1 Introduction to Search Theory

Readings: ch 6 of Lungqvist and Sargent.

Search theory is a way to capture the idea that it may take time and/or effort for an agent to contact other agents in order to engage in a product interaction. For example

1. Shopping for some consumer durable. It takes time to search around to find an acceptable combination of quality, features, and price. If you look long enough eventually you will find a really good deal on a TV, computer, stereo, ... The cost is that you will go longer without being able to enjoy the use of the desired product.

2. Marriage market. If you wait/look for a long enough period of time, eventually you will find the ideal (best) mate, who will provide a very high level of utility. There are two big cost of waiting for the ideal mate. The first is the forgone pleasure you would enjoy in a match. The second is that your ideal match might find someone else who is "good enough" before you have a chance to meet him. In this environment, with searchers on both sides, obviously both parties must agree to the match. Thus, matches only have positive probability of forming if both parties are above the lower bound of the other.

3. Labour market. An unemployed worker might find a very high paying job, if only she looks for a long enough period of time. The cost, again, is that the longer she looks, the more current income she is giving up.

We will be thinking quite a bit about this third application. Perhaps we will spend some time on the 2nd since it is fun.

- Today, and then in detail in the problem sets we will consider formally the decision problem of a worker (Partial Equilibrium)
- In the 2nd half of the course we will return and bring in the firm side of the model and talk about what a labour market equilibrium looks like with search frictions (IE) information is not costless to obtain).

1.1 McCall (1970) model of inter temporal job search

A formal model of unemployed search

- Unemployed worker
  - seeks to maximize discounted lifetime income (over an infinite planning horizon)
    - Here we can justify the focus on lifetime income by either assuming complete markets or risk neutral agents.
  - At the beginning of each period the worker receives one wage offer. The wage offer is drawn from the distribution

    $$F(W) = \Pr\{w \le W\}, \qquad 0 \le w_{min} < w_{max} \le B$$

    where $F(w)$ is the cdf and $f(w)$ is the pdf.
    - $F(w)$ is time invariant and known to the worker. In other words, the worker knows the distribution of possible wages and that this distribution will not change over time.
  - After receiving the wage offer, the worker may choose to either
    1. accept the offer and work at this wage forever (we assume no quiting or firing for the moment)

2. reject the offer, receive unemployment benefit this period, and draw a new (independent) wage next period.

– Formally:

Objective:

$$V(w) = \max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t y_t, \qquad \beta \in (0,1)$$

where β is the discount factor and y_t is income in period t with

$$y = \begin{cases} b & \text{if unemployed} \\ w & \text{if employed at wage } w \end{cases}$$

where the maximization is obtained by choosing which wage offers to reject and which to accept.

– Consider the value of accepting a wage offer w and working at this wage forever, this is equal to:

