series that we cover in this book. We have seen it in a few examples before, but let us start at the beginning.

11.8.1 Trigonometric polynomials

A *trigonometric polynomial* is an expression of the form

$$a_0 + \sum_{n=1}^{N} \bigl(a_n \cos(nx) + b_n \sin(nx)\bigr),$$

or equivalently, thanks to Euler's formula ($e^{i\theta} = \cos(\theta) + i\sin(\theta)$):

$$\sum_{n=-N}^{N} c_n e^{inx}.$$

The second form is usually more convenient. If $z \in \mathbb{C}$ with $|z| = 1$, we write $z = e^{ix}$, and so

$$\sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} c_n z^n.$$

So a trigonometric polynomial is really a rational function of the complex variable z (we are allowing negative powers) evaluated on the unit circle. There is a wonderful connection between power series (actually Laurent series because of the negative powers) and Fourier series because of this observation, but we will not investigate this further.

Another reason why Fourier series are important and come up in so many applications is that the functions are eigenfunctions† of various differential operators. For example,

$$\frac{d}{dx}\bigl[e^{ikx}\bigr] = (ik)e^{ikx}, \qquad \frac{d^2}{dx^2}\bigl[e^{ikx}\bigr] = (-k^2)e^{ikx}.$$

That is, they are the functions whose derivative is a scalar (the eigenvalue) times itself. Just as eigenvalues and eigenvectors are important in studying matrices, eigenvalues and eigenfunctions are important when studying linear differential equations.

The functions $\cos(nx)$, $\sin(nx)$, and e^{inx} are 2π-periodic and hence trigonometric polynomials are also 2π-periodic. We could rescale x to make the period different, but the theory is the same, so let us stick with the period of 2π. The antiderivative of e^{inx} is $\frac{e^{inx}}{in}$ and so

$$\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 2\pi & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Named after the French mathematician Jean-Baptiste Joseph Fourier (1768–1830).

†Eigenfunction is like an eigenvector for a matrix, but for a linear operator on a vector space of functions.

Consider

$$f(x) := \sum_{n=-N}^{N} c_n e^{inx},$$

and for $m = -N, \dots, N$ compute

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=-N}^{N} c_n e^{i(n-m)x} \right) dx = \sum_{n=-N}^{N} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = c_m.$$

We just found a way of computing the coefficients c_m using an integral of f. If $|m| > N$, the integral is just 0: We might as well have included enough zero coefficients to make $|m| \leq N$.

Proposition 11.8.1. *A trigonometric polynomial $f(x) = \sum_{n=-N}^{N} c_n\, e^{inx}$ is real-valued for real x if and only if $c_{-m} = \overline{c_m}$ for all $m = -N, \dots, N$.*

Proof. If $f(x)$ is real-valued, that is $\overline{f(x)} = f(x)$, then

$$\overline{c_m} = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x) e^{-imx}}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{imx}\,dx = c_{-m}.$$

The complex conjugate goes inside the integral because the integral is done on real and imaginary parts separately.

On the other hand if $c_{-m} = \overline{c_m}$, then

$$\overline{c_{-m}\, e^{-imx} + c_m\, e^{imx}} = \overline{c_{-m}}\, e^{imx} + \overline{c_m}\, e^{-imx} = c_m\, e^{imx} + c_{-m}\, e^{-imx},$$

which is real valued. Also $c_0 = \overline{c_0}$, so c_0 is real. By pairing up the terms we obtain that f has to be real-valued. $\square$

The functions e^{inx} are also linearly independent.

Proposition 11.8.2. *If*

$$\sum_{n=-N}^{N} c_n\, e^{inx} = 0$$

for all $x \in [-\pi, \pi]$, then $c_n = 0$ for all n.

Proof. The result follows immediately from the integral formula for c_n. $\square$

11.8.2 Fourier series

We now take limits. We call the series

$$\sum_{n=-\infty}^{\infty} c_n\, e^{inx}$$

the *Fourier series*. The numbers c_n are called *Fourier coefficients*. Using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we could also develop everything with sines and cosines, that is, as the series $a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$, but it is equivalent and slightly more messy.

