1 Cosmology

1.1 Cosmological Parameters

$$\Omega_x(t) = \frac{\rho_x(t)}{\rho_{\text{crit}}(t)}$$

Sum to one: $\Omega_m + \Omega_\lambda + \Omega_\kappa = 1$
Values measured today:

Matter: $\Omega_m \approx 30\%$

Dark matter: $\Omega_{DM} = \Omega_m - \Omega_b = 25\%$

Baryonic density: $\Omega_b \approx 5\%$

Cosm. const.: $\Omega_\Lambda = 70\%$

Curvature: $|\Omega_\kappa| < 1\%$

Measured via standard candles and standard rulers.

1.2 Standard Candles

SNIa explode at same Chandrasekhar mass. High luminosity fluctuations exist.
Phillips: SNIa with higher peak M take longer to fade. Gives luminosity distance d_L up to an overall factor.

1.3 Distances

Luminosity distance:

$$F = \frac{L}{4\pi d_L^2(z)}$$

Angular diameter distance

$$d_A(z) = \frac{l_{\text{phys}}}{\delta\theta}$$

where l_{phys} is the physical size of the small edge of the triangle.

units the angle is measured in radians!

Their relation is

$$d_L = d_A(1+z)^2$$

where the angular diameter distance is smaller.
Scale factor relates physical and comoving distances

$$r_{\text{phys}} = a(t) r_{\text{comoving}}$$

For the local group ($z \ll 1$), all distances agree:

$$d_{\text{phys}} = d_{\text{comoving}} = d_A = d_L$$

1.4 Magnitudes

Bigger magnitude means fainter object. Dimming: magnitude increases $m + \Delta m_{15}$
Apparent magnitude: $m = -2.5 \log_{10}(\frac{F}{F_{\text{ref}}})$
Absolute magnitude is apparent magnitude at distance of 10pc

$$M = -2.5 \log_{10}(\frac{L}{L_{\text{ref}}})$$

Distance modulus is difference between apparent and absolute magnitudes:

$$\mu = m - M = 5 \log_{10}\left(\frac{d_L}{\text{Mpc}}\right) + 25$$

1.5 Scale Factor

Hubble's law:

$$v = H_0 d$$

with $H(t) = \frac{\dot{a}}{a}$
Relationship between redshift and scale factor

$$1 + z = \frac{1}{a}$$

Blackbody temperature and scale factor

$$T = \frac{T_0}{a}$$

1.6 Curvature

flat: $\kappa = 0$ closed: $\kappa > 0$ open: $\kappa < 0$

For a flat universe, $\rho = \rho_{crit} = \frac{H^2(t)}{(8\pi G/3)}$.

1.7 Matter

matter: $\rho_m = \dfrac{\rho_{m,0}}{a^3}$ radiation: $\rho_r = \dfrac{\rho_{r,0}}{a^4}$

Matter-radiation equality z_{eq}

1.8 CMB

Ionised early universe. Thomson scattering of photons with free electrons gives Photon-baryon fluid. At recombination, $z = 1100$, hydrogen forms and photons travel freely $\Rightarrow$ CMB.
Temperature $T_{CMB} = 2.72$K
Recombination $z_{\text{rec}} = 1100$ at energy scale $T_{CMB}(z+1) = 3000K$. Later than ionisation energy due to Wien tail.
BAO: photon pressure against grav. collapse $\Rightarrow$ sound waves with wavelength: Comoving sound horizon: $\lambda_S \approx 150$Mpc
Standard ruler (2-point correlation in CMB), first peak, used to measure d_A and thus cosmological parameters (curvature).

1.9 Large Scale Structure

CMB anisotropies $\frac{\delta T}{T} \sim 10^5$
Perturbations (relative matter density contrast) grow as

- matter: $\delta \sim t^{2/3} \sim a$
- radiation domination: $\delta \sim \ln t$, $a \sim t^{1/2}$
- cosmological constant: $\delta \sim 1$

Need dark matter (not affected by BAO's) to explain structures today.

