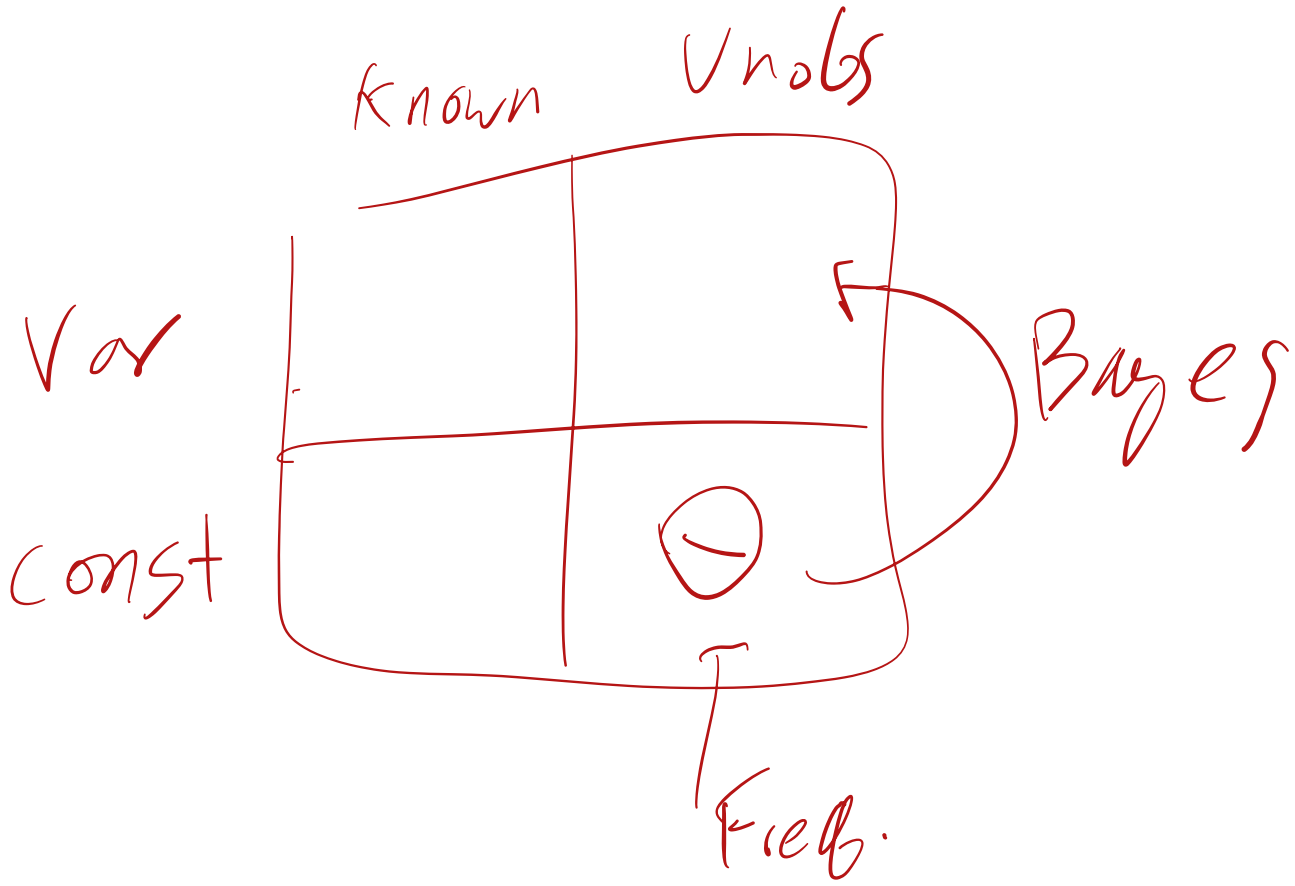
Lecture 3: One Parameter Models

Professor Alexander Franks

2025-10-06


Known
Unobs
Var
const
Bayes
Fieb.

Announcements

- Reading: Chapter 2 and 3, Bayes Rules

- Homework 1 due Sunday (turn in on gradescope!)

- Quiz 1 Monday in Section.

- OH after class.

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Bayesian Inference

- In frequentist inference, θ is treated as a fixed unknown constant
- In Bayesian inference, θ is treated as a random variable
- Need to specify a model for the joint distribution $p(y, \theta) = p(y \mid \theta)p(\theta)$

Setup

- The *sample space* $\mathcal{Y}$ is the set of all possible datasets. We observe one dataset y from which we hope to learn about the world.
  - Y is a random variable, y is a realization of that random variable
- The *parameter space* Θ is the set of all possible parameter values θ
  - θ encodes the population characteristics that we want to learn about!

Bayesian Inference in a Nutshell

1. The *prior distribution* $p(\theta)$ describes our belief about the true population characteristics, for each value of $\theta \in \Theta$.
2. Our *sampling model* $p(y \mid \theta)$ describes our belief about what data we are likely to observe when the true population parameter is θ.
3. Once we actually observe data, y, we update our beliefs about θ by computing *the posterior distribution* $p(\theta \mid y)$. We do this with Bayes' rule!

$$p(\theta \mid y) \propto L(\theta) p(\theta)$$

Bayes' Rule

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

- $P(A \mid B)$ is the conditional probability of A given B
- $P(B \mid A)$ is the conditional probability of B given A
- $P(A)$ and $P(B)$ are called the marginal probability of A and B (unconditional)

Bayes' Rule for Bayesian Statistics

$$P(\theta \mid y) = \frac{P(y \mid \theta)P(\theta)}{P(y)}$$

- $P(\theta \mid y)$ is the posterior distribution (Belief after seeing y)
- $L(\theta) \propto P(y \mid \theta)$ is the likelihood (Sampling model)
- $P(\theta)$ is the prior distribution (Belief before seeing data)
- $P(y) = \int_\Theta p(y \mid \tilde{\theta})p(\tilde{\theta})d\tilde{\theta}$ is the model evidence

Computing the Posterior Distribution

$$
\begin{aligned}
P(\theta \mid y) &= \frac{P(y \mid \theta) P(\theta)}{P(y)} \\
&\propto P(y \mid \theta) P(\theta) \\
&\propto L(\theta) P(\theta)
\end{aligned}
$$

- Start with a subjective belief (prior)
- Update it with evidence from data (likelihood)
- Summarize what you learn (posterior)

The posterior is proportional to the likelihood times the prior!

Bayesian vs Frequentist

- In frequentist inference, unknown parameters treated as constants
  - Estimators are random (due to sampling variability) Capital Y
  - Asks: what would I expect to see if I repeated the experiment?"

(counterfactual world)

Bayesian vs Frequentist

- In frequentist inference, unknown parameters treated as constants
  - Estimators are random (due to sampling variability)
  - Asks: what would I expect to see if I repeated the experiment?"

- In Bayesian inference, unknown parameters are random variables.
  - Need to specify a prior distribution for θ (not easy)
  - Asks: "what do I *believe* are plausible values for the unknown parameters given the data?"
  - Who cares what might have happened, focus on what *did* happen by conditioning on observed data.

$P(\theta \mid Y = y)$

Example

- Assume we sample the a point on the Earth and record whether it is land or water
- Let $Y \sim \text{Bin}(n, \theta)$ where θ corresponds to ~~his true skill~~
- Frequentist inference tells us that the maximum likelihood estimate is simply $\frac{y}{n}$
- What would our estimates be if we use Bayesian inference?
  - What properties do we want for our prior distribution?

Cromwell's Rule

The use of priors placing a probability of 0 or 1 on events should be avoided except where those events are excluded by logical impossibility.

If a prior places probabilities of 0 or 1 on an event, then no amount of data can update that prior.

