1 Part 1 - Vacuum

Coulomb force on charge q_2 caused by an electric field associated with charge q_1:

$$\boldsymbol{F} = q_2\boldsymbol{E}(\boldsymbol{r}_2) = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} \tag{1}$$

Lorentz force on moving charge in electric and magnetic fields:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{2}$$

Retarded time (time it takes EM field / light to move from source $\boldsymbol{r}_2$ to $\boldsymbol{r}$):

$$\tau = \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{c} \tag{3}$$

1.1 Vector Calculus

- Spherical polars
$$\boldsymbol{\nabla} r = \frac{\boldsymbol{r}}{r} = \hat{\boldsymbol{r}} \tag{4}$$
- The gradient of a scalar field ϕ is irrotational:
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = 0 \tag{5}$$
- The curl of a vector field $\boldsymbol{F}$ is solenoidal:
$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0 \tag{6}$$
- Curl of a curl (when solving ME in vacuum)
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{E}) \tag{7}$$
where $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\boldsymbol{E}) = \nabla^2\boldsymbol{E}$ is the vector Laplacian.
- Divergence of a cross product (when deriving Poynting's Theorem):
$$\boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{E} \bullet (\boldsymbol{\nabla} \times \boldsymbol{B}) - \boldsymbol{B} \bullet (\boldsymbol{\nabla} \times \boldsymbol{E}) \tag{8}$$
- Flux through a closed surface and Gauss' theorem:
$$\phi = \oiint_S \boldsymbol{F} \bullet d\boldsymbol{S} = \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{F} dV \tag{9}$$
so divergence is total flux through a point, or flux density.
- Stokes' theorem
$$\iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \bullet d\boldsymbol{S} = \oint_C \boldsymbol{F} \bullet d\boldsymbol{l}, \tag{10}$$
so curl is circulation density.
- Helmholtz' theorem: Any twice differentiable vector field $\boldsymbol{F}$ can be decomposed into a sum of an irrotational a solenoidal vector field:
$$\boldsymbol{F} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{A} \tag{11}$$

1.2 Maxwell's equations

- Gauss' law
$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \qquad \oiint_S \boldsymbol{E} \bullet d\boldsymbol{S} = \frac{Q_V}{\varepsilon_0} \tag{12}$$
- No magnetic monopoles
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \qquad \oiint_S \boldsymbol{B} \bullet d\boldsymbol{S} = 0 \tag{13}$$
- Faraday's law (evolves $\boldsymbol{B}$):
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \frac{\partial \phi_B}{\partial t} = -\oint \boldsymbol{E} \bullet d\boldsymbol{l} = -\mathcal{E} \tag{14}$$
- Ampére's law (evolves $\boldsymbol{E}$):
$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial \boldsymbol{E}}{\partial t} \tag{15}$$

1.2.1 Source-free

In a vacuum free of charges, we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{16}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t\boldsymbol{B}, \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0\varepsilon_0\partial_t\boldsymbol{E} \tag{17}$$

taking the curl of Faraday's law, we find the vector wave equation:

$$\nabla^2\boldsymbol{E} = \mu_0\varepsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2} \tag{18}$$

so $\mu_0\varepsilon_0 = 1/c^2$ is the speed of EM waves / light. In 1D, assuming fields only vary in $\hat{\boldsymbol{z}}$, solutions are spherical waves

$$\boldsymbol{E}(z,t) = E_0\hat{\boldsymbol{x}}\cos(kz - \omega t + \psi) = \mathrm{Re}\left\{\boldsymbol{E}_0 e^{i(kz-\omega t+\psi)}\right\} \tag{19}$$

$$\boldsymbol{B}(z,t) = B_0\hat{\boldsymbol{y}}\cos(kz - \omega t + \psi) = \mathrm{Re}\left\{\boldsymbol{B}_0 e^{i(kz-\omega t+\psi)}\right\} \tag{20}$$

with phase velocity $c = z/t = \omega/k$ and phase shift ψ.

1.3 Charge conservation

Volume: Rate of change of charge in V is charge leaving V through surface S per unit time:

$$\frac{\mathrm{d}}{\mathrm{d}t}Q = \oiint \boldsymbol{J} \bullet (-d\boldsymbol{S}) \tag{21}$$

Point: the rate of change of current density is equal to the incoming current flux density:

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \tag{22}$$

1.4 Energy Density

$$w_E = \frac{1}{2}\varepsilon_0 E^2, \quad w_B = \frac{1}{2\mu_0}B^2 \tag{23}$$

and integrating over all space we find the total energy stored in the EM fields (no need to derive)

$$W_E = \frac{1}{2}CV^2, \quad W_B = \frac{1}{2}LI^2 \tag{24}$$

of a capacitor $C = Q/V$ and an inductor $L = \phi/I$.

1.4.1 Poynting's Theorem

$$\frac{\partial}{\partial t}\iiint_V w dV = -\oiint_S \boldsymbol{S} \bullet d\boldsymbol{A} - \iiint_V \boldsymbol{E} \bullet \boldsymbol{J} dV \tag{25}$$

saying that the rate of change of EM energy in V is the flux entering through the surface S minus the power delivered to the charges in V. The Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \frac{\boldsymbol{B}}{\mu_0} \tag{26}$$

is EM energy flux. The intensity of an EM wave is $\langle S \rangle$.

