Lecture 6: Optimization in Continuous Time

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Roadmap: Continuous-time problems

- So far, we analyzed dynamic problems formulated in discrete time.
- Today: continuous time.
  Why? sometimes more convenient, some computational advantages
- Choice should be context-dependent, not a force of habit.
- We will run through the theory somewhat analogous to discrete time, and go over several applications.
- But first some background: derive continuous-time problem as limit of discrete-time problem.

Outline

NGM in discrete time

- Recall the optimal growth problem in discrete time:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

$$\text{s.t.} \quad c_t + k_{t+1} = f(k_t) + k_t \text{ for each } t,$$

$$c_t, k_{t+1} \geq 0, \text{ and given } k_0.$$

  - I assumed away depreciation for now, $\delta = 0$ (will bring back later).
  - Here, $f(k_t)$, c_t, $u(c_t)$ capture flow variables: they capture the amount of production and consumption that happens in period t, and the amount of "utility obtained" in period t.
  - The discount factor, β, represents time discounting over one period.

Interpreting time steps

- We haven't said anything about the time-length of the period: it could be a year, a quarter, a day... Our analysis is general.
- The implicit assumption is that $f(k_t)$ and β etc are appropriately chosen to fit the time-length for the economic problem.
- If we switch the time-length, then we also have to appropriately renormalize the exogenous parameters of the problem.
    - For instance, if we use $f(k_t) = Ak_t^{\alpha}$ for a yearly analysis, then we might want to use $f(k_t) = (A/4)\,k_t^{\alpha}$ for a quarterly analysis.
    - Likewise, if we use β for the yearly analysis, then we might want to use $\tilde{\beta}$ such that $\tilde{\beta}^4 = \beta$ for the quarterly analysis.

Choosing time length

- This suggests considering a discrete time problem in which we account for the time-length as an explicit parameter, say Δt, in terms of a fixed unit, say years.
    - We could then do exercises as in the previous slide by simply changing Δt: the yearly (resp. quarterly) analysis would be $\Delta t = 1$ (resp. $\Delta t = 1/4$).
- How would you formulate that type of discrete problem?

Choosing time length

- One possible formulation is to divide $[0, \infty)$ into intervals:

$$\underbrace{[0, \Delta t)}_{\text{period } n=0 \text{ starts at time } 0} \quad \cup \quad \underbrace{[\Delta t, 2\Delta t)}_{\text{period } n=1 \text{ starts at time } \Delta t} \quad \cup \ldots \quad \underbrace{[n\Delta t, (n+1)\,\Delta t)}_{\text{period } n \text{ starts at time } n\Delta t} \quad \ldots$$

- We can then consider the following discrete problem:

$$\begin{aligned} & \max_{\{c_n, k_{n+1}\}_{n=0}^{\infty}} \sum_{n=0}^{\infty} (\beta(\Delta t))^n \, u(c_n)\, \Delta t, \qquad (2) \\ \text{s.t.} \quad & c_n \Delta t + k_{n+1} = f(k_n)\,\Delta t + k_n \text{ for each } n, \\ & c_n, k_{n+1} \geq 0, \text{ and given } k_0. \end{aligned}$$

  - $f(k_n)$, c_n, $u(c_n)$ still correspond to flow variables = rate of production/consumption/utility in period n. Need to multiply by Δt
  - $\beta(\Delta t)$ denotes the discount factor as a function of the time-length parameter, Δt

Exponential discounting

What should the function $\beta(\bullet)$ look like?

- It would be desirable if $\beta\left(\Delta t\right)$ satisfied the following property,

$$\left(\beta\left(\Delta t\right)\right)^{n} = \left(\beta\left(\frac{\Delta t}{2}\right)\right)^{2n}$$

  so that changing Δt to $\Delta t/2$ doesn't change the discounting.

- Check that the exponential form will do the job:

$$\beta\left(\Delta t\right) = e^{-\rho\Delta t} \text{ for some } \textbf{discount rate } \rho > 0.$$

  - With this choice, the utility in period n is discounted by,

$$\beta\left(\Delta t\right)^{n} = e^{-\rho n\Delta t} = e^{-\rho\times(\text{time from zero})}.$$

- Alternative way to see this is that we can write $\beta^{t} = \left(\frac{1}{1+\rho}\right)^{t}$ and

$$\lim_{n\to\infty}\left(1+\frac{\rho}{n}\right)^{-nt} = e^{-\rho t}$$

From discrete to continuous time

- If we set $\Delta t = 1$ in problem (2), then we recover the original discrete-time problem.
- If we consider the limit as $\Delta t \to 0$, then we obtain a problem in which decisions are made and flows are realized continuously. This is essentially the continuous-time formulation.
- Let's see how that formulation looks like...

From discrete to continuous time

- To derive this limit heuristically, let $t = n\Delta t$ denote the time from zero.
- The objective function is a sum,

$$\sum_{n=0}^{\infty} e^{-\rho \times \overbrace{n\Delta t}^{\text{time from zero, } t}} u(c_n)\,\Delta t,$$

which (heuristically) limits to the integral,

$$\int_0^{\infty} e^{-\rho t} u(c(t))\,dt.$$

- Here, $c(t)$ is *the rate* of the consumption flow at time t.

From discrete to continuous time

- Appropriately normalized changes in the stock variables limit to derivatives:
$$\frac{k_{n+1}-k_n}{\Delta t}=\frac{k(t+\Delta t)-k(t)}{\Delta t}\to \dot{k}(t).$$

- Here, $k(t)$ is the capital stock at time t, and $\dot{k}(t)=\frac{dk(t)}{dt}$ is its time derivative.

- Using this observation, the resource constraint,
$$c_n\Delta t+k_{n+1}=f(k_n)\Delta t+k_n,$$
limits to:
$$\dot{k}(t)=f(k(t))-c(t).$$

From discrete to continuous time

- Putting everything together, **the continuous-time formulation** is:

$$\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u(c(t))\, dt$$

$$\text{s.t.} \qquad \dot{k}(t) = f(k(t)) - c(t) \text{ for each } t,$$

$$c(t), k(t) \geq 0 \text{ and given } k_0.$$

- Finally let's re-introduce depreciation:
  - One natural way of adjusting the resource constraint (in line with our discussion of discounting) is

$$c_n \Delta t + k_{n+1} = f(k_n)\Delta t + e^{-\delta \Delta t} k_n$$

  or

$$\frac{k_{n+1} - k_n}{\Delta t} = f(k_n) + \frac{e^{-\delta \Delta t} - 1}{\Delta t} k_n - c_n$$

  - Taking the limit:

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$$

Canonical SP in continuous time

In this lecture we consider the following canonical sequence problem:

$$
\begin{aligned}
& \max_{[y(t),x(t)]_{t=0}^{T}} \int_0^T f\left(t, x\left(t\right), y\left(t\right)\right) dt \qquad (3) \\
\text{s.t.} \quad & \dot{x}\left(t\right) = g\left(t, x\left(t\right), y\left(t\right)\right), \\
& x\left(t\right) \in \mathcal{X}, y\left(t\right) \in \mathcal{Y}, \text{ given } x\left(0\right).
\end{aligned}
$$

- State variable: $x : [0, T] \to \mathcal{X}$; Control variable: $y : [0, T] \to \mathcal{Y}$.
    - For now $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}$. But results naturally generalize to $\mathbb{R}^{K_x}, \mathbb{R}^{K_y}$.
- Suppose that f and g are continuously differentiable.
    - Note that f is allowed to depend on time—can embed discounting.
- We start with T finite, then we do $T = \infty$.
- We will also often add a terminal constraint that applies at time T:
    - We start with no constraint (to state a Theorem in Acemoglu), and then add constraints such as $x\left(T\right) = \underline{x}$ and $x\left(T\right) \geq \underline{x}$.

Continuous time: discussion

- The continuous-time formulation captures similar economic trade-offs.
- But it is sometimes more tractable. The choice is driven by convenience.
- The optimization theory in continuous time is more challenging:
    1. We are optimizing over functions $[c(t), k(t)]_{t=0}^{\infty}$ as opposed to sequences. So even with $T < \infty$, maximization is wrt an infinitely-dimensional object
    2. Constraints are formulated as differential equations, not equalities (so we can't easily solve out)
- We will be more heuristic. The objective is to state the formal results and provide economic intuition for those results.
    - Using the results is not too difficult (hence the tractability).
- Our main reference for this part of the course is Acemoglu, Chapter 7.

Optimal control: Roadmap

- Our main tool to attack these problems is **optimal control**.
  - This is a particular variational approach in continuous time.
    Slightly different precisely because we can't easily solve out controls.
- Same roadmap as in discrete time:
  1. Derive necessary conditions for optimality: **Maximum Principle**.
  2. Establish a sufficiency result under concavity assumptions.

Outline

Outline

Heuristic derivation of optimality conditions

Approach: we will use analogy with Lagrangian optimality conditions from finite dimensional optimization to obtain (i.e., heuristically derive) the FOCs. I find this more intuitive than the derivation in the Acemoglu textbook.

- Suppose $[x^*(t), y^*(t)]_{t=0}^{\infty}$ are interior and continuous functions.
- We seek *necessary* conditions for $[x^*(t), y^*(t)]_{t=0}^{\infty}$ to be optimal.
- Let $\lambda(t)$ denote the Lagrange multiplier corresponding to the constraint,
$$g(t, x(t), y(t)) = \dot{x}(t).$$
- Understanding $\lambda(t)$ provides much of the intuition for the optimality conditions...

What does the Lagrange multiplier capture?

- Note that the constraint can be approximately rewritten as,

$$x(t) + \Delta t g(t, x(t), y(t)) \simeq x(t + \Delta t).$$

  - The marginal values at t and $t + \Delta t$ become the same as $\Delta t \to 0$.

- So in continuous time, $\lambda(t)$ is the marginal value of $x(t)$. In the continuous time context, $\lambda(t)$ is also known as the **costate variable**.

