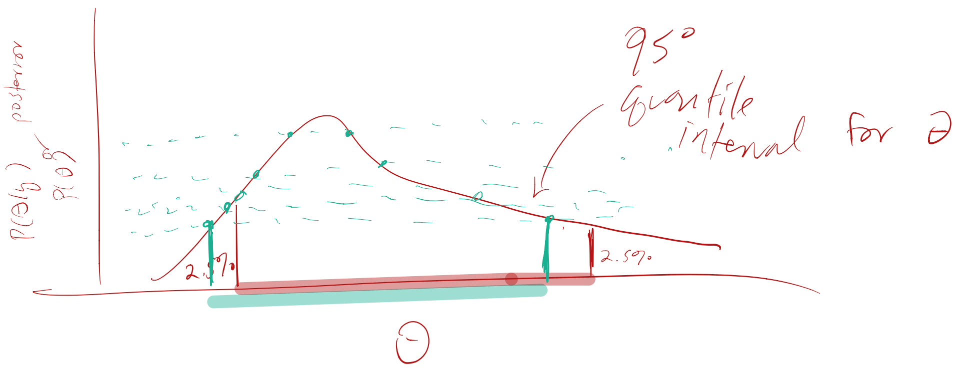
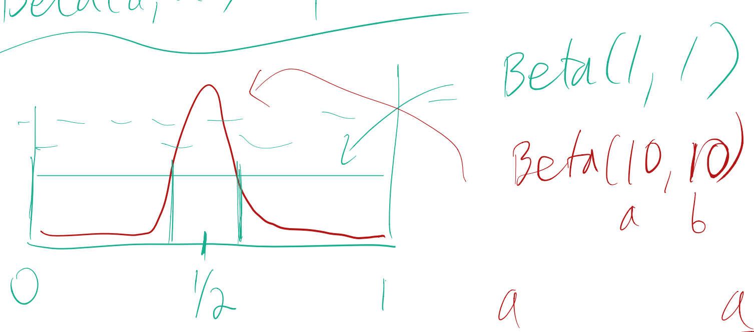


95% quantile

Beta(a, a) prior



$$\text{mean} = \frac{a}{a+b} \qquad \frac{a}{a+a} = 1/2$$

$$y_1, \ldots, y_n \sim N(\mu, 13^2)$$

$$\mu \sim N\left(100, \frac{13^2}{\kappa_0}\right)$$

$$P(\mu | y) \sim N\left(w\bar{y} + (1-w)100, \frac{13^2}{\kappa_0 + n}\right)$$

$$w = \frac{n}{n + \kappa_0}$$

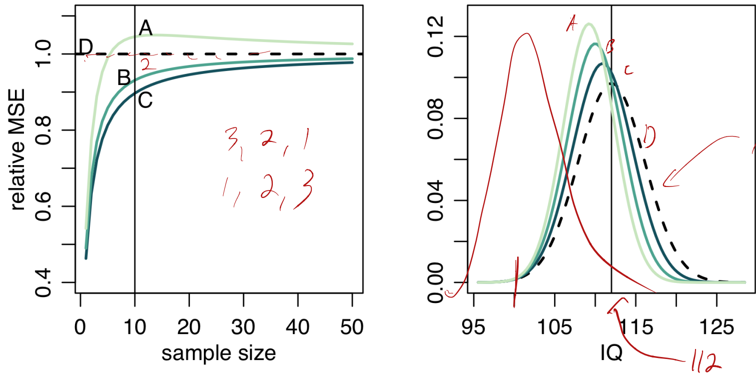
Shift:

ML Estimator: $\bar{Y}$

Posterior Mean Estimator: $w\bar{Y} + (1-w)100$

7. First try this without referring to the lecture notes. Consider the following figure from the IQ example discussed in class. This figure is based on the following model: $p(y \mid \mu, \sigma^2) \sim N(\mu, 13^2)$ and $p(\mu) \sim N(100, \frac{13^2}{\kappa_0})$.

$$\frac{MSE(w\bar{Y}+(1-w)100)}{MSE(\bar{Y})}$$



$\kappa_0 = 0$

112

a. The left figure shows the mean squared error (MSE) of the posterior mean estimator relative to the maximum likelihood estimator. Fill in the blanks with the number 0, 1, 2, or 3. For line A $\kappa_0 =$ __3__, for line B $\kappa_0 =$ __2__, for line C, $\kappa_0 =$ __1__, and for line D $\kappa_0 =$ __0__.

b. Circle one. The right figure depicts:

   i. The posterior distribution for μ for each value of κ_0.
   ii. The sampling distribution of the Bayes estimator, $\hat{\mu}$ for each value of κ_0.
   iii. The likelihood of μ for each value of κ_0.
   iv. The prior distribution of μ for each value of κ_0.

