Lecture 4: Intervals

Uncertainty?

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Announcements

- Reading: Chapter 8.1 (intervals), 8.3 (posterior prediction)
- Homework 2 out today
- Quiz 1, today.

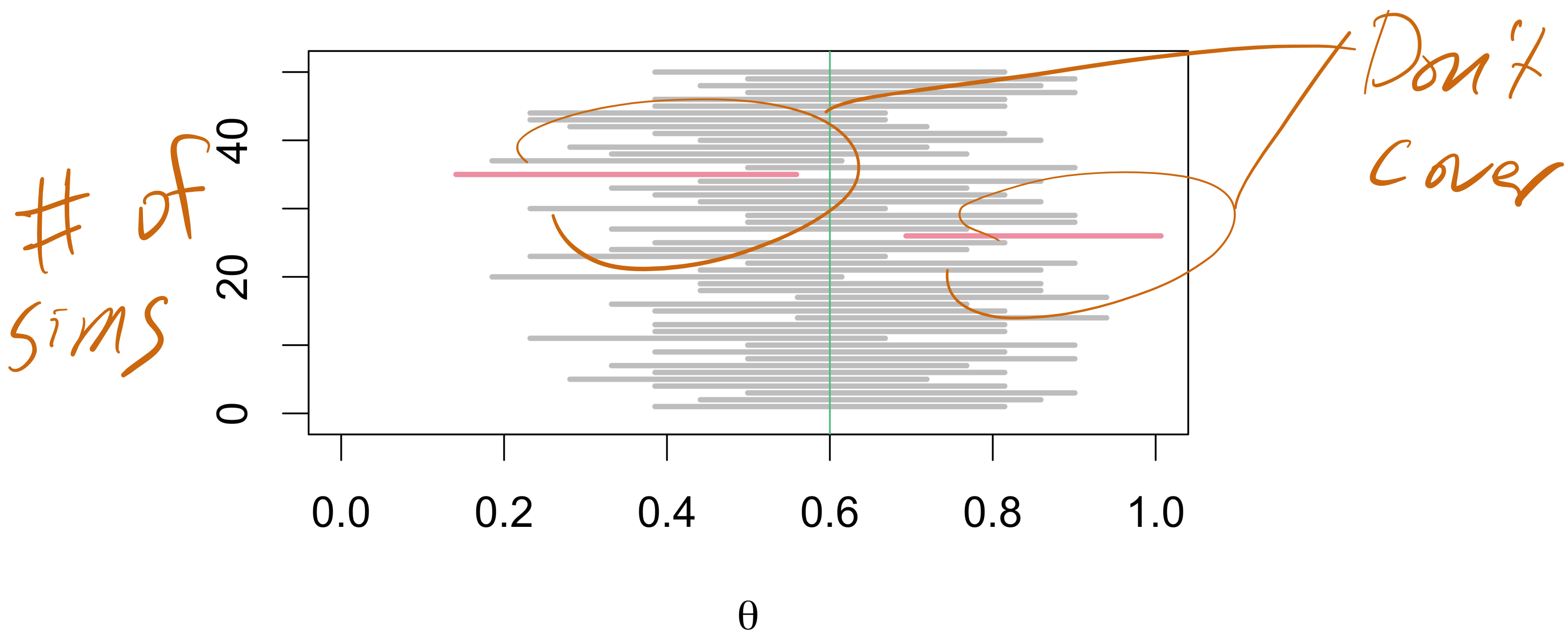
Reminder: Frequentist confidence interval

Non-Bayesian

- Frequentist interval: $Pr(l(Y) < \theta < u(Y) \mid \theta) = 0.95$
  - Probability that the interval will cover the true value *before* the data are observed.
  - Interval is random since Y is random

