1 Differential Geometry

1.1 Tensors

…are important in GR because:

- spacetime is curved in GR, captured by (1,3) Riemann tensor
- gravity is a consequence of geometry given by (0,2) Lorentzian metric tensor
- there is no special coordinate system
- physical laws are tensor equations that hold in all coordinate systems
- equations of motion of spacetime are Einstein tensor equation

Notes:

- use different indices when transforming a contraction!

$$T^{\mu}{}_{\mu}(x) \to T'^{\mu'}{}_{\mu'}(x') = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\mu'}} T^{\mu}{}_{\rho}(x)$$

- careful when transforming derivatives: product rule!

$$\text{velocity is a vector:} \quad \frac{\mathrm{d}x'^{\mu'}}{\mathrm{d}\lambda} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}$$

$$\text{but its derivative is not:} \quad \frac{\mathrm{d}^2 x'^{\mu'}}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\right)$$

1.2 Local Inertial Frames (LIF)

There are coordinates $\{x^{\mu}\}$ centered at point p in which

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + O(x^2)$$

Proof.

1. Shift coords to have p at origin.
2. Ansatz: quadratic change of coordinates

$$x'^{\mu'} = M^{\mu'}{}_{\mu} x^{\mu} = B^{\mu'}{}_{\mu}(x^{\mu} + \frac{1}{2} C^{\mu}{}_{\rho\sigma} x^{\rho} x^{\sigma})$$

   note: B is invertible, C is symmetric

3. find change of coordinate matrices

$$M^{\mu'}{}_{\mu} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}}, \qquad \overline{M}^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x'^{\mu'}}$$

   note: when proving inverse, check left and right inverse!

4. Taylor expand $g_{\mu\nu}(x)$
5. Choose $C^{\mu}{}_{\rho\sigma} = \Gamma^{\mu}{}_{\rho\sigma}(0)$, find a term $\nabla_{\sigma} g_{\mu\nu} = 0$
6. $g_{\mu\nu}(0)$ symmetric and invertible $\implies$ diagonalisable

□

Properties:

- at the origin p of a LIF, $g_{\mu\nu} = \eta_{\mu\nu}$ and so

$$\partial_{\rho} g_{\mu\nu} = 0, \quad \Gamma^{\mu}{}_{\rho\sigma} = 0, \quad \nabla_{\rho} = \partial_{\rho} \quad \text{but} \quad \partial_{\nu}\Gamma^{\mu}{}_{\rho\sigma} \neq 0$$

- local LT: $x'^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu}$ is also a LIF at p

  $\Rightarrow$ can find rest frame where velocity is: $v^{\mu}|_p = (1,0,0,0)$

- Riemann Normal Coordinates: $O(x^2) = -\frac{1}{3} R_{\mu\nu\rho\sigma} x^{\rho} x^{\sigma}$

1.3 Covariant derivative

Definition. Covariant derivative ∇_{μ}

$$\nabla_{\mu} V^{\nu}(x) = \partial_{\mu} V^{\nu}(x) + \Gamma^{\nu}{}_{\mu\sigma} V^{\sigma}(x)$$

- $\Gamma^{\mu}{}_{\rho\sigma}$ is the Christoffel symbol (not a tensor)

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2} g^{\mu\alpha}(g_{\alpha\sigma,\rho} + g_{\alpha\rho,\sigma} - g_{\rho\sigma,\alpha})$$

  useful to know: $\Gamma^{\mu}{}_{\mu\mu} = \frac{1}{2} g^{\mu\mu}(-g_{\mu\mu,\mu})$

- metric connection for vector spaces at x^{μ} and $x^{\mu} + \delta x^{\mu}$

$$V^{\mu}(x + \delta x) = V^{\mu}(x) - \Gamma^{\mu}{}_{\rho\sigma}(x) V^{\rho}(x) \delta x^{\sigma}$$

