1 Abstract Group Theory

1.1 Lagrange's Theorem

- For a subgroup $H \subset G$: $\frac{|G|}{|H|} \in \mathbb{N}^+$

1.2 Conjugate Classes

- $g \sim g'$ are conjugate elements if $\exists x \in G$ s.t.
$$xgx^{-1} = g'$$
- equivalence relation: reflexivity, transitivity, symmetry
- conjugate classes $C(g) = \{xgx^{-1} \mid x \in G\}$
- same order: $g^k = E \implies (g')^k = E$
- identity is its own class
- centraliser: $Z(g) = \{g' \in G \mid g'g = gg'\} \subset G$
  (i) $xgx^{-1} = x'g(x')^{-1} \iff x^{-1}x' \in Z(g)$
  (ii) $E \in Z(g)$
  $\therefore$ $xgx^{-1} = x'g(x')^{-1} \iff x, x' \in xZ$

1.3 Normal (Invariant) Subgroup

- A normal subgroup $N \subset G$ satisfies $\forall g \in G$,
$$gNg^{-1} = N$$
- fully contains conjugate classes: $g \in N \implies C(g) \subset N$
- left/right cosets are equal: $gN = Ng$
- factor/quotient group: $G/N = \{gN \mid g \in G\}$
- by Lagrange's Theorem: $|G/N| = |G|/|N|$

1.4 Homomorphism

- for any $a, b, c \in G$, there are $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \widetilde{G}$, s.t.
$$ab = c \iff \widetilde{a}\widetilde{b} = \widetilde{c}$$
- $\{g \in G \mid g \leftrightarrow \widetilde{E}\}$ form a normal subgroup N of G
  (i) elements in coset gN correspond to same elt $\widetilde{g}\widetilde{E} = \widetilde{g}$
  (i.i) distinct cosets gN correspond to distinct elements $\widetilde{g}$
  (ii) each $\widetilde{g} \in \widetilde{G}$ corresponds to $|gN|$ elements $g \in G$
  $\therefore$ $\widetilde{G}$ is isomorphic to the factor group G/N

1.5 Cayley's Theorem

- Any finite group G is isomorphic to a subgroup of $S_{|G|}$
- Symmetric group S_n: permutations of n indices
  - can represent elements of S_n as $2 \times n$ matrices
$$\begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \in S_n$$
- Since S_n is finite, number of subgroups of S_n is finite

$\therefore$ number of distinct (non-isomorphic) groups G of order $|G| = n$ is finite

1.6 Point Groups

- C_n axes and σ planes cross at a point to keep G finite
- reflections: σ
  1. σ_v containing C_n axis
  2. $\sigma_h \perp C_n$ axis
  3. σ_d diagonal (the rest)
- rotations: $C_n = \{C_n, C_n^2, \dots C_n^n = E\}$ (cyclic Abelian)
$$C_n = R(\varphi = \frac{2\pi}{n}) \tag{1}$$
- improper rotations: rotation and reflection
$$S_n = C_n\sigma_h = \sigma_h C_n \tag{2}$$
  - inversion (about origin): $i = S_2$

(i) Rotations of a point group form a subgroup

(ii) (Im)proper rotations form conjugate classes
$$gC_ng^{-1} = C_n^{-1}$$

$\therefore$ a point group's rotational subgroup is normal

- point group hierarchy:
  - $C_{nv} : (C_n, n\sigma_v)$
    - non-Abelian for $n > 2$
  - $C_{nh} : (C_n, \sigma_n \perp C_n)$
  - dihedral group: $D_n : (C_n, nC_2 \perp C_n)$
  - $D_{nh} : (C_n, nC_2 \perp C_n, \sigma_h \perp C_n, n\sigma_v)$
  - $D_{nd} : (C_n, nC_2 \perp C_n, i = S_2, n\sigma_d)$

edges/vertices	3	4	5
3	tetrahed. T_d	cube O_h	dodecahed. I_h
4	octahed. O_h		
5	icosahed. I_h		