$w = \frac{n}{n+\kappa_0} = 1$

ML Estimator: $\bar{Y}$

Posterior Mean Estimator: $w\bar{Y} + (1-w)100$

$E[\bar{Y}] = \frac{1}{n}\sum E[Y] = \frac{1}{n}\, n \times \mu = \mu$

$MSE(\bar{Y}) = Bias(\bar{Y})^2 + Var(\bar{Y})$
$= 0^2 + \frac{13^2}{n} = \frac{13^2}{n}$

PM

$E[w\bar{Y} + (1-w)100] = w\mu + (1-w)100$

$Bias(PM) = w\mu + (1-w)100 - \mu$
$= (1-w)100 - (1-w)\mu$
$= (1-w)(100-\mu)$

$MSE(PM) = ((1-w)(100-\mu))^2 + w^2\frac{13^2}{n}$

$P(\hat{\mu}_{PM\,estimator} \mid \mu)$

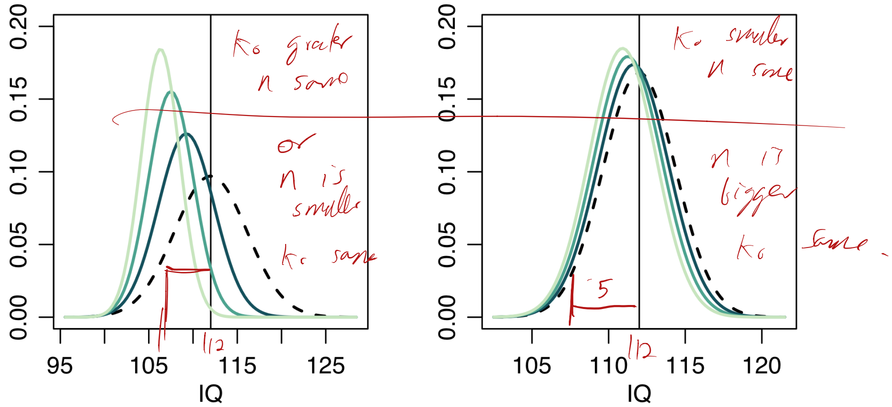
$w\bar{Y} + (1-w)(100) \sim N\left(w\mu + (1-w)100,\ w^2\frac{13^2}{n}\right)$

$P(\mu \mid y) \sim N\left(w\bar{y} + (1-w)100,\ \frac{13^2}{n+\kappa_0}\right)$

$N\left(\bar{Y}, \frac{13^2}{n}\right)$

$$w = \frac{n}{n+k_0}$$

c. Circle all options that could describe the differences between the two figures below.



i. In the left figure, the values of κ_0 for each corresponding line are larger than they are for the right figure. Both have the same sample size, n.
ii. In the left figure, the values of κ_0 for each corresponding line are smaller than they are for the right figure. Both have the same sample size, n.
iii. In the left figure, n is larger than it is for the right figure. Both have the same values of κ_0.
iv. In the left figure, n is smaller than it is for the right figure. Both have the same values of κ_0.

$$w\mu + (1-w)100$$

$$w = \frac{n}{n+k_0}$$

$$P(\theta) = k_0(20 \times \theta(1-\theta)^3 + 1)$$

$$L(\theta) = \cancel{\binom{n}{y}} \theta^y (1-\theta)^{n-y}$$

$$P(\theta/y) = k_0(20 \times \theta^1(1-\theta)^3 + 1) \times \theta^y (1-\theta)^{n-y}$$

$$k_0\left(20\theta^{y+1}(1-\theta)^{3+n-y} + \theta^y(1-\theta)^{n-y}\right)$$

$$\int P(\theta/y)\,d\theta = 1$$

$$\mathrm{Beta}(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$$

$$\cancel{K_0}\int \left(20\theta^{y+1}(1-\theta)^{3+n-y} + \theta^{y}(1-\theta)^{n-y}\right)d\theta = \frac{1}{K_0}$$

$$\underbrace{20\int_0^1 \theta^{y+1}(1-\theta)^{3+n-y}\,d\theta}_{\mathrm{Beta}(y+2,\ 4+n-y)} + \underbrace{\int_0^1 \theta^{y}(1-\theta)^{n-y}\,d\theta}_{\mathrm{Beta}(y+1,\ n-y+1)}$$

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$20\,\frac{\Gamma(y+2)\Gamma(4+n-y)}{\Gamma(6+n)} + \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(2+n)} = \frac{1}{K_0}$$