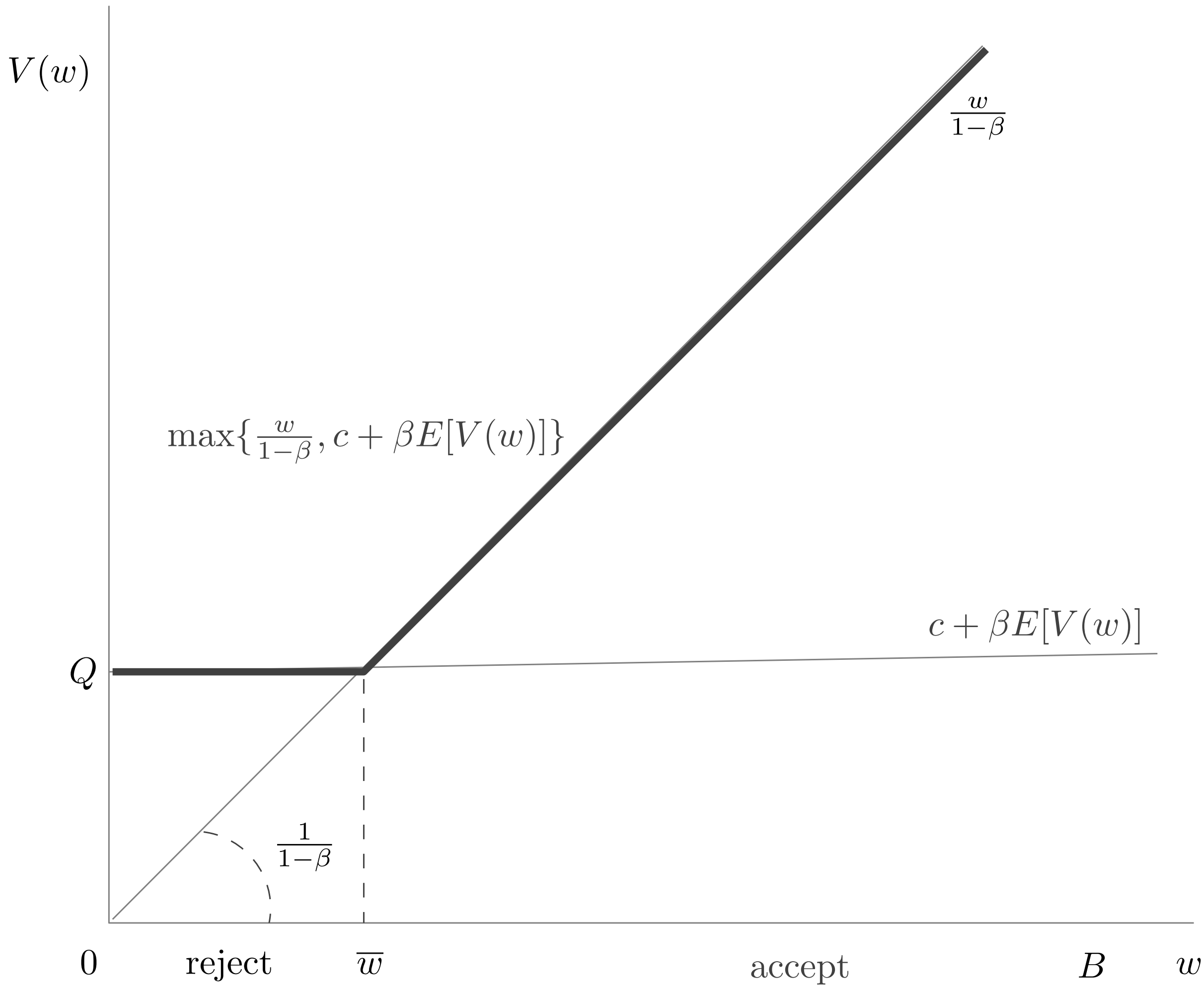
$$\begin{aligned} V^A(w) &= \sum_{t=0}^{\infty} \beta^t w \\ &= w \sum_{t=0}^{\infty} \beta^t \\ &= \frac{w}{1-\beta} \end{aligned}$$

– Consider the value of rejecting the wage offer w , collecting unemployment and then obtaining a new offer tomorrow:

$$\begin{aligned} V^R(w) &= b + \beta \mathbb{E}\left[V(w')\right] \\ &= b + \beta \int V(w') f(w') \, dw' \\ & \quad\; b + \beta \int V(w') \, dF(w') \end{aligned}$$

– So, we can write the value function as

$$V(w) = \max \left\{ \underbrace{\frac{w}{1-\beta}}_{\text{accept}}, \underbrace{b + \beta \int V(w')\, dF(w')}_{\text{reject}} \right\}$$



– The wage at which it is optimal to reject all lower offers and accept all higher offers is called the reservation wage: $\overline{w}$.

– So

$$V(w) = \begin{cases} \frac{\overline{w}}{1-\beta} = b + \beta \int_0^B V(w')\, dF(w') & \text{if } w < \overline{w} \\ \frac{w}{1-\beta} & \text{if } w \geq \overline{w} \end{cases}$$

Now, we want to rewrite the first row of the above expression in a way that allows us to talk about what affects the reservation wage $\overline{w}$. The reservation wage is defined by the equation

$$\begin{aligned}
\frac{\overline{w}}{1-\beta} &= b+\beta\int_0^B V\left(w'\right)dF\left(w'\right) \\
&= b+\beta\int_0^{\overline{w}} V\left(w'\right)dF\left(w'\right)+\beta\int_{\overline{w}}^B V\left(w'\right)dF\left(w'\right) \\
&= b+\beta\int_0^{\overline{w}} \frac{\overline{w}}{1-\beta}dF\left(w'\right)+\beta\int_{\overline{w}}^B \frac{w'}{1-\beta}dF\left(w'\right)
\end{aligned}$$