Table 1: Point Groups of the Platonic Solids

2 Representation Theory

- homomorphism from G to a set of linear operators $\hat{T}(G)$ on a vector space L.
$$g_ig_j = g_k \iff \hat{T}(g_i)\hat{T}(g_j)\vec{x} = \hat{T}(g_k)\vec{x}$$
- basis $\{\hat{e}_i\}$ of L: decompose $\vec{x} = x_i\hat{e}_i$
$$\hat{T}(g)e_i = \sum_{j=1}^{\dim(L)} D_{ji}(g)e_j$$

- matrices $D(G)$ form a representation of G on L

$$D(g_i)D(g_j) = D(g_i g_j) \quad \text{(homomorphism)}$$

  - exact / faithful rep.: isomorphism $g \leftrightarrow D(g)$
  - fully symmetric / trivial IRREP.: $A_1(g) = 1$
  - unitary: $D^\dagger = D^{-1} \implies \langle D\vec{x}|D\vec{y}\rangle = \langle \vec{x}|\vec{y}\rangle$
  - equivalent: $D' = MDM^{-1}$

- Can find an equivalent unitary rep. for any $D(G)$ on $\mathbb{C}^n$.

$\therefore$ WLOG assume D is unitary

2.1 (Ir)reducible Representations

- reducible: D leave some subspace of L invariant
  - in special basis: D are block diagonal
    - construct this basis with projection operators

$$\hat{P}_{ik}^{(j)} = \frac{\dim D}{|G|} \sum_g (D^{(j)})^*_{ik} \hat{T}(g)$$

  - blocks are IRREPs $D = \sum_i^{\oplus} r_i D^{(i)}$

- number of inequiv. IRREPs is number of conjugate classes
- Character: $\chi^{(j)}(g) = \operatorname{Tr} D^{(j)}(g_i)$
  - equivalent IRREPs have same character
  - conjugate elements have the same character

- character of product rep. is product of characters

$$\chi(D^{(i)} \otimes D^{(j)}) = \chi^{(i)}\chi^{(j)} \tag{3}$$

  - (direct) product rep. is in general reducible
  - basis of product rep. is product of original bases

- character of reducible rep is sum of IRREP characters

$$\chi(D^i \oplus D^j) = \chi^{(i)} + \chi^{(j)} \tag{4}$$

2.2 Schur's Lemmas

$$D^{(i)}(G)M = MD^{(j)}(G) \implies M = \delta_{ij}\alpha I$$

2.3 Orthogonality Theorems

$$\sum_{g\in G} \left(D^{(i)}(g)\right)^*_{\mu\nu} \left(D^{(j)}(g)\right)_{\alpha\beta} = \frac{|G|}{\dim(D)}\delta_{ij}\delta_{\mu\alpha}\delta_{\nu\beta}$$

Proof. Use Shur's lemmas on $M = \sum_g D^{(i)} X D^{(j)}$ □

- Number of inequivalent IRREPs is finite
- character row orthogonality (inner product)

$$(\chi^{(i)}, \chi^{(j)}) = \sum_g (\chi^{(i)}(g))^* \chi^{(j)}(g) = |G|\delta_{ij}$$

  - can sum over conjugate classes

$$(\chi^{(i)}, \chi^{(j)}) = \sum_C n_C \left(\chi^{(i)}(C)\right)^* \chi^{(j)}(C) = |G|\delta_{ij}$$

- column orthogonality

$$\sum_i n_C (\chi^{(i)}(C))^* \chi^{(i)}(C') = |G|\delta_{CC'} \tag{5}$$

2.4 Reducing a Representation

- Reducible rep: $D^{(red)}(g) = \sum_i^{\oplus} I_{r_i} \otimes D^{(i)}(g)$
- Decompose character: $\chi^{(red)} = \sum_i r_i \chi^{(i)}$
- Orthogonality: find coefficient r_i

$$r_i = \frac{1}{|G|}(\chi^{(i)}, \chi(D)) \quad \text{(reduction formula)}$$

2.5 Regular Representation

- obtained from multiplication table of group

$$\left.\begin{array}{c|ccc} G & E & A & B \\ \hline E & E & A & B \\ B & B & E & A \\ A & A & B & E \end{array}\right\} R(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- rearrangement $\implies$ 0 along diagonal for $R(g \neq E)$

