Lecture 4: Stochastic Dynamic Programming

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Introducing uncertainty

- So far, our problems were deterministic. Many economic problems however feature uncertainty.
- Now: discrete time problems with uncertainty.
    1. Stochastic version of the dynamic programming approach. Introduce today for NGM with shocks.
- Main takeaway from this:
    1. *Global* solutions: On computer need to discretize shocks, math is then $\approx$ same as w/o uncertainty. [Will see this today.] But run into curse of dimensionality for VFI solution more quickly.
    2. *Local* solutions: no particular challenge, stochastic = deterministic. Follows because linearization $\rightarrow$ certainty equivalence. More on this next term with Vincent.

Outline

Outline

NGM with productivity shocks

- Consider the NGM with shocks to productivity.
  - As before focus on version with exogenous labor (simplicity).
  - With endogenous labor, this would be the baseline RBC model.
- Output is now given by

$$y_t = z_t f\left(k_t\right)$$

  where k_t is capital as before, and z_t is the TFP shock.
- We assume the TFP shock follows a Markov chain...

Outline

We model uncertainty with Markov chains

- At each time, z_t is drawn from the set $Z = \{\zeta^1, ..., \zeta^m\} \subset \mathbb{R}_+$.
  - Suppose wlog the set Z is ordered: $\zeta^1 < .. < \zeta^m$.
- The transition probabilities are denoted by $\pi(z_{t+1}|z_t)$ and can be stacked together in an $m \times m$ matrix of the form:

$$P = \begin{bmatrix} \pi(\zeta^1|\zeta^1) & \pi(\zeta^2|\zeta^1) & ... \\ \pi(\zeta^1|\zeta^2) & \pi(\zeta^2|\zeta^2) & ... \\ ... & ... & ... \end{bmatrix}.$$

- Row i: Probabilities of next state given that current state is ζ^i.
- The key feature is **Markov property:** Relevant information for the future realizations of uncertainty are captured by the latest realization, $z_t = \zeta^i$. Put differently, the process is "memoryless." The history does not matter except for the most recent realization.

Examples of Markov chains

For example, suppose $Z = \{0.5, 1, 1.5\}$:

- **Fully temporary shocks:**
  - Suppose z_t is i.i.d. with $\pi(0.5) = \pi(1.5) = 1/4$ and $\pi(1) = 1/2$.
  - Then the corresponding matrix is given by:

$$P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

- **Fully permanent shocks:**
  - Suppose $\pi(0.5|1) = \pi(1.5|1) = 1/4$ and $\pi(1|1) = 1/2$ as before.
  - But also suppose $\pi(0.5|0.5) = 1$ and $\pi(1.5|1.5) = 1$.
  - Then the corresponding matrix is given by:

$$\Pi = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}.$$

- We could also accommodate other possibilities in between.

Markov chains – invariant distribution

Unconditional probability distribuions evolve according to:

$$\pi'_{t+1} = \pi'_t P \tag{1}$$

An unconditional distribution is called *stationary* or *invariant* if it satisfies:

$$\pi_{t+1} = \pi_t \tag{2}$$

$$\pi' = \pi' P \tag{3}$$

$$\pi'(I - P) = 0 \tag{4}$$

or

$$(I - P')\pi = 0 \tag{5}$$

Markov Chains - invariant distribution

- π is an eigenvector (appropriately normalized) associated with a unit eigenvalue of P.
- The fact that P has non-negative elements and satisfies $\sum_j P_{ij} = 1$ for all i guarantees that P has at least one unit eigenvalue and that there is at least one eigenvector π that satisfies equation (5).
- The stationary distribution is not necessarily unique.

Proposition

Let P a stochastic matrix with $P_{ij} > 0, \quad \forall (i, j)$. Then P has a unique stationary distribution, and the process is asymptotically stationary.

Outline

History and choice variables

- Let z^t denote the history of shocks up to period t:

$$z^t \equiv (z_0, z_1, ..., z_t).$$

- We also let $\pi(z^t|z_0) \equiv \pi(z_t|z_{t-1})\pi(z_{t-1}|z_{t-2})....\pi(z_1|z_0)$ denote the probability of the history z^t conditional on the initial state.
- Note that the allocations in period t are contingent on the particular realization of z^t. Hence, we should now be thinking of allocations as random variables: the uncertainty in TFP introduces uncertainty in output, consumption, etc.
- Accordingly, the planner's problem involves choosing, not deterministic time sequences, but rather sequences of random variables (i.e., stochastic processes) of the form,

$$\{c_t(z^t), k_{t+1}(z^t)\}_{t=0}^{\infty}.$$

The sequence problem

- The planner's sequence problem can be written as:

$$
\begin{aligned}
V^*(k_0, z_0) \quad &= \quad \max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \sum_{z^t} \pi(z^t | z_0) \beta^t u(c_t(z^t)), && (6) \\
\text{s.t.} \quad & \quad c_t(z^t) + k_{t+1}(z^t) = (1-\delta) k_t(z^{t-1}) + z_t f\left(k_t(z^{t-1})\right), \\
& \quad \text{with } c_t \geq 0, k_{t+1} \geq 0 \text{ and } \left(k_0 > 0, z_0 \in Z\right) \text{ given.}
\end{aligned}
$$

- Note that the initial "state" is now capital, k_0, and current productivity, z_0.
- The planner is maximizing the sum of **expected** discounted utilities.