now, split the LHS the same way (making use of $\int_0^B dF(w')=1$ which implies $\int_0^{\overline{w}} dF\left(w'\right)+\int_{\overline{w}}^B dF\left(w'\right)=1$)

$$\underbrace{\int_0^{\overline{w}} \frac{\overline{w}}{1-\beta}dF\left(w'\right)}_{a}+\underbrace{\int_{\overline{w}}^B \frac{\overline{w}}{1-\beta}dF\left(w'\right)}_{b}$$

$$=b+\underbrace{\beta\int_0^{\overline{w}} \frac{\overline{w}}{1-\beta}dF\left(w'\right)}_{c}+\underbrace{\beta\int_{\overline{w}}^B \frac{w'}{1-\beta}dF\left(w'\right)}_{d}$$

$$\underbrace{\overline{w}\int_0^{\overline{w}} dF\left(w'\right)-b}_{a-c}=\underbrace{\frac{1}{1-\beta}\int_{\overline{w}}^B \left(\beta w'-\overline{w}\right)dF\left(w'\right)}_{d-b}$$

now add $\overline{w}\int_{\overline{w}}^B dF\left(w'\right)=\frac{1-\beta}{1-\beta}\int_{\overline{w}}^B \overline{w}dF\left(w'\right)$ to each side

$$\boxed{\left(\overline{w}-b\right)=\frac{\beta}{1-\beta}\int_{\overline{w}}^B \left(w'-\overline{w}\right)dF\left(w'\right)}$$

this characterizes the reservation wage. Note: it is not a solution since $\overline{w}$ appears on both sides, and in the integrand.

- LHS is the cost of searching one more period when we have offer $\overline{w}$ in hand (it is $(\overline{w}-b)$ because we only give up the amount in excess of b).

- RHS is the benefit from searching one more period.
  - β because we are discounting to the current period
  - $\frac{1}{1-\beta}$ because we will receive it forever
  - $\int_{\overline{w}}^{B} (w' - \overline{w})\, dF(w')$ because the gain is the expected difference between the offer and the reservation wage (the worker receives the flow value $\overline{w}$ even without taking the job).
- The reservation wage is chosen to equate the benefit and cost of searching one more period.

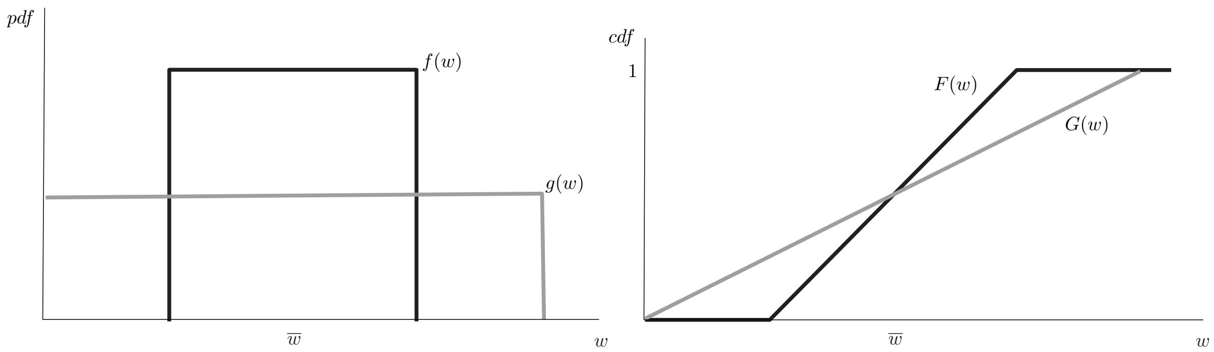
1.1.1 How does a change in the unemployment benefit affect the reservation wage?

- We can see this directly from the diagram.
- An increase in b will lead to an increase in $\overline{w}$.
- See Sargent and Lungqvist p 145-148 for a formal derivation that may be of some use in the problem sets.

1.1.2 What is the effect of a mean preserving spread on the reservation wage?

See Lungqvist and Sargent for a more technical derivation.

- The intuition is quite simple.
- $F(w)$ and $G(w)$ are two possible wage offer distributions.
- They have the same mean, but $G(w)$ is a *mean preserving spread* of $F(w)$. For example, see the following figure with uniformly distributed random variables.