Several questions arise. What functions are expressible as Fourier series? Obviously, they have to be 2π-periodic, but not every periodic function is expressible with the series. Furthermore, if we do have a Fourier series, where does it converge (where and if at all)? Does it converge absolutely? Uniformly? Also note that the series has two limits. When talking about Fourier series convergence, we often talk about the following limit:

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx}.$$

There are other ways we can sum the series that can get convergence in more situations, but we refrain from discussing those.

Conversely, we start with an integrable function $f\colon [-\pi,\pi] \to \mathbb{C}$, and we call the numbers

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

its *Fourier coefficients*. Often these numbers are written as $\hat{f}(n)$.* We then formally write down a Fourier series. As you might imagine such a series might not even converge. We write

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n\, e^{inx},$$

although the $\sim$ doesn't imply anything about the two sides being equal in any way. It is simply that we created a formal series using the formula for the coefficients.

A few sections ago, we proved that the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

converges uniformly and hence converges to a continuous function. This example and its proof can be extended to a more general criterion.

Proposition 11.8.3. *Let $\sum_{n=-\infty}^{\infty} c_n\, e^{inx}$ be a Fourier series, and C, $\alpha > 1$ constants such that*

$$|c_n| \leq \frac{C}{|n|^\alpha} \qquad \textit{for all } n \in \mathbb{Z} \setminus \{0\}.$$

Then the series converges (absolutely and uniformly) to a continuous function on $\mathbb{R}$.

The proof is to apply the Weierstrass M-test (Theorem 11.2.4) and the p-series test, to find that the series converges uniformly and hence to a continuous function (Corollary 11.2.8). We can also take derivatives.

Proposition 11.8.4. *Let $\sum_{n=-\infty}^{\infty} c_n\, e^{inx}$ be a Fourier series, and C, $\alpha > 2$ constants such that*

$$|c_n| \leq \frac{C}{|n|^\alpha} \qquad \textit{for all } n \in \mathbb{Z} \setminus \{0\}.$$

Then the series converges to a continuously differentiable function on $\mathbb{R}$.

*The notation seems similar to Fourier transform for those readers that have seen it. The similarity is not just coincidental, we are taking a type of Fourier transform here.

The trick is to first notice that the series converges first to a continuous function by the previous proposition, so in particular it converges at some point. Then differentiate the partial sums

$$\sum_{n=-N}^{N} inc_n \, e^{inx}$$

and notice that for all nonzero n

$$\lvert inc_n \rvert \leq \frac{C}{\lvert n \rvert^{\alpha-1}}.$$

The differentiated series converges uniformly by the M-test again. Since the differentiated series converges uniformly, we find that the original series $\sum c_n \, e^{inx}$ converges to a continuously differentiable function, whose derivative is the differentiated series (see Theorem 11.2.14).

We can iterate the same reasoning. Suppose there is some C and $\alpha > k+1$ ($k \in \mathbb{N}$) such that

$$\lvert c_n \rvert \leq \frac{C}{\lvert n \rvert^{\alpha}}$$

for all nonzero integers n. Then the Fourier series converges to a k-times continuously differentiable function. Therefore, the faster the coefficients go to zero, the more regular the limit is.

11.8.3 Orthonormal systems

Let us abstract away some of the properties of the exponentials, and study a more general series for a function. One fundamental property of the exponentials that make Fourier series what it is that the exponentials are a so-called *orthonormal system*. Let us fix an interval $[a,b]$. We define an *inner product* for the space of functions. We restrict our attention to Riemann integrable functions since we do not have the Lebesgue integral, which would be the natural choice. Let f and g be complex-valued Riemann integrable functions on $[a,b]$ and define the inner product

$$\langle f, g \rangle := \int_a^b f(x) \overline{g(x)} \, dx.$$

If you have seen Hermitian inner products in linear algebra, this is precisely such a product. We have to put in the conjugate as we are working with complex numbers. We then have the "size," that is the L^2 norm $\lVert f \rVert_2$ by (defining the square)

$$\lVert f \rVert_2^2 := \langle f, f \rangle = \int_a^b \lvert f(x) \rvert^2 \, dx.$$

Remark 11.8.5. Notice the similarity to finite dimensions. For $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, we define

$$\langle z, w \rangle := \sum_{k=1}^{n} z_k \overline{w_k}.$$

Then the norm is (usually denoted by simply $\lVert z \rVert$ in $\mathbb{C}^n$ rather than $\lVert z \rVert_2$)

$$\lVert z \rVert^2 = \langle z, z \rangle = \sum_{k=1}^{n} \lvert z_k \rvert^2.$$

This is just the euclidean distance to the origin in $\mathbb{C}^n$ (same as $\mathbb{R}^{2n}$).

