1 Basic Principles

Degenerate spectral decomposition

$$F(\hat{A}) = \sum_i \sum_{j_i=1}^{d_i} F(A_i) \left|A_i, j_i\right\rangle\!\left\langle A_i, j_i\right|$$

provided $F(A_i)$ is well defined.

1.1 Tensor Product

Direct product $\mathcal{H}_A \otimes \mathcal{H}_B$ has basis

$$|i, I\rangle = |B_i\rangle \otimes \left|\widetilde{B}_I\right\rangle \qquad \langle i, I| = \langle B_i| \otimes \left\langle \widetilde{B}_I\right|$$

Rule for combining tensor products of operators, bras, and kets:

$$(A \otimes B)(C \otimes D) = AC \otimes BC$$

rule: left stays left, right stays right

1.2 Trace and Partial trace

Trace of operator: $\mathrm{Tr}(\hat{O}) = \sum_i \langle B_i|\hat{O}|B_i\rangle$
Properties:

- cyclic: $\mathrm{Tr}\left(\hat{A}\hat{B}\hat{C}\right) = \mathrm{Tr}\left(\hat{C}\hat{A}\hat{B}\right) = \mathrm{Tr}\left(\hat{B}\hat{C}\hat{A}\right)$
- linear: $\mathrm{Tr}\left(a\hat{A} + \hat{B}\right) = a\,\mathrm{Tr}\left(\hat{A}\right) + b\,\mathrm{Tr}\left(\hat{B}\right)$
- trace of unity operator: $\mathrm{Tr}\left(\hat{1}\right) = \dim(\mathcal{H})$

Trace over $\mathcal{H}_A \otimes \mathcal{H}_B$ is

$$\mathrm{Tr}_{\mathcal{H}_A \otimes \mathcal{H}_B} = \mathrm{Tr}_{\mathcal{H}_A} \mathrm{Tr}_{\mathcal{H}_B} = \mathrm{Tr}_{\mathcal{H}_B} \mathrm{Tr}_{\mathcal{H}_A}$$

with partial traces

$$\mathrm{Tr}_{\mathcal{H}_B}(\hat{O}) = \sum_K \left\langle \widetilde{B}_K \middle| \hat{O} \middle| \widetilde{B}_K \right\rangle = \hat{O}_A^{\text{reduced}}$$

with reduced operator acting on $\mathcal{H}_A$ with matrix elements

$$\langle i|\hat{O}_A^{\text{red}}|j\rangle = \sum_K \langle i, K|\hat{O}|j, K\rangle$$

- if $\hat{O} = \hat{O}_A \otimes \hat{1}_B$ then $\mathrm{Tr}\left(\hat{O}\right) = \mathrm{Tr}_{\mathcal{H}_A}(\hat{O}_A)$
- reduced density operator $\hat{\rho}_A = \mathrm{Tr}_{\mathcal{H}_B}(\hat{\rho})$ describes probabilities in $\mathcal{H}_A$

hint: only orthogonal terms survive:

$$\mathrm{Tr}_{\mathcal{H}_A \otimes \mathcal{H}_B} |abc\rangle\!\langle a'b'c'| = |a\rangle\!\langle a'| \underbrace{\langle b'c'|bc\rangle}_{\delta_{b'b}\delta_{c'c}}$$

pitfall: have to factor out products:

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_B}\left[(|ab\rangle + \gamma\,|cd\rangle)(\langle ab| + \gamma^*\,\langle cd|)\right] \\
&\qquad = |a\rangle\!\langle a| + |\gamma|^2\,|c\rangle\!\langle c| \\
&\qquad \neq \left[(|a\rangle + \gamma\,|c\rangle)(\langle a| + \gamma^*\,\langle c|)\right] \mathrm{Tr}(\ldots)
\end{aligned}$$

1.3 Stone's Theorem

Every continuous symmetry $a \to a + \Delta a$ (automorphism) is enacted by a unitary operator

$$\hat{U}(\Delta a) = e^{i\Delta\hat{A}}$$

where the generator $\hat{A}$ is Hermitian and time-independent.

