1 Rotating Frames

- Apply twice to Newton II: e.o.m in rot. frame:

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}r^2}\bigg|_R = \boldsymbol{F}\underbrace{-2m\boldsymbol{\omega}\times\boldsymbol{v}_R}_{\text{Coriolis Force}}\underbrace{-m\boldsymbol{\omega}\times(\omega\times\boldsymbol{r})}_{\text{Centrifugal Force}}$$

2 Rigid Bodies

1. to obtain centre of mass at origin, solve:

$$\boldsymbol{R} = \frac{1}{M}\sum_a m_a\boldsymbol{r}_a = 0, \qquad \boldsymbol{R} = \frac{1}{M}\int_V \rho(\boldsymbol{r})\boldsymbol{r}\,\mathrm{d}^3r = 0$$

2. tensor of inertia:

$$\text{discrete} \quad I_{ij} = \sum_a m_a(|\boldsymbol{r}_a|^2\delta_{ij} - r_{ai}r_{aj})$$

$$\text{continuous} \quad I_{ij} = \int_V \rho(\boldsymbol{r})(|\boldsymbol{r}|^2\delta_{ij} - r_ir_j)\,\mathrm{d}^3r$$

hint: use symmetry → diagonal in principal axes basis. I_1 along rot. axis; $I_2 = I_3$ along any perpendiculars

strategy: calculate $X = \sum_a m_a x_a^2$, $Y = \dots$, $Z = \dots$ then $I_x = Y + Z$, $I_y = X + Y$, $I_z = \dots$

hint: if the body is totally symmetric: $I_1 = I_2 = I_3$, calculate the sum $3I = 2\rho\int \mathrm{d}V\,r^2$

3. angular momentum: $\boldsymbol{L} = \bar{\bar{\boldsymbol{I}}}\boldsymbol{\omega}$ and torque $\boldsymbol{\tau} = \dot{\boldsymbol{L}}|_I = \boldsymbol{r}\times\boldsymbol{F}$

state: in object's frame $\dot{\bar{\bar{\boldsymbol{I}}}}|_R = 0 \implies \dot{\boldsymbol{L}}|_R = \bar{\bar{\boldsymbol{I}}}\dot{\boldsymbol{\omega}}$

use: relation between inertial and rotating frame:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\bigg|_I = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}\bigg|_R + \boldsymbol{\omega}\times\boldsymbol{L} \tag{1}$$

4. kinetic energy: $T = \frac{1}{2}m\dot{\boldsymbol{R}}^2 + \frac{1}{2}\sum_i I_i\omega_i^2$

3 Lagrangian Mechanics

1. Conservative system: $L = T - V, \quad H = T + V$

T**:** total kinetic energy: $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

V**:** total potential energy

2. conservation laws

$$\frac{\partial L}{\partial t} = 0 \iff H = \sum_i p_i\dot{q}_i - L \quad \text{is conserved}$$

$$\frac{\partial L}{\partial q_i} = 0 \iff p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{is conserved}$$

3. Euler-Lagrange equations (no need to derive)

$$\frac{\delta L}{\delta q_i} = \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

3.1 Small Oscillations

- equilibrium points $\boldsymbol{q}(t) = \bar{\boldsymbol{q}}$ satisfy $\frac{\partial V}{\partial q_i}\Big|_{\bar{\boldsymbol{q}}} = 0$
- perturbations about eq. points: $q_i(t) = \bar{q}_i + \varepsilon\delta q_i(t)$

1. expand L to second order in ε, $\boldsymbol{q} = (\delta q_1, \dots, \delta q_N)$

   hint: ignore $\bar{q}_i$ and ε cancels, so only $O(\varepsilon^2)$ matters
2. diagonalise kinetic term $T = \frac{1}{2}\dot{\boldsymbol{q}}\cdot\dot{\boldsymbol{q}}$

   hint: build perfect squares, define normalised $\boldsymbol{q}$
3. Lagrangian allows identification of $\underline{\underline{\boldsymbol{k}}}$

$$L = \frac{1}{2}\dot{\boldsymbol{q}}\cdot\dot{\boldsymbol{q}} - \frac{1}{2}\boldsymbol{q}\cdot\underline{\underline{\boldsymbol{k}}}\boldsymbol{q}, \qquad k_{ij} = \frac{\partial^2 V}{\partial q_i\partial q_j}\bigg|_{\boldsymbol{q}=\bar{\boldsymbol{q}}}$$