- using Leibnitz product rule, can deduce

$$\begin{aligned} \nabla_{\mu} T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \ldots \nu_m} &= \partial_{\mu} T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \ldots \nu_m} \\ &\left.+ \sum_{i=1}^{n} \Gamma^{\mu_i}{}_{\mu\alpha} T^{\mu_1 \ldots \alpha \ldots \mu_n}{}_{\nu_1 \ldots \nu_m} \right\} \text{upper } (+) \\ &\left.- \sum_{i=1}^{m} \Gamma^{\alpha}{}_{\mu\nu_i} T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \ldots \alpha \ldots \nu_m} \right\} \text{lower } (-) \end{aligned}$$

- For example

$$\nabla_{\mu} T^{\nu}{}_{\rho} = \partial_{\mu} T^{\nu}{}_{\rho} + \Gamma^{\nu}{}_{\mu\alpha} T^{\alpha}{}_{\rho} - \Gamma^{\alpha}{}_{\mu\rho} T^{\nu}{}_{\alpha}$$

- $\nabla_{\mu} \delta^{\rho}{}_{\sigma} = \nabla_{\mu} g_{\rho\sigma} = \nabla_{\mu} g^{\rho\sigma} = 0$
  - Can raise and lower indices freely inside $\nabla_{\mu}(\ldots)$

1.4 Parallel Transport

V^{μ} is parallelly propagated / transported along $x^{\mu}(\lambda)$ if

$$\frac{D}{D\lambda} V^{\mu} := \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda} \nabla_{\nu} V^{\mu} = 0.$$

Along an affinely parametrised geodesic, the tangent vector $v^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}$ is parallelly propagated along the curve:

$$v^{\mu} \nabla_{\mu} v^{\nu} = 0 \quad \text{(geodesic equation)}$$

- $v \cdot v$ is constant along a geodesic

$\Rightarrow$ a geodesic is either timelike, null, or spacelike

For a general, non-affine parameter, the geodesic condition is that the parallel transport of the tangent vector is proportional to itself:

$$\boxed{v^{\mu} \nabla_{\mu} v^{\nu} = f(\lambda) v^{\nu}}$$

For affine parameters, $f(\lambda) = 0$ vanishes.

1.5 Curvature

- Riemann Curvature tensor

$$R^{\nu}{}_{\mu\rho\sigma} = \partial_\rho \Gamma^{\nu}{}_{\mu\sigma} + \Gamma^{\nu}{}_{\rho\alpha}\Gamma^{\alpha}{}_{\mu\sigma} - (\rho \leftrightarrow \sigma)$$

  - Symmetries:

$$\begin{aligned} \text{Sym in (12)(34)} : R_{\alpha\mu\sigma\nu} &= R_{\sigma\nu\alpha\mu} \\ \text{ASym in 34} : R_{\alpha\mu\sigma\nu} &= -R_{\alpha\mu\nu\sigma} \\ \text{cyclic in 234} : R_{\alpha\mu\sigma\nu} + R_{\alpha\sigma\nu\mu} + R_{\alpha\nu\mu\sigma} &= 0 \end{aligned}$$

  - Bianchi Identities (cyclic in (34;5))

$$R_{\alpha\mu\sigma\nu;\lambda} + R_{\alpha\mu\nu\lambda;\sigma} + R_{\alpha\mu\lambda\sigma;\nu} = 0$$

    proof: write down first tensor (LIF), then permute $\mu, \sigma, \nu, \mu, \ldots$ in mind

  - gives commutator of (co-)vector covariant derivatives

$$[\nabla_\sigma, \nabla_\rho] W_\mu = R^{\nu}{}_{\mu\rho\sigma} W_\nu$$

    for a general tensor, have one term for each index (- up, and + down)

  - encodes curvature via difference of parallel transport along different paths

    flat: vanishes for flat manifold: $R^{\nu}{}_{\mu\rho\sigma} = 0$

- Ricci Tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$
- Ricci Scalar $R = R^{\alpha}{}_{\alpha}$
- Einstein Tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$
  - $G^{\alpha}{}_{\mu;\alpha} = 0$

  proof: start with Bianchi. Contract over the first tensor twice.

