Microeconomics Midterm

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Question 1

(i)

We want to prove σ_i admits an APU representation $\implies$ σ_i satisfies Ordinal IIA
σ_i admits an APU representation implies that we can express σ_i as follows

$$\sigma_i(\cdot|S_i) = argmax_{p_i \in \Delta(S_i)} \sum_{s_i \in S_i} p_i(s_i)u_i(s_i) - c_i(p_i(s_i))$$

We set the Lagrangian

$$\mathcal{L} = \sum_{s_i \in S_i} p_i(s_i)u_i(s_i) - c_i(p_i(s_i)) + \lambda(S_i)[\sum_{s_i \in S_i} p_i(s_i) - 1]$$

where λ is the Lagrangian multiplier on the constraint that the probability of choosing s_i from menu S_i should sum up to 1. Then σ_i can be solved by setting FOC to 0 with respect to $p_i(s_i)$:

$$u_i(s_i) + \lambda(S_i) = c_i'(p_i(s_i)|S_i)$$

For any $s_i, s_i' \in S_i$, we have $c_i'(p_i(s_i)|S_i) - u_i(s_i) = c_i'(p_i(s_i')|S_i) - u_i(s_i') = \lambda(S_i)$ by combining the FOCs.
Similarly we have $c_i'(p_i(s_i)|S_i') - u_i(s_i) = c_i'(p_i(s_i')|S_i') - u_i(s_i') = \lambda(S_i')$ for any $s_i, s_i' \in S_i'$. Combining the above two equations, we have

$$c_i'(p_i(s_i)|S_i) - c_i'(p_i(s_i')|S_i) = c_i'(p_i(s_i)|S_i') - c_i'(p_i(s_i')|S_i') = u_i(s_i) - u_i(s_i')$$

Now we define our function $\phi(p_i) = \exp(c_i'(p_i))$, then

$$\frac{\phi(\sigma_i(s_i|S_i))}{\phi(\sigma_i(s_i'|S_i))} = \frac{\exp(c_i'(\sigma_i(s_i|S_i)))}{\exp(c_i'(\sigma_i(s_i'|S_i)))} = \exp(c_i'(\sigma_i(s_i|S_i)) - c_i'(\sigma_i(s_i'|S_i))) = \exp(u_i(s_i) - u_i(s_i'))$$
$$\frac{\phi(\sigma_i(s_i|S_i'))}{\phi(\sigma_i(s_i'|S_i'))} = \frac{\exp(c_i'(\sigma_i(s_i|S_i')))}{\exp(c_i'(\sigma_i(s_i'|S_i')))} = \exp(c_i'(\sigma_i(s_i|S_i')) - c_i'(\sigma_i(s_i'|S_i'))) = \exp(u_i(s_i) - u_i(s_i'))$$

Hence, we have

$$\frac{\phi(\sigma_i(s_i|S_i))}{\phi(\sigma_i(s_i'|S_i))} = \frac{\phi(\sigma_i(s_i|S_i'))}{\phi(\sigma_i(s_i'|S_i'))}$$

for all S_i, S_i' and $s_i, s_i' \in S_i \cap S_i'$.
To finally get Ordinal IIA property, we remain to show that our function ϕ is continuous, monotone, mapping $[0,1]$ to $\mathbb{R}_+ \cup \{\infty\}$ and with $\phi(0) = 0$.
Since the co domain of $c(\cdot)$ is $\mathbb{R} \cup \{\infty\}$ and $p_i \in [0,1]$, it is easy to see that $\phi(p_i) : [0,1] \to \mathbb{R}_+ \cup \{\infty\}$.
Since $c(\cdot)$ and $\exp(\cdot)$ are both monotone and continuous, $\phi(\cdot)$ is also monotone and continuous.
Finally, $\phi(0) = \exp(c_i'(o)) = \exp(-\infty) = 0$. Our constructed function $\phi(\cdot)$ satisfies all conditions.
Hence, σ_i satisfies Ordinal IIA.

