ECON0107 2022
Lukasz Rachel

Problem Set 1

1 Interiority and finite value of candidate path

Consider the candidate path in the sequence formulation of the consumption-savings problem with isoelastic utility that we derived in the lecture.

1. Derive the conditions on the parameters that ensure that the candidate path is interior and yields finite value.
2. What happens if these conditions are not satisfied?

2 Analytical Solution for the Cake Eating Problem

1. Consider the infinite horizon version of the cake-eating problem (denote the amount of cake left at t with y_t, with $y_0 = x$ given). Show that the period-by-period budget constraint, together with the initial condition and the transversality condition, imply the lifetime budget constraint.
2. Assume $U(c) = -1/c$. Does this utility function satisfy assumptions V1-V3 given in the lecture? Solve for the policy function (i.e. solve for consumption as a function of the state) using the sequential formulation.
3. *This part of the question looks ahead to week 2 lecture on dynamic programming, but you should attempt it this week - it'll be good prep for next week's lecture. You may (or should) rely on your past knowledge and on the readings.* Solve for the value and policy functions by Guess-and-Verify using the recursive formulation. Hint: Guess that $V(y) = -Ay^{-1}$

3 Golden rule and optimality

1. In the context of the growth model, define a sustainable level of the capital stock to be a value of k such that if $\hat{k}_0 = k$ then $k_t = k$ for all t is feasible.

   (a) Prove that the set of sustainable capital stocks is given by an interval $[0, \hat{k}]$ for some value $\hat{k}$. Derive an expression that implicitly defines $\hat{k}$.

(b) Each sustainable level of capital k is associated with a sustainable level of consumption, defined by the function $c^S(k) = f(k) - \delta k$. Prove that the function $c^S(k)$ is single peaked. Derive an implicit expression for the value of k at which $c^S(k)$ attains its maximum. This is what is referred to in the literature as the golden rule level of the capital stock. Denote it by k^G.

(c) Let k^* be the steady state level of the capital stock corresponding to the Social Planner's problem that we considered to find Pareto efficient allocations. Show that $k^* < k^G$. If we let c^* and c^G denote the corresponding levels of consumption, it follows that $c^* < c^G$. Note that c^G is the highest sustainable level of consumption in this economy. If we started the economy with $k_0 = k^G$, it is feasible to have consumption of c^G forever by maintaining capital at k^G forever. Given that c^G is the highest sustainable level of consumption, this would seem to be an appealing outcome. Explain why it is that the Social Planner does not choose this option.

4 Feasibility and transversality in the growth model

1. Consider the discrete-time version of the growth model. As we showed in the lecture, the optimality conditions were

$$u'(c_t) = \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta) \tag{1}$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \tag{2}$$

for all t, with k_0 given and the transversality conditioon

$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0 \tag{TC'}$$

For this question I want you to solve for the dynamic paths for consumption and the capital stock that result from iterating on the optimality conditions for the growth model, after making an arbitrary guess for the value of k_1 (note that choosing a value for k_1 is the same as choosing a value for c_0). Assume the following specification: $f(k) = k^{1/3}$, $\delta = .08$, $u(c) = \log(c)$, and $\beta = .96$. (As we will see later in the course, this choice of functional forms and parameter values is fairly standard in the literature, and corresponds to choosing a period length of one year). Use MATLAB or the programming language of your choice. You should submit the results as well as the code.

(a) Solve for the value of the (positive) steady state level of k.

(b) Solve for the maximum sustainable capital stock.

(c) Assume that $k_0 = .1k^*$ where k^* is the steady state level computed in (a). What range of values for c_0 are feasible? Denote this interval of values by $[c_{\min}, c_{\max}]$.

(d) Consider twenty equally spaced values of c lying inside the open interval $(c_{\min}, c_{\max})$. For each of these values iterate on the FOC as discussed in class, and solve for the values of c_t and k_t for many periods. In some cases the iterative procedure will fail because it moves you to a negative value for k at some point. If this does not happen then continue for one hundred periods. In each case produce a plot of the values for $\{c_t\}$ and $\{k_t\}$.