1.6 Matrices

Inverse of a block diagonal matrix:

$$\begin{pmatrix} \mathbf{A}_1 & 0 & \ldots & 0 \\ 0 & \mathbf{A}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{A}_n \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}_1^{-1} & 0 & \ldots & 0 \\ 0 & \mathbf{A}_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{A}_n^{-1} \end{pmatrix}$$

Inverse of 2×2 matrix:

$$\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

1.7 Spherical Polars

Euclidean metric is

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2) \tag{1}$$

and

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

2 Special Relativity

...is physics (without gravity) in Minkowski space $\mathbb{M}^4$.
In inertial coordinates $x^\mu = (t, x, y, z)$ the metric has the form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

Proper time is defined by $d\tau^2 = -ds^2$ and so

$$g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -1$$

velocity of massive particle satisfies $v^\mu v_\mu = -1$

note: can always swap velocity normalisation (or in the null case $g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$ for one of the geodesic equations

2.1 Timelike, Spacelike, Null

Types of spacetime intervals in $\mathbb{M}^4$:

timelike: $ds^2 < 0$ null: $ds^2 = 0$ spacelike: $ds^2 > 0$

- classify vectors v^μ similarly

  timelike: $v^\mu v_\mu < 0$ null: $v^\mu v_\mu = 0$ spacelike: $v^\mu v_\mu > 0$

- classify curves $x^\mu(\lambda)$ by their tangent vector $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$

  timelike curve $v^\mu v_\mu < 0$ everywhere along the curve

- for any event $O \in \mathbb{M}^4$, spacetime is divided into

  causal future events on and in the future light cone; events reachable by future directed timelike or null geodesics starting on O

  causal past events on and in the past light cone

  spacelike events causally unrelated to O

2.2 Future and Past Pointing

only for timelike and null vectors v^μ:

$$\text{future pointing: } v^0 > 0 \qquad \text{past pointing: } v^0 < 0 \tag{2}$$

Can always find an inertial frame in which

timelike: then in some inertial frame $v^\mu = (\pm k, 0, 0, 0)$ where k is given by the norm $v^\mu v_\mu = -k^2 \neq 0$

null: then in some inertial frame $v^\mu = (1, 1, 0, 0)$

2.3 Lorentz Transformations

...are linear coordinate changes $x'^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^\mu$ between inertial frames that keep the origin fixed.

- "between inertial frames" $\Rightarrow \quad \eta_{\mu\nu} = \eta_{\mu'\nu'} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu}$
- "linear": components of vector transform in the same way that coordinates do (not true e.g. for transform from Cartesian to polar)
- Lorentz group $O(1,3)$: rotations, reflections, boosts, and their compositions.
- $SO(1,3)$ is the subgroup with $\det \Lambda = \pm 1$
- proper orthochronous Lorentz group $SO^+(1,3)$ preserves temporal and spatial orientation (i.e. no reflections)

2.3.1 Lorentz Boosts

...are the most important LTs.

$$\left.\begin{array}{l} t \longmapsto t' = \gamma(t - vx) \\ x \longmapsto x' = \gamma(x - vt) \\ y \longmapsto y' = y \\ z \longmapsto z' = z \end{array}\right\} \gamma = \frac{1}{\sqrt{1-v^2}}$$

2.4 Geodesics

- Tangent vector to curve $\Gamma : x^\mu(\lambda)$ is

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$$

Definition. Geodesic: curve that extremises proper time

$$\frac{Dv^\mu}{D\lambda} = v^\nu \nabla_\nu v^\mu = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\sigma\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = \kappa\underbrace{\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}}_{\text{vanishes for affine parameter}}$$

Derivation of affinely parametrised geodesic equation:

1. use $\mathrm{d}\tau^2 = -\,\mathrm{d}s^2$

$$\tau = \int_A^B \mathrm{d}\tau = \int_{\lambda_A}^{\lambda_B} \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\,\mathrm{d}\lambda = \int_{\lambda_A}^{\lambda_B}\sqrt{L}\,\mathrm{d}\lambda$$