> I beseech you, in the bowels of Christ, think it possible that you may be mistaken.
>
> — Oliver Cromwell

$$P(\theta|y) \propto L(\theta) P(\theta)$$

Cromwell's Rule

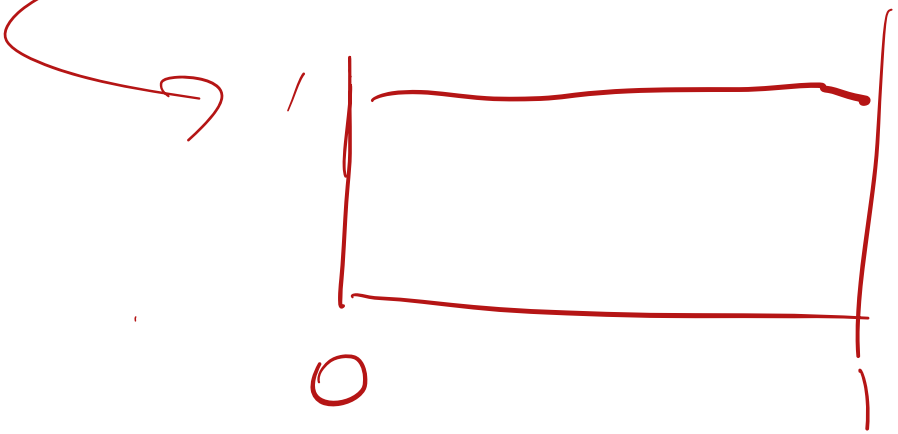
> Leave a little probability for the moon being made of green cheese; it can be as small as 1 in a million, but have it there since otherwise an army of astronauts returning with samples of the said cheese will leave you unmoved.
> — Dennis Lindley (1991)

If $p(\theta = a) = 0$ for a value of a, then the posterior distribution is always zero, regardless of what the data says

$$p(\theta = a|y) \propto p(y|\theta = a)p(\theta = a) = 0$$

The Binomial Model

- The uniform prior: $p(\theta) = \text{Unif}(0, 1) = \mathbf{1}\{\theta \in [0, 1]\}$
  - A "non-informative" prior
- Posterior: $p(\theta \mid y) \propto \underbrace{\theta^y (1-\theta)^{n-y}}_{\text{likelihood}} \times \underbrace{\mathbf{1}\{\theta \in [0, 1]\}}_{\text{prior}}$
- The above posterior density is is a density over θ.



The Binomial Model

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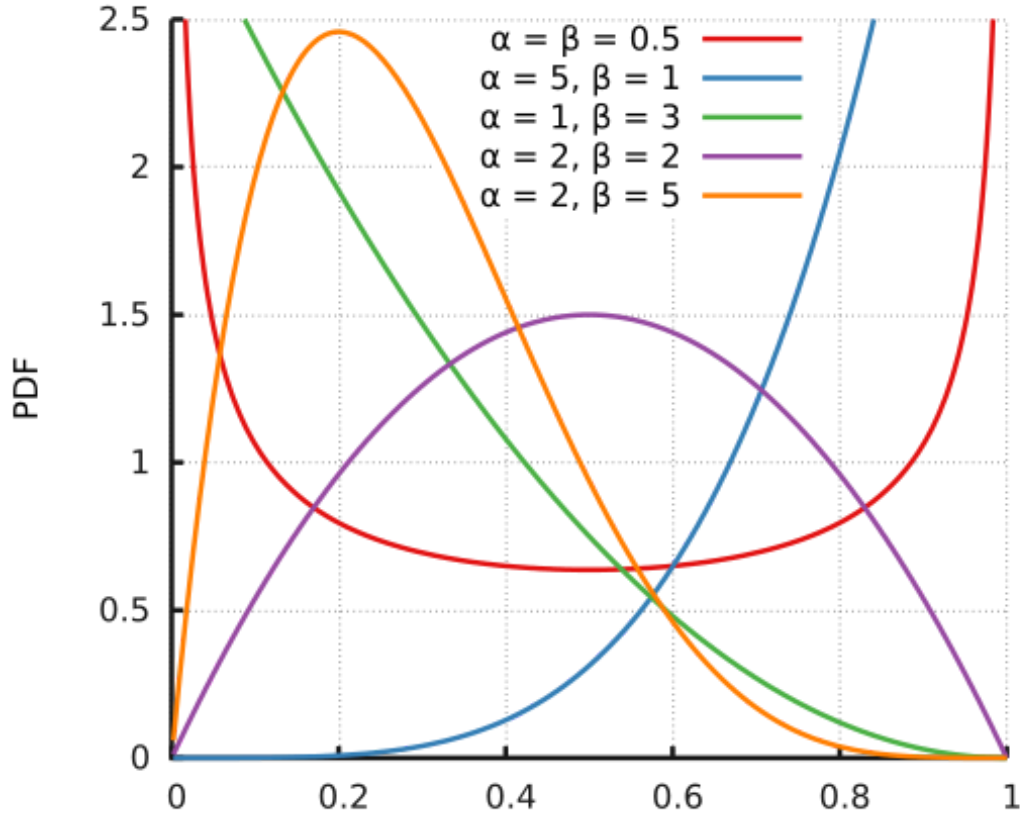
$$p(\theta \mid y) \sim \text{Beta}(y + 1, n - y + 1)$$
$$= \frac{\Gamma(n)}{\Gamma(n-y)\Gamma(y)} \theta^y (1-\theta)^{n-y}$$

Beta Distributions



$$\text{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

normalizing constant

Binomial likelihood + uniform prior

$$\Rightarrow \theta \sim \text{Beta}(y+1,\ n-y+1)$$

$y \quad s$

Sequential Bayesian Updating

$$\underbrace{p(\theta)}_{\text{prior}} \rightarrow \underbrace{p\left(\theta \mid y_1\right)}_{\text{reveal the first observation}} \rightarrow \underbrace{p\left(\theta \mid y_1, y_2\right)}_{\text{reveal the second observation}}$$

$$\rightarrow \cdots \rightarrow \underbrace{p\left(\theta \mid y_1, \ldots, y_n\right)}_{\text{reveal all } n \text{ observations}}$$

When data are i.i.d., final posterior is the same, regardless of whether we analyze data sequentially or as a single batch.

Demo

Example

- On November 18, 2017, an NBA basketball player, Robert Covington, had made 49 out of 100 three point shot attempts.
- At that time, his three point field goal percentage, 0.49, was the best in the league and would have ranked in the top ten all time
- How can we estimate his true shooting skill?
  - Think of “true shooting skill” as the fraction he would make if he took infinitely many shots

Example

- Assume every shot is independent (reasonable) and identically distributed (less reasonble?)
- Let $Y \sim \text{Bin}(n, \theta)$ where θ corresponds to his true skill
- Frequentist inference tells us that the maximum likelihood estimate is simply $\frac{y}{n} = 49/100 = 0.49$
- What would our estimates be if we use Bayesian inference?