Let us get back to function spaces. In what follows, we will assume all functions are Riemann integrable.

Definition 11.8.6. Let $\{\varphi_n\}$ be a sequence of integrable complex-valued functions on $[a,b]$. We say that this is an *orthonormal system* if

$$\langle \varphi_n, \varphi_m \rangle = \int_a^b \varphi_n(x)\,\overline{\varphi_m(x)}\,dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\|\varphi_n\|_2 = 1$ for all n. If we only require that $\langle \varphi_n, \varphi_m \rangle = 0$ for $m \neq n$, then the system would be called an *orthogonal system*.

We noticed above that

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}$$

is an orthonormal system. The factor out in front is to make the norm be 1.

Having an orthonormal system $\{\varphi_n\}$ on $[a,b]$ and an integrable function f on $[a,b]$, we can write a Fourier series relative to $\{\varphi_n\}$. We let

$$c_n := \langle f, \varphi_n \rangle = \int_a^b f(x)\overline{\varphi_n(x)}\,dx,$$

and write

$$f(x) \sim \sum_{n=1}^{\infty} c_n \varphi_n.$$

In other words, the series is

$$\sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n(x).$$

Notice the similarity to the expression for the orthogonal projection of a vector onto a subspace from linear algebra. We are in fact doing just that, but in a space of functions.

Theorem 11.8.7. *Suppose f is a Riemann integrable function on $[a,b]$. Let $\{\varphi_n\}$ be an orthonormal system on $[a,b]$ and suppose*

$$f(x) \sim \sum_{n=1}^{\infty} c_n \varphi_n(x).$$

If

$$s_n(x) := \sum_{k=1}^{n} c_k \varphi_k(x) \quad \textit{and} \quad p_n(x) := \sum_{k=1}^{n} d_k \varphi_k(x)$$

for some other sequence $\{d_k\}$, then

$$\int_a^b |f(x) - s_n(x)|^2\,dx = \|f - s_n\|_2^2 \leq \|f - p_n\|_2^2 = \int_a^b |f(x) - p_n(x)|^2\,dx$$

with equality only if $d_k = c_k$ for all $k = 1, 2, \ldots, n$.

In other words, the partial sums of the Fourier series are the best approximation with respect to the L^2 norm.

Proof. Let us write

$$\int_a^b |f - p_n|^2 = \int_a^b |f|^2 - \int_a^b f\overline{p_n} - \int_a^b \bar{f} p_n + \int_a^b |p_n|^2.$$

Now

$$\int_a^b f\overline{p_n} = \int_a^b f \sum_{k=1}^n \overline{d_k}\overline{\varphi_k} = \sum_{k=1}^n \overline{d_k} \int_a^b f\overline{\varphi_k} = \sum_{k=1}^n \overline{d_k} c_k,$$

and

$$\int_a^b |p_n|^2 = \int_a^b \sum_{k=1}^n d_k \varphi_k \sum_{j=1}^n \overline{d_j}\overline{\varphi_j} = \sum_{k=1}^n \sum_{j=1}^n d_k \overline{d_j} \int_a^b \varphi_k \overline{\varphi_j} = \sum_{k=1}^n |d_k|^2.$$

So

$$\int_a^b |f - p_n|^2 = \int_a^b |f|^2 - \sum_{k=1}^n \overline{d_k} c_k - \sum_{k=1}^n d_k \overline{c_k} + \sum_{k=1}^n |d_k|^2 = \int_a^b |f|^2 - \sum_{k=1}^n |c_k|^2 + \sum_{k=1}^n |d_k - c_k|^2.$$

This is minimized precisely when $d_k = c_k$. □

When we do plug in $d_k = c_k$, then

$$\int_a^b |f - s_n|^2 = \int_a^b |f|^2 - \sum_{k=1}^n |c_k|^2$$

and so

$$\sum_{k=1}^n |c_k|^2 \leq \int_a^b |f|^2$$

for all n. Note that

$$\sum_{k=1}^n |c_k|^2 = \|s_n\|_2^2$$

by the calculation above. We take a limit to obtain the so-called *Bessel's inequality*.