$$\chi^{(R)} = (|G|, 0, \dots)$$

- Since $r_i = n_j$, decomposing R into IRREPs shows

$$|G| = \sum_j n_j^2$$

2.6 Character Table Completion

1. # IRREPs = # Conjugate Classes
2. Trivial IRREP
3. First column $\chi(E)$ using $\sum_j n_j^2 = |G| = \sum_C n_C$
4. homomorphism: $g^n = E \iff D(g)^n = I$
   - Abelian $\implies$ 1D IRREP $\implies \chi(g) = \exp\left(i\frac{2\pi}{n}k\right)$
   - g, g^{-1} in same class $\implies \chi(g) = \chi(g)^*$ is real
   - then, if n odd (e.g. C_5) then it is $+1$
5. Row and column orthogonality

3 Applications to QM

3.1 Unitary Reps in QM

1. Have N degenerate wavefunctions $\{\psi_i\}$ which solve

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi_i(\vec{r}) = E\psi_i(\vec{r})$$

2. $V(\vec{r})$ is invariant under orthogonal $\vec{r} \to U\vec{r}$

- Then $\psi_i(U^{-1}\vec{r}) = \hat{T}(U)\psi_i(\vec{r})$ also solves TISE

→ Has to be superposition of degenerate eigenfunctions:

$$\hat{T}(U)\psi_i(\vec{r}) = \sum_{j=1}^{N} D_{ji}(U)\psi_j(\vec{r})$$

- N-degenerate eigenfns $\{\psi_i\}$ form basis for a unitary IRREP $D(G)$ of the symmetry group G of the Hamiltonian
- scalar product of basis functions of different IRREPs is zero

3.2 Wigner's Theorem

- $[\hat{T}(G), \hat{H}] = 0 \implies HD = DH$

$$D^{(i)}(g)H_{ij} = H_{ij}D^{(j)}(g) \implies H_{ij} = \delta_{ij}\lambda_{ij}I_{n_i}$$

- degeneracy n_i for each $n_i \times n_i$ IRREP $D^{(i)}$
- additional accidental degeneracy (e.g. if $H_{11} = H_{22}$)

3.3 Perturbations and Degeneracy Lifting

A $n \times n$ IRREP has n degenerate eigenfunctions ψ as a basis.

- Perturbation can only decrease symmetry $G' \subset G$
- new rep: select from $D(G)$ the matrices $D(g' \in G')$
  - If new rep. is reducible: lift degeneracy; split level
  - guess or use character orthogonality to decompose

trick: In questions about lifting degeneracy, use only rotational subgroup SO(3). For selection rules, use full O(3).

pitfall: Example $\Gamma^2 = A_1 \oplus 2E$, then the 2 E have different energies in general (each with 2-fold degeneracy).

3.4 Selection Rules

Let $\{\psi_i\}$ be a basis for IRREP $D^{(i)}$ and let the operator $\hat{O}$ transform as $\hat{T}_g\hat{O}_\alpha\hat{T}_g^{-1} = \sum_\beta D'_{\beta\alpha}\hat{O}_\beta$

- $\langle\psi_i|\,\hat{O}_\alpha\,|\psi_j\rangle \neq 0$ only if $D^{(i)} \otimes D' \otimes D^{(j)}$ contains A_{1g}
  - Calculate $\chi^{(D^{(i)}\otimes D'\otimes D^{(j)})}(g) = \chi^{(i)}(g)\chi'(g)\chi^{(j)}(g)$
  - Scalar product with $\chi^{(A_1)}$ is 0 $\implies$ forbidden

trick (PS10) direct product of same parity IRREPs is even, i.e. a direct sum of only even IRREPs; direct product of different parity is odd, i.e. a direct sum of odd IRREPs only

$$\chi_g \otimes \chi_g = \chi_u \otimes \chi_u = \bigoplus \chi_g$$
$$\chi_g \otimes \chi_u = \bigoplus \chi_u$$

- fundamental transition $|0\rangle \to |1\rangle$: ground state transforms as totally symmetric IRREP A_{1g}