Heuristic derivation of optimality conditions

- Construct the Lagrangian as we do for a finite dimensional problem:
  Recall Lecture 1!

$$\mathcal{L} = \left(\int_0^T \left(f\left(t, x\left(t\right), y\left(t\right)\right) + \lambda\left(t\right)\left(g\left(t, x\left(t\right), y\left(t\right)\right) - \dot{x}\left(t\right)\right)\right) dt \right).$$

  Not easy to take FOCs since it has $\dot{x}\left(t\right)$ terms.

- Using integration by parts (assuming $\lambda\left(t\right)$ differentiable), we have:

$$\int_0^T \lambda\left(t\right)\dot{x}\left(t\right) dt = -\int_0^T x\left(t\right)\dot{\lambda}\left(t\right) dt + x\left(T\right)\lambda\left(T\right) - x\left(0\right)\lambda\left(0\right).$$

- This in turn implies

$$\begin{aligned}\mathcal{L} = &\int_0^T f\left(t, x\left(t\right), y\left(t\right)\right) dt + \lambda\left(t\right) g\left(t, x\left(t\right), y\left(t\right)\right) dt \\ &+ \int_0^T x\left(t\right)\dot{\lambda}\left(t\right) dt + x\left(0\right)\lambda\left(0\right) - x\left(T\right)\lambda\left(T\right)\end{aligned}.$$

- We can now take the FOCs with respect to $\left[y\left(t\right), x\left(t\right)\right]_{t=0}^{T}$...

Optimality conditions via Lagrangian

- The first order conditions (before the end time, T) are:

  1a. $\frac{\partial \mathcal{L}}{\partial y(t)} = f_y(t, x^*(t), y^*(t)) + \lambda(t) g_y(t, x^*(t), y^*(t)) = 0.$

  2. $\frac{\partial \mathcal{L}}{\partial x(t)} = f_x(t, x^*(t), y^*(t)) + \lambda(t) g_x(t, x^*(t), y^*(t)) + \dot{\lambda}(t) = 0.$

- We also have the constraints that describe the state evolution:

  3. $\dot{x}^*(t) = g(t, x^*(t), y^*(t))$ for $t \in [0, T]$

- Finally, we have an end-value condition (for the end time, T):

  4a. $\frac{\partial \mathcal{L}}{\partial x(T)} = -\lambda(T) = 0$ (since we assumed no constraint on $x(T)$)

The Hamiltonian

Define a new function, **the Hamiltonian**, by:

$$H(t, x, y, \lambda) = f(t, x, y) + \lambda g(t, x, y)$$

so that the FOCs and the constraints can be written more memorably:

1. $H_y(t, x^*(t), y^*(t), \lambda(t)) = 0$ for each $t \in [0, T]$
2. $H_x(t, x^*(t), y^*(t), \lambda(t)) = -\dot{\lambda}(t)$ for each $t \in [0, T]$
3. $H_\lambda(t, x^*(t), y^*(t), \lambda(t)) = \dot{x}^*(t)$ for each $t \in [0, T]$
4. $\lambda(T) = 0$

Maximum principle for finite horizons

Theorem (A7.4, Simplified Maximum Principle)

Suppose there is an interior and continuous solution: $\left[x^*(t), y^*(t)\right]_{t=0}^{T}$ *to problem* (9)*. Then, there exists a continuously differentiable function* $\lambda(t)$ *such that* $\left[x^*(t), y^*(t), \lambda(t)\right]_{t=0}^{T}$ *jointly satisfy conditions 1, 2, 3, 4.*

- See Acemoglu (Ch. 7) for a proof based on first order deviations.

Interpreting the optimality conditions: 1

- The most important condition is 1 which is effectively

$$y^*(t) \in \arg\max_{y \in \mathcal{Y}} H(t, x^*(t), y, \lambda(t))$$

$$\Longrightarrow \quad y^*(t) \in \arg\max_{y \in \mathcal{Y}} \overbrace{f(t, x^*(t), y)}^{\text{immediate value}} + \overbrace{\lambda(t)\, g(t, x^*(t), y)}^{\text{changes in future value, } \lambda(t)\dot{x}^*(t)} .$$

  - Once we interpret $\lambda(t)$ as the marginal value of the state variable, this has a natural economic intuition: trade off immediate gains with future losses due to a depletion of the stock variable (or vice versa).
  - So the result has the flavor of DP (even if it is variational).

Interpreting the optimality conditions: 1

- To see the intuition cleanly it might be useful to go over this condition in the consumption-savings example (that we analyze in greater detail later).
- There, the period utility is given by,

$$f(t, a, c) = e^{-\rho t} \log(c),$$

and the evolution of the state variable is described by,

$$\dot{a} = g(t, a, c) = (ra + w - c).$$

- So the Hamiltonian is given by,

$$H(t, a, c, \lambda) = e^{-\rho t} \log(c) + \lambda (ra + w - c).$$

- The first optimality condition then becomes,

$$c^*(t) \in \arg\max_c \overbrace{e^{-\rho t} \log(c)}^{\text{immediate value}} + \overbrace{\lambda(t)(ra^*(t) + w - c)}^{\text{changes in future value}}.$$

Interpreting the optimality conditions: 2

- The second condition is, $H_a(t, a^*(t), c^*(t), \lambda(t)) = -\dot{\lambda}(t)$.
- In our application, this can approximately be written as,

$$\lambda(t - \Delta t) \simeq H_a + \lambda(t) \simeq \underbrace{\lambda(t)\, r\, \Delta t}_{\text{flow marginal value}} + \underbrace{\lambda(t)}_{\text{continuation value}}.$$

- For intuition, suppose you had one unit of asset at the end of time $t - \Delta t$ (i.e., excluding the interest it earns during time $t - \Delta t$).
- Suppose, instead of consuming it, you decide to save it for an instant.
- This would give you some interest, which is valued at λ, as well as your asset at the next instant, valued at $\lambda(t)$.
- So this condition pins down how the marginal values are linked through time.

Interpreting the optimality conditions: 4

- Condition 4 is an end-value optimality condition similar to the complementary slackness or transversality conditions in discrete time.
- When there is no constraint on the choice of $x(T)$, the relevant constraint is

$$\lambda(T) = 0.$$

  - In the consumption-saving problem, this is the main reason why we had to put a terminal constraint. Absent the constraint, the consumer would deplete the assets to $-\infty$.
  - General intuition: if you are free to pick the terminal state, then giving you an additional unit of the state can't be valued. You already set it to the optimal value anyway

Generalizations to different terminal conditions

- This intuition suggests that we could also accommodate various terminal constraints. We just need to change the optimality condition appropriately:

  4a. If $x(T)$ is free, then $\lambda(T) = 0$ as above.

  4b. If $x(T) = \underline{x}$, then $\lambda(T)$ is free (and $x^*(T) = \underline{x}$).

  4c. If $x(T) \geq \underline{x}$ then $\lambda(T) \geq 0$ and $(x(T) - \underline{x})\lambda(T) = 0$

    - Note $\lambda(T)$ is allowed to be strictly positive, but only if $x(T)$ is already at the constraint, $x(T) = \underline{x}$. What is the intuition?

- Note that these cases are exactly analogous to our discrete-time models. Want to push a_{T+1} or k_{T+1} to the lowest possible level.

Outline

Consider the consumption-savings problem

- Consider the consumption-savings problem in continuous time:

$$\max_{[c(t),a(t)]_{t=0}^{T}} \int_0^T e^{-\rho t} \log(c(t))\, dt$$

$$\text{s.t.} \quad \dot{a}(t) = r a(t) + w - c(t),$$

$$\text{with } a(0) \text{ given and } a(T) \geq 0.$$

- Assets are given by $a(t)$. They now evolve continuously.
  - Recall the discrete version for comparison: $a_{t+1} = a_t(1+r) + w - c_t$.
- r is instantaneous interest rate. Assumed to be constant.
- w is instantaneous wages. Assumed to be constant.
- Let's approach problem with the Maximum Principle...

FOCs for consumption-saving problem

- Note that $x = a$, $y = c$ and thus $f = e^{-\rho t} \log(c)$ and $g = ra + w - c$.
- Thus, the Hamiltonian is,

$$H = e^{-\rho t} \log(c) + \lambda (ra + w - c).$$

- FOCs for an interior solution:

1. $H_c = 0$, which implies $e^{-\rho t}/c(t) = \lambda(t)$. Intuition?
   Value of a dollar at time t is discounted marginal utility at t, evaluated at 0.

2. $H_a = -\dot{\lambda}(t)$, which implies $\lambda(t) r + \dot{\lambda}(t) = 0$. Intuition?
   Value of an additional dollar decreases at rate r.

3. $H_\lambda = \dot{a}(t)$, i.e., the state evolution, $\dot{a}(t) = ra(t) + w - c(t)$.

4. $\lambda(T) \geq 0, a(T) \geq 0$ and $\lambda(T) a(T) = 0$. Intuition?

Solving the differential equations

- We can substitute $c(t)$ or $\lambda(t)$ out. Let's proceed just as in discrete time, so we keep the control variable, $c\,(t)$, around.
- Conditions 2 and 1 are:

$$\lambda\,(t)\,r + \dot{\lambda}\,(t) = 0, \text{ and } \lambda\,(t) = \frac{e^{-\rho t}}{c\,(t)}.$$

- Differentiating the second wrt time and substituting in from the first, we derive **the continuous-time Euler equation**,

$$\frac{\dot{c}\,(t)}{c\,(t)} = r - \rho \tag{4}$$

  - Solving this equation (how?), we can also obtain,

$$c\,(t) = c\,(0)\,e^{(r-\rho)t}.$$

The continuous-time Euler equation

- Recall that the discrete time Euler equation with log utility was:

$$\frac{c_{t+1}}{c_t} = \beta\left(1+r\right), \text{ which implies } c_t = c_0\left(1+g\right)^t.$$

- The continuous-time analogue with log utility is,

$$\frac{\dot{c}\left(t\right)}{c\left(t\right)} = r-\rho, \text{ which implies } c\left(t\right) = c\left(0\right)e^{gt}.$$

- We will develop a more general version of the continuous-time Euler equation in the next lecture. For now, note that it essentially captures the same economic trade-off as its discrete-time counterpart.