Theorem 11.8.8 (Bessel's inequality*)**.** *Suppose f is a Riemann integrable function on $[a,b]$. Let $\{\varphi_n\}$ be an orthonormal system on $[a,b]$ and suppose*

$$f(x) \sim \sum_{n=1}^\infty c_n \varphi_n(x).$$

Then

$$\sum_{k=1}^\infty |c_k|^2 \leq \int_a^b |f|^2 = \|f\|_2^2.$$

In particular (given that a Riemann integrable function satisfies $\int_a^b |f|^2 < \infty$), we get that the series converges and hence

$$\lim_{k\to\infty} c_k = 0.$$

*Named after the German astronomer, mathematician, physicist, and geodesist Friedrich Wilhelm Bessel (1784–1846).

11.8.4 The Dirichlet kernel and approximate delta functions

Let us return to the trigonometric Fourier series. Here we note that the system $\{e^{inx}\}$ is orthogonal, but not orthonormal if we simply integrate over $[-\pi,\pi]$. We can also rescale the integral and hence the inner product to make $\{e^{inx}\}$ orthonormal. That is, if we replace

$$\int_a^b \qquad \text{with} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi},$$

(we are just rescaling the dx really)*, then everything works and we obtain that the system $\{e^{inx}\}$ is orthonormal with respect to the inner product

$$\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx.$$

Suppose $f\colon \mathbb{R}\to\mathbb{C}$ is 2π-periodic and integrable on $[-\pi,\pi]$. Let

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

Write

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n\,e^{inx}.$$

Define the *symmetric partial sums*

$$s_N(f;x) := \sum_{n=-N}^{N} c_n\,e^{inx}.$$

The inequality leading up to Bessel now reads:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |s_N(f;x)|^2\,dx = \sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx.$$

The *Dirichlet kernel* is the sum

$$D_N(x) := \sum_{n=-N}^{N} e^{inx}.$$

We claim that

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin\bigl((N+1/2)x\bigr)}{\sin(x/2)},$$

at least for x such that $\sin(x/2)\neq 0$. We know that the left-hand side is continuous and hence the right-hand side extends continuously to all of $\mathbb{R}$ as well. To show the claim we use a familiar trick:

$$(e^{ix}-1)D_N(x) = e^{i(N+1)x} - e^{-iNx}.$$

*Mathematicians in this field sometimes simplify matters by making a tongue-in-cheek definition that $1=2\pi$.

Multiply by $e^{-ix/2}$

$$(e^{ix/2} - e^{-ix/2}) D_N(x) = e^{i(N+1/2)x} - e^{-i(N+1/2)x}.$$

The claim follows.

We expand the definition of s_N

$$s_N(f;x) = \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt\; e^{inx}$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-N}^{N} e^{in(x-t)}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t)\, dt.$$

If you replace $x-t$ with $t-x$ (D_N is even), we see that convolution strikes again! As D_N and f are 2π-periodic, we may also change variables and write

$$s_N(f;x) = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x-t) D_N(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t)\, dt.$$

See Figure 11.10 for a plot of D_N for $N = 5$ and $N = 20$.

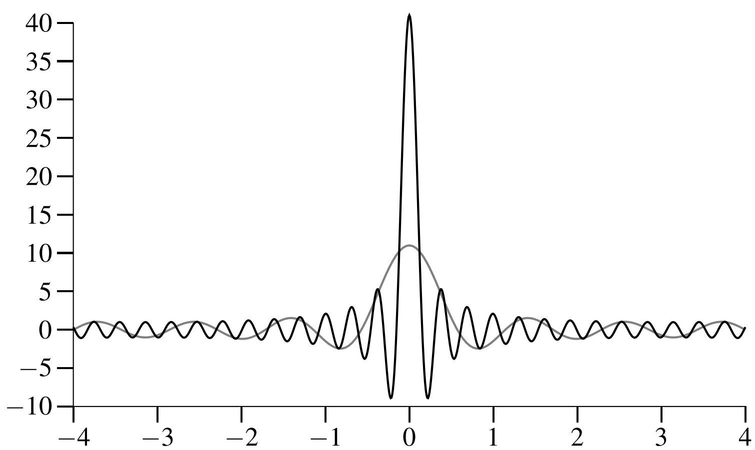

Figure 11.10: Plot of $D_N(x)$ for $N = 5$ (gray) and $N = 20$ (black).