(e) For those cases in which you were able to continue for 100 periods without violating feasibility, compute the period by period value of the expression that enters the transversality condition and make a plot of its value over time. Does it look like this expression is going to zero?

5 Ak model

Consider the Social Planner's problem that we considered in class. We assumed that the function $f(k)$ is strictly concave and satisfies the Inada conditions. This question asks you to consider the case where $f(k) = Ak$.

1. Formulate the Social Planner's problem and derive first order conditions.

2. Argue that the Social Planner's problem will not have a positive steady state value of k except under very special conditions, and that if it has one positive steady state then all positive values of k are steady states.

6 A linear dynamic system

Consider the following linear system:

$$x_{t+1} = (1 - \alpha)x_t + \lambda(1 - x_t)$$

where $0 < \alpha < 1$, $0 < \lambda < 1$, and $0 \leq x_0 \leq 1$.

This type of dynamical system can arise in many economic situations of interest. For example, some models of labor market dynamics produce this type of equation for the evolution of the unemployment rate. In this context x_t would be the unemployment rate in period t, α would represent the fraction of unemployed workers that find jobs in each period, and λ would represent the fraction of employed workers that lose their jobs each period.

1. Show that $0 \leq x_t \leq 1$ for all t.
2. Determine how many steady states the above equation has. Solve for these steady states in terms of the parameters α and λ.
3. Show that starting from any x_0 that the sequence of values for x_t will converge to a steady state value.
4. Find conditions such that the convergence is monotone.

7 Balanced growth preferences

This exercise introduces a class of preferences which is very useful in macroeconomics: balanced growth preferences. These preferences are defined as follows:

$$\begin{aligned} U(c,z) &= \frac{(cv(z))^{1-1/\sigma}-1}{1-1/\sigma} \text{ if } \sigma \in (0,\infty) \\ U(c,z) &= \ln(c) + \ln(v(z)) \text{ if } \sigma = 1 \end{aligned}$$

where v is increasing and concave. $z = 1-l$ is leisure time (and l is hours worked, with time endowment normalized to 1).

1. Consider first a one-period model with no capital and linear technology $F(l) = Al$. Solve the social planning problem

   $$\max U(c, 1-l)$$

   s.t.

   $$c \leq Al$$

   Let $\{c^*(A), l^*(A)\}$ be the solution.

2. Show that l^* does not depend on A. Explain why this is the case.

3. *As we mentioned in class, the neoclassical growth model features no economic growth in steady state. We will now put growth back into the NGM.* Consider the neoclassical growth model with exogenous growth where in period t, output is given by $A_t k_t^{\alpha} l_t^{1-\alpha}$, where $A_t = A_0 e^{gt}$ where $g > 0$ is the exogenous growth rate of TFP. The sequence problem is given by

$$V(k) = \max_{\{c_t, l_t k_t\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t)$$

s.t.

$$c_t + k_{t+1} = A_t k_t^{\alpha} l_t^{1-\alpha}$$

$$k_0 = k$$

4. Let $\gamma = g/(1-\alpha)$, $\hat{k}_t = k_t/(A_0 e^{\gamma t})$, $\hat{c}_t = c_t/(A_0 e^{\gamma t})$ and $\hat{l}_t = l_t$. Perform a change of variable in the planning problem from $\{c_t, l_t, k_t\}_{t\geq 0}$ to $\left\{\hat{c}_t, \hat{l}_t, \hat{k}_t\right\}_{t\geq 0}$ and write down the corresponding planning problem.

5. Show that there exists a non-trivial steady-state for the variables $\hat{c}_t, \hat{l}_t, \hat{k}_t$ and compute it. Actually, one can show that the hat economy always converges to this steady-state, as in the model without growth.

6. If $\hat{c}_t, \hat{l}_t, \hat{k}_t$ are in this steady-state, how do c_t, l_t, k_t evolve over time? This is called a balanced growth path.

7. Explain why balanced growth preferences are key for the existence of a balanced growth path.