2. varying gives $\delta\tau = \int \frac{1}{2\sqrt{L}}\,\mathrm{d}\lambda\,\delta L$
3. choose $\tau = \lambda$, then $\sqrt{L} = 1$ and $\delta\tau = \frac{1}{2}\delta(\int \mathrm{d}\tau\, L)$
4. E-L eqns: $\frac{\mathrm{d}}{\mathrm{d}\tau}\underbrace{(-2g_{\mu\nu}v^\nu)}_{\partial L/\partial v^\mu} + \underbrace{g_{\sigma\nu,\mu}v^\sigma v^\nu}_{\partial L/\partial x^\mu} = 0$
5. use chain rule: $\frac{\mathrm{d}}{\mathrm{d}\tau}g_{\mu\nu}(x) = g_{\mu\nu,\rho}v^\rho$
6. split and relabel: $g_{\mu\nu,\rho}v^\rho v^\nu = \frac{1}{2}(g_{\mu\nu,\rho} + g_{\mu\rho,\nu})v^\rho v^\nu$
7. finally, raise with $g^{\mu\alpha}$ to isolate a^ν

- affine parameter: any λ such that $\kappa = \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}\lambda} = 0$
- affine transformation: $\lambda' = a\lambda + b$, where a, b constant
- proper time τ is an affine parameter
- derivative w.r.t. affine parameter: chain rule

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\partial}{\partial x^\mu} = v^\mu\partial_\mu$$

- geodesics are straight lines $x^\mu(\lambda) = a^\mu\lambda + b^\mu$ in inertial coordinates

2.4.1 Massive Particles

Tangent vector $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ is timelike and future pointing. Using $\lambda = \tau$:

- tangent vector is the (four-)velocity of the particle

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \qquad v\cdot v = \left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 = -1$$

  - In an inertial frame $\{t, x, y, z\}$

$$v^\mu = \gamma\big(1, u^i\big), \qquad \gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-u^2}},$$

    where $u^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}$ is the Newtonian three-velocity
  - Instantaneous rest frame: $v^\mu = (1, 0, 0, 0)$.

- geodesic equations are E-L eqns for the action

$$S = \int d\tau L \qquad L = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

- When there is a force, $F^\mu = ma^\mu = \frac{Dp^\mu}{D\tau}$.
  - Since $a\cdot v = 0$, acceleration and force are spacelike
- (four-) momentum: $p^\mu = mv^\mu$.
- $v\cdot v = -1 \Rightarrow p^2 + m^2 = 0$. Total momentum (without external forces) is conserved.
- energy of particle depends on velocity v_{obs} of observer

$$E = -v_{obs}\cdot p$$

  scalar is physically meaningful

- "energy" only physically meaningful if there is a meaningful inertial frame
  - in particle's rest frame $E = p^0 = m$.
  - in observer's rest frame, $E = p^0 = m\gamma$

2.4.2 Massless Particles

...travel on null geodesics satisfying

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0,$$

and

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$$

where $\lambda \neq \tau$ since $\mathrm{d}\tau = 0$ on null worldlines.
A photon has null momentum p^μ proportional to the tangent vector. The observed energy is again $E = -v_{\text{observer}}\cdot p$

- no physical meaning for free, non-interacting photon

2.5 Spacetime diagrams

If you have coordinates $x^\mu(\lambda)$ in terms of parameter λ, then to draw a spacetime diagram of t vs x, use

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}x}{\mathrm{d}\lambda}\right)^{-1}$$

hint: for null geodesic, slope is $\frac{\mathrm{d}t}{\mathrm{d}r} = \pm 1$

3 General Relativity

To deduce a law in GR from SR:

1. express law in inertial frame in $\mathbb{M}^4$ in terms of $\eta_{\mu\nu}$ & ∂_μ
2. make covariant: $\eta_{\mu\nu} \to g_{\mu\nu}$, $\partial_\mu \to \nabla_\mu$
3. change metric (Minkowski) $\to$ (curved spacetime)

- Equivalence principle: in small region, LIF, approx. $\mathbb{M}^4$
- Free particles follow geodesics