The central peak gets taller and taller as N gets larger, and the side peaks stay small. We are convolving (again) with *approximate delta functions*, although these functions have all these oscillations away from zero. The oscillations on the side do not go away but they are eventually so fast that we expect the integral to just sort of cancel itself out there. Overall, we expect that $s_N(f)$ goes to f. Things are not always simple, but under some conditions on f, such a conclusion holds. For this reason people write

$$2\pi\, \delta(x) \sim \sum_{n=\infty}^{\infty} e^{inx},$$

where δ is the "delta function" (not really a function), which is an object that will give something like "$\int_{-\pi}^{\pi} f(x-t)\delta(t)\, dt = f(x)$." We can think of $D_N(x)$ converging in some sense to $2\pi\,\delta(x)$. However, we have not defined (and will not define) what kind of an object the delta function is, nor what does it mean for it to be a limit of D_N or have a Fourier series.

11.8.5 Localization

If f satisfies a Lipschitz condition at a point, then the Fourier series converges at that point.

Theorem 11.8.9. *Let x be fixed and let f be a 2π-periodic function Riemann integrable on $[-\pi,\pi]$. Suppose there exist $\delta > 0$ and M such that*

$$|f(x+t)-f(x)| \leq M|t|$$

for all $t \in (-\delta,\delta)$, then

$$\lim_{N\to\infty} s_N(f;x) = f(x).$$

In particular, if f is continuously differentiable at x, then we obtain convergence (exercise). We state an often used version of this corollary. A function $f\colon [a,b] \to \mathbb{C}$ is *continuous piecewise smooth* if it is continuous and there exist points $x_0 = a < x_1 < x_2 < \cdots < x_k = b$ such that f restricted to $[x_j,x_{j+1}]$ is continuously differentiable (up to the endpoints) for all j.

Corollary 11.8.10. *Let f be a 2π-periodic function Riemann integrable on $[-\pi,\pi]$. Suppose there exist $x \in \mathbb{R}$ and $\delta > 0$ such that f is continuous piecewise smooth on $[x-\delta,x+\delta]$, then*

$$\lim_{N\to\infty} s_N(f;x) = f(x).$$

The proof of the corollary is left as an exercise. Let us prove the theorem.

Proof of Theorem 11.8.9. For all N,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N = 1.$$

Write

$$\begin{aligned} s_N(f;x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt - f(x)\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(x-t)-f(x)\big)D_N(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-t)-f(x)}{\sin(t/2)} \sin\big((N+1/2)t\big)\,dt. \end{aligned}$$

By the hypotheses, for small nonzero t we get

$$\left|\frac{f(x-t)-f(x)}{\sin(t/2)}\right| \leq \frac{M|t|}{|\sin(t/2)|}.$$

As $\sin(\theta) = \theta + h(\theta)$ where $\frac{h(\theta)}{\theta} \to 0$ as $\theta \to 0$, we notice that $\frac{M|t|}{|\sin(t/2)|}$ is continuous at the origin and hence $\frac{f(x-t)-f(x)}{\sin(t/2)}$ must be bounded near the origin. As $t = 0$ is the only place on $[-\pi,\pi]$ where the denominator vanishes, it is the only place where there could be a problem. The function is also Riemann integrable. We use a trigonometric identity

$$\sin\big((N+1/2)t\big) = \cos(t/2)\sin(Nt) + \sin(t/2)\cos(Nt),$$

so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(x-t)-f(x)}{\sin(t/2)}\sin\bigl((N+1/2)t\bigr)\,dt = \\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f(x-t)-f(x)}{\sin(t/2)}\cos(t/2)\right)\sin(Nt)\,dt+\frac{1}{2\pi}\int_{-\pi}^{\pi}\bigl(f(x-t)-f(x)\bigr)\cos(Nt)\,dt.$$