(ii)

Redefine the APU game $G =< \Gamma, c >$ as $\tilde{\Gamma} =< I, \Sigma, \tilde{u} >$, where $\Sigma_i = [0,1]^{|S_i|}$, $\tilde{u_i}(\sigma) = u_i(\sigma) - \sum_{s_i \in S_i} c_i(\sigma_i'(s_i))$
We want to use the Theorem 3 (Existence of PSNE) in part 3 in Navin Kartik's Notes to show the existence of NE.
Since S_i is finite, $\Sigma_i = [0,1]^{|S_i|}$ is a finite dimensional vector on $[0,1] \in \mathbb{R}$. So Σ_i is non-empty, compact and convex.
By the linearity of probability, $u_i(\sigma)$ is continuous and concave in σ. By assumption, c_i is continuous and convex, which implies $-c_i$ is continuous and concave. Hence, $\tilde{u_i} : \Sigma_i \to \mathbb{R}$ is continuous and concave (thus quasi-concave) in σ.
By Theorem 3, we can conclude there exists a PSNE in the game $\tilde{\Gamma}$. Hence there exists a NE in every APU game with additive perturbed utility G.

(iii)

$\tilde{\Gamma} =< I, \Sigma, \tilde{u} >$: the original game
$\tilde{\Gamma'} =< I, \Sigma, \tilde{u'} >$: the game with increasing $u_i(1,1)$

(iv)

Yes.
σ is a NE of Γ can basically comes from that the correspondence $S^{NE}(\lambda) : \Lambda \rightrightarrows S$ is uhc at $\lambda = 0$.
So our remaining goal is to prove $S^{NE}(\lambda)$ is uhc at $\lambda = 0$.
We use the Proposition we proved in class:
Proposition: If $S := \times S_i$, $S_i \subset \mathbb{R}^m$ is non-empty, compact and convex. The random parameter $T \subseteq \mathbb{R}^m$ is convex. $u_i : S \times T \to \mathbb{R}$ is continuous in (s,t). And the set of Nash equilibrium $S^{NE}(t)$ is not empty for any t' in the neighbourhood of t^*. Then S^{NE} is uhc at t^*.

As we verified in part (ii), Σ_i is non-empty, compact and convex. $\lambda \in \mathbb{R}$ is convex. As $u_i(\sigma)$ is continuous in σ and $c_i^n = \lambda^n c_i$ is continuous in λ, $\tilde{u}_i(\sigma, \lambda) = u_i(\sigma_i) - \sum_{s_i \in S_i} \lambda^n c_i$ is continuous in (σ, λ). Also, we have proved in part (ii) that every APU game with additive perturbed utility has a NE. Hence $S^{NE}(\lambda) \neq \emptyset$.
So by this proposition, we can conclude that the set of NE of APU game with additive pertubed utility parameter λ $S^{NE}(\lambda)$ is uhc at $\lambda = 0$.
Then by the sequential definition of uhc and $S^{NE}(\lambda) \subseteq \Sigma$ is compact-valued, for any sequence $((\lambda^n)_n, (\sigma^n)_n) \to (0, \sigma)$, we have $\sigma \in S^{NE}(0)$, which is the set of NE of original normal-form game Γ since $\lambda = 0$, there is no perturbation.

Question 2

(i)

First note that there is no symmetric PSNE.
If all three players bid s^1, all bids get cancelled and each player get utility $-s^1 < 0$. There is an incentive to deviate to s^2, which leads to a positive payoff $v - s^2$.
Similarly, if all players bid s^2, they all get negative payoff $-s^2$. s^1 is a profitable deviation that generates positive payoff $v - s^1$.

Now consider symmetric MSNE.
Denote the probability of playing s^1 for each player as p. Then the probability of playing s^2 is $1 - p$. Since the game is symmetric and we are considering the symmetric NE, WLOG we consider player 1.

$$u_1(s^1, \sigma_{-1}) = (v - s^1)(1-p)^2 + (-s^1)[1 - (1-p)^2] = v(1-p)^2 - s^1$$
$$u_1(s^2, \sigma_{-1}) = (v - s^2)p^2 + (-s^2)(1 - p^2) = vp^2 - s^2$$