3.5 Normal Modes

- if potential energy $V = \frac{1}{2}y_i\Omega_{ij}y_j$ has symmetry G

$$D^{(3N)}\Omega = \Omega D^{(3N)}$$

1. normal modes transform as (form basis of) IRREPs of the symmetry group of the potential
2. degeneracy of energy = degeneracy of frequency

- basis: 3N-component N-atom displacement vectors
- representation matrices $D^{(3N)} = R \otimes A$
  - R exchanges equilibrium positions of N atoms $R_{ij}(g) = \delta_{jk}$, where g interchanges $i \leftrightarrow k$

$$R(C_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for} \quad \begin{matrix} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \\ 4 \to 4 \end{matrix}$$

  - $A(g)$ is action on each atom (e.g. rotation)
- decompose $D^{(3N)} = D^{(\text{vib})} \oplus D^{(\text{rot})} \oplus D^{(\text{trn})}$ using
- character $\chi(D^{(3N)}) = \chi(R)\chi(A) = n_g\chi(A)$
  - n_g: number of atoms that stay in place after g

$$\chi^{(3N)}(g) = \begin{cases} n_g(2\cos\varphi + 1), & \text{proper} \\ n_g(2\cos\varphi - 1), & \text{improper} \end{cases}$$

  - ∗ trn: polar, just like A
  - ∗ rot: axial, times (-1) for improper rotations
- $\chi^{(vib)} = \chi^{(3N)} - \chi^{(trn)} - \chi^{(rot)}$

$$\chi^{(\text{vib})}(g) = \begin{cases} (n_g - 2)(2\cos\varphi + 1), & \text{proper} \\ n_g(2\cos\varphi - 1), & \text{improper} \end{cases}$$

In practice, find normal modes IRREPs as follows:

1. identify symmetry G of system
2. for each $g \in G$, identify n_g atoms staying in place
3. Calculate $\chi^{(vib)}(g)$ (or $\chi^{(trn)}$, $\chi^{(rot)}$ or total $\chi^{(3N)}$)
   - identity E = proper rot with $\varphi = 0$
   - reflections σ = improper rot with $\varphi = 0$
   - improper rot $S_n = \sigma C_n$ with $\varphi = 2\pi/n$
   - C_n proper rot with $\varphi = 2\pi/n$
4. decompose / reduce into IRREPs

3.6 Spherical Harmonics

For $Y_{l,m}$, have $2l + 1$ orthogonal functions, so representation are $2l + 1 \times 2l + 1$ matrices. These are in general reducible, e.g. for C_{5v}, $Y_{1,0}$ is basis for 1×1 IRREP and $Y_{1,\pm 1}$ for 2×2.

Rotation axis is z (for θ, φ convention), then to see how group operations transform the spherical harmonics into each other, use

- θ invariant under C_n or σ_v
- $C_n : (\varphi \to \varphi + \frac{2\pi}{n})$
- $\sigma_v : (\varphi \to -\varphi)$

4 Lie Groups

- continuous group: parameters $\vec{a} = (a_1, \ldots, a_r)$ s.t.

$$\vec{a} \to \vec{a} + \delta\vec{a} \implies g(\vec{a}) \to g(\vec{a}) + \delta g$$

- $g(\vec{c}) = g(\vec{a})g(\vec{b})$ with continuous function $\vec{f}(\vec{a}, \vec{b}) = \vec{c}$
- Lie Group: $f(\vec{a})$ analytic

4.1 Classification

- $GL(2,R)$: 2x2 matrices with $\det A \neq 0$, 4 parameters
- $SL(2,R)$: $\det A = 1$, 3 parameters
- $O(2)$: $A^T = A^{-1} \implies \det A = \pm 1$, 1 parameter
- $SO(2)$: $\det A = 1$, one parameter (angle of rotation)

4.2 Generators

- finite groups: generate C_{3v} from products of C_3 and σ_v
- SO(2):
$$A(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$
- Taylor expand:
$$A(\varphi) = I_2 + X\varphi + \cdots = e^{\varphi X} = I\cos\varphi + X\sin\varphi$$
with generator
$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
- SO(3): 9-6 = 3 free parameters (Euler Angles)
  - 6 constraints from $A^TA = I_3$
- generators of x, y, z rotations satisfy
$$[X_i, X_j] = \varepsilon_{ijk}X_k$$