$$P(\theta) = K_0\left(20\times\theta(1-\theta)^3+1\right)$$

$$K_0 20\int \theta(1-\theta)^3 d\theta + K_0 = 1$$

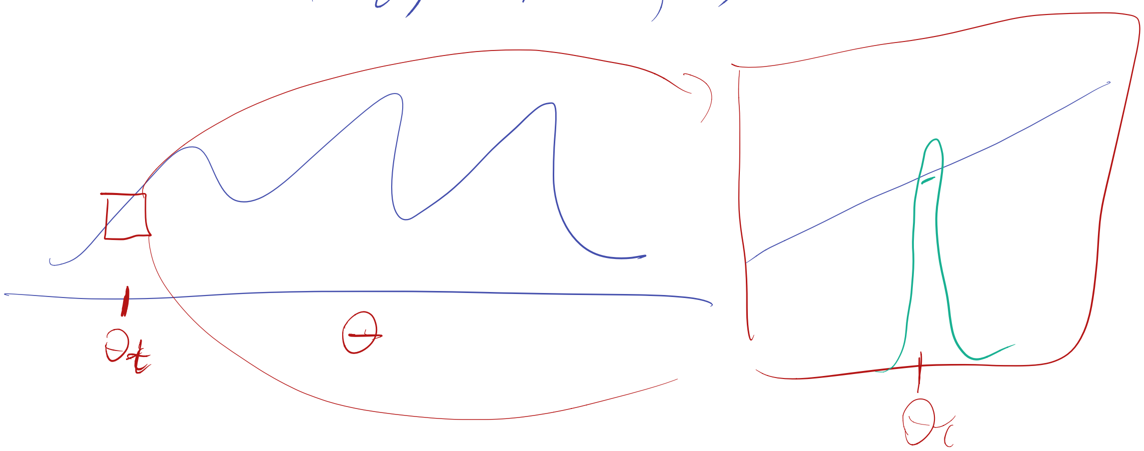
$$\int \theta(1-\theta)^3 d\theta = \frac{1-K_0}{20K_0}$$

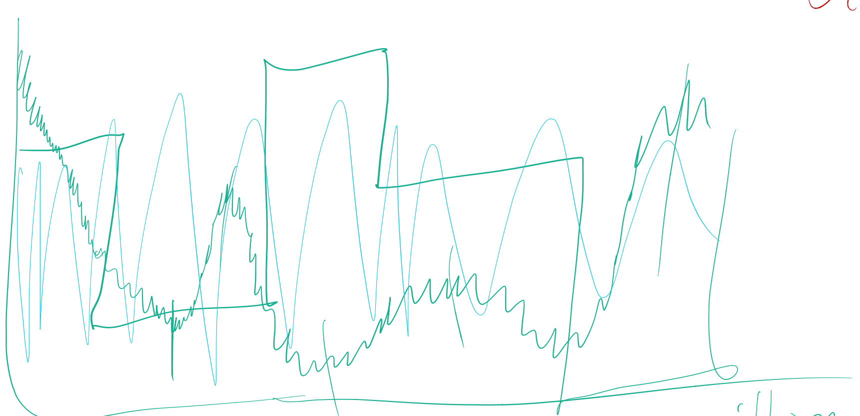
$$\mathrm{Beta}(2,4) \rightarrow \frac{\Gamma(2)\Gamma(4)}{\Gamma(6)} = \frac{1-K_0}{20K_0}$$

$$r = \min\left(1, \left[\frac{P(\theta^* \mid y)}{P(\theta_t \mid y)}\right]\right)$$

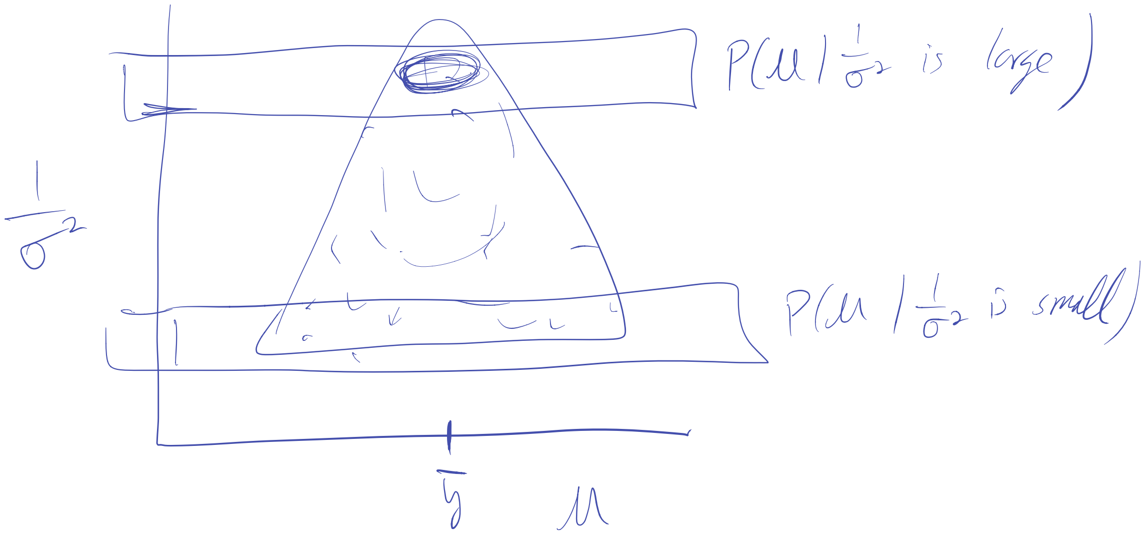
→ close to 1 always → rejection rate close to 0.

$$J(\theta^* \mid \theta_t) \sim N(\theta_t, \sigma^2)$$





$P(\mu, \frac{1}{\sigma^2} | y)$



$y \sim N(\mu, \sigma^2)$