CS 111: Homework 3: Due by 6:00pm Friday, January 25

NEW TURNIN RULES: Homework must be submitted online as a PDF file to GradeScope. When you turn in your homework, tell GradeScope which page(s) of your PDF contain each individual problem. Doing this correctly will be worth 2 points.

1. How many arithmetic operations (total of additions, subtractions, multiplications, divisions) are required to do each of the following? (You can omit lower-order terms in n.)

1a. Compute the sum of two n-vectors?

1b. Compute the product of an n-by-n matrix with an n-vector?

1c. Compute the product of two n-by-n matrices?

1d. Solve an n-by-n upper triangular linear system $Ux = y$?

2. Suppose A and B are n-by-n matrices, with A nonsingular, and c is an n-vector. Describe the steps you would use to *efficiently* compute the product $A^{-1}Bc$. Describe a different, less efficient, sequence of steps.

3. Suppose that A is a square, nonsingular, nonsymmetric matrix, b is an n-vector, and that you have called

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L, U, p = LUfactor(A)
```

(using the routine from the lecture files). Now suppose you want to solve the system $A^T x = b$ (not $Ax = b$) for x. Show how to do this using calls to `Lsolve()` and `Usolve()`, without modifying either of those routines or calling `LUfactor()` again. Test your method in `numpy` on a randomly generated 6-by-6 matrix (see `np.random.rand()`).

4. Do problem 2.3 on pages 32–33 of the NCM book, showing the `numpy` code you use and its output. Note: To understand intuitively what the problem means by "assume that joint 1 is rigidly fixed both horizontally and vertically and that joint 8 is fixed vertically," think of the truss as a (2-dimensional) drawbridge across a river, with the left end being a hinge and the right end lying on the ground.

5. Consider the linear system

$$\begin{pmatrix} \alpha & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \alpha + 2 \\ 3 \end{pmatrix},$$

for some $\alpha < 1$. Clearly the solution is $(x_0, x_1)^T = (1, 2)^T$. For each value of $\alpha = 10^{-4}, 10^{-8}, 10^{-16}, 10^{-20}$, solve this system using the routines `LUfactor()`, `Lsolve()`, and `Usolve()` from `LUsolve.ipynb` in the lecture files. For each α, do this twice, first with `pivoting = True` in `LUfactor()` and then with `pivoting = False`. Show your `numpy` code and its output. Comment on your results.

6. Recall that a symmetric matrix A is *positive definite* (SPD for short) if and only if $x^T Ax > 0$ for every nonzero vector x.

6a. Find a 2-by-2 matrix A that (1) is symmetric, (2) is not singular, and (3) has all its elements greater than zero, but (4) is *not* SPD. Show a nonzero vector x such that $x^T Ax < 0$.

6b. Let B be a nonzero matrix, of any size, not necessarily symmetric. Prove that the matrix $A = B^T B$ is SPD.