By the indifference condition in mixed strategy, we have

$$u_1(s^1, \sigma_{-1}) = u_1(s^2, \sigma_{-1}) \implies v(1-p)^2 - s^1 = vp^2 - s^2$$

$$\implies p = \frac{s^2 - s^1 + v}{2v}$$

which is well-defined since $v > s^2 > s^1 \geq 0 \implies 0 < p < 1$
Hence, there is a unique symmetric Nash equilibrium which characterized as:

$$(\sigma_1(s^1), \sigma_2(s^1), \sigma_3(s^1)) = (\frac{s^2 - s^1 + v}{2v}, \frac{s^2 - s^1 + v}{2v}, \frac{s^2 - s^1 + v}{2v})$$

(ii)

Suppose the auctioneer does not value the object. Then all of her revenue comes from the bids.
All players pay the auction whether they win the object or not. So the revenue $\pi(s) = s_1 + s_2 + s_3$. At the symmetric MSNE, the revenue of the auctioneer is given by

$$\pi = 3s^1p^3 + 3s^2(1-p)^3 + (s^1 + 2s^2)3p(1-p)^2 + (2s^1 + s^2)3p^2(1-p)$$

Substitute $s^1 = 1, s^2 = 2$ and $p = \frac{s^2 - s^1 + v}{2v} = \frac{1+v}{2v}$

$$\pi = 6 - 3p = \frac{9v - 3}{2v}$$

(iii)

First we consider symmetric PSNE.

1. (s^1, s^1, s^1)

   This is a NE if deviation to s^2 is not profitable.

$$\frac{1}{3}v - s^1 = \frac{1}{3}v - 1 = u(s^1, s^1, s^1) \geq u(s^2, s^1, s^1) = v - s^2 = v - 2$$

$$\implies v \leq \frac{3}{2}$$

   But $v > s^2 = 2$, v cannot be below $\frac{3}{2}$. (s^1, s^1, s^1) is not a symmetric NE.

2. (s^2, s^2, s^2)

   By playing s^2 under this strategy profile, each player get a utility $\frac{1}{3}v - s^2$, whereas deviating to s^1 gives the utility $-s^1$.

   There is no profitable deviation if

$$\frac{1}{3}v - s^2 \geq -s^1 \iff \frac{1}{3}v - 2 \geq -1$$

$$\implies v \geq 3$$

   Hence, (s^2, s^2, s^2) is a symmetric NE when $v \geq 3$.

   The expected payoff of the auctioneer is given by $\pi'((s^2, s^2, s^2)) = 3s^2 = 6$

   Since the payoff in part (ii) $\frac{9v-3}{2v} < \frac{9}{2} < 6$, the auctioneer's payoff under (s^2, s^2, s^2) is strictly greater than that in setup (i).

Now we consider symmetric MSNE. Denote $(\sigma_i(s^1), \sigma_i(s^2)) = (p, 1-p)$
Again by the symmetry, WLOG we consider player 1.

$$u_1(s^1, \sigma_2, \sigma_3) = (\frac{1}{3}v - s^1)p^2 + (-s^1)(1-p^2) = \frac{1}{3}vp^2 - s^1 = \frac{1}{3}vp^2 - 1$$
$$u_1(s^2, \sigma_2, \sigma_3) = vp^2 + \frac{1}{2}v2p(1-p) + \frac{1}{3}v(1-p)^2 - s^2 = vp^2 + \frac{1}{2}v2p(1-p) + \frac{1}{3}v(1-p)^2 - 2$$

By the indifference condition of mixed strategy, we have

$$u_1(s^1, \sigma_2, \sigma_3) = u_1(s^2, \sigma_2, \sigma_3) \implies \frac{1}{3}vp^2 - 1 = vp^2 + \frac{1}{2}v2p(1-p) + \frac{1}{3}v(1-p)^2 - 2$$

$$\implies p = \frac{3-v}{v}$$

p is well-defined when $\frac{3}{2} < v < 3 \implies 0 < p < 1$.
The expected payoff of the auctioneer is given by

$$\pi'(\sigma) = 3s^1p^3 + 3s^2(1-p)^3 + (s^1 + 2s^2)3p(1-p)^2 + (2s^1 + s^2)3p^2(1-p) = \frac{9(v-1)}{v}$$

Since

$$\frac{9(v-1)}{v} \geq \frac{9v-3}{2v} \iff v \geq \frac{5}{3}$$

Under symmetric MSNE, the payoff of the auctioneer in set (iii) is always greater as we have $v > 2 \implies v > \frac{5}{3}$.

