Lecture 4: Intervals

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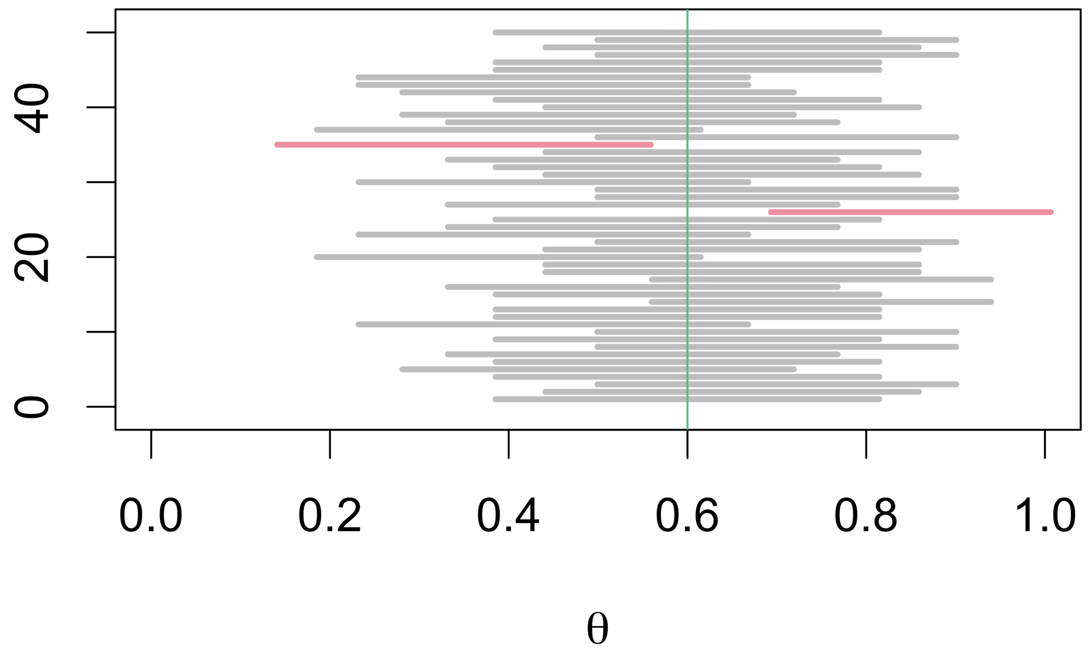
Announcements

- Reading: Chapter 8.1 (intervals), 8.3 (posterior prediction)
- Homework 2 out today

Reminder: Frequentist confidence interval

- Frequentist interval: $Pr(l(Y) < \theta < u(Y) \mid \theta) = 0.95$
  - Probability that the interval will cover the true value *before* the data are observed.
  - Interval is random since Y is random

Reminder: Frequentist confidence interval



We expect 0.05 x 50 = 2.5 will *not* cover the true parameter 0.6

Posterior Credible Intervals

- Frequentist interval: $Pr(l(Y) < \theta < u(Y) \mid \theta) = 0.95$
  - Probability that the interval will cover the true value *before* the data are observed.
  - Interval is random since Y is random
- **Bayesian Interval**: $Pr(l(y) < \theta < u(y) \mid Y = y) = 0.95$
  - Information about the the true value of θ *after* observeing $Y = y$.
  - θ is random (because we include a prior), y is observed so interval is non-random.

Posterior Credible Intervals (Quantile-based)

- The easiest way to obtain a confidence interval is to use the quantiles of the posterior distribution.

If we want $100 \times (1-\alpha)$ interval, we find numbers $\theta_{\alpha/2}$ and $\theta_{1-\alpha/2}$ such that:

1. $p(\theta < \theta_{\alpha/2} \mid Y = y) = \alpha/2$
2. $p(\theta > \theta_{1-\alpha/2} \mid Y = y) = \alpha/2$

$$p(\theta \in [\theta_{\alpha/2}, \theta_{1-\alpha/2}] \mid Y = y) = 1 - \alpha$$

- Use quantile functions in R, e.g. `qbeta`, `qpois`, `qnorm` etc.