$a = t$: time translation: $\hat{A} = -\hat{H}/\hbar$

- differentiating gives TDSE

$a = x$: spatial translation: $\hat{A} = -\hat{p}_x/\hbar$

- expanding $\hat{x}'$ to $O(\Delta x)$ gives CCR: $[\hat{x}, \hat{p}_x] = i\hbar$

$a = \theta$: rotation: $\hat{A} = \hat{L}\theta/\hbar\hat{A} = \hat{L}/\hbar$

1.4 Automorphisms

Unitary transform between representations

$$|\psi'\rangle = \hat{X}\,|\psi\rangle\,, \qquad \hat{O}' = \hat{X}\hat{O}\hat{X}^\dagger$$

- preserves $\langle A|\hat{O}|B\rangle$ and $\hat{A}\hat{B} = \hat{C}$

$$\langle A|X^\dagger X\hat{O}X^\dagger X|B\rangle$$

1.5 Schrödinger picture

Unitary time evolution operator evolves states:

$$|\psi(t)\rangle_S = \hat{U}(t)\,|\psi(0)\rangle\,, \qquad \hat{U}(t) = e^{-i\hat{H}_S t/\hbar}$$

Equations of motion:

$$\underbrace{i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle_S = \hat{H}_S\,|\psi(t)\rangle_S,}_{\text{Schrödinger equation*}} \qquad \frac{\partial}{\partial t}\hat{A}_S = 0$$

1.6 Heisenberg picture

Automorphism $\hat{X} = \hat{U}^\dagger(t) = \exp\left(it\hat{H}_S/\hbar\right)$ gives:

$$\hat{O}_H(t) = \hat{U}^\dagger(t)\hat{O}_S\hat{U}(t)$$

preserving matrix elements

$$\langle A(t)|\hat{O}|B(t)\rangle_S = \langle A|\hat{U}^\dagger(t)\hat{O}\hat{U}(t)|B\rangle = \langle A|\hat{O}(t)|B\rangle_H$$

Equations of motion:

$$i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle_H = 0, \qquad \underbrace{\frac{\partial}{\partial t}\hat{A}_H(t) = \frac{i}{\hbar}\left[\hat{H}_H, \hat{A}_H\right]}_{\text{Heisenberg equation of motion}}$$

1.7 Interacting (Dirac) picture

Interaction Hamiltonian: $\hat{H}_I = \hat{H}_I^0 + \hat{H}_I^1$

- operators evolve with $\hat{H}_I^0$, states with $\hat{H}_I^1$

From (S), automorphism $\hat{X} = \exp\left(it\hat{H}_S^0/\hbar\right)$

$$\underbrace{i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle_I = \hat{H}_I^1(t)\left|\psi(t)\right\rangle_I,}_{\text{Schrödinger with } \hat{H}_I^1(t)} \qquad \underbrace{\frac{\partial}{\partial t}\hat{A}_I(t) = -\frac{i}{\hbar}\left[\hat{A}_I(t), \hat{H}_I^0(t)\right]}_{\text{Heisenberg with } \hat{H}_I^0(t)}$$

- Dirac reduces to Heisenberg when $\hat{H}_S^1 = 0$
- perturbation theory: $\hat{H}^0 \gg \hat{H}^1$: expand in powers of $\hat{H}^1$

2 Position and Momentum Rep

In the continuous case, projection operators are

$$\hat{P}_{x,x+\Delta x} = \int_x^{x+\Delta x} \mathrm{d}x' \left|x'\right\rangle \left\langle x'\right|$$

- Wavefunction: $\psi(x) = \mathcal{A}(x|\psi) = \langle x|\psi\rangle$

Momentum eigenstates $\hat{p}\left|k\right\rangle = p\left|k\right\rangle = \hbar k\left|k\right\rangle$

- Completeness:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi}\left|k\right\rangle\left\langle k\right| = \hat{1}$$

- Orthonormality: $\langle k|k'\rangle = (2\pi)\delta(k-k')$
- Orthogonality to $\left|x\right\rangle$ and CCR:

$$\langle x|k_x\rangle = e^{ik_x x} = e^{ip_x x/\hbar}, \qquad [\hat{x},\hat{p}] = i\hbar$$

- in position basis: $\langle x|F(\hat{p}_x)|\psi\rangle = F\big(-i\hbar\partial_x\big)\psi(x)$