Likelihood.

$$P(\theta \mid y) \propto \theta^{y}(1-\theta)^{n-y}\underbrace{1[\theta \in [0,1]]}_{\text{Uniform Prior}}$$

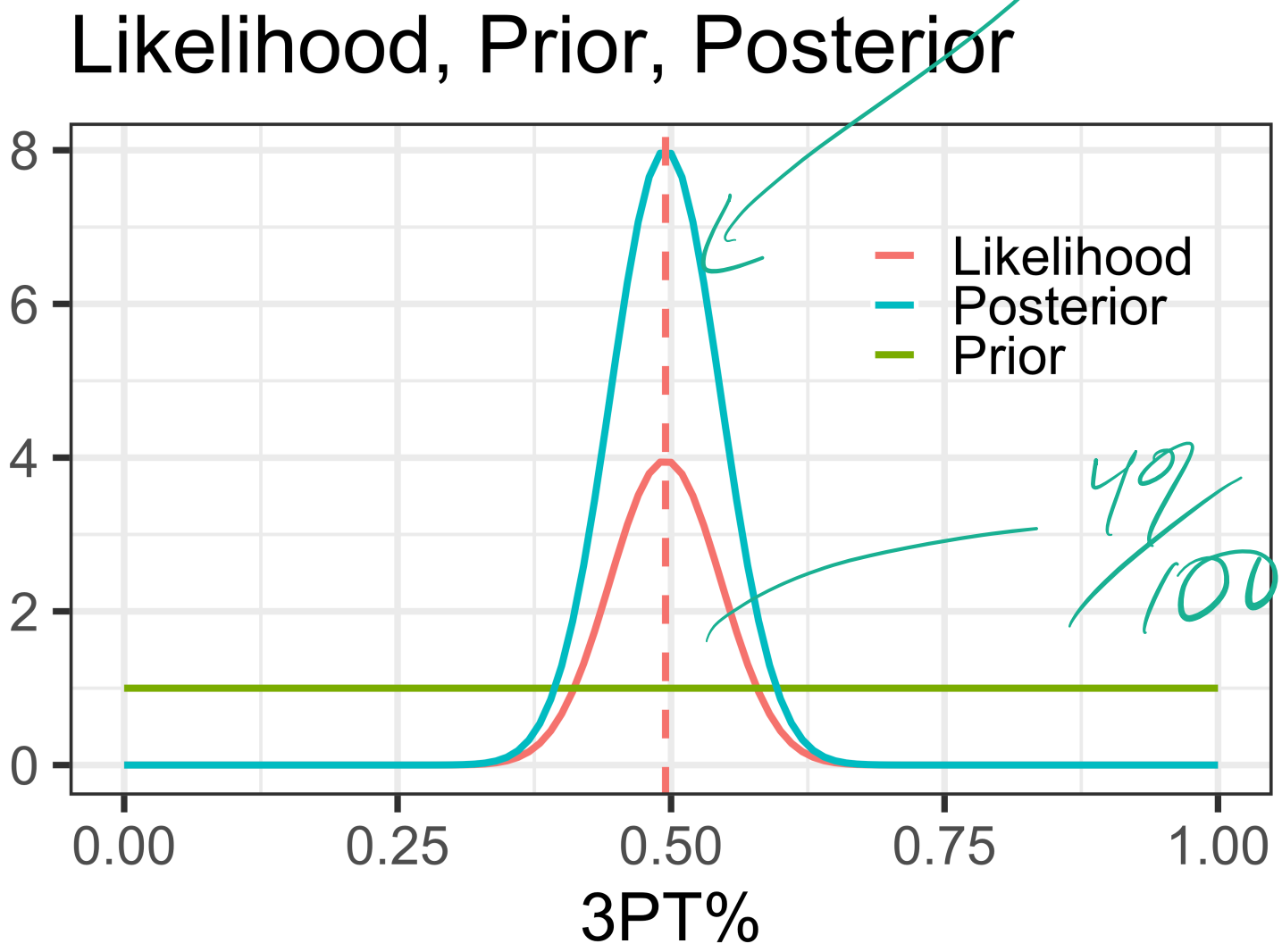
$$\propto \theta^{\alpha+1}(1-\theta)^{\beta+1}$$

which is a $\text{Beta}(\alpha, \beta)$

$$\alpha = y+1$$

$$\beta = n-y+1$$

Example



Posterior is proportional to the likelihood

Example

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- Frequentist inference tells us that the maximum likelihood estimate is simply $\frac{y}{n} = 49/100 = 0.49$
- What would our estimates be if we use Bayesian inference?
  - If our prior reflects “complete ignorance” about basketball?
  - What if we want to incorporate prior domain knowledge?

Informative prior distributions

- At that time, his three point field goal percentage, 0.49, was the best in the league and would have ranked in the top ten all time
- It seems very unlikely that this level of skill would continue for an entire season of play.
- A uniform prior distribution doesn't reflect our known beliefs. We need to choose a more *informative* prior distribution.

Options:
- Previous play
- Other players in the league

(most similar type)

Informative prior distributions

- When $p(\theta) \sim U(0, 1)$ then the posterior was a Beta distribution

$\text{Unif}(0,1) \equiv \text{Beta}(1,1)$

- Remember: the binomial likelihood is $L(\theta) \propto \theta^y(1-\theta)^{n-y}$
- Choose a prior with a similar looking form: $p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$

$\text{Beta}(\alpha, \beta)$

$$P(\theta|y) \propto \theta^y(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$\propto \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}$$

$$\Rightarrow \text{Beta}(y+\alpha, n-y+\beta)$$

Informative prior distributions

- Remember: the binomial likelihood is $L(\theta) \propto \theta^y (1-\theta)^{n-y}$
- Choose a prior with a similar looking form: $p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$
- Then $p(\theta \mid y) \propto \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}$ is a $\text{Beta}(y+\alpha, n-y+\beta)$
- For the binomial model, a beta prior distribution implies a beta posterior distribution!
- The family of Beta distributions is called a **conjugate prior** distribution for the binomial likelihood.

Conjugate Prior Distributions

Definition: A class of prior distributions, $\mathcal{P}$ for θ is called *conjugate* for a sampling model $p(Y|\theta)$ if
$p(\theta) \in \mathcal{P} \implies p(\theta|y) \in \mathcal{P}$

- The prior distribution and the posterior distribution are in the same family
- Conjugate priors are very convenient because they make calculations easy
- The parameters for conjugate prior distribution have nice interpretations

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Bin Lik + Beta Prior ⇒ Beta post.

Beta is conjugate for the Binomial.

Summarizing Posterior Results

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- *Point estimates:* posterior mean or mode:
  - $E[\theta \mid y] = \int_\Theta \theta p(\theta \mid y) d\theta$ (the posterior mean)
  - $\arg\max p(\theta \mid y)$ (*maximum a posteriori* estimate)

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- $\mathrm{Var}[\theta \mid y] = \int_\Theta (\theta - E[\theta \mid y])^2 p(\theta \mid y) d\theta$

posterior Variance

Summarizing Posterior Results

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- $\mathrm{Var}[\theta \mid y] = \int_\Theta (\theta - E[\theta \mid y])^2 p(\theta \mid y) d\theta$

- Posterior credible intervals: for any region $R(y)$ of the parameter space compute the probability that θ is in that region: $p(\theta \in R(y))$