3.1 Vacuum Einstein Equations

$$R_{\mu\nu} = 0$$

Motivation: Analogies to Newtonian Physics

1. connecting vector $\eta^i = x_2^i - x_1^i \longrightarrow \eta^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$
2. time derivative $\longrightarrow$ covariant derivative along geodesic
$$\frac{\mathrm{d}}{\mathrm{d}t} \longrightarrow \frac{D}{D\tau} = v^\rho \nabla_\rho$$
3. Newtonian tidal forces $\longrightarrow$ geodesic deviation
$$\frac{\mathrm{d}^2\eta^i}{\mathrm{d}t^2} = -\Phi_{,ik}\eta^k \longrightarrow \frac{D^2\eta^\nu}{D\tau^2} = R^\nu{}_{\sigma\rho\mu} v^\sigma v^\rho \eta^\mu$$
4. Laplace's equation
$$\delta^{ij}\Phi_{,ji} = 0 \longrightarrow -R^\nu{}_{\sigma\rho\nu} v^\sigma v^\rho = 0$$

4 Schwarzschild Solution

...is spacetime metric outside spherically symmetric star of surface $r = R > 2M$.
In Schwarzschild coordinates $\{t, r, \theta, \phi\}$, with $G_N = c = 1$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

- Properties:
  - Birkoff: The only spherically sym. soln of $R_{\mu\nu} = 0$.
  - asymptotically flat: $ds^2 \to \mathbb{M}^4$ as $r \to \infty$
  - static: indep. of t, inv. under $t \to -t$
  - coordinate singularity at $r_S = 2M$
- for constant radial and angular coords, use $\mathrm{d}tau^2 = -\mathrm{d}s^2 = f(R)\,\mathrm{d}t^2$ to find $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ and $a^\mu = v^\nu \nabla_\nu v^\mu$

4.1 Newtonian (Weak Field) Limit

In Newtonian frame, to order $\mathcal{O}(\Phi^{3/2})$

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) + \dots$$

$$g_{\mu\nu} = \eta_{\mu\nu} - 2\Phi\delta_{\mu\nu} + \dots \qquad g^{\mu\nu} = \eta^{\mu\nu} + 2\Phi\delta^{\mu\nu} + \dots$$

Work out Christoffels in index form:

$$\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2}\eta^{\mu\alpha}(-2\delta_{\alpha\nu}\Phi_{,\sigma} - 2\delta_{\alpha\sigma}\Phi_{,\nu} + 2\delta_{\nu\sigma}\Phi_{,\alpha}) + \dots$$

- results are not tensor equations! only valid in Newtonian frame
  - in particular, $\delta_{\mu\nu}$ are just numbers and so
$$\eta^{\mu\rho}\delta_{\mu\sigma} = \eta^{\sigma\rho} \tag{3}$$

Non-relativistic limit $|v_3|^2 = \mathcal{O}(\Phi)$:

$$v^\mu = \frac{\mathrm{d}t}{\mathrm{d}\tau}(1, v_3^i) \approx (1 - \Phi + \frac{1}{2}|v_3|^2, v_3^i) + \mathcal{O}(|v_3|^3)$$

1. relativistic and gravitational time dilation
$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1 - \frac{1}{2}|v_3|^2 + \Phi + \dots$$
2. $\mu = t$ geodesic equation: energy conservation
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|v_3|^2 + \Phi\right) + \dots = 0$$
3. $\mu = i$ geodesic equation: Newtonian gravitation
$$a_3^i = \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = -\Phi_{,i} + \dots$$

- To first order, only nonzero component of Einstein tensor is $G_{tt} = 2\delta^{ij}\delta_i\delta_j\Phi + \cdots \implies \Phi$ satisfies Poisson's equation.