Now $\frac{f(x-t)-f(x)}{\sin(t/2)}\cos(t/2)$ and $\bigl(f(x-t)-f(x)\bigr)$ are bounded Riemann integrable functions and so their Fourier coefficients go to zero by Theorem 11.8.8. So the two integrals on the right-hand side, which compute the Fourier coefficients for the real version of the Fourier series go to 0 as N goes to infinity. This is because $\sin(Nt)$ and $\cos(Nt)$ are also orthonormal systems with respect to the same inner product. Hence $s_N(f;x)-f(x)$ goes to 0, that is, $s_N(f;x)$ goes to $f(x)$. □

The theorem also says that convergence depends only on local behavior.

Corollary 11.8.11. *Suppose f is a 2π-periodic function, Riemann integrable on $[-\pi,\pi]$. If J is an open interval and $f(x)=0$ for all $x\in J$, then $\lim s_N(f;x)=0$ for all $x\in J$.*

In particular, if f and g are 2π-periodic functions, Riemann integrable on $[-\pi,\pi]$, J an open interval, and $f(x)=g(x)$ for all $x\in J$, then for all $x\in J$, the sequence $\bigl\{s_N(f;x)\bigr\}$ converges if and only if $\bigl\{s_N(g;x)\bigr\}$ converges.

That is, convergence at x is only dependent on the values of the function near x. To prove the first claim, take $M=0$ in the theorem. The "In particular" follows by considering the function $f-g$, which is zero on J and $s_N(f-g)=s_N(f)-s_N(g)$. On the other hand, we have seen that the rate of convergence, that is how fast does $s_N(f)$ converge to f, depends on global behavior of the function.

There is a subtle difference between the corollary and what can be achieved by the Stone–Weierstrass theorem. Any continuous function on $[-\pi,\pi]$ can be uniformly approximated by trigonometric polynomials, but these trigonometric polynomials need not be the partial sums s_N.

11.8.6 Parseval's theorem

Finally, convergence always happens in the L^2 sense and operations on the (infinite) vectors of Fourier coefficients are the same as the operations using the integral inner product.

Theorem 11.8.12 (Parseval*)**.** *Let f and g be 2π-periodic functions, Riemann integrable on $[-\pi,\pi]$ with*

$$f(x)\sim\sum_{n=-\infty}^{\infty}c_n e^{inx}\qquad\text{and}\qquad g(x)\sim\sum_{n=-\infty}^{\infty}d_n e^{inx}.$$

Then

$$\lim_{N\to\infty}\|f-s_N(f)\|_2^2=\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-s_N(f;x)|^2\,dx=0.$$

Also

$$\langle f,g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\overline{g(x)}\,dx=\sum_{n=-\infty}^{\infty}c_n\overline{d_n},$$

and

$$\|f\|_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx=\sum_{n=-\infty}^{\infty}|c_n|^2.$$

*Named after the French mathematician Marc-Antoine Parseval (1755–1836).

Proof. There exists (exercise) a continuous 2π-periodic function h such that

$$\|f-h\|_2 < \varepsilon.$$

Via Stone–Weierstrass, approximate h with a trigonometric polynomial uniformly. That is, there is a trigonometric polynomial $P(x)$ such that $|h(x)-P(x)| < \varepsilon$ for all x. Hence

$$\|h-P\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(x)-P(x)|^2\,dx} \leq \varepsilon.$$

If P is of degree N_0, then for all $N \geq N_0$

$$\|h-s_N(h)\|_2 \leq \|h-P\|_2 \leq \varepsilon,$$

as $s_N(h)$ is the best approximation for h in L^2 (Theorem 11.8.7). By the inequality leading up to Bessel, we have

$$\|s_N(h)-s_N(f)\|_2 = \|s_N(h-f)\|_2 \leq \|h-f\|_2 \leq \varepsilon.$$

The L^2 norm satisfies the triangle inequality (exercise). Thus, for all $N \geq N_0$,

$$\|f-s_N(f)\|_2 \leq \|f-h\|_2 + \|h-s_N(h)\|_2 + \|s_N(h)-s_N(f)\|_2 \leq 3\varepsilon.$$

Hence, the first claim follows.