More intuitive-

Summarizing Posterior Results

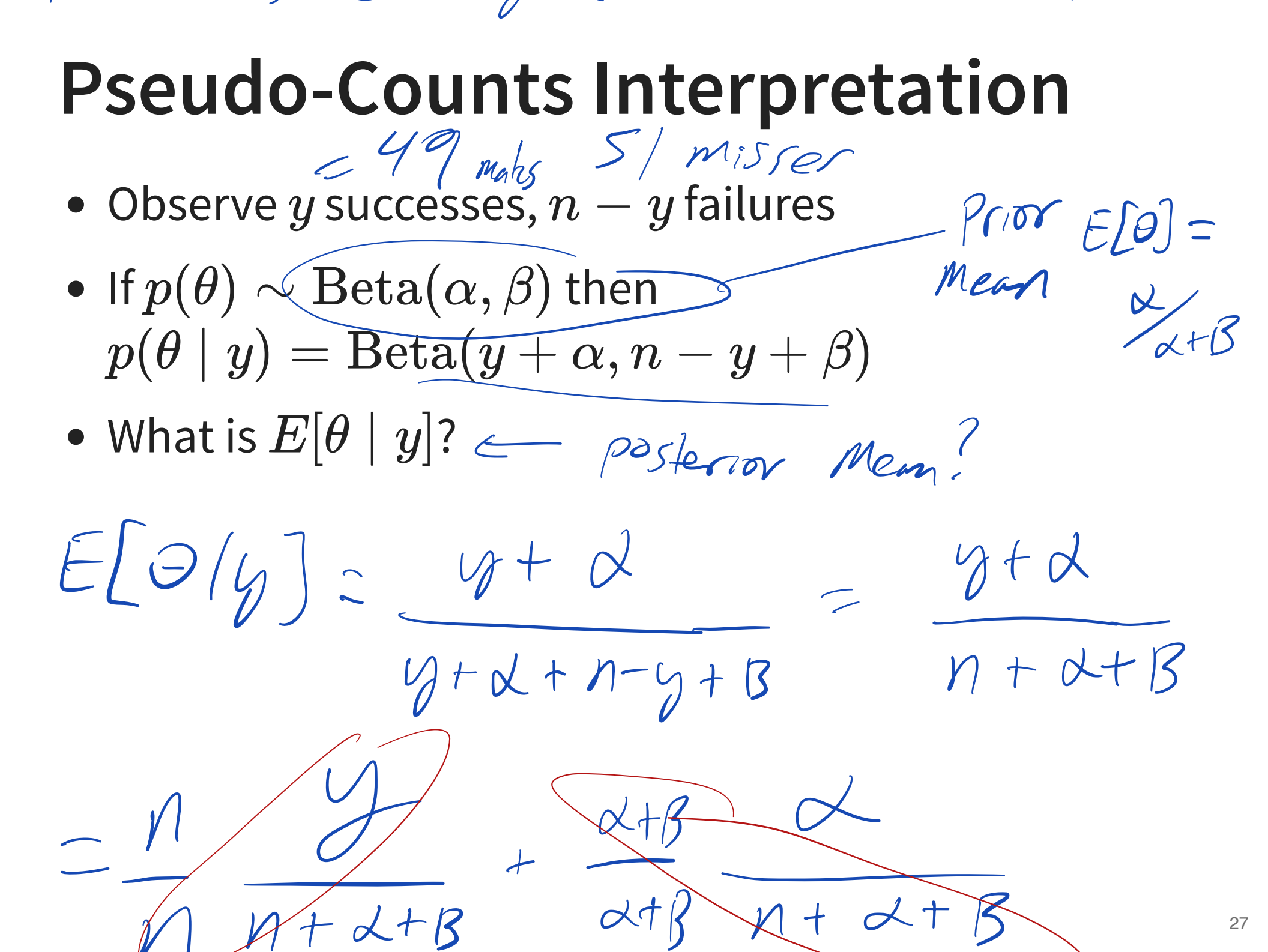
- $\text{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$ (Density)
- The mean of a $\text{Beta}(\alpha, \beta)$ distribution r.v. $\frac{\alpha}{\alpha+\beta}$
- The mode of a $\text{Beta}(\alpha, \beta)$ distributed r.v. is $\frac{\alpha-1}{\alpha+\beta-2}$
- The variance of a $\text{Beta}(\alpha, \beta)$ r.v. is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- In R: `dbeta`, `rbeta`, `pbeta`, `qbeta`

density RNG CDF Quantile Memorize

E.g. Beta(50, 20) then Mean is $\frac{50}{50+20}$

Pseudo-Counts Interpretation

- Observe y successes, $n - y$ failures
- If $p(\theta) \sim \text{Beta}(\alpha, \beta)$ then
  $p(\theta \mid y) = \text{Beta}(y + \alpha, n - y + \beta)$
- What is $E[\theta \mid y]$?



$$= \frac{n}{n+\alpha+\beta} \boxed{\frac{y}{n}} + \frac{\alpha+\beta}{n+\alpha+\beta} \boxed{\frac{\alpha}{\alpha+\beta}}$$

$\frac{n}{n+\alpha+\beta}$: w — $\frac{y}{n}$: MLE — $\frac{\alpha+\beta}{n+\alpha+\beta}$: $1-w$ — $\frac{\alpha}{\alpha+\beta}$: prior mean

$$= w\,\hat{\theta}_{MLE} + (1-w)\,\hat{\theta}_{\text{prior mean}}$$

where $w = \frac{n}{n+\alpha+\beta}$

$$P(\theta \mid y) \sim \text{Beta}(y+\alpha,\ n-y+\beta)$$

y: makes shots — α: "prior" shots made — $n-y$: misses — β: "prior" misses

α: prior/pseudo makes

β: pseudo misses

$\alpha+\beta$: prior n

$$\frac{\alpha}{\alpha+\beta} = \frac{\text{"prior mean"}}{\text{"prior n"}}$$

$\alpha+\beta$ increasing $\Rightarrow$ variance decreases (more pseudo data)