To summarize, there are two symmetric NE: one PSNE that everyone is playing $s^2 = 2$ and one MSNE that everyone is mixing with probability $\frac{3-v}{v}$ to play s^1. In both NEs, the payoff of auctioneer is greater than that in setup (i).

(iv)

1. setup (i)

   Substitute $p = \frac{1+v}{2v}$ into $u_i(\sigma) = u_i(s^2, \sigma_{-i}) = vp^2 - s^2 = vp^2 - 2 = \frac{v^2-6v+1}{4v}$

2. setup (iii)

   (a) (s^2, s^2, s^2) when $v \geq 3$
   $u_i(s) = \frac{1}{3}v - s^2 = \frac{1}{3}v - 2$
   Since
   $$\frac{v^2-6v+1}{4v} \geq \frac{1}{3}v - 2 \iff v \in (3-2\sqrt{3}, 3+2\sqrt{3})$$
   We have that the bidders in setup (i) are better off if $3 \leq v < 3+2\sqrt{3}$, and are worse off if $v > 3+2\sqrt{3}$.

   (b) $(\sigma_i(s^1)) = (\frac{3-v}{v})$ when $2 < v < 3$
   Bidders' expected payoff is given by $u_i(\sigma) = u_i(s^1, \sigma_{-i}) = \frac{1}{3}vp^2 - 1 = \frac{v^2-9v+9}{3v}$
   Since
   $$\frac{v^2-9v+9}{3v} \geq \frac{v^2-6v+1}{4v} \iff v < 9-4\sqrt{3} \text{ or } v > 9+4\sqrt{3}$$
   We have the bidders are better off in (iii) if $2 < v < 9-4\sqrt{3}$, and better off in (i) if $9-4\sqrt{3} < v < 3$.

(v)

When the auctioneer values the item at $\tilde{v}$, there is an additional term $-\tilde{v}$ when there is a winner.

In setup (i), $\pi(p = \frac{1+v}{2v}) = 3s^1p^3 + 3s^2(1-p)^3 + (s^1+2s^2-\tilde{v})3p(1-p)^2 + (2s^1+s^2-\tilde{v})3p^2(1-p) = \frac{9v-3}{2v} - \frac{3(v^2-1)}{4v^2}\tilde{v}$
In setup (iii)
when $v \geq 3$, $\pi'((s^2, s^2, s^2)) = 3s^2 - \tilde{v} = 6 - \tilde{v}$.
We want the auctioneer is better off in set (i). Let

$$\pi(p = \frac{1+v}{2v}) > \pi'((s^2, s^2, s^2))$$
$$\implies \frac{9v-3}{2v} - \frac{3(v^2-1)}{4v^2}\tilde{v} > 6 - \tilde{v}$$
$$\implies \tilde{v} > \frac{6v(v+1)}{v^2+3}$$

when $v < 3$, $\pi'(p = \frac{3-v}{v}) = 3s^1p^3 + 3s^2(1-p)^3 + (s^1+2s^2)3p(1-p)^2 + (2s^1+s^2)3p^2(1-p) - \tilde{v} = \frac{9(v-1)}{v} - \tilde{v}$
Let
$$\pi(p = \frac{1+v}{2v}) > \pi'(p = \frac{3-v}{v})$$
$$\implies \frac{9v-3}{2v} - \frac{3(v^2-1)}{4v^2}\tilde{v} > \frac{9(v-1)}{v} - \tilde{v}$$
$$\implies \tilde{v} > \frac{6v(3v-5)}{v^2+3}$$

Comment:

(vi)

Question 3

We denote the price that the buyer offers in period 1 as p_1,the price in period 2 as p_2, and the threshold acceptance cost of the seller as c_0.
To characterize the wPBE, we first try to solve the game by sequential rationality(SR) (or Backward Induction).