Solving the differential equations

- So have the Euler equation that represents conditions 2+1

$$\frac{\dot{c}(t)}{c(t)} = r - \rho.$$

- We also have condition 3, the resource constraint, as before,

$$\dot{a}(t) = ra(t) + w - c(t).$$

- So we need to solve an (equivalent) ODE in terms of $c(t), a(t)$ with initial constraint, $a(0) = 0$, and end-value constraint, $a(T) = 0$.
- As in discrete time, the main help comes from converting the flow budget constraint (condition 3) into a lifetime budget constraint...

The LBC in continuous time

- The following identity can be useful in applications:

$$\frac{d}{dt}\left(x\left(t\right)e^{bt}\right)=\dot{x}\left(t\right)e^{bt}+x\left(t\right)be^{bt}=e^{bt}\left(\dot{x}\left(t\right)+x\left(t\right)b\right).$$

- To use this identity, rewrite the flow budget constraint as,

$$\underbrace{e^{-rt}\left(\dot{a}\left(t\right)-ra\left(t\right)\right)}_{\frac{d}{dt}\left(a(t)e^{-rt}\right)}=e^{-rt}\left(w-c\left(t\right)\right).$$

- We can now integrate this over $[0,T]$ to obtain,

$$e^{-rT}a\left(T\right)-a\left(0\right)=\int_0^T e^{-rt}\left(w-c\left(t\right)\right)dt.$$

The LBC in continuous time

- Rearranging, we obtain **the lifetime budget constraint**,

$$e^{-rT}a(T) + \int_0^T e^{-rt}c(t)\,dt = a(0) + \int_0^T e^{-rt}w\,dt. \tag{5}$$

- Note that from the Euler equation we also have

$$c(t) = c(0)\,e^{gt} \text{ where } g = r - \rho.$$

- We can plug this into Eq. (5) and obtain an equation in which the only unknown variable is $c(0)$.
- Does this remind you of anything we have done before?
  - Complete the rest of the analysis. Show that as $T \to \infty$, the solution approximates what we have found in discrete time.
- What is the intuition for the optimal choice of initial consumption, $c(0)$? What goes wrong if choose lower or higher $c(0)$?

Interpreting the FOCs and the "solution"

- Solving these systems gives us an interior path that satisfies FOCs.
- This narrows down possible solutions. Eliminates other interior paths.
- But how do we know that this is "the optimum"?
  - What if there is no optimum? Existence not so easy.
  - What if the optimum is not interior? FOCs need modification.
- As before, need a sufficiency theorem.

Outline

Our next goal: A sufficiency theorem

- Not surprisingly, sufficiency requires concavity assumptions.
- To state the result, define the **maximized Hamiltonian**:

$$M(t,x,\lambda) \equiv \max_{y\in\mathcal{Y}} H(t,x,y,\lambda) = \max_{y} f(t,x,y) + \lambda g(t,x,y).$$

- The relevant assumption is the concavity of $M(t,x,\lambda(t))$ in x...

Arrow's sufficiency theorem

Theorem (A7.6, Arrow's Sufficiency Conditions)

Suppose there is an interior and continuous path, $([x^*(t), y^*(t)]_{t=0}^{T}$*, and a continuously differentiable costate variable,* $[\lambda(t)]_{t=0}^{T}$*, that satisfy the optimality conditions 1,2,3,4 for problem* (9)*. Suppose further that the Maximized Hamiltonian,* $M(t, x, \lambda(t))$ *is concave in* x *for all* t*. Then, the path* $[x^*(t), y^*(t)]_{t=0}^{T}$ *is optimal. If in addition,* M *is strictly concave, then the path is the unique optimal path.*

- See Acemoglu (Ch. 7) for the proof.

Application to consumption-savings problem

- For the consumption-savings problem, maximized Hamiltonian is:

$$\begin{aligned} M(t, a, \lambda) &= \max_c e^{-\rho t} \log(c) + \lambda (ra + w - c) \\ &= e^{-\rho t} \log\left(\frac{e^{-\rho t}}{\lambda}\right) + \lambda \left(ra + w - \frac{e^{-\rho t}}{\lambda}\right). \end{aligned}$$

- This is linear in a, and thus, is weakly concave in a.
- Thus, the solution we have characterized is an optimum.

Outline

Consumption-savings with infinite horizon

- Consider the infinite-horizon version of the consumption-savings problem:

$$\begin{aligned} &\max_{[c(t),a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u\left(c\left(t\right)\right) dt \\ \text{s.t.} \quad &\dot{a}\left(t\right) = ra\left(t\right) + w - c\left(t\right), \\ &+ \text{ a terminal condition, and given } a\left(0\right). \end{aligned}$$

- Note: **infinite horizon** and **exponential discounting**.
- We will now focus on problems that have these features.

Terminal constraints for consumption-savings problem

- What's a sensible restriction on asset choice?
- In discrete time, we worked with the rather stringent constraint:

$$a_t \geq 0, \text{ in continuous time } a(t) \geq 0, \text{ for each } t. \tag{6}$$

- Alternatives that allow for some borrowing:

$$\lim_{t\to\infty} \frac{a_{t+1}}{(1+r)^t} \geq 0,$$

$$\text{and} \qquad a_{t+1} \geq -\frac{w}{r} \text{ for each } t.$$

- The continuous time counterparts would be

$$\lim_{t\to\infty} e^{-rt} a(t) \geq 0 \tag{7}$$

$$\text{and} \qquad a(t) \geq -\frac{w}{r} \text{ for each } t \tag{8}$$

- We want a canonical problem that could accommodate these.

Canonical problem with discounting

- Our canonical problem with infinite horizon and discounting is,

$$\begin{aligned} & \max_{[y(t),x(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} f\left(x\left(t\right), y\left(t\right)\right) dt \qquad (9) \\ \text{s.t.} \quad & \dot{x}\left(t\right) = g\left(t, x\left(t\right), y\left(t\right)\right), \\ & x\left(t\right) \in \mathcal{X}, y\left(t\right) \in \mathcal{Y}, \\ \text{and} \quad & \lim_{t\to\infty} b\left(t\right) x\left(t\right) \geq \underline{x} \text{ given } x\left(0\right), \end{aligned}$$

  where $b\left(t\right)$ is a given function $b : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} b(t) < \infty$.

- As before $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}$. But results generalize to $\mathbb{R}^{K_x}, \mathbb{R}^{K_y}$.

- $f\left(x\left(t\right), y\left(t\right)\right)$ now denotes the flow payoff before discounting.

- Everything should be intuitive except for the terminal constraint...

Terminal constraints in the canonical problem

$$\lim_{t\to\infty} b(t)\,x(t) \geq \underline{x}.$$

- We can accommodate (7) with $b(t) = e^{-rt}$ and $\underline{x}= 0$.
- We can also accommodate the weaker limit versions of (6) and (8):

$$\lim_{t\to\infty} a(t) \geq 0 \text{ and } \lim_{t\to\infty} a(t) \geq -\frac{w}{r}.$$

  - We will state a sufficiency result for the canonical problem (9).
  - But note that, with a sufficiency result, we can throw in additional non-binding constraints. For instance, we can strengthen the limit constraint, $\lim_{t\to\infty} a(t) \geq 0$, to the borrowing constraint (6). If the candidate path does not violate the additional constraints, then it is also optimal for the more constrained problem.

Outline

Maximum Principle & necessary conditions

- Consider a candidate path for a solution $[x^* (t), y^* (t)]_{t=0}^{\infty}$, where $x^* (t)$ and $y^* (t)$ are interior and continuous functions.
- As before, a version of the Maximum Principle applies in this context and provides the necessary optimality conditions.
    - Approach: we will simply state – and not prove, or even derive – the natural extension to the infinite-horizon setting
- The Hamiltonian in this context looks like

$$H(t, x, y, \lambda) = e^{-\rho t} f(x, y) + \lambda g(t, x, y). \tag{10}$$

    - Recall that the Hamiltonian captures a trade-off between the flow utility at time t, $e^{-\rho t} f(x, y)$, with the continuation value, $\lambda g(t, x, y)$.
    - Trade-off concerns date t choices but it is described in terms of date 0 utility/value (since the objective is in terms of date 0 utility).

Current-value costate variable

- Could now state FOCs as before. But it will turn out to be convenient to slightly re-state things first.
- Define the **current-value costate variable** as

$$\mu(t) = e^{\rho t}\lambda(t), \text{ which implies } \lambda(t) = e^{-\rho t}\mu(t). \tag{11}$$

- Recall that $\lambda(t)$ captures the marginal value of the state variable at time t evaluated in date 0 utility (the objective function).
- So $\mu(t)$ can be thought of as capturing the marginal value of the state variable at time t, but evaluated in date t utility.
  - We will see momentarily the dynamic programming formulation and illustrate that $\mu(t)$ is in fact $\frac{dV(x(t))}{dx}$ (as we might have expected given the previous discussion)

Current-value Hamiltonian

- Armed with $\mu(t)$, we can also define the **current-value Hamiltonian** as a date-t-utility version of the Hamiltonian:

$$\hat{H}(t, x, y, \mu) = f(x, y) + \mu g(t, x, y).$$

- Combining Eqs. (10) and (11), we also obtain:

$$H(t, x, y, \mu) = e^{-\rho t} \hat{H}(t, x, y, \mu).$$

The first optimality condition

1. Recall that the first condition was

$$\begin{aligned} & H_y\left(t, x^*(t), y^*(t), \lambda(t)\right)=0, \\ \Longrightarrow \quad & e^{-\rho t} f_y\left(x^*(t), y^*(t)\right)+\lambda(t) g_y\left(t, x^*(t), y^*(t)\right)=0 \end{aligned}$$

After substituting $\lambda(t)=e^{-\rho t} \mu(t)$, this becomes:

$$\begin{aligned} & \hat{H}_y\left(t, x^*(t), y^*(t), \mu(t)\right)=0, \\ \Longrightarrow \quad & f_y\left(x^*(t), y^*(t)\right)+\mu(t) g_y\left(t, x^*(t), y^*(t)\right)=0 \end{aligned}$$

Intuitively, maximizing the Hamiltonian is the same thing as maximizing the CV Hamiltonian. Same trade-off, different units.

