1 Calculus of Variations

Any action integral S of the form

$$S = \int_a^b L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)dt,$$

where $q_n(t)$ and $\dot{q}_n(t)$ are the generalised positions and velocities respectively, and $q(a)$ and $q(b)$ are held fixed, is extremised if the q_n satisfy the Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \dots, n. \tag{E-L}$$

Recognising $p_n = \partial_{\dot{q}_n} L$ as the generalised momentum, this reduces down to Newton's II. law if $L = T - V$ for conservative forces. If L does not depend on t explicitly then Beltrami's formula,

$$H = \sum_n (\dot{q}_n \frac{\partial L}{\partial \dot{q}_n}) - L$$

holds, where the Hamiltonian H is a constant of the motion (often - but not always - the energy of the system).

1.1 Shortest curve and Brachistochrone problems

1.2 Noether's theorem

If L does not depend on a coordinate q_i, then by (E-L), we have a conserved quantity,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial \dot{q}_i}) = \frac{\mathrm{d}p_i}{\mathrm{d}t} = 0, \tag{1}$$

called a constant of the motion. Such a coordinate is called *cyclic*.

Baby Noether's theorem: If q_i is cyclic, then p_i is a constant of the motion and no change of coordinates can destroy the conservation law: If q_i cyclic, L is invariant under replacement $q_i' = q_i + \varepsilon$, $\varepsilon =$ constant

$$L(q_1 + \varepsilon, q_2, \dot{q}_1, \dot{q}_2) = L(q_1, q_2, \dot{q}_1, \dot{q}_2) \tag{2}$$

More general (PS2): Suppose L is invariant under the replacement

$$q_i' = q_i + \varepsilon Q_i(q_1, q_2, \dots, q_n), \quad \varepsilon \text{small}, \tag{3}$$

(replacing all (or some if $Q_i = 0$ for some i) generalised coordinates in this fashion) then

$$\sum_{i=1}^n p_i Q_i \tag{4}$$

is a constant of the motion.

1.3 Charged Particle

The Lagrangian for the non-relativistic Lorentz force on a single charged particle q in an Electric field $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \partial_t \boldsymbol{A}$ and Magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ is (see PS2):

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 + q(\boldsymbol{A} \bullet \dot{\boldsymbol{r}} - \phi) \tag{5}$$

For a 1D relativistic particle

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \approx -mc^2 + T, \quad v^2 \ll c^2 \tag{6}$$

So the Lagrangian for the relativistic Lorentz Force law is

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + q(\boldsymbol{A} \bullet \boldsymbol{v} - \phi). \tag{7}$$

1.4 Constrained Extremisation

Lagrange multiplier: Want to optimise $f(x, y)$ subject to constaint $g(x, y) = 0$, then add extra variable λ to enforce constraint. Define

$$h(x, y, \lambda) = f(x, y) + \lambda g(x, y) \tag{8}$$

and find stationary points of h.

1.4.1 Hanging Rope Problem

Want to minimise potential energy V of the mass distributed over the rope

$$V = \frac{mg}{l} \int_{-a}^a y(x)\sqrt{1 + (y'(x))^2} dx, \tag{9}$$

hanging from the fixed points $y(a) = y(-a) = h$, subject to the constraint that the length $l > 2a$ of the rope is fixed

$$g = l - \int_{-a}^a \sqrt{1 + y'^2} dx = 0 \tag{10}$$

Dropping the constant factors as we are only interested in the shape, the integral we have to optimise becomes

$$S = V + \lambda g = \int_{-a}^a [y\sqrt{1 + y'^2} + \lambda(\frac{l}{2a} - \sqrt{1 + y'^2})] dx \tag{11}$$

Note, as L has no explicit x-dependence we can use the Beltrami formula. Isoperimetric problems:

- Maximise area A for fixed length L
- Minimise length L for fixed area A

$$A = \int_a^b y(x)dx, \quad L = \int_a^b \sqrt{1 + y'^2} dx > (b - a) \tag{12}$$

2 Tensors

2.1 Cartesian Tensors

R orthogonal and $\boldsymbol{e}_i$ orthonormal.