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why not

$$y_i \sim \text{Bin}(n_i, \theta)$$

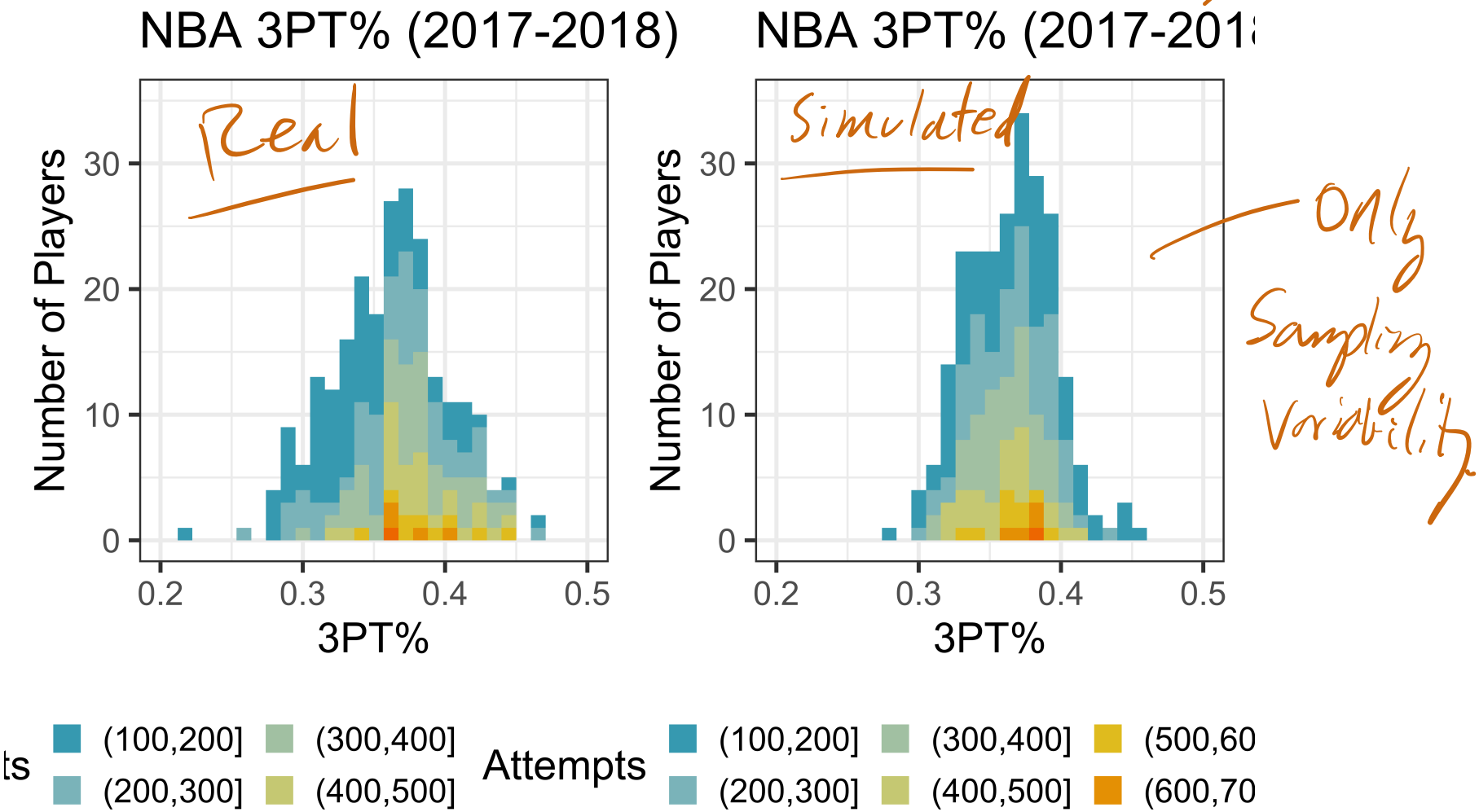
$$i = 1, \ldots, 9.$$

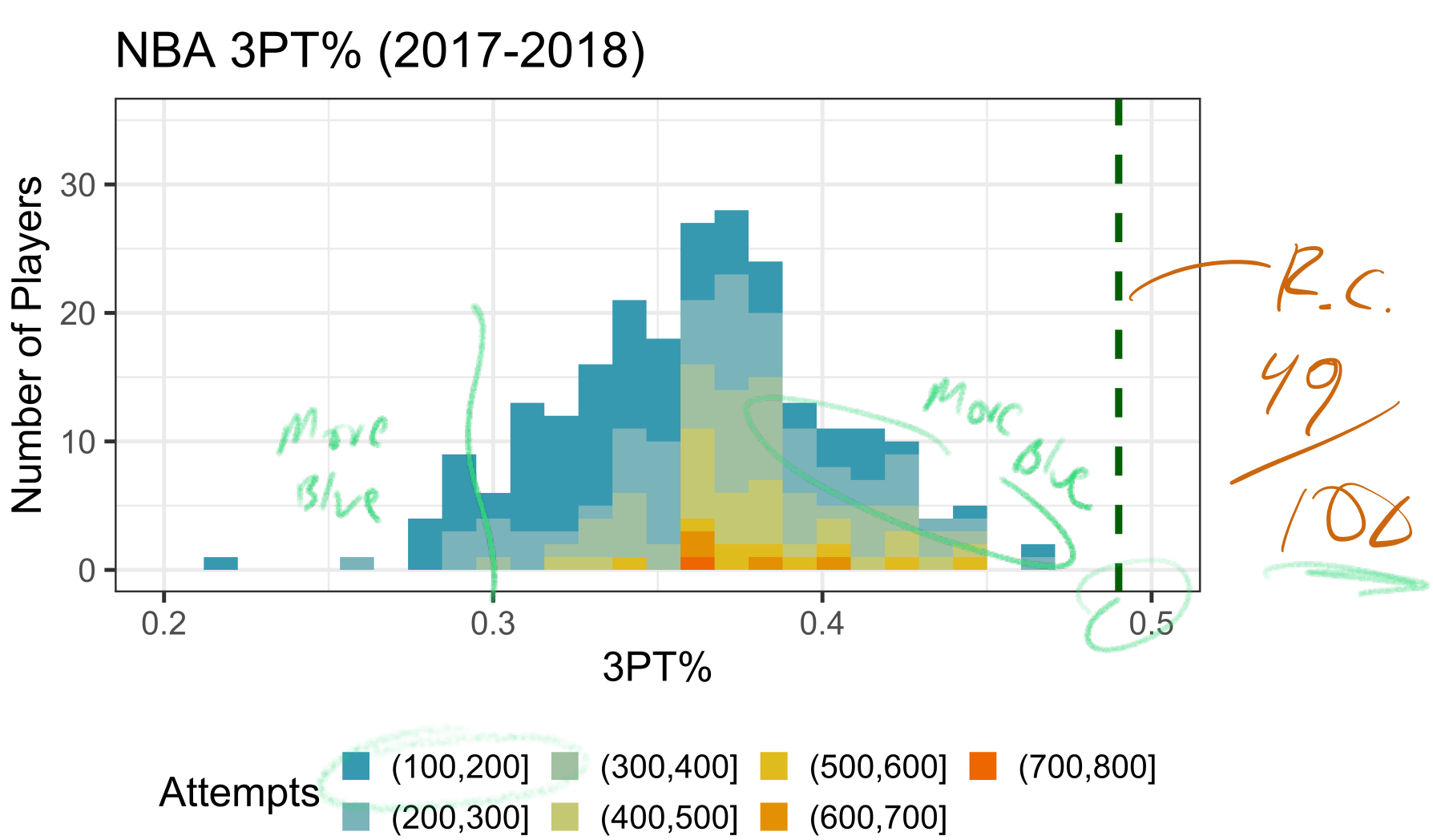
Example

- On November 18, 2017, an NBA basketball player, Robert Covington, had made 49 out of 100 three point shot attempts.
- At that time, his three point field goal percentage, 0.49, was the best in the league and would have ranked in the 10 ten all time
- Prior knowledge tells us it is unlikely this will continue!
- How can we use Bayesian inference to better estimate his true skill?