The second optimality condition

2. Recall that the second condition was

$$H_x\left(t,^*\left(t\right),y^*\left(t\right),\lambda\left(t\right)\right)=-\dot{\lambda}\left(t\right),$$
$$\Longrightarrow\quad e^{-\rho t}f_x\left(x^*\left(t\right),y^*\left(t\right)\right)+\lambda\left(t\right)g_x\left(t,x^*\left(t\right),y^*\left(t\right)\right)=-\dot{\lambda}\left(t\right).$$

After substituting $\lambda\left(t\right)=e^{-\rho t}\mu\left(t\right)$, this becomes,

$$\hat{H}_x\left(t,x^*\left(t\right),y^*\left(t\right),\mu\left(t\right)\right)=\rho\mu\left(t\right)-\dot{\mu}\left(t\right),$$
$$\Longrightarrow\quad f_x\left(x^*\left(t\right),y^*\left(t\right)\right)+\mu\left(t\right)g_x\left(t,x^*\left(t\right),y^*\left(t\right)\right)=\rho\mu\left(t\right)-\dot{\mu}\left(t\right).$$

- To obtain the term on the right hand side, first note:

$$-\dot{\lambda}\left(t\right)=-\frac{d\lambda\left(t\right)}{dt}=-\frac{d}{dt}\left(e^{-\rho t}\mu\left(t\right)\right)=\rho e^{-\rho t}\mu\left(t\right)-e^{-\rho t}\dot{\mu}\left(t\right).$$

  Then cancel $e^{-\rho t}$ since it shows up on both sides of the equation

Intuition for second optimality condition

- As before, the second equation has a valuation intuition. To see this, plug $\dot{\mu}(t) \simeq (\mu(t) - \mu(t - \Delta t)) / \Delta t$ into the condition,

$$\hat{H}_x = \rho \mu(t) - \dot{\mu}(t)$$

  to obtain the approximate version,

$$\mu(t - \Delta t) \simeq \hat{H}_x \Delta t + \mu(t) - \rho \Delta t \mu(t).$$

- As before, $\mu(t - \Delta t)$ captures the value of having a state variable at (the beginning of) time t, now evaluated in terms of time t utility.
- If you bring it to next instant, then you obtain $\hat{H}_x$ in time-t utility.
- You also obtain one more state variable at time $t + \Delta t$.
  - $\mu(t)$ would be the marginal value evaluated in time $t + \Delta t$ utility.
  - $\mu(t) - \rho \Delta t \mu(t) \simeq e^{-\rho \Delta t} \mu(t)$ is the approximate value in time-t utility.

Intuition for second optimality condition

- A related intuition for the condition comes from writing it as,

$$\rho\mu(t) = \hat{H}_x + \dot{\mu}(t).$$

- Imagine the marginal unit of capital as an asset and imagine $\mu(t)$ at its "price" of the asset at time t evaluated in time t utility.
- The left side can be viewed as the "required return on the asset":
  - This is the amount that will be lost (in CV units) due to discounting.
  - For $\mu(t)$ to capture the price, the loss needs to be compensated.
- The right hand side captures two different forms of compensation:

$$\rho\mu(t) = \overbrace{\hat{H}_x}^{\text{dividend gain/loss}} + \overbrace{\dot{\mu}(t)}^{\text{capital gain/loss (due to price change)}}.$$

The third condition

Let's go back to the optimality conditions.

3. The third condition is the resource constraint,

$$H_\lambda\left(t, x^*\left(t\right), y^*\left(t\right), \mu\left(t\right)\right) = \dot{x}\left(t\right).$$

   Check that this has the same form in the new notation,

$$\hat{H}_\mu\left(t, x^*\left(t\right), y^*\left(t\right), \mu\left(t\right)\right) = \dot{x}\left(t\right).$$

   - Check that this boils down to the resource constraint $\dot{x}\left(t\right) = g\left(t, x, y\right)$, since $\hat{H}\left(t, x, y, \mu\right) = f\left(x, y\right) + \mu g\left(t, x, y\right)$.

The TVC

- With $T = \infty$, the necessity of the transversality condition 4 is trickier.
- Taking a limit of the finite T condition is intuitive. But does not hold in some pathological cases. See Theorems 7.12-7.13 in Acemoglu and the discussion there.
- But we are more interested in sufficient conditions, in which case a stronger and intuitive transversality condition does the job.

Outline

A sufficient transversality condition

4. The optimal path satisfies,

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)\,x^*(t) = 0.$$

In addition, *any* feasible path satisfies

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)\,x(t) \geq 0.$$

- Note that the second part of the condition should be checked for all feasible paths (not just the candidate optimal path).
- This might sound daunting. But it will not be too difficult to check thanks to the terminal constraint, $\lim_{t\to\infty} b(t)\,x(t) \geq \underline{x}$. Will see this in examples.
- Note that, together, these two are a very natural sufficient condition. Best path leaves behind less value than any other path:

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)\,(x^*(t) - x(t)) \leq 0$$

A sufficient concavity condition

- As before, sufficiency also requires concavity assumptions.
- Define the **maximized current-value Hamiltonian**:

$$\hat{M}(t, x, \mu) \equiv \max_{y \in \mathcal{Y}} \hat{H}(t, x, y, \mu) = \max_{y} f(x, y) + \mu g(t, x, y).$$

- The relevant assumption was the concavity of $M(t, x, \lambda(t))$ in x.
- Since $M = e^{-\rho t}\hat{M}$, equivalent to the concavity of $\hat{M}(t, x, \mu(t))$ in x.
- As before, implied by concavity of f, g in (x, y) and $\mu(t) > 0$.

Summary: The sufficient optimality conditions

Collecting the conditions together, we have:

1. $\hat{H}_y(t, x^*(t), y^*(t), \mu(t)) = 0$ for each $t \in [0, \infty)$,
2. $\hat{H}_x(t, x^*(t), y^*(t), \mu(t)) = \rho\mu(t) - \dot{\mu}(t)$ for each $t \in [0, \infty)$,
3. $\hat{H}_\mu(t, x^*(t), y^*(t), \mu(t)) = \dot{x}(t)$ for each $t \in [0, \infty)$,
4. (i) $\lim_{t\to\infty} e^{-\rho t}\mu(t)x^*(t) = 0$ for the candidate path, and
   (ii) $\lim_{t\to\infty} e^{-\rho t}\mu(t)x(t) \geq 0$ for any feasible path.

And the concavity of the maximized Hamiltonian, $\hat{M}(t, x, \mu(t))$ in x.

A sufficiency theorem for infinite horizons

Theorem (Sufficient Conditions for Discounted Infinite Horizon, A7.14)

Suppose there is an interior and continuous path, $[x^*(t), y^*(t)]_{t=0}^{\infty}$, and a continuously differentiable current-value costate variable, $[\mu(t)]_{t=0}^{\infty}$, that satisfy the optimality conditions 1,2,3,4 for problem (9). Suppose further that the maximized Hamiltonian $\hat{M}(t, x, \mu(t))$ is concave in x for all t. Then, the path $([x^*(t), y^*(t)]_{t=0}^{\infty}$ is optimal. If in addition, $\hat{M}$ is strictly concave, then the path is the unique optimal path.

- See Acemoglu, Chapter 7, for proof.

A sufficiency theorem suggests a 2-step approach

- This theorem is very useful. Suggests a two step approach:
  1. **Characterization:** Use (sufficient) FOCs to reduce problem into finding a solution to an ordinary differential equation with boundary conditions.
  2. **Solving the ODE:** If there is a solution, it is the (unique) optimum by the sufficiency theorem.
- Let's see how this works in an application.

Outline

Neoclassical growth model

The optimal growth problem in continuous time is,

$$\begin{aligned} & \max_{[c(t),k(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u\left(c\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{k}\left(t\right) = f\left(k\left(t\right)\right) - \delta k\left(t\right) - c\left(t\right) \text{ for each } t, \\ & c\left(t\right), k\left(t\right) \geq 0 \text{ and given } k\left(0\right), \end{aligned}$$

with the same assumptions imposed on $u\left(c\right)$ and $f\left(k\right)$ as in discrete time.

Concavity condition for the NGM

- The CV Hamiltonian is

$$\hat{H}(t, c, k, \mu) = u(c) + \mu(f(k) - \delta k - c).$$

- The maximized Hamiltonian is then given by

$$\hat{M}(t, k, \mu(t)) = \max_{c} u(c) + \mu(t)(f(k) - \delta k - c).$$

- When $\mu(t) > 0$ (which will be the case for the candidate path, see next slide), the objective value is strictly concave in (c, k). Because of $f(k)$.
- So the maximized Hamiltonian is strictly concave in k.
- Given concavity, the Sufficiency Theorem approach looks promising.

Optimality conditions for the NGM

The optimality conditions are then given by:

1. $\hat{H}_c = 0$, which implies
$$u'(c) = \mu.$$
(Note that greater μ implies smaller c. Intuition?)

2. $\hat{H}_k = \rho\mu - \dot{\mu}$, which implies
$$\rho\mu = \mu[f'(k) - \delta] + \dot{\mu}.$$
(What are the two sources of value that provide the required return?)

Optimality conditions for the NGM

3. $\hat{H}_\mu = \dot{k}$, which implies

$$\dot{k} = f(k) - \delta k - c,$$

4. (i) $\lim_{t\to\infty} e^{-\rho t}\mu(t)\,k(t) = 0$ for the candidate path, and (ii) $\lim_{t\to\infty} e^{-\rho t}\mu(t)\,\tilde{k}(t) \geq 0$ for any other feasible path $\left[\tilde{k}(t)\right]_{t=0}^{\infty}$.