4.3 Angular Momentum

- $SO(2)$ infinitesimal angle $\delta\varphi$ on $F(x,y)$:
$$F(x', y) \approx F(x,y) + \frac{i}{\hbar}\hat{L}_z F\delta\varphi$$
with $\hat{L}_z = -i\hbar(x\partial_y - y\partial_x)$
- SO(3): all components of $\hat{L}$ with
$$[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k$$
- in general $[\hat{x}_i, \hat{x}_j] = \sum_k c_{ij}^{(k)}\hat{x}_k$
  - structure constants $c^{(k)}$
  1. AS of commutator: $c^{(k)} = -c^{(k)}$
  2. Jacobi identity
$$[A,[B,C]] + (\text{cyclic}) = 0$$

4.4 Representations of SO(2)

- integration over group:
$$\int f(R)dR = \int f(R)\frac{\mathrm{d}R}{\mathrm{d}a}da = \int f(R)g(R)da$$
  - density: $g(R)$
  - Rearrangement: If $g(R)da = g(R'R)da$, then
$$\int f(R)dR = \int f(R'R)dR$$
- SO(2): $g(\varphi) = 1$, Abelian $\implies$ 1D: only need characters
  1. $\chi(\varphi)\chi(\varphi') = \chi(\varphi + \varphi')$
  2. $\chi(0) = 1$
  3. $\chi(\varphi) = \chi(\varphi + 2\pi)$

  $\implies \chi^{(m)}(\varphi) = e^{im\varphi}$

SO(2)	E	$R(\varphi)$	basis
$\chi^{(\pm m)}$	1	$e^{\pm im\varphi}$	$(x+iy)^m$

- orthogonality: (2π = "volume" of SO(2))
$$\int_0^{2\pi} (\chi^{(m)})^*\chi^{(n)}\,\mathrm{d}\varphi = 2\pi\delta_{mn}$$

4.5 SO(3) IRREPs

$$A\vec{x} = \lambda\vec{x} \implies \lambda = 1, e^{\pm i\varphi}$$

- $\lambda = 1$: axis of rotation $A\vec{n} = \vec{n}$
  - Use $A^T\vec{n} = A^TA\vec{n} = \vec{n} \implies (A - A^T)\vec{n} = 0$ and $|\vec{n}|^2 = 1$ to find rot. axis $\vec{n}$
- φ angle of rotation around $\vec{n}$
  - use $\mathrm{Tr}\,A = \mathrm{Tr}\,\Lambda$ to find φ
  - sphere of rad. φ = representation of rotations of φ around possible $\vec{n}$
- parametrisation $(n_1\varphi, n_2\varphi, n_2\varphi)$ $\quad n_1^2 + n_2^2 + n_3^2 = 1$
$$(\varphi\cos\phi\sin\theta, \varphi\sin\phi\sin\theta, \varphi\cos\theta)$$
- rotations about different axes:
$$R(\vec{n},\varphi) = U^{-1}(\vec{n},\vec{n}')R(\vec{n}',\varphi)U(\vec{n},\vec{n}')$$
- rearrangement theorem
$$2\int (1-\cos\varphi)F(\varphi,\theta,\phi)\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\phi$$
- For characters:
$$\int_0^{\pi} (1-\cos\varphi)(\chi^{(\mu)}(\varphi))^*\chi^{(\nu)}(\varphi)\,\mathrm{d}\varphi = \pi\delta_{\mu\nu}$$
- To find characters: Use $\hat{H}$ with SO(3) symmetry. Then $\{\psi_{lm} = Y_{lm}\}$ form basis of IRREP
$$RY_{lm} = \sum_{m'=-l}^{+l} \Gamma_{m,m'}^{(l)} Y_{lm'}$$
- choosing e.g. z-axis: $R_z(\varphi)Y_{lm} = e^{-im\varphi}Y_{lm}$
$$\Gamma^{(l)}(R_z(\varphi)) = \begin{pmatrix} e^{-il\varphi} & & \\ & \ddots & \\ & & e^{+il\varphi} \end{pmatrix}$$
- $\chi^{(l)}(R) = \mathrm{Tr}\,\Gamma^{(l)}(R) = \frac{\sin[(l+1/2)\varphi]}{\sin(\varphi/2)}$ (geometric series)