Three point shooting in 2017-2018

$y_i \sim Bin(N_i, .35)$




NBA 3PT% (2017-2018)
Number of Players
30
20
10
0
0.2
0.3
0.4
0.5
3PT%
More Blue
More Blue
R.C.
49/100
Attempts
(100,200]
(200,300]
(300,400]
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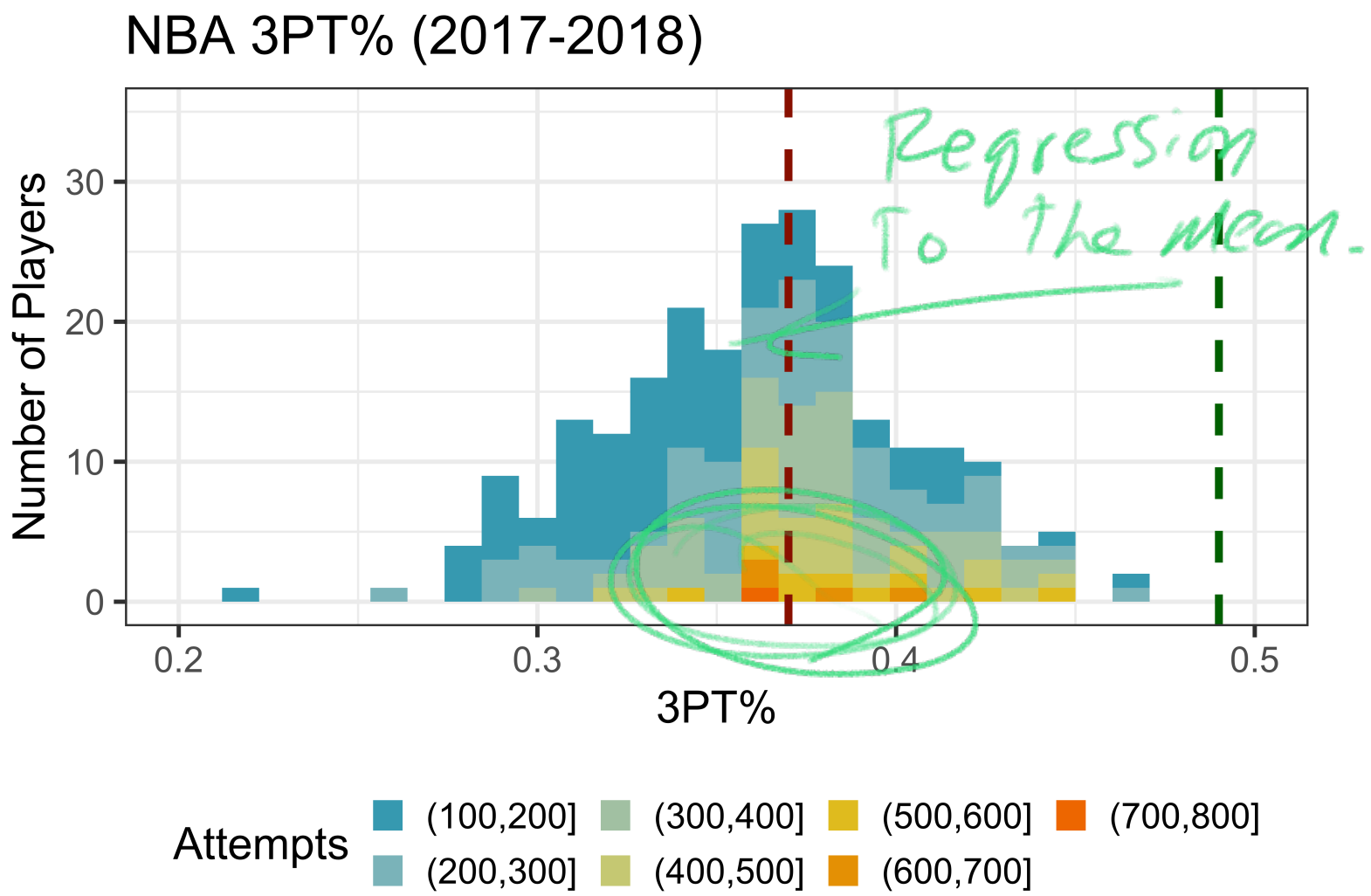
NBA 3PT% (2017-2018)
Regression
To the mean.
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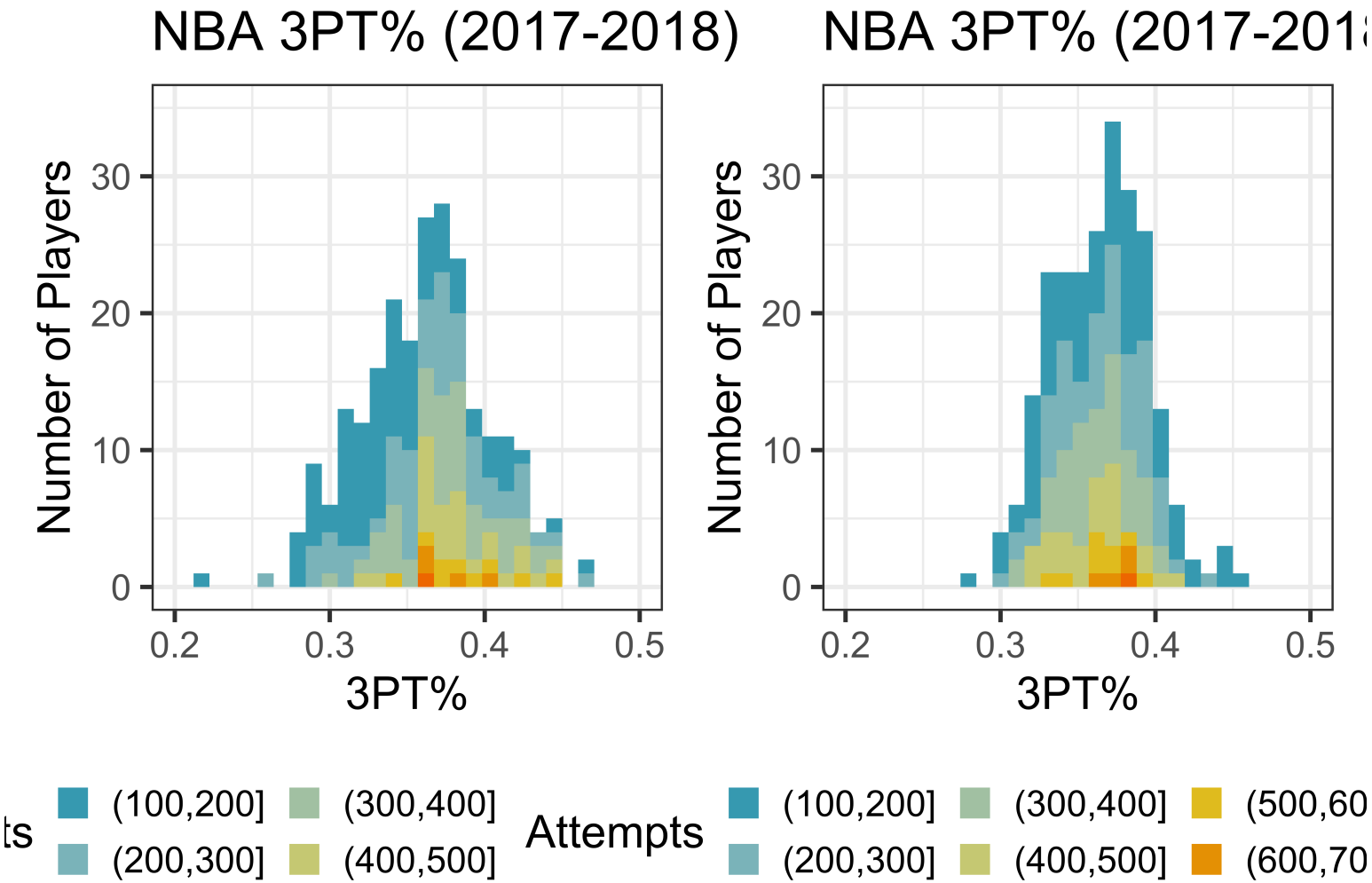
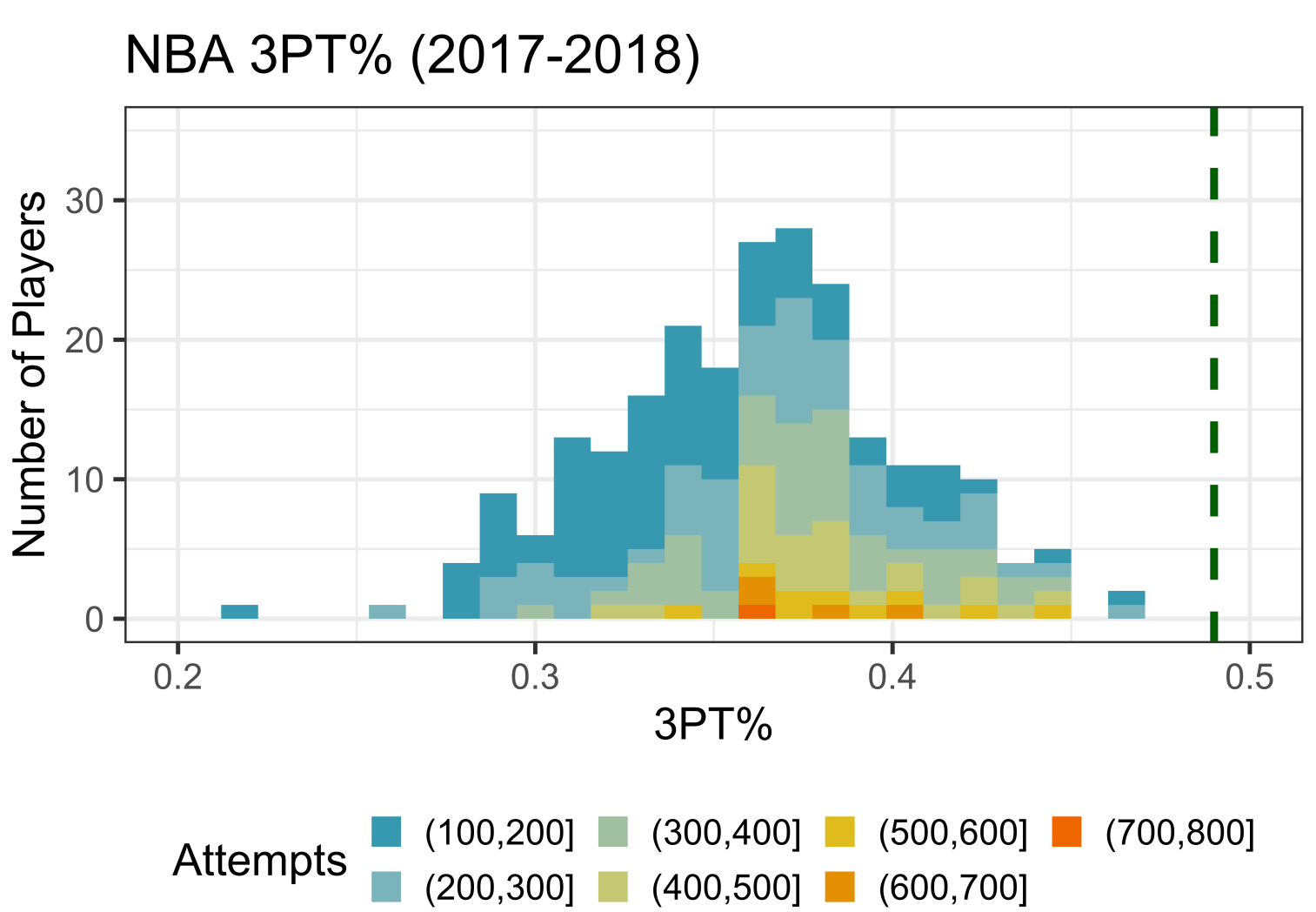
Three point shooting in 2017-2018