Euler equation for the NGM

- Plugging condition 1 into 2, we obtain condition 1+2,

$$
\begin{aligned}
\rho u'(c) &= u'(c)[f'(k)-\delta] + \underbrace{\frac{du'(c)}{dt}}_{\dot{\mu}} \\
&= u'(c)[f'(k)-\delta] + u''(c)\underbrace{\frac{dc}{dt}}_{\dot{c}}.
\end{aligned}
$$

- Rearranging terms, we obtain an Euler equation,

$$
\frac{\dot{c}}{c} = \mathcal{E}(c)\left[f'(k)-\delta-\rho\right] \text{ where } \mathcal{E}(c) = \frac{-u'(c)}{cu''(c)}.
$$

Intuition for the Euler equation

- Here, $\mathcal{E}(c) = \frac{-u'(c)}{cu''(c)}$ captures the elasticity of intertemporal substitution when the consumption is at c. Check that:
  - If utility is CES, $u(c) = \frac{\varepsilon}{\varepsilon-1}\left(c^{\frac{\varepsilon-1}{\varepsilon}} - 1\right)$, then $\mathcal{E}(c) = \varepsilon$ for each c.
  - If utility is log (special case with $\varepsilon = 1$), then $\mathcal{E}(c) = 1$ for each c.
- So the Euler equation is similar to before.
- What is the intuition for $f'(k) - \delta$, ρ, and $\mathcal{E}(c)$ in the Euler equation?

Differential equations for the NGM

- Combining conditions 1+2 (Euler) and 3, we obtain,

$$\begin{aligned}\frac{\dot{c}}{c} &= \mathcal{E}(c)\,[f'(k)-\delta-\rho],\\ \dot{k} &= f(k)-\delta k - c.\end{aligned}$$

- The second part of condition 4 trivially holds since $\tilde{k}(t) \geq 0$.
- But we also need the first part of condition 4 (TC) to hold:

$$\lim_{t\to\infty} e^{-\rho t} u'(c(t))\,k(t) = 0.$$

- So we have an ordinary differential equation system in (c, k), given the initial-value $k(0)$ and an end-value constraint.
- Note that we could also obtain a similar ODE in terms of (μ, k).

Differential equations for the NGM

- Since the concavity condition is satisfied, if we find a (feasible and interior) path that solves these systems, then we are done.
- This completes step 1. We have "characterized" the solution.
- Step 2 is to solve the differential equation system.
    - This is in general not easy. Next: A general approach based on the phase diagram.
    - But the solution turns out to be simple when $k(0)$ happens to be at the steady state...

The steady state for the NGM

- Show that the steady-state (k^*, c^*) is given by,

$$f'(k^*) = \delta + \rho \text{ and } c^* = f(k^*) - \delta k^*.$$

Result

If $k(0) = k^*$, then the optimum is unique and given by $k(t) = k^*, c(t) = c^*$ for each t.

- Why? Check that all optimality conditions hold. Uniqueness follows from strict concavity.

Transitional dynamics for the NGM

- Now consider $k(0) > 0$ such that $k(0) \neq k^*$. We conjecture that the solution is globally stable, so that $\lim_{t\to\infty} k(t) = k^*$.

- Under this conjecture, the solution is characterized by the ODE,

$$\begin{aligned}\frac{\dot{c}}{c} &= \mathcal{E}(c)\left[f'(k) - \delta - \rho\right] \\ \dot{k} &= f(k) - \delta k - c,\end{aligned}$$

  with $k(0)$ given and $\lim_{t\to\infty} k(t) = k^*$.

- Next: qualitative properties of the solution.

Outline

A complementary approach: Phase diagram

- We can do local stability analysis at this point. We will not cover here, but see Acemoglu Ch 7.
- The phase diagram is an additional, complementary tool to study stability in continuous time.
    - For certain problems, such as the NGM, this enables us to establish not only local but also global stability.
    - It also enables to visualize the solution and conduct comparative statics/dynamics exercises on the diagram.
- Let's see how this works in the context of the NGM.

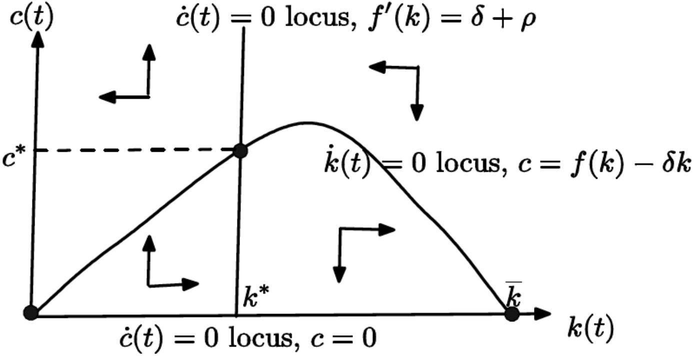
The phase diagram for the neoclassical model

- Recall the original nonlinear system for the NGM,

$$\begin{aligned}\dot{c} &= c\varepsilon[f'(k) - \delta - \rho] \\ \dot{k} &= f(k) - \delta k - c\end{aligned}$$

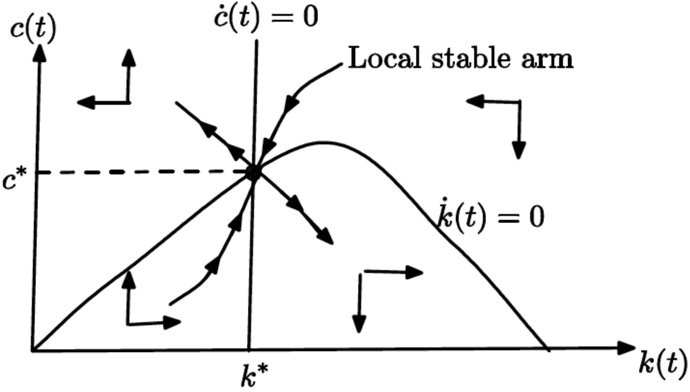
- We can visualize this nonlinear law-of-motion in (k, c) space.
- The $\dot{c} = 0$ locus is (k, c) such that $f'(k) = \delta + \rho$, and also $c = 0$.
    - What happens if $f'(k) > \delta + \rho$, i.e., if $k < k^*$? What if $k > k^*$?
- The $\dot{k} = 0$ locus is given by (k, c) such that $c = f(k) - \delta k$.
    - What happens if $c > f(k) - \delta k$? What if $c < f(k) - \delta k$?

The phase diagram for the neoclassical model



- The horizontal arrows indicate law-of-motion for $k(t)$. The vertical arrows indicate law-of-motion for $c(t)$.
- There is a positive steady-state at (k^*, c^*).
- This steady-state is locally saddle-path stable (see Acemoglu page 304). The arrows confirm this.

Local saddle-path stability on phase diagram



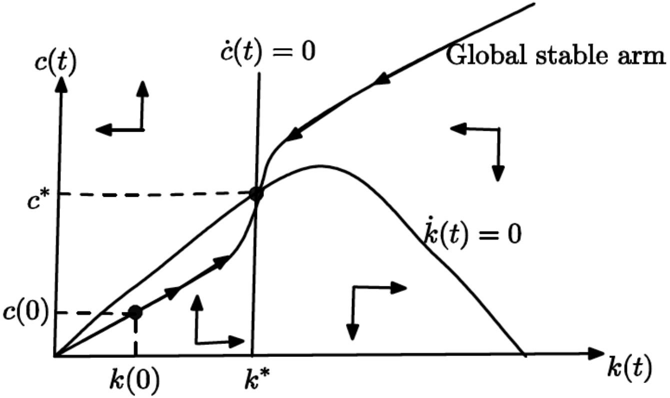
- There is a local stable and unstable arm around (k^*, c^*).

Extending the local arm

$$\begin{aligned}\dot{c} &= c\varepsilon[f'(k) - \delta - \rho] \\ \dot{k} &= f(k) - \delta k - c\end{aligned}$$

- We can use the phase diagram to characterize *global* dynamics.
  - Solve the system backward from a neighborhood of k^*. This gives the global stable arm.
- What does the stable arm look like? E.g. below goes to $(0, 0)$:
  - Cannot cross $c(t) = 0$ since then would go forward to $\bar{k}$.
  - Cannot cross $\dot{k}(t) = 0$ since from there $k(t) \to 0$.

Global saddle-path stability on phase diagram



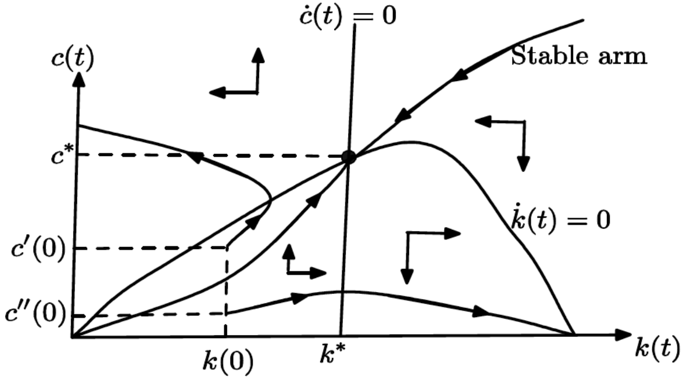
- For any $k(0)$, the global stable arm gives $c(0)$ s.t. we go to (k^*, c^*).
- Since initial condition, TVC and optimality conditions hold, we know from sufficiency theorem that this is a solution.

The solution to the continuous time NGM

- Remains to argue that this is also the unique solution. Two ways of doing this:

  1. The Maximized Hamiltonian is strictly concave (check this), so we know that it is the unique solution.
  2. Use the phase diagram. Argue that other paths – i.e., starting above or below the stable arm – cannot be solutions.

Ruling out other candidate paths



- What would happen if $c\,(0)$ were to start above stable arm?
  Violate resource constraint in finite time.
- What would happen if $c\,(0)$ were to start below stable arm?
  Go to $\bar{k}$ with $c = 0$, so violate TVC.