Fixed, Random

Reminder: Frequentist confidence interval



We expect 0.05 x 50 = 2.5 will *not* cover the true parameter 0.6

Posterior Credible Intervals

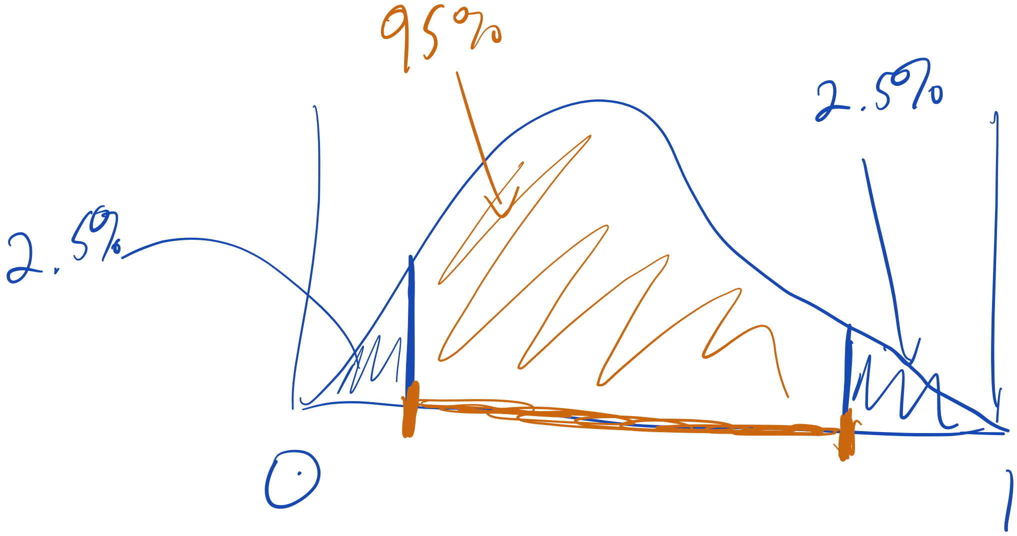
- Frequentist interval: $Pr(l(Y) < \theta < u(Y) \mid \theta) = 0.95$
  - Probability that the interval will cover the true value *before* the data are observed.
  - Interval is random since Y is random

Conditional on Data

- **Bayesian Interval**: $Pr(l(y) < \theta < u(y) \mid \underline{Y = y}) = 0.95$
  - Information about the the true value of θ *after* observeing $Y = y$.
  - θ is random (because we include a prior), y is observed so interval is non-random.

Constant

Random.



$P(\theta \mid y) \sim \text{Beta}(\alpha, \beta)$

Quantile Based Interval

$$l(y) \leftarrow \text{qbeta}(.025, \alpha, \beta)$$

$$u(y) \leftarrow \text{qbeta}(1-.025, \alpha, \beta)$$

Fixed Quantities

Posterior Credible Intervals (Quantile-based)

- The easiest way to obtain a confidence interval is to use the quantiles of the posterior distribution.

If we want $100 \times (1 - \alpha)$ interval, we find numbers $\theta_{\alpha/2}$ and $\theta_{1-\alpha/2}$ such that:

e.g. $\alpha = 5\% = .05$

1. $p(\theta < \underbrace{\theta_{\alpha/2}}_{l(y)} \mid Y = y) = \alpha/2$ lower
2. $p(\theta > \underbrace{\theta_{1-\alpha/2}}_{u(y)} \mid Y = y) = \alpha/2$ upper

$$p(\theta \in [\overbrace{\theta_{\alpha/2}}^{l(y)}, \overbrace{\theta_{1-\alpha/2}}^{u(y)}] \mid Y = y) = 1 - \alpha$$

- Use quantile functions in R, e.g. `qbeta`, ~~`qpois`~~, `qnorm` etc.

qgamma,

Example: interval for shooting skill

- The posterior distribution for Covington's shooting percentage is a

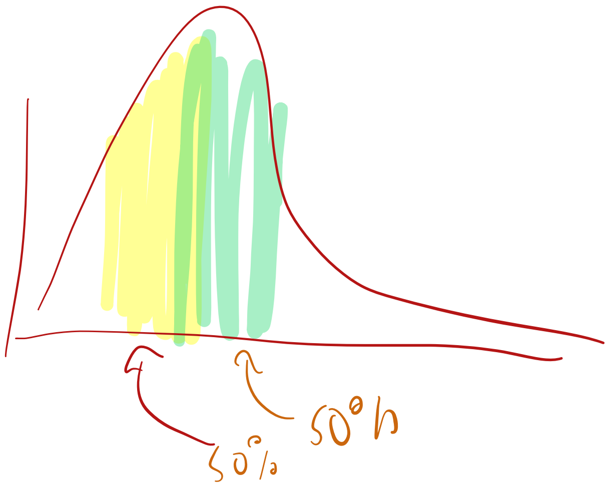
$$\text{Beta}(49 + 478, 50 + 873) = \text{Beta}(528, 924)$$

- For a 95% *credible* interval, $\alpha = 0.05$
  - Lower endpoint: `qbeta(0.025, 528, 924)`
  - Upper endpoint: `qbeta(0.975, 528, 924)`
  - $[\theta_{\alpha/2}, \theta_{1-\alpha/2}]$ = [0.34, 0.39]
- Compared to frequentist *confidence* interval without prior information: [0.39, 0.59]
- End-of-season percentage was 0.37
- Credible intervals and confidence intervals have different