SO(3)	$E = R(0)$	$R(\varphi)$	basis
$\chi^{(l)}$	$2l+1$	$\frac{\sin[(2l+1)\,\varphi/2]}{\sin(\varphi/2)}$	Y_{lm}

4.6 Perturbation of spherically symmetric $\hat{H}$

- O(3): SO(3) & improper rotations
  - Since $iC_2 = S_2C_2 = \sigma_h C_2^2 = \sigma_h$, can build improper rotations with i as $S_n = \sigma_h C_n = iC_2C_n$
- Since $\hat{i}Y_{lm} = (-1)^l Y_{lm}$, have extra factor $(-1)^l$

$$\chi^{(l)}(i \otimes R(\varphi)) = (-1)^l \chi^{(l)}(R(\varphi))$$

- Can then calculate characters, e.g.:
  - $\chi(\sigma_h) = \chi(i \otimes C_2) = \chi(i \otimes R(\pi))$
  - $\chi(S_4) = \chi(i \otimes C_4^{-1}) = \chi(i \otimes R(\frac{\pi}{2}))$
  - $\chi(S_n) = \chi(i \otimes C_2 \otimes C_n) = \chi(i \otimes R(\pi + \frac{2\pi}{n}))$

O(3)	$E = R(0)$	$R(\varphi)$	$i = i \otimes E$	$i \otimes R(\varphi)$	basis
$\chi^{(l)}$	$2l+1$	$\frac{\sin[(2l+1)\,\varphi/2]}{\sin(\varphi/2)}$	$(-1)^l(2l+1)$	$(-1)^l\chi^{(l)}(R(\varphi))$	Y_{lm}

- perturbation example: Octahedral arrangement
  - new $\hat{H}$ has symmetry of rotational subgroup of O_h
  - calculate characters $\chi^{(l)}(O_h)$ via table
  - decompose into IRREPs of O_h
    - $l = 2$ level is split into $\chi^{(2)} = E \oplus T_2$
  - selection rules between split states
    - use symmetry (g / u) arguments

4.7 Wigner-Eckart Theorem

- decompose product representation $D^{(j_1)} \otimes D^{(j_2)} = \sum_J^{\oplus}$
- Clebsch-Gordan Series for SO(3):

$$\chi^{(j_1 \otimes j_2)} = \frac{\sin[(j_1+1/2)\varphi]\sin[(j_2+1/2)\varphi]}{\sin^2(\varphi/2)} \tag{6}$$

$$= \sum_{J=|j_1-j_2|}^{j_1+j_2} \frac{sin[(J+1/2)\varphi]}{\sin(\varphi/2)} \tag{7}$$

$$= \sum_{J=|j_1-j_2|}^{j_1+j_2} \chi^{(J)} \quad \text{simply reducible} \tag{8}$$

- direct product basis $Y_{m_1}^{j_1} \otimes Y_{m_2}^{j_2}$
- basis in which $D^{(j_1)} \otimes D^{(j_2)}$ is block diagonal is linear combination

$$\omega_M^{Jj_1j_2} = \sum_{m_1m_2} C_{m_1m_2M}^{Jj_1j_2} Y_{m_1}^{j_1} Y_{m_2}^{j_2}$$

where C are (tabulated) Clebsch-Gordan coefficients

- $(2\omega+1)$-dimensional tensor vector operator $\hat{T}_\mu^{(\omega)}$ transforms under the $D^{(l=\omega)}$ IRREP of SO(3):

$$R\hat{T}_\mu^{(\omega)}R^{-1} = \sum_{\mu'} \hat{T}_{\mu'}^{(\omega)} D_{\mu\mu'}^{(\omega)}(R) \tag{9}$$

- Wigner-Eckart theorem:

$$\langle N'j'm'|\,\hat{T}_\mu^{(\omega)}\,|Njm\rangle = C_{m\mu m'}^{j\omega j'} \underbrace{\langle N'j'|\,\hat{T}^{(\omega)}\,|Nj\rangle}_{\text{reduced matrix element}}$$