We can write a 3D vector in index notation as

$$\boldsymbol{V} = \sum_{i=1}^3 V_i \boldsymbol{e}_i = V_i \boldsymbol{e}_i, \tag{13}$$

where using the Einstein summation convention, the summation sign can be omitted whenever an index is repeated. The basis is orthonormal if

$$\boldsymbol{e}_i \bullet \boldsymbol{e}_j = \delta_{ij} \tag{14}$$

- Dot product
$$\boldsymbol{U} \bullet \boldsymbol{V}_i = U_i V_i \tag{15}$$
- Cross product
$$\boldsymbol{U} \times \boldsymbol{V} = (\boldsymbol{U} \times \boldsymbol{V})_i \boldsymbol{e}_i = (\varepsilon_{ijk} a_j b_k)\boldsymbol{e}_i \tag{16}$$

2.2 Transformation properties

Two types of transformation:

- R: transforms coordinates
$$x^{i\prime} = R^i{}_j x^j \tag{17}$$
- S: transforms basis vectors
$$\boldsymbol{e}_i{}' = S_i{}^j \boldsymbol{e}_j \tag{18}$$

Two kinds of vector:

- Contravariant vector: transforms like coordinates:
$$V^{i\prime} = R^i{}_j V^j \tag{19}$$
- Covariant vector: transforms like basis vectors:
$$V_i' = S_i{}^j V_j \tag{20}$$

A tensor of type (p, q) is 3^{p+q} numbers which transform as

$$T'^{i_1 \dots i_p}{}_{j_1 \dots j_q} = R^{i_1}{}_{k_1} \dots R^{i_p}{}_{k_p} S_{j_1}{}^{l_1} \dots S_{j_q}{}^{l_q} T^{k_1 \dots k_p}{}_{l_1 \dots l_q},$$

where $i_1, \dots, i_p$ are contravariant and $j_1, \dots, j_q$ are covariant indices.

The rank is $p + q$, so vectors are tensors of rank 1, scalars are tensors of rank 0.

2.3 Important Tensors

The gradient operator

$$\partial_i' = \frac{\partial}{\partial x^{i\prime}} = S_i^j \boldsymbol{\partial}_j \tag{21}$$

transforms like a covariant vector.

The Krönicker delta

$$\delta^i{}_j = \frac{\partial x^i}{\partial x^j} = \partial_j x^i \tag{22}$$

is a tensor of type (1,1). The metric

$$g_{ij} = \boldsymbol{e}_i \bullet \boldsymbol{e}_j \tag{23}$$

is a symmetric tensor of type (0,2).

2.4 Orthogonal Transformations

- Orthogonal matrix/transformation:
$$R^T R = I \tag{24}$$
- orthogonal matrices have the property
$$\det(R) = \pm 1 \tag{25}$$
as transposition does not change the determinant.
- proper rotation: $\det(R) = 1$.
Else: improper rotation (rotation and parity transformation)

2.5 Cartesian Tensors

Linear change of coordinates and basis vectors:

$$x_i' = R_{ij} x_j, \quad \boldsymbol{e}_i' = S_{ij} \boldsymbol{e}_j. \tag{26}$$

Requiring that the vector is unchanged (we just represent it differently in different bases) gives

$$\boldsymbol{r} = x_i \boldsymbol{e}_i = x_i' \boldsymbol{e}_i' = R_{ij} x_j S_{ik} \boldsymbol{e}_k = x_k \boldsymbol{e}_k, \tag{27}$$

which leads to

$$R_{ij} S_{ik} = (R^T)_{ji} S_{ik} = \delta_{ik} \tag{28}$$

And if $\boldsymbol{e}_i$ and $\boldsymbol{e}_i'$ are orthonormal then R_{ij} must be an orthogonal matrix. So $S = R$.

Then there is no distinction between covariant and contravariant as both transform in the same way. The metric becomes

$$g_{ij} = \boldsymbol{e}_i \bullet \boldsymbol{e}_j = \delta_{ik} \tag{29}$$

2.6 Levi-Civita Symbol

The Levi-Civita Symbol ε_{ijk} is 27 numbers defined by

- ε_{ijk} is totally anti-symmetric under exchange of neighbouring indices
- $\varepsilon_{123} = 1$

Hence $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$ and $\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{312} = -1$ and all the other numbers are 0.

2.7 Vector Calculus

- Gradient
$$\boldsymbol{\nabla}\phi = (\partial_i \phi)\boldsymbol{e}_i \tag{30}$$
- Divergence
$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \partial_i F_i \tag{31}$$
- Curl
$$\boldsymbol{\nabla} \times \boldsymbol{F} = (\varepsilon_{ijk} \partial_j F_k)\boldsymbol{e}_i \tag{32}$$
- Laplacian
$$\nabla^2 \phi = \partial_i \partial_i \phi \tag{33}$$

Physical examples of cartesian tensors.