Highest Posterior Density (HPD) region

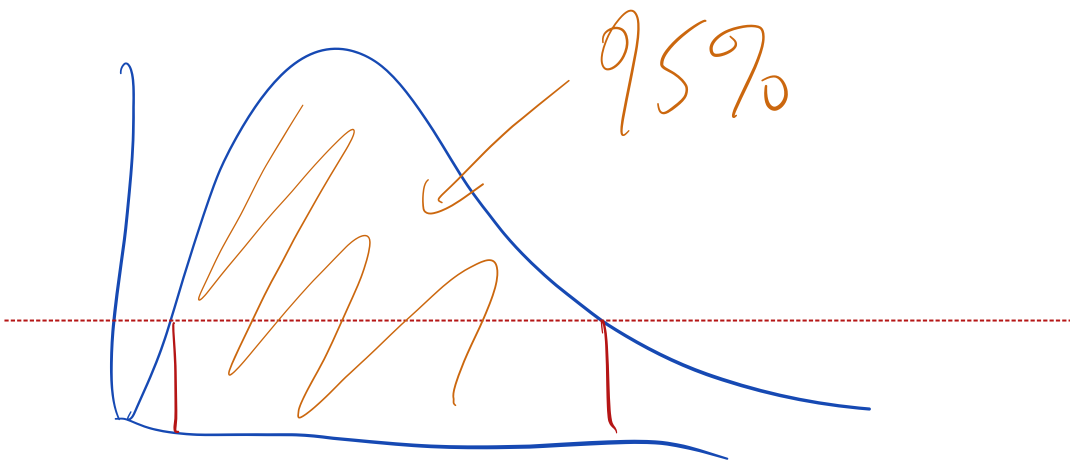
Definition: (HPD region) A $100 \times (1 - \alpha)$ HPD region consists of a subset of the parameter space, $R(y) \in \Theta$ such that

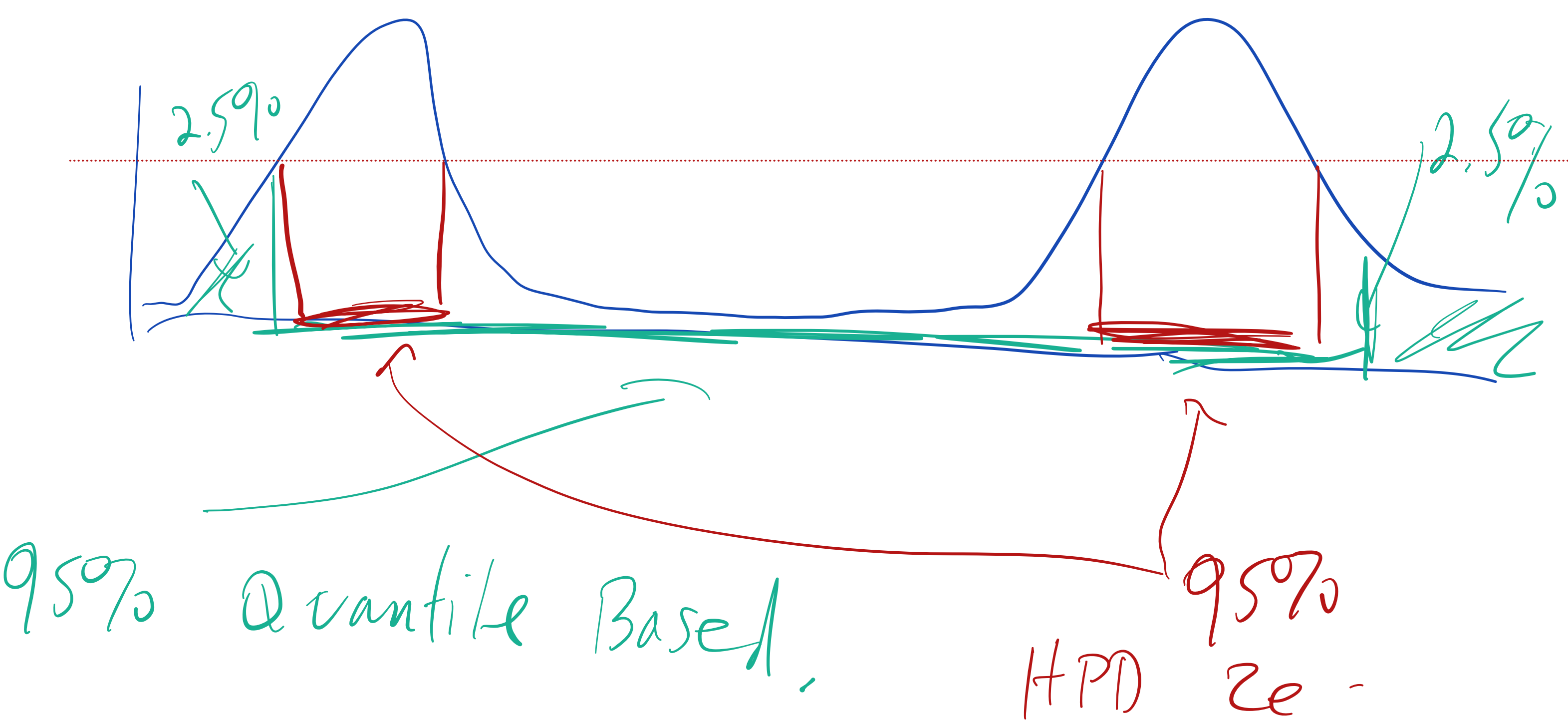
1. $\Pr(\theta \in R(y)|Y = y) = 1 - \alpha$ = 95%
    - The probability that θ is in the HPD region is $1 - \alpha$
2. If $\theta_a \in R(y), \text{ and } \theta_b \notin R(y)$ then $p\left(\theta_a|Y = y\right) > p\left(\theta_b|Y = y\right)$ Math
    - All points in an HPD region have a higher posterior density than points out- side the region.