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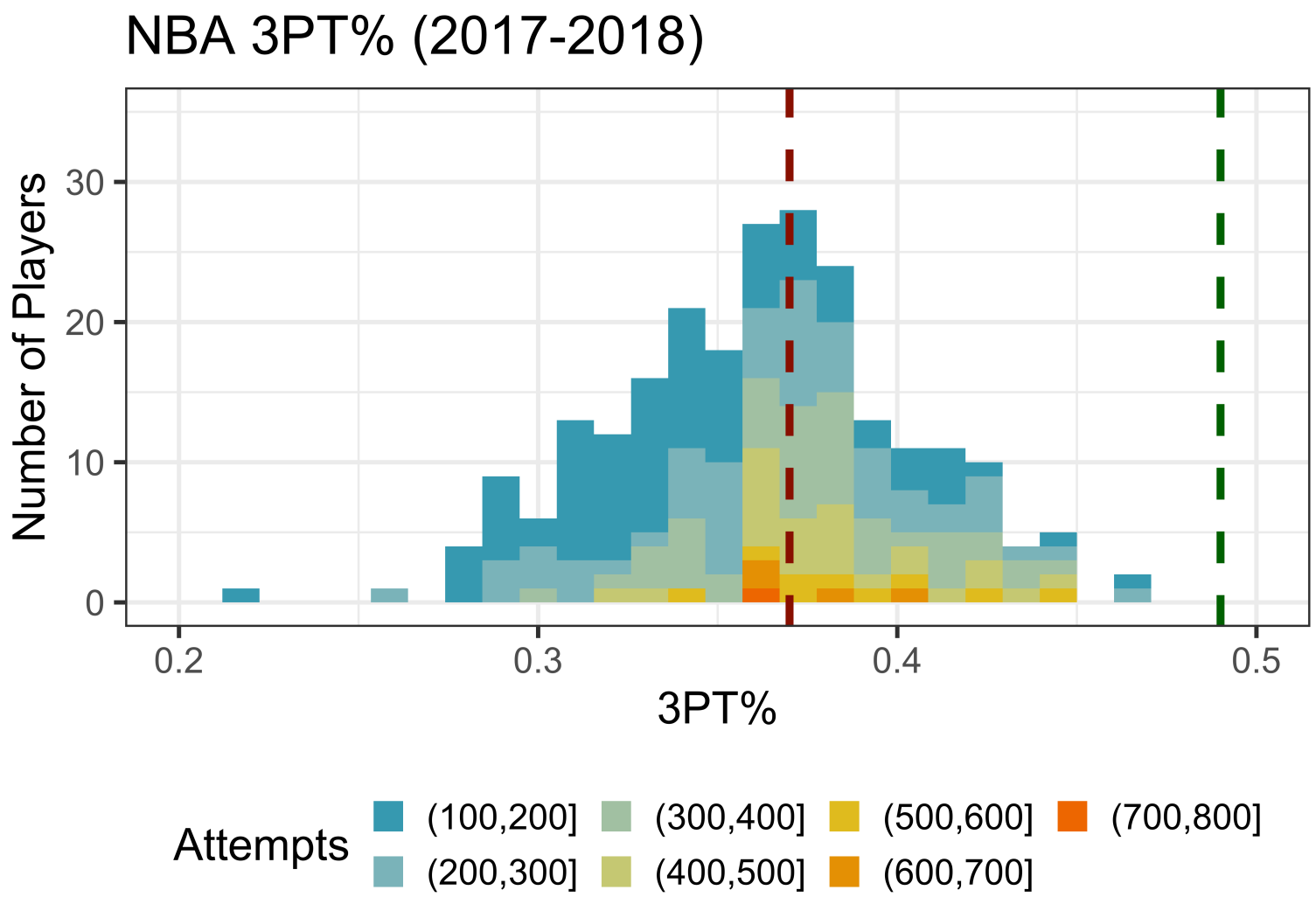


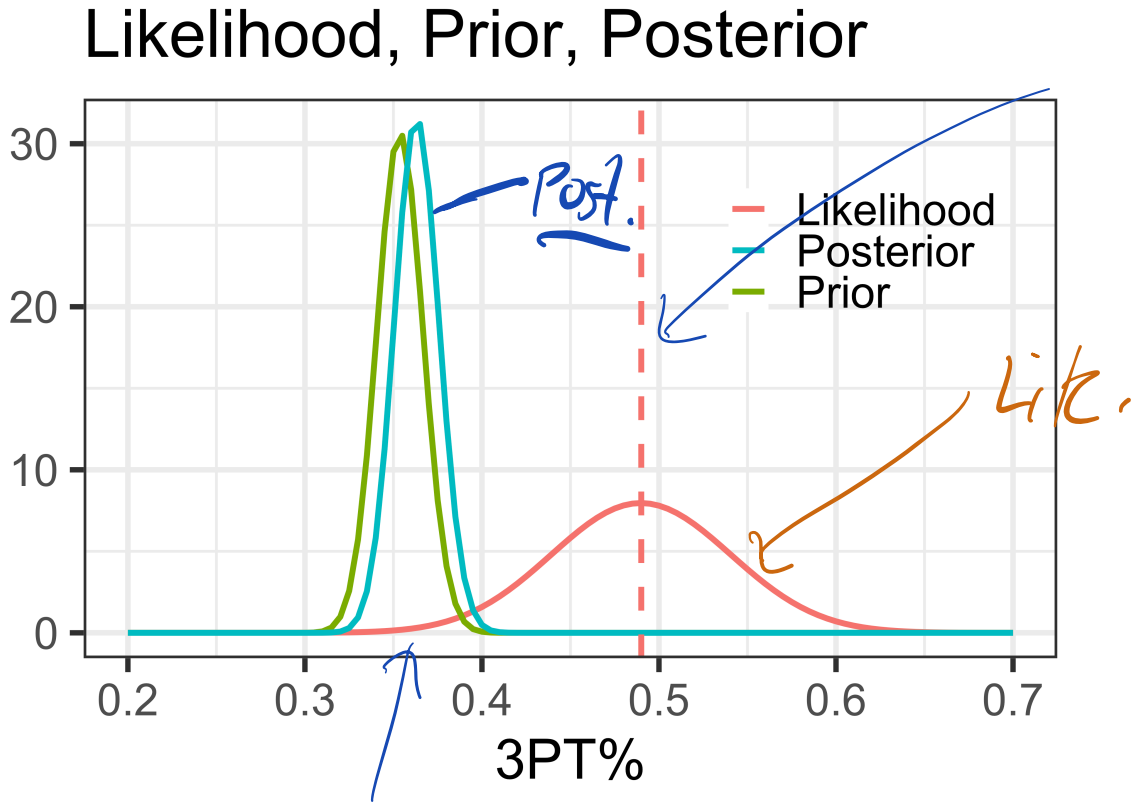
Regression Toward the Mean

What is a reasonable model?

