PSTAT 115 Section 3 Solutions

Oct 17, 2018

1 Part 1

1. Binomial density:
$$\binom{n}{y} p^y (1-p)^{n-y}$$

2.
$$\begin{aligned} p(\theta|y) &\propto p(\theta) * p(y|\theta) \\ &= \binom{n}{y} p^y (1-p)^{n-y} \text{ (in this context } \theta \text{ is } p) \\ &\propto p^y (1-p)^{n-y} \end{aligned}$$

   The above corresponds to the functional form of a beta distribution. Specifically, it is $Beta(y+1, n-y+1)$.

3.
$$\begin{aligned} p(\theta|y) &\propto p(\theta) * p(y|\theta) \\ &= p^{2-1} * (1-p)^{2-1} \binom{n}{y} p^y (1-p)^{n-y} \text{ (in this context } \theta \text{ is } p) \\ &\propto p^{y+1} (1-p)^{1+n-y} \end{aligned}$$

   The above corresponds to the functional form of a beta distribution. Specifically, it is $Beta(y+2, n-y+2)$.

2 Part 2

There is no problem for this part. But you are encouraged to derive the conjugate pairs listed in the table to appreciate this beautiful idea.

3 Part 3

1. Following the above formula, we can see the posterior is a $Beta(44, 56)$ distribution. Therefore by the formula of Beta distribution we know the mean is $\frac{44}{(44+56)} = 0.44$.

2. Consider the density of the posterior distribution, which is

$$\frac{\Gamma(44+56)}{\Gamma(44)\Gamma(56)}p^{44-1}(1-p)^{56-1}.$$

   Since the MAP is the mode of the posterior distribution, it is the MLE of p. To maximize the above density, it is equivalent to maximizing the log-likelihood after simplification, which is

$$43\log(p) + 55\log(1-p).$$

   By taking the derivative with respect to p and setting it to 0, we get the MAP of p is $\frac{43}{98}$.

   Alternatively, you can check the formula for the mode of the Beta distribution directly.

3. No. The posterior distribution might be bimodal or multimodal, therefore we might encounter multiple MAPs.

4 Part 4

1. It means there is 0.95 probability that the probability of female birth lies in the interval [0.3445430, 0.5377312].