When applying the weak field limit with

$$\Phi = \frac{-G_N M}{r} = \frac{-G_N M}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}$$

to massless particles we obtain new physics (light bending & redshift)

4.2 Gravitational Redshift

Photons emitted at regular intervals $\Delta\tau_e$ travel between constant radial coordinates from r_e to r_o

1. Metric gives: $\Delta\tau^2 = (1 - \frac{2M}{r})\Delta t^2$
2. static metric $\Rightarrow \Delta t_A = \Delta t_B$.
3. since $\phi(x^i)$ const. along stationary worldlines,
$$\frac{\nu_o}{\nu_e} = \frac{\Delta\tau_e}{\Delta\tau_o} = \sqrt{\frac{1 - \frac{2M}{r_o}}{1 - \frac{2M}{r_e}}} \to 0 \quad \text{as} \quad r_e \to 2M$$

The photon loses energy as it climbs out of the potential, so o loses sight of e. Photons sent at horizon $r_e = 2M$ will not escape.

4.3 Gravitational Collapse

At $r = 2M$

- Kretschmann curvature scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 48M^2/r^6$ which measures tidal force strength is finite
- $\mathrm{d}r/\mathrm{d}\tau < 0$ is finite

so particle keeps on falling; no physical singularity.

4.3.1 Eddington-Finkelstein Coordinates

- Change of coords $v = t + r_*(r)$ where

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \left(1 - \frac{2M}{r}\right) = (1 + 2\Phi)$$

  removes coordinate singularity

- metric regular at $r = 2M$

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2 d\Omega_2^2$$

- $r = 0$ is curvature singularity (see Kretschmann scalar)

Massive particle falling radially from $r = R_0 > 2M$:

1. Lagrangian: $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$
2. E-L eqn for v gives conserved quantity ϵ
3. use $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$ and substitute out $\dot{v}$
4. use initial condition to find value of ϵ in the end

4.4 Radial Null Geodesics

- null: $g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$
- radial: $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0$
- black hole: even outgoing radial null geodesics starting from $r < 2M$ hit the singularity

5 General Relativity II

5.1 Einstein Field Equations

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}}$$

- Recovering vacuum equations

$$T_{\mu\nu} = 0 \implies G_{\mu\nu} = 0 \implies R_{\mu\nu} = 0$$

- energy momentum tensor $T_{\mu\nu}$
  - symmetric: $G_{\mu\nu} = G_{\nu\mu} \implies T_{\mu\nu} = T_{\nu\mu}$
  - divergence-free: $\nabla_\mu G^{\mu\nu} = 0 \implies \nabla_\mu T^{\mu\nu} = 0$

5.2 General Perfect Fluids

The perfect fluid stress tensor is:

$$\boxed{T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}}$$

with density ρ, pressure P, and 4-velocity u^μ with $g_{\mu\nu}u^\mu u^\nu = -1$.

- the fluid velocity is parallel to the flow lines
- isotropic metric: in comoving coordinates $u^\mu = (1, 0, 0, 0)$, $T^{\mu\nu} = \mathrm{diag}(\rho, P, P, P)$

5.3 Equation of State

$$P = P(\rho) = \omega\rho$$

where the factor ω depends on the type of fluid:

dust: $\omega = 0$ radiation: $\omega = 1/3$ dark energy: $\omega = -1$

5.4 Pressure Free Fluids (Dust)

$$P = 0 \implies T^{\mu\nu} = \rho u^\mu u^\nu$$

- energy density $\rho(x)$ measured by co-moving observer (velocity u^μ)
- inertial coords ($\mathbb{M}^4$), $u^\mu = \gamma(1, u_3^i)$, energy density measured by observer at rest in inertial frame is

$$T^{tt} = \gamma^2\rho = \widetilde{\rho} \tag{4}$$

- divergence-free $\nabla_\mu T^{\mu\nu} = 0 \implies$ fluid equations

$$\begin{aligned} \nu = t: &\quad \partial_t\tilde{\rho} + \partial_i(\tilde{\rho}u_3^i) = 0 \quad \text{(energy)} \\ \nu = i: &\quad \partial_t(\tilde{\rho}u_3^i) + \partial_i(\tilde{\rho}u_3^j u_3^i) = 0 \quad \text{(momentum)} \end{aligned}$$