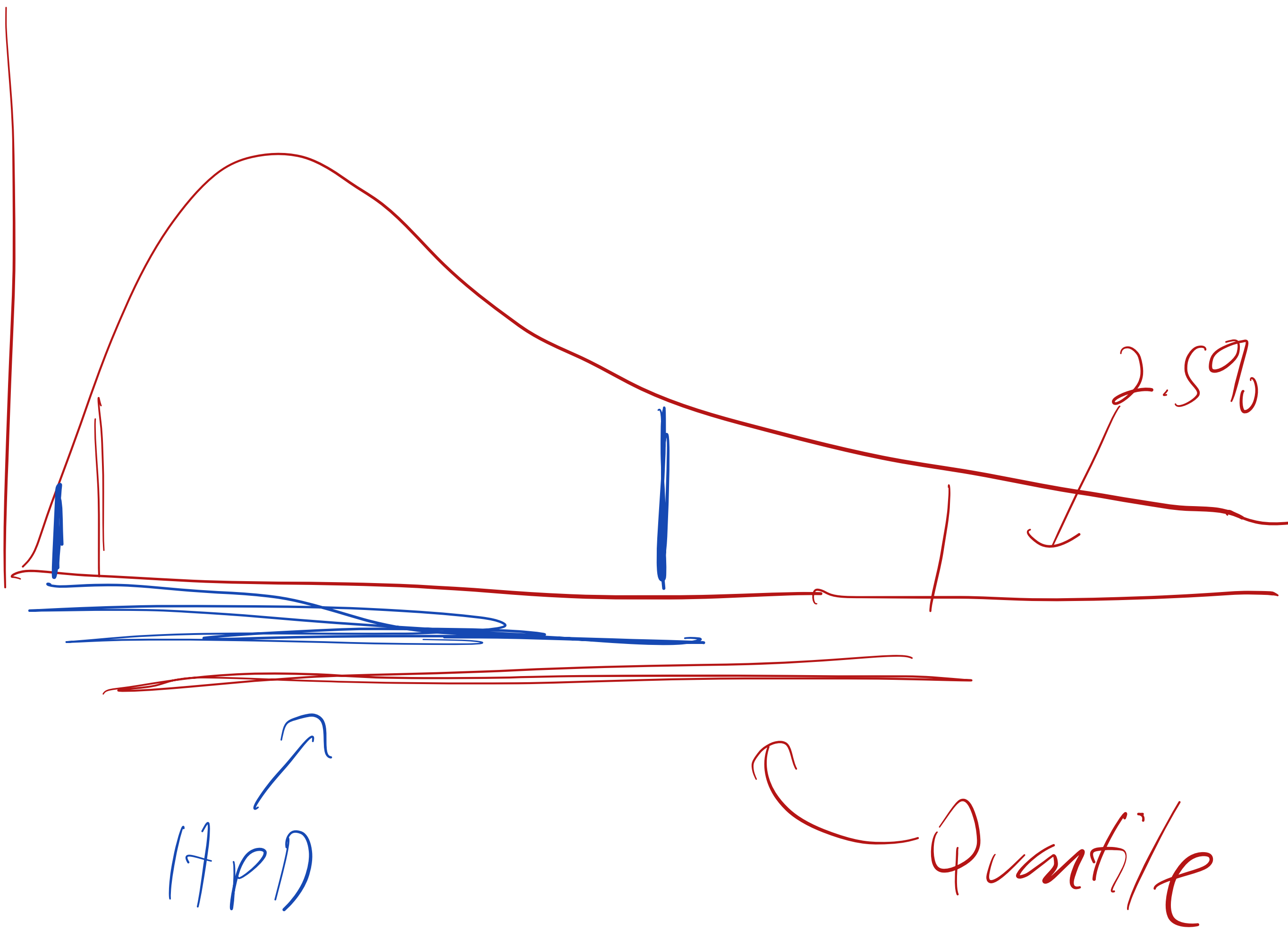


HPD interval (region) is the shortest possible

$1-\alpha\%$ interval possible.






2.5%
HPD
Quantile

Highest Posterior Density (HPD) region

1. $p(\theta \in s(y) \mid Y = y) = 1 - \alpha$
2. If $\theta_a \in s(y)$, and $\theta_b \notin s(y)$, then $p(\theta_a \mid Y = y) > p(\theta_b \mid Y = y)$.
   - All points in an HPD region have a higher posterior density than points out- side the region.

The HPD region is the *smallest* region with prob $(1 - \alpha)$%

Calibration: Frequentist Behavior of Bayesian Intervals

Match Reality.

- A credible interval is calibrated if it has the right frequentist coverage
- Bayesian credible intervals usually won't have correct frequentist coverage
  to the real world
- If our prior was well-calibrated and the sampling model was correct, we'd have well-calibrated credible intervals
- Specifying *nearly* calibrated prior distributions is hard!

Calibration of political predictions