1.10 Equations

Acceleration Equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(1+3w)$$

Friedmann Equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{a^2} = H_0^2\left[\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \frac{\Omega_\kappa}{a^2} + \Omega_\Lambda\right]$$

1.11 Horizon and Flatness Problem

Horizon: Horizon distance at recombination with $\Omega_m = 0.3$ subtends 1° on the sky. These places were never in causal contact but still have the same CMB temperature.
Flatness: In order for $\Omega = \frac{\rho}{\rho_{crit}} \approx 1$, the initial density must have a particular fine-tuned value giving a flat universe.
Inflation solves these by enlarging a small patch in causal contact to larger than the observable universe, flattening any curvature.
Number of e-folds N_i:

$$\frac{a(t_{\text{end}})}{a(t_{\text{start}})} = e^{N_i}$$

Energy density of inflation: $\rho_i^{ed} = 10^{101}[\frac{TeV}{m^3}]$

1.12 Neutron Freeze-out

Weak force falls as $\Gamma_W \sim t^{-5/2}$, Hubble (expansion) parameter as $H \sim t^{-1}$. When $Gamma_W < H$, weak interactions do not keep up with expansion and comoving baryon number density is constant.
Helium Mass fraction: 25% of baryonic mass is primordial ^4He

$$Y = \frac{(\text{Helium mass density})}{(\text{total baryon mass density})} = \frac{4n_{He}}{4n_{He} + n_H} \approx 25\%$$

Can measure Ω_b and thus $\Omega_{DM} = \Omega_m - \Omega_b$

1.13 Dark Matter

Evidence:

- flat rotation curves: expect $v^2 \sim r^{-1}$, but observe $v \sim const.$ $\Rightarrow$ dark halo
- gravitational lensing $\Rightarrow$ total mass determined via strong lensing much larger than baryonic mass
- structure formation
- BBN and CMB
- absence of microlensing rules out MACHOs

WIMPS: non-relativistic (cold) at freeze-out. Neutrinos: relativistic (hot) at freeze-out.
WIMP miracle: relic density of right magnitude.
Cosmological constant domination for $z \ll z_\star$ matter-Λ equality

2 Particle Physics

2.1 Feynman Diagrams

- electrical quark charge is written e_q at vertex

2.2 Scattering

Measure $R = \frac{\sigma(\text{hadrons})}{\sigma(\mu^+\mu^-)}$ to cancel out errors in beam intensity and detector efficiency.

2.3 Proton Structure

Each parton carries xp momentum, then

1. initially, have $E_{cm} = E_1 + E_2 = 2pc$ and $p_T = 0$
2. partons carry $E'_T = (x_1 + x_2)pc$ and
$$p'_T = p_1 - p_2 = (x_1 - x_2)p$$
3. Centre of mass energy is then
$$E'_{cm} = \sqrt{s} = \sqrt{(x_1 + x_2)^2 - (x_1 - x_2)^2}pc = \sqrt{x_1 x_2}E_{cm}$$
4. for creation of 2 particles, condition is $E'_{cm} \geq 2mc^2$, so
$$\sqrt{x_1 x_2} \geq 2\frac{mc^2}{E_{cm}}$$

3 Fluid Dynamics

3.1 Fundamentals

- Mass conservation / continuity equation
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = 0$$
- material / Lagrangian derivative following a fluid parcel
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})$$

3.2 Types of Flow

- angular momentum conserving flow
$$u_\theta = \frac{L}{r}$$
- source / sink flow
$$u_r = \frac{Q}{2\pi r}$$
- solid body rotation
$$u_r = u_z = 0 \quad u_\theta = \Omega r$$

