1 Even and Odd functions

Tipp: write $z = x - x_0$, then if f(z) is even or odd, so is $f(x)$ around x_0.

2 Orthogonality

Inner product

$$< f, g >= \int_a^b f(x) g^*(x) dx \tag{1}$$

Set $\{f_i\}$ orthonormal if $< f_n, f_m >= \delta_{nm}$. These then form a basis in which to write other functions $g(x)$:

$$g(x) = \sum_{n=-\infty}^{\infty} a_n f_n(x) \tag{2}$$

3 Fourier series

Complex exponentials

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi}{L} x} \tag{3}$$

with

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{n\pi}{L} x} dx \tag{4}$$

Trigonometric

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \tag{5}$$

with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \tag{6}$$

3.1 Differentiation and Integration

- If $f(x)$ continuous, can differentiate term-by-term.
- If k^{th} derivative is discontinuous, Fourier coefficients will be proportional to $\frac{1}{n^{1+k}}$.
- Can integrate term-by-term if the average value over the interval $a_0 = 0$.

3.2 Dirichlet Conditions

Fourier series converges if $f(x)$:

- has period $2L$
- is single valued and absolutely integrable
- does not have infinite extrema or discontinuities

Note: FS will converge to average value at discontinuities.

3.3 Gibb's Phenomena

Truncated FS overshoots at discontinuities by about five percent.

3.4 Power Spectrum

Define (analogous to polar coordinates):

$$a_n = \alpha_n \cos \theta_n, \quad b_n = \alpha \sin \theta_n \tag{7}$$

such that

$$a_n \cos nx + b_n \sin nx = \alpha_n \cos nx - \theta_n. \tag{8}$$

The set $\{\alpha_n\}$ tells us how much power the frequency components have.

3.5 Parseval's Identity for FS

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 + b_n^2 \tag{9}$$

and $a_n^2 + b_n^2 = \alpha_n^2$.

4 Fourier Transform

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} dt \tag{10}$$

4.1 Properties

- Linearity
- Conjugation
$$\mathscr{F}(f^*(t)) = g^*(-w)$$
- Translation
$$\mathscr{F}(f(t - t_0)) = e^{i\omega t_0} g(w)$$
- Scaling
$$\mathscr{F}(f(\alpha t)) = \frac{1}{|\alpha|} g(\frac{\omega}{\alpha})$$
- Differentiation
$$\mathscr{F}(f'(t)) = -i\omega g(w)$$

4.2 Parseval's equality for Fourier Tranforms

Inner products are equal:

$$< f_1, f_2 >=< g_1, g_2 >, \tag{11}$$

taking $f_1 = f_2$ implies that the norms are equal as well.

4.3 Proof that $\mathscr{F}(\mathscr{F}^{-1}(f)) = f$

Use Sifting property of delta-function and the fact that

$$\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)} d\omega \tag{12}$$

4.4 Proof that FT of Gaussian is Gaussian

Use completion of squares in exponent. Then use standard integral:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \tag{13}$$

5 Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau \tag{14}$$

When calculating convolutions. Substitute $s = t - \tau$ for $g(t-\tau)$ after $f(\tau)$ has cut down the limits of integration.

5.1 Properties

- Commutativity
$$f * g = g * f$$
- FT of convolution is the product of the individual transformed functions
$$\mathscr{F}(f * g) = 2\pi \mathscr{F}(f)\mathscr{F}(g)$$
- FT of product is convolution
$$\mathscr{F}(fg) = \frac{1}{2\pi}\mathscr{F}(f) * \mathscr{F}(g)$$

5.2 Special Convolutions

Convolution of Delta function does not change function

$$f * \delta = f \tag{15}$$

Convolution of two Gaussians is a Gaussian with mean $\mu = \mu_1 + \mu_2$ and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$.