The sequence problem looks complicated

- Think of the event tree. The planner is effectively choosing one (c_t, k_{t+1}) at date 0, m such pairs at date 1, m^2 such pairs at date 2, and so on.

- Clearly, the dimensionality of the problem is exploding with time. Even if there are only two states ($m = 2$), by date $t = 20$ we have more than 2×2^{20} $\approx$2 million choice variables!

Outline

The recursive formulation is more parsimonious

- The main benefit of the DP formulation will be its much simpler representation of the problem and the solution. Instead of stochastic processes, we will just have to think about policy functions,

$$k^{next} = g(k, z) \text{ and } c = C(k, z).$$

- These functions embed all the relevant information. The sequences of random variables $\{c(z^t), k_{t+1}(z^t)\}$ can be obtained from these functions.
    - Starting with k_0 and z_0, we have $c_0 = C(k_0, z_0)$ and $k_1 = g(k_0, z_0)$.
    - Given k_1 and z_1, we have $c_1 = C(k_1, z_1)$ and $k_2 = g(k_1, z_1)$...

Outline

1. Stochastic problem setup
    - NGM with productivity shocks
    - Markov chains
    - The sequence problem

2. Recursive formulation
    - Bellman equation

Dynamic programming formulation

- The stationary structure allows for a recursive formulation.
- Note that

$$
\begin{aligned}
V^*(k_0, z_0) &= \sum_{t=0}^{\infty} \sum_{z^t} \pi(z^t|z_0)\beta^t u(c_t(z^t)) \\
&= u(c_0(z_0)) + \beta \sum_{t=1}^{\infty} \sum_{z^t} \pi(z^t|z_0)\beta^{t-1} u(c_t(z^t)) \\
&= u(c_0(z_0)) + \beta \sum_{t=1}^{\infty} \sum_{z_1 \in Z} \sum_{z^t|z_1} \overbrace{\pi\left(z^t|z_1\right) \pi(z_1|z_0)}^{\text{this is equal to } \pi(z^t|z_0)} \beta^{t-1} u(c_t(z^t)) \\
&= u(c_0(z_0)) + \beta \sum_{z_1 \in Z} \pi(z_1|z_0) \underbrace{\sum_{t=1}^{\infty} \sum_{z^t|z_1} \pi\left(z^t|z_1\right) \beta^{t-1} u(c_t(z^t))}_{\text{this is equal to } V^*(k_1, z_1)}
\end{aligned}
$$

The Bellman equation with uncertainty

- We could repeat the above steps starting at z^t as opposed to z_0.
- The payoff relevant state is (k_t, z_t), where z_t is the latest realization.
- Hence, the dynamic programming formulation (or the Bellman equation) is:

$$\begin{aligned} V(k,z) &= \max_{c,k^{next}} u(c) + \beta \sum_{z^{next}\in Z} \pi\left(z^{next}|z\right) V\left(k^{next}, z^{next}\right), \quad (7) \\ \text{s.t.} \quad & c + k^{next} \leq \phi(k,z) = zf(k) + (1-\delta)k \\ & \text{with } c \geq 0,\ k^{next} \geq 0 \text{ and } (k,z) \text{ given.} \end{aligned}$$

The Bellman equation with uncertainty

- Note the problem is mathematically similar to the deterministic case.
- The uncertainty has been captured by expanding the state space.
- With finite Z and bounded payoffs (our focus), the dynamic programming approach remains very similar to the non-stochastic case.
- Computationally:
    - Once z is discretized: no particular challenge. Value function with two arguments but only one choice – one state (z) evolves completely exogenously
    - Curse of dimensionality: $n_z = 10$ and $n_k = 100$ already gives 1,000 states
    - Alternative: linearization

Euler equation

- The envelope condition is

$$
\begin{aligned}
V_k(k,z) &= \phi_k(k,z)\,u'(\phi(k,z)-g(k,z)) \\
&= (zf'(k)+1-\delta)\,u'(\phi(k,z)-g(k,z))
\end{aligned}
$$

- The optimality condition is:

$$
\begin{aligned}
u'\left(\phi(k,z)-k^{next}\right) &= \beta \sum_{z^{next}\in Z} \pi\left(z^{next}\right) V_k\left(k^{next},z^{next}\right) \qquad (8) \\
&= \beta E\left[V_k\left(k^{next},z^{next}\right)\right]
\end{aligned}
$$

- By substituting the expression for V_k, we can recover the Euler equation.

  $\Rightarrow$ As before, the DP formulation embeds the FOCs at each t.

What is “the steady state” in this model?

- To see if there is a steady-state, consider the evolution of capital,

$$\begin{aligned} k_1 &= g(k_0, z_0), \\ k_2 &= g(k_1, z_1), \\ &\ldots \\ k_{t+1} &= g(k_t, z_t). \end{aligned}$$

- Clearly, $k_t = k^*$ cannot satisfy these equations for all shocks.
- Shocks keep perturbing the economy forever. So we cannot possibly expect a steady-state or convergence to it.