Aside: numerical solution

- Knowing these sub-optimal paths can also help you build efficient shooting algorithms for finding the optimal path from any given $k(0)$.
- In particular, take an arbitrary $k(0) < k^*$. The optimal path can be approximated numerically as follows:

  1. Start with an arbitrary guess for the initial level of consumption $c(0) < f(k(0)) - \delta k(0)$.
  2. Start constructing the path for $k(t)$ and $c(t)$ using the ODE system.
  3. If either $\dot{k}(t)$ or $\dot{c}(t)$ turns negative (= not on saddle path), stop constructing the path; adjust your initial guess for $c(0)$ downwards if $\dot{k}(t) < 0$ and upwards if $\dot{c}(t) < 0$; and restart.
  4. Continue this process until either $\hat{t}$ is "sufficiently" large or you end up within a "sufficiently" small neighborhood of the steady state.

  An alternative that we will mention next: numerically solving this model in continuous time using dynamic programming techniques.

Outline

Comparative dynamics

- Phase diagram is also useful to study comparative dynamics.
- We illustrate this with a few examples within the NGM.
- Suppose the economy starts at the steady-state $k(0) = k^*$ and $c(0) = c^*$. We consider a change in underlying environment and trace its effect.
  - **Comparative statics**: What happens to (k^*, c^*)? Can do this analytically.
  - **Comparative dynamics:** What happens to $[c(t), k(t)]_{t=0}^{\infty}$?
    - **Long-run:** What happens as $t \to \infty$?
    - **Short-run:** What happens at $t = 0$ and along the transition?

  Behavior along the transition also known as **transitional dynamics**. Generally do this on the computer. Can illustrate using the phase diagram.

Experiment I: Decrease in the discount rate

- Imagine the economy as initially being at its (old) steady state.
- Suppose, at time $t = 0$, there is a permanent reduction in ρ.
  - People becoming "more patient." All else equal, more willing to save.
  - This exercise might sound strange if you take it literally. But imagine this as capturing in reduced form a reform that increases the incentives to save. E.g. improved property rights, which reduces the risk of expropriation.
- The "reform" happens at date 0, and comes as a surprise.
- How would this affect the steady state (c^*, k^*)?
- How would it affect, $[c(t), k(t)]_{t=0}^{\infty}$, starting at $c^{*,old}, k^{*,old}$?

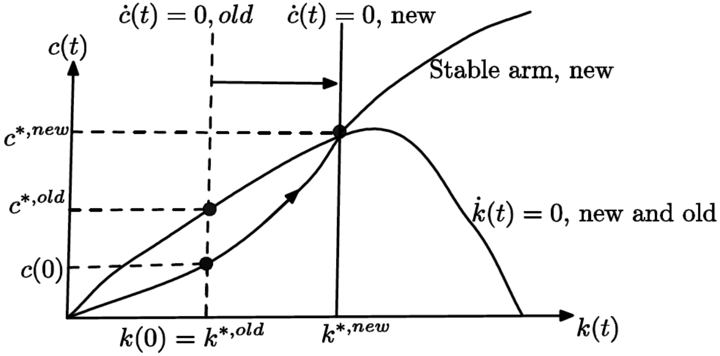
Experiment I: Decrease in the discount rate

- Note that k^* and c^* would both increase since:

$$f'(k^*) = \delta + \rho \text{ and } c^* = f(k^*) - \delta k^*.$$

- What is the intuition? Why is the steady state k^* higher?
- Comparative dynamics can be seen on the phase diagram...

Experiment I: Decrease in the discount rate



- What happens to $k(t)$, $c(t)$ in the short-run? In the long-run?

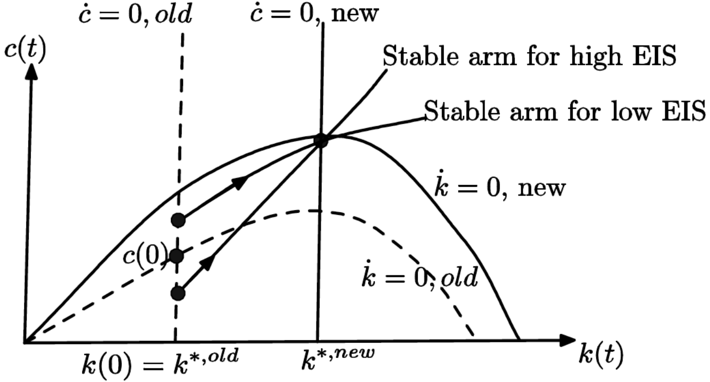
Experiment I: Decrease in the discount rate

- The initial jump in consumption might seem strange since the Euler equation suggests consumption should change continuously.
  - Usual intuition: the consumer does not like jumpy consumption in view of the concavity of the utility function.
- Here, the jump is the optimal response to the unexpected and discrete change in the primitives of the problem that happen at time 0.
- If the change was anticipated before time 0, then the solution would be different. The agent would not be at the old steady-state when the change happens. She would in fact be somewhere on the new saddle path since she would have made arrangements before time 0 (when she learns the change) so as to land on this saddle path at time 0.

Experiment II: Permanent productivity shock

- Now consider a shock to the productivity of the economy.
- Suppose that technology is given by $y = Af(k)$, where A parameterizes total factor productivity (TFP).
- The economy is initially at the steady-state.
- At date 0, the agent learns that the TFP increases (unexpectedly, immediately, and permanently) from $A = A_{low}$ to $A = A_{high}$ (with $A_{high} > A_{low} > 0$).
    - Arrival of a new global purpose technology (electricity/IT),
    - Reduced form model of a reform that improves productivity.
- What happens to $k(t), c(t)$ in the short-run? In the long-run?

Permanent TFP shock



Permanent TFP shock

- Both $\dot{k} = 0$ and $\dot{c} = 0$ loci shift out. Steady-state shifts out.
- Draw stable arm from lower left to upper right quadrant. Clearly this can't uniquely sign $c(0) - c^{*,old}$ – two cases to consider.
- Look at the differential equation: if ε is large, the Euler equation is steep, so the saddle path is steep.
- Economic intuition is related to income and substitution effects:
  - **Income:** Higher TFP $\Longrightarrow$ More resources/income $\Longrightarrow$ Higher current (and future) consumption.
  - **Substitution:** Higher TFP $\Longrightarrow$ Higher MPK $(Af'(k))$ $\Longrightarrow$ Temporarily great savings opportunity, so savings $\uparrow$
- The strength of the substitution effect is parameterized by the EIS, so high EIS implies consumption jumps down.

Dynamics in non-stationary problems

- So far, we considered dynamics in stationary systems.
  - Problem and the FOCs remain the same over time.
  - So the phase diagram also remains the same over time.
  - We allowed for changes at time $t = 0$, but once the change happens we are still in a stationary system (with new parameters).
- But some economic problems are inherently nonstationary.

A non-stationary problem: Transitory shock

- If the shocks are transitory, then the problem will be "nonstationary".
  Here non-stationary = change in primitives. Not the time series sense.

- For example, consider a **transitory TFP shock**: There exists a finite $\hat{t}$ such that A goes up to A_{high} for all $t \in [0, \hat{t})$, but jumps back to A_{low} for all $t \geq \hat{t}$. What is the dynamic response of the economy?
  Interpretation: temporary productivity boom. Sharp jump is unrealistic, but insights will apply more generally.

A non-stationary problem: News shocks

- News about future shocks provides another source of non-stationarity.
- For example, suppose the agent learns at date 0 that a permanent TFP change will happen (with certainty) at a future date $\hat{t}$.

  Interpretation: news about future progress that will only be implemented after a while.

Detour: Generalized FOCs

Before we attack these problems, we need to revisit our formal results.

- Nonstationarity *per se* is not an issue for us since we are using the variational approach, and we allowed the payoff and state evolution functions to depend on time, $f(t, x, y), g(t, x, y)$.
- We did, however, require f, g to be continuously differentiable, and thus, continuous in t.
- This is a problem: in the scenarios above, there is a discontinuity in g at $t = \hat{t}$ since parameters change.
  - $\dot{k} = \begin{cases} A_{high} f(k) - \delta k - c, & \text{for } t < \hat{t} \\ A_{low} f(k) - \delta k - c, & \text{for } t \geq \hat{t} \end{cases}.$
- Luckily, the sufficiency result continues to apply in this case as long as we require our candidate path $[x(t), y(t), \mu(t)]_{t=0}^{\infty}$ to satisfy intuitive regularity conditions...

Detour: Generalized sufficient FOCs

The following are sufficient FOCs when $f(t,x,y), g(t,x,y)$ are piecewise continuous in t (and continuously differentiable in x, y):

- $x(t)$ and $\mu(t)$ **are required to be continuous everywhere**, and continuously differentiable at all t at which f, g are continuous.
- Conditions 1,2,3 hold at all t at which f, g are continuous.
- Condition 4 holds and $\hat{M}(t, x, \mu(t))$ is concave in x.

Understanding the generalized sufficient FOCs

- The key condition above is that $x(t), \mu(t)$ are **continuous everywhere**.
- Solution proceeds in two steps:
  1. The optimality conditions still apply at t for which f, g are continuous, and so give differential equations in terms of $x(t), \mu(t)$.
  2. The continuity of $x(t), \mu(t)$ then enables us to *stitch together* the solutions to the differential equations that apply in different regions of continuity.

Understanding the generalized sufficient FOCs

- In the examples, this means we have one system of differential equations that applies for $t < \hat{t}$, and another one that applies for $t > \hat{t}$.
- We need four conditions to solve these differential equations.
- We already have the conditions $k(0)$ and $\lim_{t\to\infty} k(t) = k^*$.
- The remaining two conditions come from the continuity requirement.
  - The solution to $\mu(t)$ in both systems should agree at $\hat{t}$.
  - The solution to $k(t)$ in both systems should agree at $\hat{t}$.

Intuition for the continuity requirement

What is the intuition for the continuity requirement for $k(t), \mu(t)$?

- The continuity of $k(t)$ follows from the resource constraint, $\dot{k}(t) = g(t, x, y)$. If g jumps at some times, this will create kinks at $k(t)$ but not discontinuity.