In polar coords, $\left|r,\phi\right\rangle = \left|x = r\cos\phi, y = r\sin\phi\right\rangle$,

$$\text{(completeness)} \qquad \int_0^\infty \mathrm{d}r\int_0^{2\pi} r\,\mathrm{d}\phi\left|r,\phi\right\rangle\!\left\langle r,\phi\right| = \hat{1}$$
$$\text{(orthonormality)} \qquad \langle r,\phi|r',\phi'\rangle = r^{-1}\delta(r-r')\delta(\phi-\phi')$$

In separable spherical coords $\left|r,\theta,\phi\right\rangle = \left|r\right\rangle\otimes\left|\theta\phi\right\rangle$ have

$$\text{(completeness)} \qquad \int_0^\infty \mathrm{d}r\int r^2\,\mathrm{d}\Omega\left|r,\phi\right\rangle\!\left\langle r,\phi\right| = \hat{1}$$

$$\int_0^\infty \mathrm{d}r\, r^2\left|r\right\rangle\!\left\langle r\right| = \hat{1}_r \implies \langle r|r'\rangle = r^{-2}\delta r - r'$$

and the CCR become $[\hat{r}_i,\hat{p}_j] = i\hbar\delta_{ij}$

3 Emergence of Classicality I

3.1 Hilbert Space to Phase Space

Associate every operator $\hat{A}\in\mathcal{H}$ to classical phase space quantity

$$A_{cl}(x,p_x) = \langle x|\hat{A}|k_x\rangle\left\langle k_x|x\right\rangle$$

- Integral over p_x gives $\langle x|\hat{A}|x\rangle$
- Integral over x gives $\langle k_x|\hat{A}|k_x\rangle$
- Integral over x and p_x gives $\mathrm{Tr}\left(\hat{A}\right)$
- Invertible via $\hat{A} = \hat{1}_x\hat{A}\hat{1}_{k_x}$

Wigner-Weyl transform guarantees that A_c is real

4 Density Matrices

Pure state density operator

$$\hat{\rho} = \left|\psi\right\rangle\left\langle\psi\right|$$

- $\hat{\rho}$ is probability op.: $p(A_i) = \left|\langle A_i|\psi\rangle\right|^2 = \langle A_i|\hat{\rho}|A_i\rangle$
- expectation value: $\langle\psi|\hat{A}|\psi\rangle = \mathrm{Tr}\left(\hat{A}\hat{\rho}\right)$
- Satisfies $\mathrm{Tr}(\hat{\rho}) = \mathrm{Tr}\big(\hat{\rho}^2\big) = 1$

4.1 Mixed States

$$\hat{\rho} = \sum_J p_J\left|\psi_J\right\rangle\left\langle\psi_J\right|$$

- Satisfies $\mathrm{Tr}(\hat{\rho}) = \sum_J p_J = 1$, but $\mathrm{Tr}\big(\hat{\rho}^2\big) < 1$
- more mixed = less information about the system
- von Neumann entropy: $S = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$ measures ignorance
  - mixed vs pure: compute $\mathrm{Tr}\big(\hat{\rho}^2\big)$
- statistical (classical) prob. op.: p_J are eigenvalues of $\hat{\rho}$
- since $\hat{\rho}$ is composed of states $\left|\psi\right\rangle$, the time evolution is governed by how the states (not operators) evolve:

$$\hat{\rho}_S(t) = \hat{U}(t)\hat{\rho}_S(0)\hat{U}^\dagger(t), \qquad \hat{\rho}_H = \hat{\rho}_S(0)$$

For degenerate states with the same probability, use spectral decomposition (e.g. for entropy)

$$F(\hat{\rho}) = \sum_i\sum_{j_i} F(p_i)\left|p_i,j_i\right\rangle\!\left\langle p_i,j_i\right|$$

For a thermal state (Boltzmann)

$$\hat{\rho} = \frac{e^{-\hat{H}/(kT)}}{\mathrm{Tr}(\dots)}$$

4.2 Wigner Quasi-Probability Distribution

. . . is classical phase space version of density operator

$$W(x,p_x) = \rho_c$$

- normalised: phase space integral is $\mathrm{Tr}(\hat{\rho}) = 1$
- $\hbar\to 0$: phase space prob. density

- probability to be in phase space volume V:

$$p_V = \int_V \frac{\mathrm{d}x\,\mathrm{d}p_x}{2\pi\hbar} W(x,p_x)$$

- expectation value of any function of $\hat{x}$ or $\hat{p}_x$ is

$$\langle F(\hat{x},\hat{p}_x)\rangle = \mathrm{Tr}\Big(\hat{F}\hat{\rho}\Big) = \int \frac{\mathrm{d}x\,\mathrm{d}p_x}{2\pi\hbar} F(x,p_x)W(x,p_x)$$