4. EL eq$^{\text{ns}}$: $\ddot{\boldsymbol{q}} = -\underline{\underline{\boldsymbol{k}}}\boldsymbol{q}$ have sol$^{\text{n}}$: $\boldsymbol{q} = \boldsymbol{A}e^{\pm i\omega t}$
5. look for eigenvalues: $\underline{\underline{\boldsymbol{k}}}\boldsymbol{A}_n = \omega_n^2\boldsymbol{A}_n$

   normal modes: eigenvectors $\boldsymbol{A}_n$

   normal frequencies: root of eigenvalues ω_n^2

   stability: $\bar{\boldsymbol{q}}$ only stable if all eigenvalues $\omega_n^2 > 0$

   hint: 2×2 matrix, stable if

$$\mathrm{Tr}\,\underline{\underline{\boldsymbol{k}}} = \sum\omega_n^2 > 0 \quad \text{AND} \quad \det\underline{\underline{\boldsymbol{k}}} = \prod\omega_n^2 > 0$$

4 Hamiltonian Mechanics

Hamilton's equations:

$$\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i} \tag{2}$$

Poisson Brackets:

$$\{F, H\} = \sum_{i=1}^N\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - (p_i \leftrightarrow q_i)\right) \tag{3}$$

- time-evolution of $F(q, p, t)$:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \{F, H\}$$

- canonical coordinates: $\{q_i, p_i\} = \delta_{ij}$

4.1 Noether's theorem

Any phase-space function $G(q, p)$ generates transformations:

$$\delta q_i = \frac{\partial G}{\partial p_i}\delta\lambda, \qquad \delta p_i = -\frac{\partial G}{\partial q_i}\delta\lambda,$$

If G is a generator of symmetry, then

$$\frac{\mathrm{d}H}{\mathrm{d}\lambda} = \{H, G\} = 0 \iff \frac{\mathrm{d}G}{\mathrm{d}t} = \{G, H\} = 0 \tag{4}$$

then $G(q, p)$ is called a conserved charge.

5 Relativistic EM $[(-+++), c=1]$

1. field strength tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

   $A_\mu = (-\phi, \boldsymbol{A})$**:** photon vector field

2. Lorentz law (tensor eq: valid in any frame)

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = qF^\mu{}_\nu u^\nu \quad \text{(provided in exam)}$$

3. Gauss' & Ampèere's laws:

$$\partial_\mu F^\mu{}_\nu = -\mu_0 J_\nu \quad \text{(provided in exam)}$$

   $J_\mu = (-\rho, \boldsymbol{J})$**:** charge density

4. Lorentz & Gauge transformation
   - physics invariant under these
   - can choose frame / gauge that simplifies calculations

5.1 Lorentz Transforms

$$x'^{\mu'} = \Lambda^{\mu'}{}_\mu x^\mu$$

Need to be able to write down LT matrix for boosts in any direction (x, y, z)

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \frac{1}{\sqrt{1-v^2}}$$

check: if $v = 0$, same frame, so $\Lambda^{\mu'}{}_\mu = \delta^{\mu'}_\mu$

- can just state that $(\Lambda^\mu{}_\nu)^{-1} = \Lambda_\mu{}^\nu$ has $v \to -v$

  hint: raising both indices changes sign in first row & col:

$$F_{\mu\nu} = \left(\begin{array}{c|c} 0 & x \\ \hline -x & y \end{array}\right) \longleftrightarrow F^{\mu\nu} = \left(\begin{array}{c|c} 0 & -x \\ \hline x & y \end{array}\right)$$

- need to be able to derive how $\boldsymbol{E}$ & $\boldsymbol{B}$ transform
- matrix multiplication: contract second with first index:

$$C = A \cdot B \iff C_{\mu\nu} = A_\mu{}^\alpha B_{\alpha\nu}$$

5.2 Gauge Transforms

$$A_\mu \xrightarrow{\chi} A_\mu + \partial_\mu \chi, \qquad F_{\mu\nu} \xrightarrow{\chi} F_{\mu\nu}$$

Lorentz Gauge: $\partial_\mu A^\mu = 0$ (by using $\Box\chi = -\partial_\mu A^\mu$)

$\partial_\mu F^\mu{}_\nu = -\mu_0 J_\nu$ becomes wave equation: $\Box A_\nu = 0$

Coulomb Gauge: $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$

Weyl / Temporal Gauge: $A^0 = \phi = 0$