- The continuity of $\mu(t)$ is less obvious but it can be understood from the economic interpretation of the Lagrange multipliers.
  - Recall $\mu(t)$ is the marginal value of the state variable at time t.
  - Recall also that the problem is deterministic by assumption. Before time $\hat{t}$, we know that the change will happen at time $\hat{t}$.
  - When the time approaches $\hat{t}$, the marginal value of the capital cannot be too different than its marginal value at $\hat{t}$.
    Intuition: the value you assign to capital can't be too different at t and $t - \Delta t$.

Non-stationary problems in the NGM

- Recall that for the NGM $\hat{H}_c = 0$ gives $c(t)^{-1/\varepsilon} = \mu(t)$. So in the NGM, the continuity of $\mu(t)$ translates into the continuity of $c(t)$.
- This enables us to visualize the generalized FOCs using the phase diagram.
- Essentially, we are looking for a continuous path on phase diagram(s).
- Let's illustrate using the TFP news (temporary TFP shock in PS3).

Example: TFP news

- Let's analyze news about future TFP.
    - At $t = 0$, the economy learns that TFP will increase at $t = \hat{t} > 0$ to $A = A_{high}$ and stay at the higher level permanently.
    - Now, the economy follows one of the unstable solutions to the ODE for $A = A_{low}$ for $t < \hat{t}$, and then follow the saddle path corresponding to $A = A_{high}$.
- Let's do it together on the board.

Outline

Dynamic programming in continuous time

- We could also attack the continuous time problem using DP techniques, which provides additional insights and tools, just like in discrete time.
- As before, DP is most useful for stationary problems.
- So consider the following version of the canonical problem,

$$\begin{aligned} V\left(x^{*}(0)\right) \quad &\equiv \quad \max_{[y(t),x(t)]_{t=0}^{\infty}} \int_{0}^{\infty} e^{-\rho t} f\left(x(t), y(t)\right) dt \\ \text{s.t.} \quad & \dot{x}(t) = g\left(x(t), y(t)\right), \\ & x(t) \in \mathcal{X}, y(t) \in \mathcal{Y}\left(x(t)\right) \text{ given } x(0) = x^{*}(0). \end{aligned} \tag{12}$$

  - Note: the terminal condition is typically embedded in $\mathcal{X}$, e.g., $\mathcal{X} \subset \mathbb{R}_{+}$
- What's next:
  - Theory: principle of optimality (and converse), HJB equation
  - Applications: NGM, q-theory of investment

Outline

Principle of optimality

- Recall that, in discrete time, the principle of optimality established that the one-shot optimization implies recursive optimization.
- By analogy, we could conjecture that an appropriate version of the principle of optimality should also apply in continuous time.
- This indeed turns out to be the case.

Principle of optimality

For any $\Delta t > 0$, we can show that:

1. The optimum $[x^*(t), y^*(t)]_{t=0}^{\infty}$ satisfies:

$$V(x^*(0)) = \int_0^{\Delta t} e^{-\rho t} f(x^*(t), y^*(t))\, dt + e^{-\rho \Delta t} V(x^*(\Delta t)).$$

2. The value function satisfies the Bellman equation

$$\begin{aligned} V(x^*(0)) &= \max_{[x(t),y(t)]_{t=0}^{\Delta t}} \int_0^{\Delta t} e^{-\rho t} f(x(t), y(t))\, dt + e^{-\rho \Delta t} V(x(\Delta t)) \\ & \quad \text{s.t. } \dot{x} = g(x(t), y(t)) \text{ and given } x(0) = x^*(0). \end{aligned}$$

Need very little in the way of assumptions to establish these. The intuition is similar to the discrete time case.

Toward a more useful Bellman equation

- We can modify these equations to obtain something more useful.
- The key observation is that these equations hold for any $\Delta t > 0$.
    - As $\Delta t \to 0$, we would obtain versions of these equations in which the agent is choosing the instantaneous control rate, $y(0)$, which determines the instantaneous payoff rate, $f(x(0), y(0))$, as well as the instantaneous rate of change of the state variable, $\dot{x}(0) = g(x(0), y(0))$
- The resulting equation is the Hamilton-Jacobi-Bellman equation.
- The next slide provides a derivation for the limit of the first equation.
- The derivation for the second equation is similar (but more involved).

Hamilton-Jacobi-Bellman equation

- Rearranging the first equation, we obtain:

$$0 = \lim_{\Delta t \to 0} \left[\begin{array}{c} \frac{1}{\Delta t} \int_0^{\Delta t} e^{-\rho t} f\left(x^*(t), y^*(t)\right) dt \\ + \frac{e^{-\rho \Delta t} V(x^*(t+\Delta t)) - V(x^*(0))}{\Delta t} \end{array} \right].$$

  - The first line is $f\left(x^*(0), y^*(0)\right)$. The second line is $\frac{d}{dt}\left(e^{-\rho t} V\left(x^*(t)\right)\right)$ evaluated at $t = 0$ (assuming V is differentiable).

- Substituting and using the chain rule, we obtain,

$$\begin{aligned} \rho V\left(x^*\right) &= f\left(x^*, y^*\right) + \frac{dV\left(x^*(t)\right)}{dt}\Big|_{t=0} \\ &= f\left(x^*, y^*\right) + V_x\left(x^*\right) g\left(x^*, y^*\right) \end{aligned} \tag{13}$$

  - Here, x^* stands for $x^*(0)$ and y^* stands for $y^*(0)$.
  - But the result also holds for $\left(x^*(t), y^*(t)\right)$, since we could also apply the same steps starting at date t (nothing in the argument relies on $t = 0$).

Hamilton-Jacobi-Bellman equation

- Taking the limit of the second equation, we would expect to find

$$\rho V(x) = \max_{y \in \mathcal{Y}(x)} f(x, y) + V_x(x)\, g(x, y). \tag{14}$$

- The following result shows that these heuristic derivations are correct.

Theorem (HJB Equation, Special Case of Theorem A7.10)

Let V denote the value function corresponding to problem (12) and $[x^*(t), y^*(t)]_{t=0}^{\infty}$ denote an optimal path. If V is differentiable at $x^*(0)$, then value function satisfies the HJB equation (14). Moreover, the corresponding $y^*(0)$ is a solution to the HJB equation, that is, it satisfies Eq. (13).

Interpreting the HJB-accounting equation

- Eq. (13) accounts for the value of the total capital stock. Its intuition is similar to the optimality condition, $\rho\mu(t) = \hat{H}_x + \dot{\mu}(t)$.
  - $V(x(t))$ measures **the value** of the total state in **current utility terms**.
  - $\mu(t)$ measured **the marginal value** of the state, also in **current utility terms**.
- As before, $\rho V(x(t))$ is the amount of value that is lost (in present value) due to discounting as we move to the next instant. The lost value needs to be compensated. Two sources of value:

$$\overbrace{\rho V(x^*)}^{\text{required return}} = \overbrace{f(x^*, y^*)}^{\text{dividend gain/loss}} + \underbrace{\overbrace{\frac{dV(x^*(t))}{dt}\Big|_{t=0}}^{\text{capital gain/loss due to “price” change}}}_{\text{this is equal to } V_x(x^*)g(x^*,y^*) \text{ by the chain rule}}$$

  - We will soon contrast this with the Hamiltonian FOC w.r.t. x, characterizing μ

Interpreting the HJB-optimization equation

$$\rho V(x) = \max_{y \in \mathcal{Y}(x)} f(x, y) + V_x(x)\, g(x, y).$$

- The second equation is similar except it also embeds optimization.
- A similar accounting identity would hold—and determine the current value—for any feasible $y(t) \in \mathcal{Y}(x)$.
- The agent is choosing the optimal $y(t)$ taking the future value function as given (as captured by the $V_x(x)$ on the right side).
    - This typically involves a trade-off between the current payoff and the future.
    - For instance, in the neoclassical model where $y = c$, higher consumption will increase current utility $u(c)$, but will also lower future value value since $\dot{k} = g(k, c) = f(k) - \delta k - c$ (consumption depletes capital) and $V_k(k) > 0$ (more capital is good).

Converse result

- As in discrete time, the converse PO results also hold if we can safely ignore the distant future.
- In particular, if we find $V(x)$ that solves Eq. (14) for each $x \in \mathcal{X}$ and that also satisfies $\lim_{t\to\infty} e^{-\rho t} V(x(t)) = 0$ for all feasible paths $[x(t)]_{t=0}^{\infty}$, then it solves the original problem.

How we will use the HJB equation

- For problems with bounded value, one can show there is a unique bounded value function that solves the HJB equation (14). One can also establish various properties of the value function.
  Exactly analogous to our discrete-time dynamic programming results.
- We will not cover this, since the analysis in this case is more involved.
- Instead, we will use the HJB approach for three purposes:
  1. Solving the problem numerically (e.g., the NGM application)
  2. Alternative derivation of optimality conditions.
     - Among other things, verify the interpretation we provided for $\mu(t)$.
  3. Guess-and-verify method (e.g., the application for the q-theory)
- Let's review these points before we illustrate with applications...