1.5 Solutions to Maxwell's equations

Time dependent case

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\boldsymbol{\nabla} V - \partial_t \boldsymbol{A} \tag{27}$$
$$\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{28}$$

Applying Maxwell's equations, this leads to

$$\frac{1}{c^2}\partial_{tt} V - \nabla^2 V = \frac{\rho(\boldsymbol{r}, t)}{\varepsilon_0} \frac{1}{c^2}\partial_{tt}\boldsymbol{A} - \nabla^2 \boldsymbol{A} \quad = \frac{\rho(\boldsymbol{r}, t)}{\varepsilon_0} \tag{29}$$

1.6 Monochromatic Plane Waves

(check section!)

$$\boldsymbol{B}_0 = \frac{\hat{k} \times \boldsymbol{E}_0}{c} \tag{30}$$

1.7 Time-harmonic case

Time dependence of source (and hence of solutions) is harmonic motion. Can write $\cos(\omega t)$ or $exp(-i\omega t)$ and take the real part. The vector potential becomes

$$\boldsymbol{A}(\boldsymbol{r}) = \iiint \frac{\mu_0 \boldsymbol{J}(\boldsymbol{r}')}{4\pi} \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{31}$$

Add in time-dependency back *before* taking real part!

2 Part 2 - Matter

Have free and bound charge density due to polarisation $\boldsymbol{P} = n\boldsymbol{p}$ (dipole moment per volume):

$$\rho = \rho_f + \rho_p = \rho_f - \boldsymbol{\nabla} \cdot \boldsymbol{P}. \tag{32}$$

Have free, bound (due to magnetisation), and polarisation current:

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_p + \boldsymbol{J}_m = \boldsymbol{J}_f + \frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{M}. \tag{33}$$

In simple materials, $\mu_r = 1$

2.1 Dielectrics

No free charges: $\rho_f = 0, \boldsymbol{J}_c = 0, \varepsilon = \varepsilon_0 \varepsilon_r$. Write Polarisation $\boldsymbol{P} = \chi_e \varepsilon_0 \boldsymbol{E}$. HIL Dielectric:

- Homogeneous: χ_e is uniform
- Isotropic: $\boldsymbol{P}$ is parallel to $\boldsymbol{E}$, χ_e is a scalar. e.g. crystals are anisotropic and easier to polarise in one direction.
- Linear: χ_e is independent of $\boldsymbol{E}$

Refractive index

$$\eta = \frac{c}{v_{\text{phase}}} = \sqrt{\varepsilon_r} \tag{34}$$

and the dispersion relation becomes

$$k = \frac{\omega}{c}\eta \tag{35}$$

- ε_r increases with ω
- $\eta = \sqrt{\varepsilon_r}$ increases with ω
- Dispersive: phase vel. $v_{\text{phase}} = \frac{c}{\eta}$ depends on frequency

In HIL dielectric, Maxwell's equations are identical to vacuum but with $\varepsilon_0 \to \varepsilon$. Clausius-Mossotti equation relates polarisability α to the relative permittivity ε_r.

2.2 Four-field form

To get equations in terms of free charges, define

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}, \quad \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}, \tag{36}$$

With $\boldsymbol{P} = \chi_e \varepsilon_0 \boldsymbol{E}$, and $\boldsymbol{M} = \chi_m \boldsymbol{H}$ can derive: Constitutive relations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \quad \boldsymbol{H} = \frac{\boldsymbol{B}}{\mu}, \tag{37}$$

where $\varepsilon = \varepsilon_0 \varepsilon_r = (1 + \chi_e)\varepsilon_0$ and $\mu = \mu_0 \mu_r = (1 + \chi_m)$.

Then Maxwell's equations become

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{38}$$
$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}_f + \partial_t \boldsymbol{D} \tag{39}$$

2.3 Conductors

Only charges on surface ($\rho_f = \boldsymbol{J}_{sc} = 0 \neq \rho_{sf}$).

- Electron collision time:
$$\tau_c = \frac{m_e \sigma}{N_e e^2}, \tag{40}$$
where N_e is the electron number density.
- Charge rearrangement time:
$$\tau^* = \frac{\varepsilon}{\sigma}. \tag{41}$$
where σ is the conductivity.
- Ohm's law:
$$\boldsymbol{J}_c = \sigma \boldsymbol{E} \tag{42}$$

In a good conductor:

$$\frac{\sigma}{\omega \varepsilon} = \frac{1}{\omega \tau^*} = \frac{|\boldsymbol{J}_c|}{|\boldsymbol{J}_d|} \gg 1 \tag{43}$$

depends on frequency!

2.4 Plasmas

Also have free electrons but unlike in a conductor, they do not collide! Also, no bound atoms so never have polarisation.

2.4.1 Maxwell's equations in a plasma

Appropriate for a plane wave

$$\boldsymbol{k} \bullet \boldsymbol{E} = \frac{i\rho_f}{\varepsilon_0}, \quad \boldsymbol{k} \bullet \boldsymbol{B} = 0 \tag{44}$$

$$\boldsymbol{k} \times \boldsymbol{E} = -\omega \boldsymbol{B}, \quad \boldsymbol{k} \times \boldsymbol{B} = i\mu_0 \boldsymbol{J}_c + \varepsilon_0 \mu_0 \omega \boldsymbol{E} \tag{45}$$

3 Optics

Reflection coefficient

$$r = \frac{1-\eta}{1+\eta} \tag{46}$$

For metals, $\eta >> 1, r \approx -1$

3.1 Spherical Mirrors

Focal Length (paraxial ray approximation):

$$f = \frac{R}{2} \tag{47}$$

Mirror equation:

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \tag{48}$$

- Concave: Bent towards object
- Convex: Bent away from object

For concave mirrors in general, or for object distances less than the focal length for convex mirrors, $v < 0$ and the image is virtual.

4 Diffraction

If Fresnel number

$$F = \frac{a^2}{R\lambda} \tag{49}$$

is small, diffraction of a plane wave by aperture $A(x)$ in far-field limit is:

$$E(k_x) = C(R)\mathcal{F}[A(x)] \tag{50}$$

where $k_x = k \sin\theta$ and $I \propto E^2$ gives intensity of diffraction pattern.