- quasi-probability density: W and p_V can become negative

4.3 Quantum Liouville equation*

From time evolution of $\hat{\rho}_S(t)$, can find

$$\frac{\partial W}{\partial t} = \frac{i}{\hbar}\Big[\hat{\rho}_S(t), \hat{H}\Big]_c$$

and to leading order in $\hbar$:

$$\frac{DW_0}{Dt} = \frac{\partial W_0}{\partial t} + \{W_0, H\} \approx 0 \quad \text{(Classical Liouville)}$$

Quantum corrections vanish for $\partial^3 V(x)/\partial x^3 = 0$:

1. free particle: $V = \text{const.}$
2. constant external force: $V = -Fx$
3. harmonic oscillator: $V = \frac{1}{2}m\omega^2 x^2$
   - W: Gaussian with $\Delta x \Delta p_x = \frac{\hbar}{2}$ & $\Delta x = \sqrt{\hbar/(2m\omega)}$

5 Uncertainty Principle

Cauchy-Schwarz:

$$\langle u|u\rangle \langle v|v\rangle \geq |\langle u|v\rangle|^2$$

- uncertainty = standard deviation ΔA with

$$\Delta A^2 = \langle\psi|(\hat{A} - \langle\hat{A}\rangle)^2|\psi\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$$

- generalised uncertainty principle

$$\Delta A \Delta B \geq \frac{1}{2}\Big|\Big\langle \Big[\hat{A}, \hat{B}\Big]\Big\rangle\Big|$$

5.1 Coherent States

Uncertainty is minimised for

1. $R = \left\langle \underbrace{\Big\{\Delta\hat{A}, \Delta\hat{B}\Big\}}_{\text{Anti-commutator}} \right\rangle = 0$
2. Cauchy-Schwarz is equality: $|u\rangle \propto |v\rangle$

$\implies$ states of minimum uncertainty

These are satisfied by coherent states which are eigenstates of

$$\underbrace{(\hat{A} + i\mu\hat{B})}_{\text{non-hermitian}} |\alpha\rangle = \alpha|\alpha\rangle$$

- For $\hat{A} = \hat{x}$, $\hat{B} = \hat{p}_x$, wavefunction $\langle x|\alpha\rangle$ is Gaussian

6 Entanglement and Mixed States

Entangled states are states that cannot be written as a direct product of two states.

- Even if $\hat{\rho}$ is pure state, reduced $\hat{\rho}_A$ is in general mixed.
  - to verify entanglement, show that $\mathrm{Tr}\big(\rho_A^2\big) < 1$
- Factorisable states are not entangled: partial traces yield pure states in reduced Hilbert space.

Bell state (normalised, pure) of maximal entanglement

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(\,|0,0\rangle + |1,1\rangle\,\big)$$

Entropy is defined as

$$S = -\,\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$$

- Maximum entropy / entanglement states $\hat{\rho} = \hat{1}/\dim(\mathcal{H})$

6.1 Maximal Qubit Entanglement

General qubit state:

$$|\psi\rangle = \frac{|0,0\rangle + \alpha|0,1\rangle + \beta|1,0\rangle + \gamma|1,1\rangle}{\sqrt{1 + |\alpha|^2 + |\beta|^2 + |\gamma|^2}}$$

1. partial trace over $\mathcal{H}_B$ gives effective state in $\mathcal{H}_A$
   - cyclic + orthonormality: partial trace picks out terms where second label is the same
2. find relation between constants in $\hat{\rho}_A$:
   - symmetry $\mathcal{H}_A \leftrightarrow \mathcal{H}_B$: $|\alpha| = |\beta|$,
   - symmetry $0 \leftrightarrow 1$: $|\gamma| = 1$
   - maximal entropy: $\hat{\rho} = \frac{1}{2}\hat{1}_{\mathcal{H}_A}$

$$\frac{\beta + \gamma\alpha*}{1 + |\alpha|^2} = 0$$

$\therefore$ $|\alpha| = |\beta| = \infty$ or $\beta = -\gamma\alpha*$, e.g. $\alpha = \beta = 0$