3 Complex Variables

Complex differentiation, analytic functions,

3.1 Cauchy-Riemann equations

If u and v are real-differentiable, then a function $f(z) = u(x, y) + iv(x, y)$ of a complex variable $z = x + iy$ is complex-differentiable (holomorphic) only if the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{C-R}$$

are satisfied.

- entire function: homomorphic at all $z \in \mathbb{C}$

harmonic property of analytic functions. Complex integration,

3.2 Fundamental Theorem of Calculus

Suppose $F(z)$ is analytic inside a region $R \in \mathbb{C}$, and let C be an oriented contour inside R, then

$$\int_C f(z)dz = F(z_2) - F(z_1), \tag{34}$$

where $F'(z) = f(z)$. F is anti-derivative or primitive for f.

Proof: Parametrise $z = z(t), a \leq t \leq b$, and use chain rule:

$$\int_C f(z)dz = \int_a^b f(z(t))\frac{\mathrm{d}z(t)}{\mathrm{d}t}dt = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(z(t))dt, \tag{35}$$

and the result follows from FToC for real variables.

3.3 Cauchy's Theorem

3.3.1 Anti-Derivative Form

If C is a closed, piecewise smooth curve inside a region $R \in \mathbb{C}$,

$$\oint_C F'(z)dz = 0 \tag{36}$$

if $F(z)$ analytic in R.

3.3.2 Standard Version

Suppose f is analytic on and inside the simple, closed, piecewise smooth contour C, then

$$\oint_C f(z)dz = 0. \tag{37}$$

Same as before, but now C cannot cross itself and f has to be analytic inside whole region bounded by C. Proof: Write

$$f(z)dz = (u + iv)(dx + idy) = pdx + qdy \tag{38}$$

and apply Green's theorem (C is boundary of R)

$$\oint_{C=\partial R} f(z)dz = \int_R (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})dxdy = 0 \tag{39}$$

using that by (C-R): if f analytic, $u_x = v_y, u_y = -v_x$.

3.4 Deformation Theorem

If C_1 and C_2 are simple, closed, piecewise smooth, anti-clockwise oriented curves, where C_1 lies inside C_2, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz \tag{40}$$

if $f(z)$ analytic on and in the region in between C_1 and C_2. Proof: Take anti-clockwise orientation of inner contour for regions with holes and use Cauchy's theorem on contour that has joining element between C_1 and C_2, where the contributions along joining lines cancel.

3.5 Cauchy's Integral Formula

If f analytic on and inside C,

$$f(w) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{w - z}dz. \tag{CIF}$$

Proof: Deform C into circle of radius R centered at w. Parametrise in terms of θ:

$$z = w + Re^{i\theta} \tag{41}$$

and let $R \to 0$. Can be used to prove following two results.

3.5.1 Liouville's Theorem

If f is entire and bounded ($|f(z)| < C, \forall z \in \mathbb{C}$), it is constant.

Proof: Differentiate (CIF) w.r.t. w. Take same parametrisation as before and use

$$|f'(w)| = \frac{1}{2\pi R}\left|\int_0^{2\pi} e^{i\theta} f(w + Re^{i\theta})d\theta\right| \tag{42}$$

$$\leq \frac{1}{2\pi R}\int_0^{2\pi} \left|e^{i\theta} f(w + Re^{i\theta})\right|d\theta \tag{43}$$

$$\leq \frac{C}{R}, \quad \forall R > 0 \tag{44}$$

provided the integral exists and is finite. Then let $R \to \infty$.

3.5.2 Analytic Functions are Infinitely Differentiable

3.6 Taylor's Theorem

Suppose f analytic inside a circle of radius R centered at $w \in \mathbb{C}$, then for $|z - w| < R$

$$f(z) = \sum_{m=0}^{\infty} c_m(z - w)^m, \tag{45}$$

$$c_m = \frac{f^{(m)}(w)}{m!} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - w)^{m+1}}dz \tag{46}$$

where C is inside and encloses the centre of the circle. Write down radius of convergence by considering distance to singularities from expansion point.

3.7 Isolated Singularities

Call $z \in \mathbb{C}$ an isolated singularity if f is complex differentiable in a punctured disc $D'_r(z) = \{w \in \mathbb{C} \mid 0 < |z - w| < r\}$ for some $r > 0$.