Next,

$$\langle s_N(f), g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_N(f;x)\overline{g(x)}\,dx = \sum_{k=-N}^{N} c_k \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}\overline{g(x)}\,dx = \sum_{k=-N}^{N} c_k\overline{d_k}.$$

We need the Schwarz (or Cauchy–Schwarz or Cauchy–Bunyakovsky–Schwarz) inequality, that is,

$$\left|\int_a^b f\bar{g}\right|^2 \leq \left(\int_a^b |f|^2\right)\left(\int_a^b |g|^2\right).$$

This is left as an exercise. The proof is not really different from the finite-dimensional version. So

$$\begin{aligned}
\left|\int_{-\pi}^{\pi} f\bar{g} - \int_{-\pi}^{\pi} s_N(f)\bar{g}\right| &= \left|\int_{-\pi}^{\pi} (f-s_N(f))\bar{g}\right| \\
&\leq \left(\int_{-\pi}^{\pi} |f-s_N(f)|^2\right)^{1/2}\left(\int_{-\pi}^{\pi} |g|^2\right)^{1/2}.
\end{aligned}$$

The right-hand side goes to 0 as N goes to infinity by the first claim of the theorem. That is, as N goes to infinity, $\langle s_N(f), g\rangle$ goes to $\langle f, g\rangle$, and the second claim is proved. The last claim in the theorem follows by using $g = f$. □

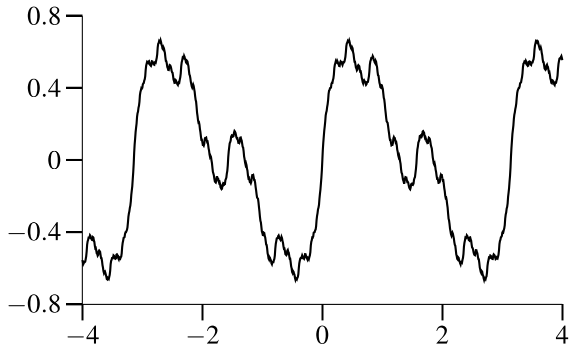

Figure 11.11: Plot of $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(2^n x)$.

11.8.7 Exercises

Exercise* 11.8.1*: *Consider the Fourier series*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(2^n x).$$

Show that the series converges uniformly and absolutely to a continuous function. Note: This is another example of a nowhere differentiable function (you do not have to prove that)[*]*. See Figure 11.11.*

Exercise* 11.8.2*: *Suppose that a 2π-periodic function that is Riemann integrable on $[-\pi,\pi]$, and such that f is continuously differentiable on some open interval (a,b). Prove that for every $x \in (a,b)$, we have $\lim\limits_{N\to\infty} s_N(f;x) = f(x)$.*

Exercise* 11.8.3*: *Prove Corollary 11.8.10, that is, suppose a 2π-periodic function is continuous piecewise smooth near a point x, then $\lim\limits_{N\to\infty} s_N(f;x) = f(x)$. Hint: See the previous exercise.*

Exercise* 11.8.4*: *Given a 2π-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ Riemann integrable on $[-\pi,\pi]$, and $\varepsilon > 0$. Show that there exists a continuous 2π-periodic function $g\colon \mathbb{R} \to \mathbb{C}$ such that $\|f-g\|_2 < \varepsilon$.*

Exercise* 11.8.5*: *Prove the Cauchy–Bunyakovsky–Schwarz inequality for Riemann integrable functions:*

$$\left| \int_a^b f\bar{g} \right|^2 \leq \left(\int_a^b |f|^2 \right) \left(\int_a^b |g|^2 \right).$$

Exercise* 11.8.6*: *Prove the L^2 triangle inequality for Riemann integrable functions on $[-\pi,\pi]$:*

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2.$$

[*]See G. H. Hardy, *Weierstrass's Non-Differentiable Function*, Transactions of the American Mathematical Society, **17**, No. 3 (Jul., 1916), pp. 301–325.