3. Bell states are natural basis of maximally entangled states

$$|\psi_{1,2}\rangle = \frac{1}{\sqrt{2}}\big(\,|0,0\rangle \pm |1,1\rangle\,\big)$$
$$|\psi_{3,4}\rangle = \frac{1}{\sqrt{2}}\big(\,|0,1\rangle \pm |1,0\rangle\,\big)$$

   - qualitatively: for each measurement A, outcome B is known

6.2 Tensor products of harmonic oscillators

- QHO ladder op. Hamiltonian: $\hat{H} = \hbar\omega(\underbrace{\hat{a}^\dagger\hat{a}}_{\text{number operator } \hat{N}} + \frac{1}{2})$

Derive thermal state / quantum Boltzmann:

1. product space $\hat{a}_+ = \hat{a} \otimes \hat{1}$, $\quad \hat{a}_- = \hat{1} \otimes \hat{a}$
2. define pure state $|\psi\rangle = A\exp\Big(\alpha\hat{a}_+^\dagger\hat{a}_-^\dagger\Big)|0,0\rangle$
   - normalisation: $A = \sqrt{1 - |\alpha|^2}$
3. partial trace $\hat{\rho}_+ = \mathrm{Tr}_{\mathcal{H}_-}(\hat{\rho})$
   - define $|\alpha|^2 = \exp\Big(-\ln\Big(1/|\alpha|^2\Big)\Big) := \exp\big(-\frac{\hbar\omega}{kT}\big)$

Entangled state is thermal from point of view of $\mathcal{H}_+$
WKB approx Path integral saddle points Quantum tunneling

7 General Theory of Measurement

Projection operator $\hat{P}_A = |A\rangle\langle A|$.
Born rule: probability of outcome A of measurement $\hat{A}$ on pure state $|\psi\rangle$ is

$$p(A|\rho) = |\langle A|\psi\rangle|^2 = \mathrm{Tr}\Big(\hat{P}_A\hat{\rho}\hat{P}_A^\dagger\Big)$$

State after measurement is

$$|\psi'\rangle = \frac{\hat{P}_A|\psi\rangle}{\Big\|\hat{P}_A|\psi\rangle\Big\|} \qquad \hat{\rho}' = \frac{\hat{P}_A\hat{\rho}\hat{P}_A^\dagger}{\mathrm{Tr}\Big(\hat{P}_A\hat{\rho}\hat{P}_A^\dagger\Big)}$$

For degenerate eigenstates, state after measurement is given by

$$|\psi'\rangle = \frac{\hat{P}_{A_i}|\psi\rangle}{\Big\|\hat{P}_{A_i}|\psi\rangle\Big\|}$$

with $\hat{P}_{A_i} = \sum_{q_i}|A_i, q_i\rangle\langle A_i, q_i|$.

7.1 Consecutive Measurements

For consecutive measurements, projection operators multiply:

$$A \text{ AND THEN } B: \quad \hat{P}_{B\cap A} = \hat{P}_B\hat{P}_A$$

For chronological set of measurements A and then B and ...and then Z:

$$\hat{\rho}' = \frac{\hat{P}_Z\cdots\hat{P}_A\hat{\rho}\hat{P}_A^\dagger\cdots\hat{P}_Z^\dagger}{\mathrm{Tr}(\ldots)}$$

- trace is same as upstairs to ensure $\mathrm{Tr}(\rho') = 1$
- denominator trace is also the probability $p(Z\cap\cdots\cap A|\rho)$
- $p(B\cap A|\rho) \neq p(A\cap B|\rho)$ and Bayes' thm does not apply

7.2 Multiple Consistent Outcomes

If $\hat{P}_A\hat{P}_B = \hat{P}_B\hat{P}_A = 0$, projection operators add:

$$A \text{ OR } B: \quad \hat{P}_{A\cup B} = \hat{P}_A + \hat{P}_B$$

Example: Measurement on d-degenerate eigenvalue A:

$$\hat{P} = \sum_{q=1}^{d}|A,q\rangle\langle A,q|$$

7.3 Two-Slit Experiment

Electron passing slit (A OR B) AND THEN measured at D:

$$p(D\cap(A\cup B)) = p(D\cap A) + p(D\cap B) + I_{D,AB}$$

with interference term $I_{D,AB} = \mathrm{Tr}\Big(\hat{P}_D\hat{P}_A\hat{\rho}\hat{P}_B^\dagger\Big) + (A \leftrightarrow B)$
There is no third or higher order interference in QM (Wick's theorem?)

