1 Terminology

- ODE/PDE: derivatives with respect to one/more independent variable/s
- order: number of differentiations
- linear combination: $ay_1 + by_2$
- linear operator: $\hat{L}(ay_1 + by_2) = a\hat{L}(y_1) + b\hat{L}(y_2)$
- homogeneous: standard form RHS is $f(x) = 0$

2 First-order linear ODEs

Standard form:

$$\frac{dy}{dx} + f(x)y = g(x)$$

Toolbox

- Separation of Variables
- Integrating factor
- Variation of parameters $y(x) = v(x)u(x)$, u is solution to HE

General recipe is using integrating factor (IF):

1. write in standard form
2. IF is $\mu(x) = \exp\left[\int^x f(x')dx'\right]$
3. solution is $y(x) = \frac{1}{\mu(x)}(\exp\left[\int^x \mu(x')g(x')dx'\right] + c)$

3 Second-order ODEs

Standard form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x)$$

1. Find CF (GS of HE), trial e^{mx}
2. Find any PI, ignoring BCs
3. GS = CF + PI
4. Set arbitrary constants using BCs

To find PI, try using the same kind as the RHS function:

- if $g(x) = ae^{\alpha x}$, then PI = $c\exp\{\alpha x\}$
- if $g(x) = a\cos\beta x$ (or sin), then PI = $c\cos\beta x + d\sin\beta x$
- if $g(x) = x^n$, then PI = $ax^n + bx^{n-1} + \dots$

If $\alpha = m_+$ or m_-, then $y = Axe^{\alpha x}$. Can use variation of parameters / Lagrange's method for general case, but guessing is usually faster.

3.1 Existence and uniqueness

If p, q, g are continuous over the interval $\alpha < x < \beta$ containing x_0 (IVP), there exists a unique solution. (no proof)

3.2 Higher order equations with constant coefficients

Not common in physics, but same method works: GS = CF + PI. For CF, try again $y = e^{mx}$ and solve resulting characteristic equation.

3.3 The Wronskian

Sufficient condition for linear independence is

$$W_{y_1,\dots,y_n} = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

Conversely, $W = 0$ on its own is not a sufficient condition for linear dependence if the functions are not analytic.

3.4 Series solutions

Useful for PDEs, only works for some ODEs. Make series expansion about a point x_0:

$$y(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

- exists if p and q are analytic at the *ordinary point* x_0
- subst. into eq. and equate coefficients of x^n to zero
- split: $y = ay_1 + by_2$, and radii of convergence obey $R \geq \min(p, q)$

3.4.1 Radius of convergence

Radius of convergence R is limiting value of $|x - x_0|$ for which the series converges.

Ratio test: Define $r_n \equiv \left|\frac{t_{n+1}}{t_n}\right|$ and let $r = \lim_{n\to\infty} r_n$, then

- $r < 1$, series converges
- $r = 1$, don't know
- $r > 1$, diverges absolutely (i.e. could still conditionally converge)

3.4.2 Associated Legendre Equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + [k - \frac{m^2}{1 - x^2}]y = 0$$

How to solve Legendre eq. ($m = 0$):

1. $x_0 = 0$ ordinary point $\rightarrow$ series expansion
2. Recurrence relation; two series with two arbitrary coefficients
3. Diverge when $x = \pm 1$, but physical! Make one series truncate $k = l(l + 1)$ and set the other coefficient to zero

Legendre polynomial solutions $P_l(x)$ form a CONS.

3.4.3 Frobenius' method

If $x_0 = 0$ is a regular singular point, i.e. $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0, then

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

Bessel's equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0, \qquad p \geq 0$$

Things to remember:

1. Do not change index of sum when taking derivative!
2. Change index of sum after inserting in the equation when grouping together x^{n+r}
3. Orphan $c_0 \neq 0$ gives indicial equation $r = \pm p$
4. if p is integer, solutions are not independent

4 Eigenvalue Problems

$$\hat{L}y(x) = \lambda\rho(x)y(x)$$

with $\hat{L}$ a linear operator, λ an eigenvalue, and weight function $\rho \in \mathbb{R} > 0$ homogeneous BCs.

4.1 Self-adjoint operators

Same as 'Hermitian' in QM:

$$\int_a^b u^*(x)\hat{L}v(x)dx = \int_a^b [\hat{L}u(x)]^* v(x)dx.$$

1. Real eigenvalues
2. Orthogonal eigenfunctions (if not degenerate)
3. Can construct orthogonal linear combinations of degenerate eigenfunctions

4.2 Sturm-Liouville Theory

$$\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)y(x) = \lambda\rho(x)y(x),$$

with all parameters continuous and real, and $\rho > 0$, $p > 0$ (except at boundaries). Sturm-Liouville problems are self-adjoint.

5 ant equations...

5.1 Wave and Heat/Diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{1}{D}\frac{\partial u}{\partial t}.$$

Replace $\partial \to \nabla$ in higher dimensions.

5.2 Laplace & Poisson equations

$$\nabla u = -\frac{\rho}{\epsilon_0} \qquad \nabla u = 0.$$

6 ...and how to solve them

6.1 Separation of Variables

$$u(x, y, z) = X(x)Y(y)Z(z)$$

1. LHS not a function of...
2. Separation constant $S = -k^2$?
3. G.S. is linear combination
4. Fourier Series for coefficients

6.2 Fourier Transform

1. For infinite systems
2. Spacial derivatives turn into factors of ik

Note: integration coefficients of the Fourier transformed solution $\tilde{u}(k, t)$ are functions of k in general, i.e. $C(k)$ because of the partial derivatives.

6.3 Green's functions

If we can solve homogeneous version of

$$\hat{L}(y(x)) = f(x),$$

then solve

$$\hat{L}(G(x, z)) = \delta(x - z)$$

1. Solve for $x \neq z$
2. Use given BCs / ICs
3. Use continuity, and discontinuity of 1 in first derivative at $x = z$
4. Find solution by integrating

$$y(x) = \int G(x, z)f(z)dz.$$

Pay attention at this integral:

- is $x > z$ or $x < z$
- insert $f(z)$ is function of z here, even though initially given in x