Exercise* 11.8.7*: *Suppose for some C and $\alpha > 1$, we have a real sequence $\{a_n\}$ with $\lvert a_n \rvert \leq \frac{C}{n^\alpha}$ for all n. Let*

$$g(x) := \sum_{n=1}^{\infty} a_n \sin(nx).$$

a) *Show that g is continuous.*

b) *Formally (that is, suppose you can differentiate under the sum) find a solution (formal solution, that is, do not yet worry about convergence) to the differential equation*

$$y'' + 2y = g(x)$$

of the form

$$y(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

c) *Then show that this solution y is twice continuously differentiable, and in fact solves the equation.*

Exercise* 11.8.8*: *Let f be a 2π-periodic function such that $f(x) = x$ for $0 < x < 2\pi$. Use Parseval's theorem to find*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise* 11.8.9*: *Suppose that $c_n = 0$ for all $n < 0$ and $\sum_{n=0}^{\infty} \lvert c_n \rvert$ converges. Let $\mathbb{D} := B(0,1) \subset \mathbb{C}$ be the unit disc, and $\overline{\mathbb{D}} = C(0,1)$ be the closed unit disc. Show that there exists a continuous function $f \colon \overline{\mathbb{D}} \to \mathbb{C}$ that is analytic on $\mathbb{D}$ and such that on the boundary of $\mathbb{D}$ we have $f(e^{i\theta}) = \sum_{n=0}^{\infty} c_n e^{i\theta}$.*
Hint: If $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$.

Exercise* 11.8.10*: *Show that*

$$\sum_{n=1}^{\infty} e^{-1/n} \sin(nx)$$

converges to an infinitely differentiable function.

Exercise* 11.8.11*: *Let f be a 2π-periodic function such that $f(x) = f(0) + \int_0^x g$ for a function g that is Riemann integrable on every interval. Suppose*

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Show that there exists a $C > 0$ such that $\lvert c_n \rvert \leq \frac{C}{\lvert n \rvert}$.

Exercise* 11.8.12*:

a) *Let φ be the 2π-periodic function defined by $\varphi(x) := 0$ if $x \in (-\pi, 0)$, and $\varphi(x) := 1$ if $x \in (0, \pi)$, letting $\varphi(0)$ and $\varphi(\pi)$ be arbitrary. Show that $\lim s_N(\varphi; 0) = 1/2$.*

b) *Let f be a 2π-periodic function Riemann integrable on $[-\pi, \pi]$, $x \in \mathbb{R}$, $\delta > 0$, and there are continuously differentiable $g \colon [x-\delta, x] \to \mathbb{C}$ and $h \colon [x, x+\delta] \to \mathbb{C}$ where $f(t) = g(t)$ for all $t \in [x-\delta, x)$ and where $f(t) = h(t)$ for all $t \in (x, x+\delta]$. Then $\lim s_N(f; x) = \frac{g(x)+h(x)}{2}$, or in other words,*

$$\lim_{N\to\infty} s_N(f;x) = \frac{1}{2} \left(\lim_{t\to x^-} f(t) + \lim_{t\to x^+} f(t) \right).$$

Further Reading

[R1] Maxwell Rosenlicht, *Introduction to Analysis*, Dover Publications Inc., New York, 1986. Reprint of the 1968 edition.

[R2] Walter Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1976. International Series in Pure and Applied Mathematics.

[T] William F. Trench, *Introduction to Real Analysis*, Pearson Education, 2003. http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_REAL_ANALYSIS.PDF.

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List of Notation

Notation	Description	Page
J_f, $\frac{\partial(f_1,f_2,\ldots,f_n)}{\partial(x_1,x_2,\ldots,x_n)}$	Jacobian determinant of f	38
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$\frac{\partial^2 f}{\partial x_2 \partial x_1}$	derivative of f with respect to x_1 and then x_2	56
$f_{x_1 x_2}$	derivative of f with respect to x_1 and then x_2	56
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$\omega_1\,dx_1 + \omega_2\,dx_2 + \cdots + \omega_n\,dx_n$	differential one-form	68
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$\underline{\int_R} f$	lower Darboux integral over rectangle R	87
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$\mathscr{R}(R)$	Riemann integrable functions on R	89, 114
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$o(f,x,\delta)$, $o(f,x)$	oscillation of a function at x	109
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