3.3 Navier-Stokes I

- body forces (long range)
$$\boldsymbol{F}_B = \iiint \rho \boldsymbol{f}_B \, \mathrm{d}V$$
  - force per unit mass
  - for gravity $\boldsymbol{f}_B = \boldsymbol{g}$
- surface forces (short range)
$$\boldsymbol{F}_S = \iint \boldsymbol{f}_S \, \mathrm{d}S = \iint (-P\boldsymbol{n} + \boldsymbol{\tau}) \, \mathrm{d}S$$
  - force per unit area (i.e. stress)
- pressure contribution
$$-\iint P \, \mathrm{d}\boldsymbol{S} = -\iiint \boldsymbol{\nabla} P \, \mathrm{d}V$$
- hydrostatic balance: fluid at rest, external forces (gravity) balanced by pressure gradient force

3.4 Viscosity

- Newtonian fluid

$$\boldsymbol{\tau} = \mu \frac{\mathrm{d}u}{\mathrm{d}z}$$

  - viscous stress is tangential (in x-direction)
  - viscous force $F = \tau A$, where A is the area of the walls
  - viscosity: $\mu > 0$
  - kinematic viscosity: $\nu = \mu/\rho$ in $m^2 s^{-1}$
    water: $\nu_{\text{water}} \sim 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$
    air: $\nu_{\text{air}} \sim 10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1}$
  - 1D viscous force (change in momentum) per unit mass is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}$$

In general, viscous force per unit mass is given by

$$\frac{\boldsymbol{F}}{m} = \nu \nabla^2 \boldsymbol{u}.$$

For divergence-free flows, $-\nu \boldsymbol{\nabla} \times \boldsymbol{\zeta}$.
Viscosity measures the momentum flow, so constant vorticity flows do not mix / rearrange fluid parcels.

3.5 Boundary Conditions

No-slip: $\boldsymbol{u} = 0$ at boundaries (e.g. strong friction)

Free-slip: $\boldsymbol{u} \neq 0$ at boundaries

no normal flow: $\boldsymbol{u} \cdot \hat{\boldsymbol{n}}$ flow cannot penetrate boundary

3.6 Navier-Stokes II

- constant density + continuity $\implies$ divergence-free

$$\frac{D\rho}{Dt} = 0 \implies \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

- for Newtonian fluid of const. density, viscous force per unit volume is

$$\mathrm{d}F = \mu \nabla^2 \boldsymbol{u}\, \mathrm{d}V$$

- Navier-Stokes equation (constant ρ, μ)

$$\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{f}_B - \frac{\boldsymbol{\nabla} P}{\rho} + \nu \nabla^2 \boldsymbol{u}$$

  - kinematic viscosity $\nu = \mu/\rho$ in $m^2 s^{-1}$

3.7 Flow in Long Channel

Assumptions:

1. Steady-state: $\partial/\partial t = 0$
2. No edge effects: $\boldsymbol{u} = u\boldsymbol{i}$
3. Flow is slab: $\partial u/\partial z = 0$
4. no slip BC: $\boldsymbol{u} = \boldsymbol{0}$

- Parabolic solution $u(y) = \frac{F}{2\mu}(1 - (y/a)^2)$
- hydrodynamic lubrication: $\Delta P = \tau\, l/a \gg \tau$

3.8 Reynolds number

$$R_e = \frac{\text{inertia}}{\text{viscosity}} = \frac{(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}}{\nu \nabla^2 u} = \frac{UL}{\nu}$$

3.8.1 Low R_e: Stokes flow

- neglect inertia compared to viscosity
- Steady-state Navier-Stokes for $R_e \ll 1$

$$\boldsymbol{0} \approx -\frac{\boldsymbol{\nabla} P'}{\rho} + \nu \nabla^2 u$$

  - gravity: background pressure $\boldsymbol{g} = \frac{1}{\rho} \boldsymbol{\nabla} P_{\text{ref}}$
  - perturbation pressure $P' = P - P_{\text{ref}}$

- solution determined by
  - continuity $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$
  - no-slip $\boldsymbol{u} = \boldsymbol{0}$ at boundary
  - $\nabla^2 \boldsymbol{\zeta} = \nabla^2(\boldsymbol{\nabla} \times \boldsymbol{u}) = \boldsymbol{0}$