1. At the node at t=2 when the seller decides whether to accept p_2 or not.

   Since c is known to the seller, c can appear in seller's strategy. By SR, she will accept if $\delta(p_2 - c) \geq 0$ (If she reject when indifference, there would be no equilibrium because the buyer could always offer a slightly lower price)

2. At the node at t=2 when the buyer offers the price p_2

   The buyer at the information set that she does not know the c. Denote her belief $P((c,p_1,r,p_2,a)|(c,p_1,r,p_2,a)\cup(c,p_1,r,p_2,r))$ as the probability that the seller accepts p_2. The buyer will choose p_2 to maximise her expected utility given the seller accepts p_2 if $\delta(p_2-c)\geq 0$.

$$\begin{aligned} p_2 &= argmax\ \ \delta(1-p_2)P((c,p_1,r,p_2,a)|(c,p_1,r,p_2,a)\cup(c,p_1,r,p_2,r)) + 0\cdot P((c,p_1,r,p_2,r)|(c,p_1,r,p_2,a)\cup(c,p_1,r,p_2 \\ &= argmax\ \ \delta(1-p_2)P(p_2\geq c|c\geq c_0) = argmax\ \ \delta(1-p_2)\frac{p_2-c_0}{1-c_0} \\ &= \frac{1+c_0}{2} \end{aligned}$$

3. At the node at t=1 when the seller decides whether to accept p_1 or not

   By SR, the seller will accept if $u_2(a,)\geq u_2(r,)$ given the optimal p_2 we calculated above. We assume acceptance among indifference by the same argument as above.

$$\begin{aligned} p_1 - c &\geq \delta(p_2) - c \\ \implies c &\leq \frac{p_1-\delta/2(1+c_0)}{1-\delta} \end{aligned}$$

   Since we denoted the acceptance threshold as c_0, we can now conclude

$$\begin{aligned} c_0 &= \frac{p_1-\delta/2(1+c_0)}{1-\delta} \\ \implies c_0 &= \frac{2p_1-\delta}{2-\delta} \end{aligned}$$

4. At the node at t=1 when the buyer offers the price p_1

   Similarly by SR, the buyer will choose p_1 to maximize her utility given the seller's strategy we showed above. Denote the belief of the buyer at this information set the probability of the seller accepting p_1 as $P((c,a)|(c,a)\cup(c,r))$. Then the optimal p_1 is given by

$$\begin{aligned} p_1 &= argmax\ \ (1-p_1)P((c,a)|(c,a)\cup(c,r)) + \delta(1-p_2)P((c,r)|(c,a)\cup(c,r))P((c,r,p_2,a)) \\ &= argmax\ \ (1-p_1)P(c\leq c_0) + \delta(1-p_2)P(c_0<c\leq p_2) \\ &= argmax\ \ (1-p_1)c_0 + \delta(1-p_2)(p_2-c_0) \end{aligned}$$

   We substitute $c_0=\frac{2p_1-\delta}{2-\delta}$ and $p_2=\frac{1+c_0}{2}$, and then obtain the optimal p_1 by setting FOC equal to 0, which gives us

$$p_1 = \frac{4-2\delta-\delta^2}{8-6\delta}$$

   Then we can back out p_2 and c_0

$$\implies c_0 = \frac{2(\delta^2-3\delta+2)}{(4-3\delta)(2-\delta)},\ \ p_2 = \frac{6-5\delta}{2(4-3\delta)}$$

   Since $6-5\delta > 4-2\delta-\delta^2$ as $\delta\in(0,1)$, we have $p_2>p_1$

   To summarize, the pure strategy wPBE in which the $p_2\geq p_1$ and the seller plays a threahold acceptance strategy is characterized as $(s_1,s_2,\mu)=((p_1,p_2),(s_2^{t=1}(c),s_2^{t=2}(c)),(\mu^1,\mu^2))$

   where

$$\begin{aligned} p_1 = \frac{4-2\delta-\delta^2}{8-6\delta}&,\ p_2 = \frac{6-5\delta}{2(4-3\delta)} \\ s_2^{t=1} &= \text{accept if } c\leq c_0;\text{reject if } c>c_0 i \\ s_2^{t=2} &= \text{accept if } c\leq p_2;\text{reject if } c>p_2 \\ \mu^1 &= P(c|c\sim U(0,1)) \\ \mu^2 &= P(c|c>c_0) \end{aligned}$$