Note that under G the worker has a more of a chance of drawing very high wages, but also a greater chance of drawing very low wages. We can call the distribution G riskier than F in the sense that it provides more variance with the same mean.

How does the worker respond to this increase in risk? Suppose, using our uniform distributions above, that c was such that $\overline{w}$ under the wage offer distribution F was equal to the median wage. Now suppose a worker is faced with the new offer distribution G, but continues to employ the same reservation wage. She will still reject wages with the same probability (because the median is the same) but now the expectation of the accepted wage is much higher under G then under F. Looking at our expression for the reservation wage we see that

$$\begin{aligned}\overline{w} - b &= \frac{\beta}{1-\beta}\int_{\overline{w}}^{B} (w' - \overline{w})\, dF(w') \\ &< \frac{\beta}{1-\beta}\int_{\overline{w}}^{B} (w' - \overline{w})\, dG(w')\end{aligned}$$

in order to restore the equality the new reservation wage $\tilde{w}$ must be higher than $\overline{w}$. This is clear as we need to increase the LHS and $\overline{w}$ is the only variable we have control over.

The worker responds to a mean preserving spread (increase in risk) by increasing her reservation wage. Indeed, her expected value function is higher than under F. This results from the fact that in this environment the worker does not have to accept the wages offered her, and thus there is no real cost to adding more low wages (they can be rejected) and the worker receives all the benefit of the additional very high wages added due to the mean preserving spread.

1.1.3 What is the expected unemployment duration?

- Define $\lambda = 1 - \int_0^{\overline{w}} dF\left(w'\right) = 1 - F\left(\overline{w}\right)$ as the probability of accepting a job offer (the probability the wage is above the reservation wage).
- $(1 - \lambda)$ is the probability of rejecting a job offer.
- Define N as the period that the worker takes a job.

$$
\begin{array}{rcl}
\Pr\left(N = 1\right) & = \lambda & = \left(1-\lambda\right)^{0}\lambda \\
\Pr\left(N = 2\right) & = \left(1-\lambda\right)\times\lambda & = \left(1-\lambda\right)^{1}\lambda \\
\Pr\left(N = 3\right) & = \left(1-\lambda\right)\times\left(1-\lambda\right)\times\lambda & = \left(1-\lambda\right)^{2}\lambda \\
\Pr\left(N = 4\right) & = \left(1-\lambda\right)\times\left(1-\lambda\right)\times\left(1-\lambda\right)\times\lambda & = \left(1-\lambda\right)^{3}\lambda \\
\vdots & \vdots & \vdots \\
\Pr\left(N = j\right) & & = \left(1-\lambda\right)^{j-1}\lambda
\end{array}
$$

$$\overline{N} = \mathbb{E}\left[N\right] = \sum_{j=1}^{\infty} j \times \left(1-\lambda\right)^{j-1}\lambda$$

- The easy way to solve this is to make use of the recursive nature of the problem (and the stationary due to the infinite horizon)
- Note that the expected number of periods remaining unemployed is independent of how long one has been unemployed so far (because of the infinite horizon and stationary

of $F(w)$). So,

$$\begin{aligned}
\overline{N} &= \underbrace{\lambda}_{\text{Pr of accepting}} \times 1 + \underbrace{(1-\lambda)}_{\text{Pr of rejecting}} \left[1 + \text{expected remaining periods}\right] \\
\overline{N} &= \lambda + (1-\lambda)\left(1+\overline{N}\right) \\
\overline{N} &= \lambda + (1-\lambda)\left(1+\overline{N}\right) \\
\overline{N} &= \frac{1}{\lambda} \\
&= \frac{1}{1-F(\overline{w})}
\end{aligned}$$

- The expected/average number of periods unemployed before accepting a job (the unemployment duration) is equal to the reciprocal of the job acceptance rate (probability).
- If $\overline{w}$ increases then $F(\overline{w})$ increases and $\overline{N}$ also increases.
- So, an increase in the unemployment benefit b leads to an increase in $\overline{w}$ and therefore to longer unemployment durations. What happens to the average wage (of the employed)? It increases since this is simply

$$\frac{\int_{\overline{w}}^{B} w dF(w)}{1-F(\overline{w})}$$

which is increasing in $\overline{w}$.