HJB equation from the Bellman equation

- To see the connection between discrete and continuous time, it may be useful to derive the HJB equation as a limit of the discrete time Bellman equation.
- For concreteness, derivation for the neoclassical growth model
- Time periods length Δ
- Discount factor: $\beta(\Delta) = e^{-\rho\Delta}$
- Discrete time Bellman equation:

$$v(k_t) = \max_{c_t} \Delta u(c_t) + e^{-\rho\Delta} v(k_{t+\Delta}) \quad \text{s.t.}$$

$$k_{t+\Delta} = \Delta(f(k_t) - \delta k_t - c_t) + k_t$$

HJB equation from the Bellman equation

- For small Δ, $e^{-\Delta\rho} = 1 - \rho\Delta$

$$v(k_t) = \max_{c_t} \Delta u(c_t) + (1-\rho\Delta)v(k_{t+\Delta})$$

- Subtract $(1-\rho\Delta)v(k_t)$ from both sides

$$\rho\Delta v(k_t) = \max_{c_t} \Delta u(c_t) + (1-\rho\Delta)(v(k_{t+\Delta}) - v(k_t))$$

- Divide by Δ and manipulate last term

$$\rho v(k_t) = \max_{c_t} u(c_t) + (1-\rho\Delta)\frac{v(k_{t+\Delta}) - v(k_t)}{k_{t+\Delta} - k_t}\frac{k_{t+\Delta} - k_t}{\Delta}$$

- Take $\Delta \to 0$

$$\rho v(k_t) = \max_{c_t} u(c_t) + v'(k_t)\dot{k}_t$$

Connection between HJB equation and Hamiltonian

- Current value Hamiltonian

$$\hat{H}(x, y, \mu) = f(x, y) + \mu g(x, y)$$

- HJB equation

$$\rho v(x) = \max_{y \in \mathcal{Y}} f(x, y) + V_x(x) g(x, y)$$

- Connection: $\mu(t) = V_x(x(t))$ i.e. co-state = shadow value
- HJB can be written as $\rho V(x) = \max_{y \in \mathcal{Y}} \hat{H}(x, y, V_x(x))$ ...
- ... hence the "Hamilton" in Hamilton-Jacobi-Bellman
- Playing around with the FOC and envelope conditions gives maximum principle conditions for optimum we have derived above (next two slides)

FOCs from the HJB equation

- It is instructive to provide a heuristic derivation of the maximum principle optimality conditions from the HJB equation.
- Recall the HJB equation,

$$\rho V(x) = \max_{y \in \mathcal{Y}(x)} f(x, y) + V_x(x)\, g(x, y).$$

- Make the following educated guess for the CV costate:

$$\mu(t) = V_x(x(t)) \text{ for each } t.$$

- Then, note that the HJB equation directly implies condition 1:

$$y^*(t) \in \arg\max_y \hat{H}(t, x^*(t), y, \mu(t)).$$

- The objective in the HJB equation is essentially the CV Hamiltonian!

FOCs from the HJB equation

- We can also get condition 2 (assuming $V(x)$ is twice differentiable).
- Using an envelope theorem on the HJB equation, we obtain,

$$\rho V_x(x) = f_x(x, y) + V_x(x(t))\, g_x(x, y) + V_{xx}(x, y)\, g(x, y).$$

- Differentiating $\mu(t) = V_x(x(t))$ with respect to time yields

$$\dot{\mu} = V_{xx}(x)\, \dot{x} = V_{xx}(x)\, g(x, y).$$

- Plugging $\mu = V_x$ and $\dot{\mu} = V_{xx} g$ into the top equation, we obtain:

$$\rho\mu(t) = f_x(x, y) + \mu(t)\, g_x(x, y) + \dot{\mu}(t),$$

or equivalently,

$$\rho\mu(t) - \dot{\mu}(t) = \hat{H}_x(t, x(t), y(t), \mu(t)).$$

- So the HJB equation embeds the date t optimality conditions!

Outline

NGM with dynamic programming

- Let us illustrate the dynamic programming approach using the NGM:

$$\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u\left(c\left(t\right)\right) dt$$

$$\text{s.t.} \qquad \dot{k}\left(t\right) = f\left(k\left(t\right)\right) - \delta k\left(t\right) - c\left(t\right) \text{ for each } t,$$
$$c\left(t\right), k\left(t\right) \geq 0 \text{ and given } k\left(0\right),$$

- The problem is stationary in k. The corresponding HJB equation is,

$$\rho V\left(k\right) = \max_{c \geq 0} u\left(c\right) + V_k\left(k\right)\left(f\left(k\right) - \delta k - c\right). \tag{15}$$

- Assume isoelastic utility $u\left(c\right) = \frac{\varepsilon}{\varepsilon-1}\left(c^{\frac{\varepsilon-1}{\varepsilon}} - 1\right)$. Then...

Optimality conditions for the NGM

- The optimality condition for problem (15) gives,

$$c^{-1/\varepsilon} = V_k(k), \text{ which implies } c = C(k) = (V_k(k))^{-\varepsilon}. \tag{16}$$

- Note that $C(k)$ is the policy function (for c as opposed to k^{next}).

Numerically solving the NGM

- For the numerical solution, we can plug $C(k) = V_k(k)^{-\varepsilon}$ from Eq. (16) back into the HJB equation (15) to get

$$\begin{aligned} \rho V(k) &= \frac{\varepsilon}{\varepsilon - 1}\left(V_k(k)^{1-\varepsilon} - 1\right) + V_k(k)\left(f(k) - \delta k - V_k(k)^{-\varepsilon}\right) \\ &= \frac{1}{\varepsilon - 1} V_k(k)^{1-\varepsilon} + V_k(k)(f(k) - \delta) - \frac{\varepsilon}{\varepsilon - 1} \end{aligned} \tag{17}$$

- This is an implicit first order ordinary differential equation for $V(k)$.
  - Implicit because it describes $V_k(k)$ as an implicit function of $k, V(k)$.
  - It would be explicit if we could write $V_k(k) =$ function of k and $V(k)$.
- This can be solved on the computer using the finite difference method.

Outline

Numerical Solution: Finite-Diff. Method

- By far the simplest and most transparent method for numerically solving differential equations.
- Approximate $k(t)$ and $c(t)$ at N discrete points in the time dimension, $t^n, n = 1, ..., N$. Denote distance between grid points by Δt.
- Use short-hand notation $k^n = k(t^n)$.
- Approximate derivatives

$$\dot{k}(t^n) \approx \frac{k^{n+1} - k^n}{\Delta t}$$

- Approximate the Euler and capital accumulation equations as

$$\frac{c^{n+1} - c^n}{\Delta t} \frac{1}{c^n} = \frac{1}{\sigma}(f'(k^n) - \rho - \delta)$$
$$\frac{k^{n+1} - k^n}{\Delta t} = f(k^n) - \delta k^n - c^n$$

Finite-Diff. Methods/Shooting Algorithm

- Or

$$
\begin{aligned}
c^{n+1} &= \Delta t c^n \frac{1}{\sigma}(f'(k^n) - \rho - \delta) + c^n \\
k^{n+1} &= \Delta t (f(k^n) - \delta k^n - c^n) + k^n
\end{aligned}
\tag{FD}
$$

  with $k^0 = k_0$ given.

- For example: how would you draw phase diagram/saddle path in MATLAB?
- Make parametric assumptions e.g.: $f(k) = Ak^\alpha$, $A = 1$, $\alpha = 0.3$, $\sigma = 2$, $\rho = \delta = 0.05$, $k_0 = \frac{1}{2}k^*$, $\Delta t = 0.1$, $N = 700$.
- Algorithm:
  1. guess c^0
  2. obtain $(c^n, k^n), n = 1, ..., N$ by running (FD) forward in time.
  3. If converges to (c^*, k^*), done. If not, back to (i) and try different c^0.
- This is a "shooting algorithm", essentially the same as the one in discrete time (you have seen this in PS2 Q4)

Shooting algorithms vs DP

- The dynamic programming (HJB) approach turns the problem into solving an ODE over the state variable k, as opposed to time t.
- Finite difference method still applies, but more structure: approximate $V_k(k_i)$ with
$$\frac{V_i - V_{i-1}}{\Delta k} \text{ or } \frac{V_{i+1} - V_i}{\Delta k}$$
depending on the drift of the state variable. This is called an upwind scheme.
- For details and applications with codes, see `http://benjaminmoll.com/wp-content/uploads/2020/06/HACT_Additional_Codes.pdf`

Outline

Points to note

- As in discrete time, the power of the dynamic approach in continuous time – the HJB equation – is most prescient when there are many state variables (note we are then solving PDEs rather than ODEs) and especially when there is uncertainty.
- Studying stochastic optimal control requires knowledge of stochastic calculus and is beyond the scope of this course.
- Here: just a little flavour of how things work.

Stochastic optimal control

- Generic problem:

$$V(x_0) = \max_{\{y(t)\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} f(x(t), y(t)) dt$$

subject to the law of motion for the state

$$dx(t) = g(x(t), y(t))dt + \sigma(x(t))dW(t) \tag{LM}$$

and $y(t) \in \mathcal{Y}$ for $t \geq 0$, $x(0) = x_0$ given.

- $\sigma(x) = 0$ gives the deterministic problem we studied so far.
- $W(t)$ is a Brownian motion, defined as $W(t + \Delta t) - W(t) = \varepsilon_t \sqrt{\Delta t}$, with $\varepsilon_t \sim \mathcal{N}(0, 1)$
    - a cont time analogue of random walk: $W_{t+1} = W_t + \varepsilon_t, \ \varepsilon_t \sim \mathcal{N}(0, 1)$
- notation $dW(t) := \varepsilon_t \sqrt{dt}$.
- (LM) is called a stochastic differential equation.

Stochastic HJB equation

- Claim: the HJB equation is

$$\rho V(x) = \max_{y \in \mathcal{Y}} f(x, y) + V'(x)\dot{x} + \frac{1}{2}V''(x)\sigma^2(x)$$

- If you want to see derivation, see e.g. chapter 2 in Stokey (2008).
- The final term comes from Ito's lemma: the most important theorem in stochastic calculus. Intuition: the value is reduced by uncertainty if there is risk aversion $v''(x) < 0$.
- Note: using Brownian motion is not restrictive.
  - By choosing functions g and σ, can get pretty much any process – without jumps
  - What about jumps?

Stochastic HJB equation: Poisson uncertainty

- But perhaps the simplest way of modelling uncertainty in continuous time: two state Poisson process (i.e. jumps).
- To fix ideas: NGM with 2-state productivity
- $z_t \in \{z_1, z_2\}$ Poisson with intensities λ_1, λ_2.
- Claim: the HJB equation is

$$\rho V_i(k) = \max_c u(c) + V_i'(k)(z_i f(k) - \delta k - c) + \lambda_i(V_j(k) - V_i(k))$$

- This is rather intuitive.
- And you may have seen this before without realising: e.g. in the context of a Diamond-Mortensen-Pissarides model where

$$\rho W = w + \sigma(U - W)$$
$$\rho U = b + f(\theta)(W - U)$$