- combining gives Navier-Stokes
- for dust ($P = 0$) flow lines are geodesics $u^\mu\nabla_\mu u^\nu = 0$
- Slowly moving pressure-free fluid in Newtonian LIF:

$$G_{tt} = 2\delta^{ij}\Phi_{,ji} \implies T_{tt} = \rho \implies \delta^{ij}\Phi_{,ji} = 4\pi G_N\rho$$

  recovering Poisson's equations / Newtonian theory

5.5 Cosmology

- Planck length $l_{\mathrm{pl}} = \sqrt{8\pi G_N\hbar/c^3} = ct_{\mathrm{pl}} = 10^{-32}\mathrm{cm}$
- Cosmological Principle: universe is homogeneous and isotropic on large scales
- Friedmann metric

$$ds^2 = -dt^2 + \tilde{a}^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2\right)$$

- curvature parameter: $k = \begin{cases} +1, & \mathbb{S}^3 \\ 0, & \mathbb{E}^3 \\ -1, & \mathbb{H}^3 \end{cases}$
- observations agree with: $k = 0$
- thm: the only possible hom. & isotr. metrics
- for $ds^2 = -dt^2 + a^2\delta_{ij}dx^i dx^j$, note that $g^{ij} = a^{-2}\delta^{ij}$
- scale factor $\tilde{a}(t) > 0$
- Hubble parameter

$$H(t) = \frac{1}{\tilde{a}}\frac{\mathrm{d}\tilde{a}}{\mathrm{d}t}$$

5.6 Conformal Coordinates

$$x^\mu = \{\eta(t), x, y, z\}$$

- Friedmann metric

$$\begin{aligned} ds^2 &= a^2(\eta)[-d\eta^2 + dx^2 + dy^2 + dz^2] \\ &= a^2(\eta)\eta_{\mu\nu}dx^\mu dx^\nu \end{aligned}$$

- conformally flat: positive fn times flat $\mathbb{M}^4$ metric
- null geodesics are straight lines at 45°: causal structure of $\mathbb{M}^4$
- co-moving geodesics: constant spatial coordinates and $t(\tau) = \tau$
  - expansion of the universe: since $\dot{a} > 0$, spatial distance ΔS between comoving observers at different time-slices $t_1 < t_2$ increases

By symmetries of Riemann, only non-zero components are

$$R_{\eta ijk} \quad R_{\eta i\eta j} \quad R_{ijkl} \tag{5}$$

and those related by symmetry.

5.7 Cosmological Redshift

- expanding universe: (observed) $a(\eta_o) > a(\eta_e)$ (emitted)
- redshift

$$z = \frac{\nu_e}{\nu_o} - 1 = \frac{a(\eta_o)}{a(\eta_e)} - 1$$

- Hubble's law: at low redshift $a(\eta_o) \approx a(\eta_e)$, Taylor expand $a(\eta_e)$ around $\eta = \eta_0$

$$z \approx \underbrace{(\eta_e - \eta_o)a(\eta_0)}_{d_{oe}} H(\eta_o)$$

5.8 Friedmann equation*

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3}\rho$$

- From fluid equation $\nabla_\mu T^{\mu t} = 0$, using $\delta^{ij}\delta_{ij} = 3$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

- Equation of state gives

$$\frac{\rho(\eta)}{\rho_0} = \left(\frac{a_0}{a(\eta)}\right)^{3(1+\omega)}$$

proof: to integrate, use $\frac{\dot{\rho}}{\rho} = \frac{\mathrm{d}}{\mathrm{d}\eta}(\ln\rho)$

big bang: for matter and radiation, as $\eta \to \eta_0$ from above, Friedmann eq gives $a \to 0$ and so $\rho \to \infty$

- data consistent with $T^{\mu\nu} = T^{\mu\nu}_{\mathrm{mat}} + T^{\mu\nu}_{\mathrm{rad}} + T^{\mu\nu}_{\Lambda}$

$$\rho = \underbrace{\rho_{\mathrm{mat}}}_{\omega=0} + \underbrace{\rho_{\mathrm{rad}}}_{\omega=1/3} + \underbrace{\rho_{\Lambda}}_{\omega=-1}$$