- features
  - long range r^{-1} effect
  - reversibility $(\boldsymbol{u}, P') \leftrightarrow (-\boldsymbol{u}, -P')$

3.8.2 High R_e

- neglect viscosity compared to inertia
- Navier-Stokes for $R_e >> 1$

$$\frac{D\boldsymbol{u}}{Dt} \approx -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{g}$$

- boundary layer $\delta \approx L/\sqrt{R_e}$
- Bernoulli function

$$B = \frac{P}{\rho} + \frac{1}{2}u^2 + \Phi$$

  - gravitational potential Φ satisfies $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$
  - conserved by fluid parcel in steady flow: $\boldsymbol{u} \cdot \boldsymbol{\nabla} B = 0$

3.9 Drag and Lift

$$\boldsymbol{F}_{\text{drag}} = \oiint_{\text{object}} (-P\boldsymbol{n} + \boldsymbol{\tau}) \cdot \boldsymbol{i}\, \mathrm{d}S$$

- Stokes' law for low R_e:

$$F_{\text{drag}} = 6\pi\mu u a$$

- reduce drag: streamline the object, or create rough surface to generate turbulent boundary layer
- viscosity is key to drag: sets pressure field and controls drag indirectly. No drag when viscosity is zero.

3.10 Circulation and Vorticity

$$C = \oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = \iint \boldsymbol{\zeta} \cdot \mathrm{d}\boldsymbol{S}$$

- generated by viscous stresses on surface of body
- Kelvin's circulation theorem: when viscous effects are neglected, for constant density fluid,

$$\boxed{DC/Dt = 0}$$

along material contour

$\implies$ for small circuit δS, $\zeta \delta S$ = constant following the flow

$\implies$ in the absence of viscosity, a fluid of constant density cannot gain circulation or vorticity

3.10.1 Hurricane Formation

$$\boldsymbol{u} = \boldsymbol{u}_R + \boldsymbol{\Omega} \times \boldsymbol{r} \implies \boldsymbol{\zeta} = \boldsymbol{\zeta}_R + 2\Omega$$

- Kelvin's circulation thm $\implies$ choosing $\delta \boldsymbol{S} \perp \boldsymbol{k}$:

$$\boldsymbol{k} \cdot (\boldsymbol{\zeta}_R + 2\boldsymbol{\Omega})\delta S = \text{const.}$$

- close to equator $f_0 = 2\boldsymbol{\Omega} \cdot \boldsymbol{k} \approx 0$, so if initially $\boldsymbol{k} \cdot \boldsymbol{\zeta}_R = 0$, the vorticity will stay that way indep. of δS

3.11 Geophysical Fluid Dynamics

- Navier-Stokes in rotating frame

$$\frac{D\boldsymbol{u}}{Dt} = -\underbrace{\boldsymbol{\nabla}\Phi'}_{\boldsymbol{f}_B} - \frac{\boldsymbol{\nabla}P}{\rho} + \nu \nabla^2 \boldsymbol{u} \underbrace{-2\boldsymbol{\Omega} \times \boldsymbol{u}}_{\text{Coriolis}}$$

  - $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$ and $\Phi' = \Phi - \Omega^2 r_H^2/2$

- Rossby number

$$R_o = \frac{\text{inertia}}{\text{Coriolis}} = \frac{U}{2\Omega L}$$

- Geostrophic balance equation

$$-fv \approx -\frac{1}{\rho}\frac{\partial P}{\partial x} + fu \approx -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

  - valid for $R_e \gg 1 \gg R_o$ (Coriolis $\gg$ inertia $\gg$ viscosity), e.g. Oceans, cyclones in atmosphere, but not hurricanes
  - also assumes steady state
  - Coriolis parameter: $f = 2\boldsymbol{\Omega} \cdot \boldsymbol{k}$

- Taylor columns: $R_e \gg 1 \gg R_0$