The best test of a probabilistic forecast is whether it's well calibrated. By that I mean: Out of all FiveThirtyEight forecasts that give candidates about a 75 percent shot of winning, do the candidates in fact win about 75 percent of the time over the long run? It's a problem if these candidates win only 55 percent of the time. But from a statistical standpoint, it's just as much of a problem if they win 95 percent of the time.

source: fivethirtyeight.com

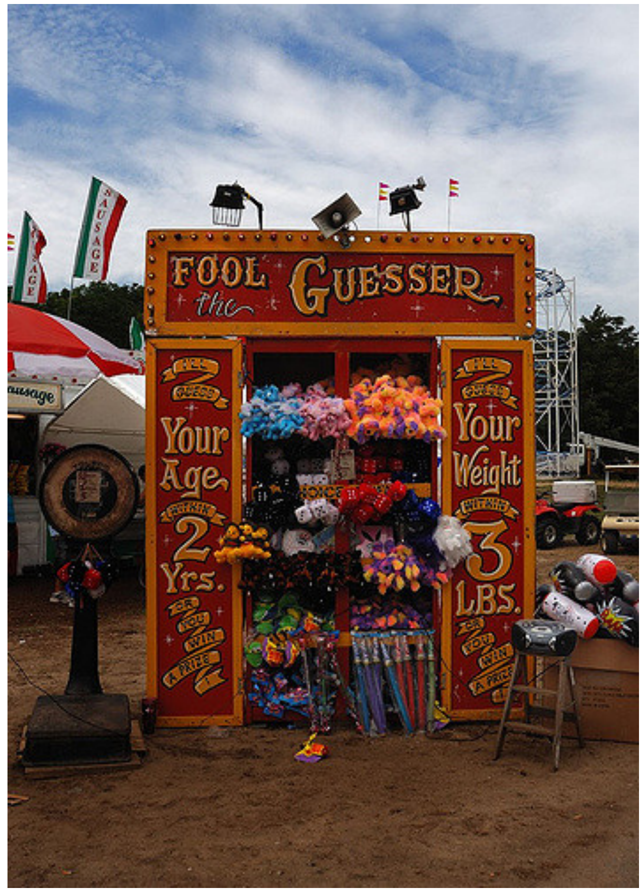
Calibration of political predictions

Calibration for FiveThirtyEight "polls-plus" forecast

WIN PROBABILITY RANGE	NO. FORECASTS	EXPECTED NO. WINNERS	ACTUAL NO. WINNERS
95-100%	27	26.7	26
75-94%	15	13.1	14
50-74%	14	8.7	11
25-49%	13	4.8	3
5-24%	27	3.1	1
0-4%	88	0.8	1

source: https://fivethirtyeight.com/features/when-we-say-70-percent-it-really-means-70-percent/

The age guessing game*



*Bayesian edition

[37 , 55]

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[37, 42)

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[39, 41]