1.1.4 What if jobs don't last forever?

Suppose that each period a worker is employed there is a probability α that she looses the job (fired). Now

$$\hat{V}(w) = \max\left\{ \underbrace{w + \beta\left((1-\delta)\hat{V}(w) + \delta\left(b + \beta\mathbb{E}\hat{V}(w')\right)\right)}_{\text{accept}}, \underbrace{b + \beta\mathbb{E}\hat{V}(w')}_{\text{reject}} \right\}$$

$$\hat{V}(w) = \begin{cases} b + \beta \mathbb{E} \hat{V}(w') & \text{if } w < \overline{w} \\ \frac{w + \delta\beta\left(b + \beta \mathbb{E} \hat{V}(w')\right)}{1 - \beta(1-\delta)} & \text{if } w \geq \overline{w} \end{cases}$$

(first row comes from simplifying

$$\hat{V}(w) = w + \beta\left((1-\delta)\hat{V}(w) + \delta\left(b + \beta \mathbb{E} \hat{V}(w')\right)\right)$$

).

$\overline{w}$ solves:

$$\begin{aligned} \frac{\overline{w} + \delta\beta\left(b + \beta \mathbb{E}\hat{V}(w')\right)}{1 - \beta(1-\delta)} &= b + \beta \mathbb{E}\hat{V}(w') \\ \overline{w} + \underbrace{\delta\beta\left(b + \beta \mathbb{E}\hat{V}(w')\right)}_{\text{cancell}} &= \left(b + \beta \mathbb{E}\hat{V}(w')\right)\left(1 - \beta + \underbrace{\delta\beta}_{\text{cancell}}\right) \\ \frac{\overline{w}}{1-\beta} &= b + \beta \mathbb{E}\hat{V}(w') \end{aligned}$$

This is exactly the same type of expression that we had in the case with no firing. BUT, $\overline{w}$ is NOT the same because

$$\hat{V}(w) \neq V(w)$$

Note that $\hat{V}(w)$ lies strictly below $V(w)$ since jobs don't last forever. Thus, the reservation wage is lower when jobs don't last forever.

1.2 An Economy (A Lake Model)

- Suppose we have an economy populated by a large number of these searchers: L
- Workers move between unemployment and employment
- δ is the probability an employed worker becomes unemployed

- $\lambda \equiv [1 - F(\overline{w})]$ is the probability an unemployed worker becomes employed (job offer is greater then her reservation wage)

- At any time period t some workers will be unemployed and others employed:
  - U_t workers are unemployed
  - $[L - U_t]$ workers are employed

- The number of unemployed workers evolves over time according to:

$$U_{t+1} = \underbrace{F(\overline{w})U_t}_{\text{workers who reject wage offer in period } t} + \underbrace{\delta[L - U_t]}_{\text{workers who lost their job in period } t}$$

- In a stationary equilibrium the number of unemployed workers remain constant: $U_{t+1} = U_t = U$

$$\begin{aligned} U &= F(\overline{w})U + \delta[L - U] \\ [1 - F(\overline{w}) + \delta]U &= \delta L \\ U &= \frac{\delta L}{\delta + 1 - F(\overline{w})} \\ \frac{U}{L} &= \frac{\delta}{\delta + \lambda} \end{aligned}$$

- This is the unemployment rate in a stationary environment

- The unemployment rate in this economy is related to the probability of loosing a job and the probability of accepting a job.

- Additionally, we can write

$$\begin{aligned}\frac{U}{L} &= \frac{\delta}{\delta+\lambda} \\ &= \frac{1}{1+\lambda/\delta} \\ &= \frac{\frac{1}{\lambda}}{\frac{1}{\lambda}+\frac{1}{\delta}} \\ &= \frac{\text{expected unemployment duration}}{\text{expected unemployment duration}+\text{expected employment durat}}\end{aligned}$$

The unemployment rate in the economy is equal to the expected fraction of live any worker will spend in unemployment

- NOTE: This is **not** and equilibrium model because the price (wages) are exogenous
  - How can we think of $F(w)$ here?
  - Why would a firm ever make a wage offer $w \leq \overline{w}$ when the firm knows all such offers will be rejected?
  - If we think that $F(w)$ represents the distribution of firms who differ by productivity (and therefore can all pay different wages) we can think of firms that offer $w \leq \overline{w}$ as representing firms that would exist, and produce, only if $\overline{w}$ were lover.
  - So, an increase in c leads to an increase in $\overline{w}$ which leads to fewer **active firms** and therefore less output.