3.8 Laurent Series

The expansion around an expansion point w of the form

$$f(z) = \sum_{-infty}^{\infty} c_m(z - w)^m \tag{LS}$$

is called a Laurent series. The principal part is all the singular terms $m < 0$ in the expression.

Laurent's theorem is like Taylor's theorem but now C lies in an annulus around w bounded by an inner and outer radius. Inner radius $R_1 = 0$ if expansion about an isolated singularity is possible. LS expansions are not unique: can take annulus excluding singularities.

3.8.1 Poles and Essential Singularities

If the number of negative powers in a LS expansion about an isolated singularity (with $R_1 = 0$) is

- infinite: essential singularity
- finite: pole

3.9 Residue Theorem

If f analytic on and inside C except for n isolated singularities at $z_1, \ldots, z_n$ inside C, then

$$\oint_C f(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res}(f, z_i), \tag{47}$$

where the residue $\text{Res}(f, w)$ is the coefficient c_{-1} in the LS expansion of f about $z = w$. Proof: For one singularity, take $m = -1$ in (LS). For $n = 2$ split contour and sum similar to deformation theorem proof.

3.10 Simple Poles on Contour

Half residue rule:

$$P \oint_C f(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res}(f, z_i) + \pi i \sum_{j=1}^{m} \text{Res}(f, z_j), \tag{48}$$

where z_i are the poles enclosed by and z_j the poles on the contour C.

3.10.1 Calculating Residues

If $f(z) = \frac{1}{g(z)}$ has a simple pole at $z = w$, then

$$\text{Res}(f, w) = \frac{1}{g'(w)}. \tag{49}$$

Proof: Taylor expand g around w. For simple pole, $g(w) = 0$.

3.11 Real Integrals

Can compute real integrals by splitting the contour integral over the semi-circle C with radius R into a part along the real axis $z = x$ and a circular part with $z = Re^{i\theta}$

$$\oint_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_0^{\pi} f(Re^{i\theta}) iRe^{i\theta} d\theta, \tag{50}$$

letting $R \to \infty$ and applying the residue theorem. Justification:

$$\int_0^{\pi} f(Re^{i\theta}) iRe^{i\theta} d\theta \leq R \int_0^{\pi} \left| f(Re^{i\theta}) \right| d\theta \tag{51}$$

vanishes as $R \to \infty$ if $f \propto 1/z^n, n \geq 2$.

Principal value of real and

complex integrals, Residue theorem with simple poles on the contour.

3.12 Miscellaneous

Hyperbolic functions are basically real functions rotated onto the imaginary axis: $\sinh(z)$ has zeros at $in\pi$ and $\cosh(z)$ at $(2n+1)\frac{\pi}{2}$.

4 Fourier Transforms

Review of Fourier transforms and Fourier integrals. Fourier Transform convention:

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx} dx \tag{52}$$

4.1 Properties

Derivatives

$$\hat{f'}(k) = ik\hat{f}(k) \tag{53}$$

Parity If $f(x)$ even, then $\hat{f}(k)$ even. If $f(x)$ odd and real, then $\hat{f}(k)$ imaginary.

Parseval's formula

$$\int_{-\infty}^{\infty} \left| \hat{\psi}(x) \right|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\psi(x)|^2 dx \tag{54}$$

Convolution theorem If $h(x) = f(x) * g(x)$, then

$$\hat{h}(k) = 2\pi \hat{f}\hat{g} \tag{55}$$

4.2 Important Transforms

Square pulse:

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(k)}{k} e^{ikx} dx = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \tag{56}$$

Computation of Fourier transforms using contour integration. Heaviside and sign function, delta function as a derivative, delta function as a limit of smooth functions, definition through the integral formula

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a). \tag{57}$$

Properties of the delta function, Fourier transform (and Fourier integral representation) of the delta function. Application of Fourier transforms to solving linear ODEs and PDEs (eg. driven oscillator ODEs and Laplaces equation in an infinite strip and half-plane).

5 Numerical Methods

Numerical integration (trapezium rule and Simpsons rule), Newton-Raphson method, Runge-Kutta algorithm.

Examination

2 hour paper with 5 questions (answer question 1 and two other questions. Question 1 comprises 8 short answer questions covering the whole module and is worth 402 to 5 are worth 30% each.

5.1 Mathematical Proofs

The examination will predominantly be a test of technical skill rather than the ability to reproduce proofs. However, students are expected to be able to give short proofs (of results given in the lectures and problem sheets or simple results not seen previously). Students will not be asked for any long proofs, e.g. Taylors theorem.