7.4 Measurements at different times

Schrödinger picture: $\hat{\rho}(t) = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{\rho}_0 e^{+\frac{i}{\hbar}\hat{H}(t-t_0)}$

- same as before but evolve state and thus $\hat{\rho}$ by $U(\Delta t)$ between each measurement
- probability is trace in denominator

Heisenberg picture:

- Absorb time evolution into projection operator:

$$\hat{P}_{A,B}(t) = e^{+\frac{i}{\hbar}\hat{H}(t-t_0)}\hat{P}_{A,B}e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}$$

7.5 Quantum Zeno Effect

1. Prepare system in state $|\phi\rangle$ with $\hat{\rho} = \hat{P} = |\phi\rangle\langle\phi|$
2. survival probability p: still in state $|\phi\rangle$ after time t/N
3. repeat N times to find combined prob. p_N after time t
4. continuous measurement: $\lim_{N\to\infty} p_N = 1$

Conclusion: continuously observed states evolve with $\hat{P}\hat{H}\hat{P}$ and remain within subspace defined by projection operator.

8 Angular Momentum

Infinitesimal rotation in SO(2): $R[\theta] \approx \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} = 1 - i\theta\sigma_2$.
Find $\hat{U}(\theta)$ by considering infinitesimal θ:

1. require position operators transform as (sim for $\hat{y}$)

$$\hat{U}(\theta)\hat{x}\hat{U}(\theta)^\dagger = \hat{x}\cos\theta - \hat{y}\sin\theta$$

2. look for soln of form $\hat{U}(\theta) \approx 1 + \frac{i}{\hbar}\hat{L}\theta + O(\theta^2)$
3. compare to find $\hat{L} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$
4. standard argument: $\hat{U}[\theta] = \lim_{N\to\infty}\hat{U}[\theta/N]^N = e^{i\theta\hat{L}/\hbar}$

8.1 Orbital Rotations

Action on position eigenkets

$$|x', y'\rangle = \hat{U}_L^\dagger(\theta)\,|x, y\rangle$$

Since $|x, y\rangle = \hat{U}_L^\dagger(2\pi)\,|x, y\rangle$ and wavefunction is continuous, require $\hat{U}_L(2\pi) = \hat{1}$:

- eigenvalues of $\hat{L}$ are integers
- find $\langle r, \phi|\hat{L}|\psi\rangle$ in polar coordinates:

$$x = r\cos\phi \qquad y = r\sin\phi$$

- eigenfn $\langle r, \phi|m\rangle = f(r)e^{im\phi}$ with eigenvalue $L = m\hbar$

8.2 Spin in 2D

$$\hat{U}_S(\theta) = e^{i\hat{S}\theta/\hbar}$$

- do not require $\hat{U}_S(2\pi) = \hat{1}$, only

$$\hat{U}_S(2\pi)\,|\psi\rangle = e^{i\alpha}\,|\psi\rangle$$

- eigenvalue equation: $S = (n + \frac{\alpha}{2\pi})\hbar$
  - $\alpha = 0$: boson
  - $\alpha = \pi$: fermion
  - else: ANYON

Total ang. mom.: $\hat{J} = \hat{L} + \hat{S} \implies \hat{U}_J = \hat{U}_S\hat{U}_L = \hat{U}_L\hat{U}_S$
Note that Pauli matrices obey $\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = \hat{1}$
Schwinger and Holstein-Primakoff representation

9 Electric and Magnetic Fields

10 Quantum Harmonic Oscillator

Ladder operators:

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

- number operator: $\hat{a}^\dagger\hat{a}$
- commutator: $[\hat{a}, \hat{a}^\dagger] = 1$

Position and momentum:

$$\hat{x} = \frac{\hbar}{2m\omega}(\hat{a} + \hat{a}^\dagger) \qquad \hat{p} = -i\omega\frac{\hbar}{2m\omega}(\hat{a} - \hat{a}^\dagger) \tag{1}$$

11 Hydrogen

For bound states, $E < 0$. This implies that there must be a maximum value of l.

12 Time-dep perturbation theory

13 Advanced topics

Spin statistics adiabatic approx Berry phase Aharanhov-Bohm effect