- If we believe that his skill doesn't change much year to year, use past data to inform prior
- In his first 4 seasons combined Robert Covington made a total of 478 out of 1351 three point shots (0.35%, just below average).
- Choose a Beta(478, 873) prior (pseudo-count interpretation)

Robert Covington 2017-2018 estimates

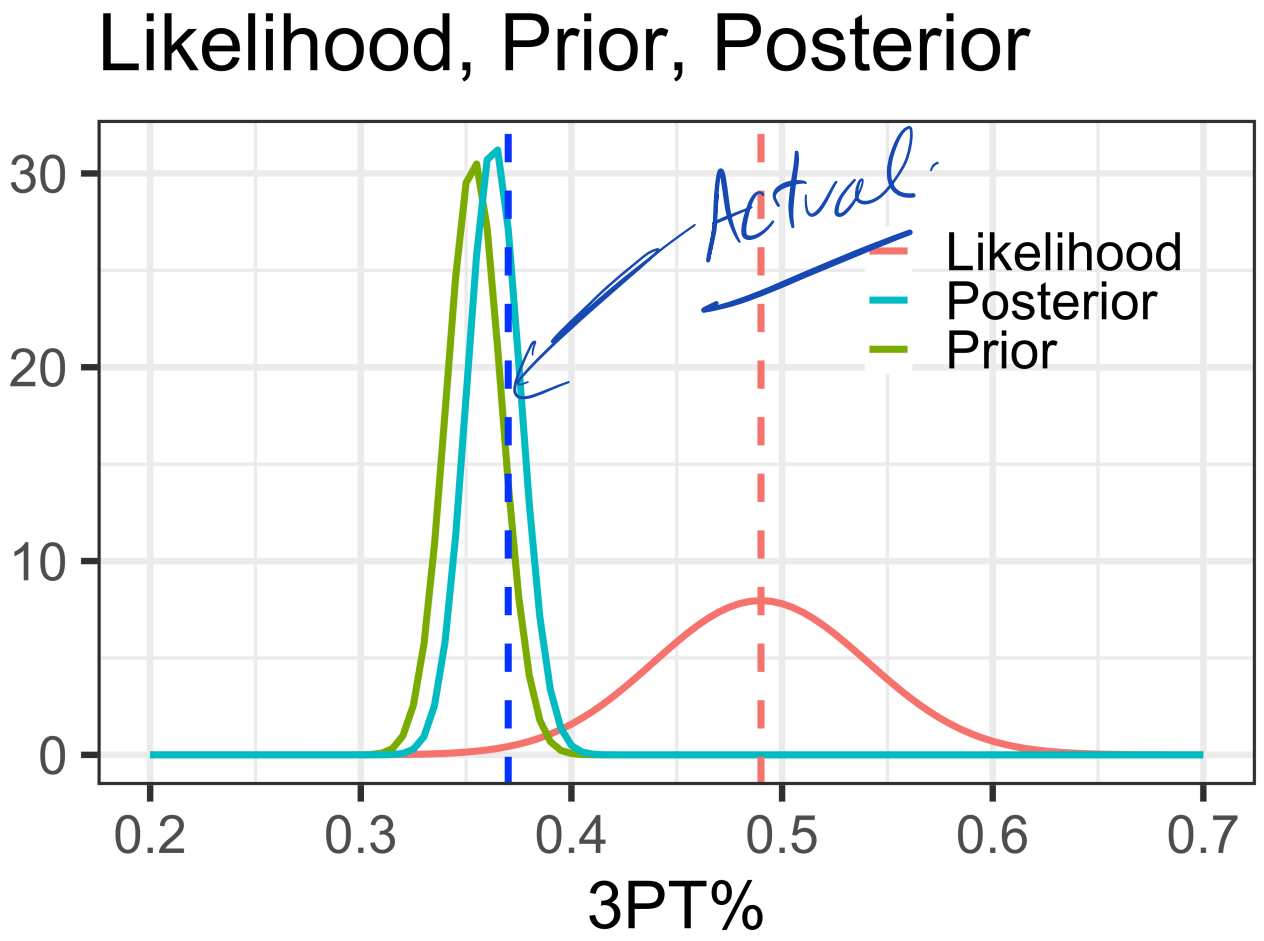
After 100 shots Robert Covington's 3PT% was 0.49



MLE = 0.49, posterior mean = 0.36

How did we do?

Robert Covington's end of season 3PT% was 0.37



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The Poisson Distribution

- Model for count data
- App.
  - # of meteorites entering Solar System.
  - # of patients → hospital
  - # of neurons firing.

$Y \sim Pois(\lambda)$, $E[Y] = Var(Y) = \lambda$

Poisson model

Assume $Y_1, \ldots, Y_n$ are n i.i.d. $\text{Pois}(\lambda)$

$$P(y_1, \ldots, y_n \mid \lambda) = \prod_{i=1}^{n} P(y_i \mid \lambda)$$

$$= \prod^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!}$$

$$\hat{\lambda}_{MLE} = \frac{\Sigma y_i}{n}$$

$$\propto \lambda^{\Sigma y_i} e^{-n\lambda}$$

$$\propto L(\lambda)$$

Poisson model with exposure

- Often times we include an "exposure" term in the Poisson model:

Known constant: "how long"

$$p(y_i \mid \nu_i \lambda) = (\nu_i \lambda)^y e^{\nu_i \lambda} / y_i!$$

- How many cars do we expect to pass an intersection in one hour? How many in two hours?
  - If we model the distribution as Poisson, we expect twice as many in two hours as in one hours.
- Homework: exposure is the length of the chapter

$$y_i \sim \text{Pois}(\lambda v_i)$$

$$P(y_i | \underbrace{\lambda v_i}) \propto \frac{(\lambda v_i)^{y_i} e^{-v_i \lambda}}{y_i!}$$

↑ expected counts per unit "time"

Poisson model example

- In a particular county 3 people out of a population of 100,000 died of asthma
- Assume a Poisson sampling model with rate λ (units are rate of deaths per 100,000 people)
- How do we specify a prior distribution for λ?
- How would our Bayesian estimate for λ differ?

$y_i \sim \text{Pois}(\lambda v_i)$ λ = Rate of deaths per 100,000

$\hat{\lambda}_{MLE}$: 3 (per 100K) $v_i = 1$

$P(1 | y=3)$ needed.

Choose $P(1)$. How?

- Air Quality of county.
- Previous Years' data
- Info from other counties.
- Regress on more info?

$P(\lambda | y_{\text{obs}})$

Conjugate Prior for the Poisson

Assume n i.i.d observations of a $\text{Poisson}(\lambda)$

$$\begin{aligned} p(\lambda \mid y_1, \dots y_n) &\propto L(\lambda) \times p(\lambda) \\ &\propto \lambda^{\sum y_i} e^{-n\lambda} \times p(\lambda) \end{aligned}$$

- A prior distribution for λ should have support on $\mathbb{R}^+$, the positive real line
- Bayesian definition of sufficiency: $p(\lambda \mid s, y_1, \dots y_n) = p(\lambda \mid s)$
  - For the Poisson, $\sum y_i$ is sufficient
- Can we find a density of the form $p(\lambda) \propto \lambda^{k_1} e^{k_2 \lambda}$?