Example: interval for shooting skill

- The posterior distribution for Covington's shooting percentage is a

$$\text{Beta}(49 + 478, 50 + 873) = \text{Beta}(528, 924)$$

- For a 95% *credible* interval, $\alpha = 0.05$
  - Lower endpoint: `qbeta(0.025, 528, 924)`
  - Upper endpoint: `qbeta(0.975, 528, 924)`
  - $[\theta_{\alpha/2}, \theta_{1-\alpha/2}]$ = [0.34, 0.39]
- Compared to frequentist *confidence* interval without prior information: [0.39, 0.59]
- End-of-season percentage was 0.37
- Credible intervals and confidence intervals have different

Highest Posterior Density (HPD) region

Definition: (HPD region) A $100 \times (1-\alpha)$ HPD region consists of a subset of the parameter space, $R(y) \in \Theta$ such that

1. $\Pr(\theta \in R(y) | Y = y) = 1 - \alpha$
   - The probability that θ is in the HPD region is $1 - \alpha$
2. $\text{If } \theta_a \in R(y), \text{ and } \theta_b \notin R(y)$ then $p\left(\theta_a | Y = y\right) > p\left(\theta_b | Y = y\right)$
   - All points in an HPD region have a higher posterior density than points out- side the region.

Highest Posterior Density (HPD) region

1. $p(\theta \in s(y) \mid Y = y) = 1 - \alpha$
2. If $\theta_a \in s(y)$, and $\theta_b \notin s(y)$, then $p(\theta_a \mid Y = y) > p(\theta_b \mid Y = y)$.
   - All points in an HPD region have a higher posterior density than points out- side the region.

The HPD region is the *smallest* region with prob $(1 - \alpha)$%

Calibration: Frequentist Behavior of Bayesian Intervals

- A credible interval is calibrated if it has the right frequentist coverage
- Bayesian credible intervals usually won't have correct coverage
- If our prior was well-calibrated and the sampling model was correct, we'd have well-calibrated credible intervals
- Specifying *nearly* calibrated prior distributions is hard!

Calibration of political predictions

The best test of a probabilistic forecast is whether it's well calibrated. By that I mean: Out of all FiveThirtyEight forecasts that give candidates about a 75 percent shot of winning, do the candidates in fact win about 75 percent of the time over the long run? It's a problem if these candidates win only 55 percent of the time. But from a statistical standpoint, it's just as much of a problem if they win 95 percent of the time.

source: fivethirtyeight.com

Calibration of political predictions

Calibration for FiveThirtyEight "polls-plus" forecast

WIN PROBABILITY RANGE	NO. FORECASTS	EXPECTED NO. WINNERS	ACTUAL NO. WINNERS
95-100%	27	26.7	26
75-94%	15	13.1	14
50-74%	14	8.7	11
25-49%	13	4.8	3
5-24%	27	3.1	1
0-4%	88	0.8	1

source: https://fivethirtyeight.com/features/when-we-say-70-percent-it-really-means-70-percent/

The age guessing game*

*Bayesian edition

Posterior Predictive Distributions

Posterior predictive distribution

- An important feature of Bayesian inference is the existence of a predictive distribution for new observations.
  - Let $\tilde{y}$ be a new (unseen) observation, and $y_1, \ldots y_n$ the observed data.
  - The Posterior predictive distribution is $p(\tilde{y} \mid y_1, \ldots y_n)$
- The predictive distribution does not depend on unknown parameters
- The predictive distribution only depends on observed data
- Asks: what is the probability distribution for new data given observations of old data?

Another Basketball Example

- I take free throw shots and make 1 out of 2. How many do you think I will make if I take 10 more?
- If my true “skill” was 50%, then $\tilde{Y} \sim \mathrm{Bin}(10, 0.50)$
- Is this the correct way to calculate the predictive distribution?

Posterior Prediction

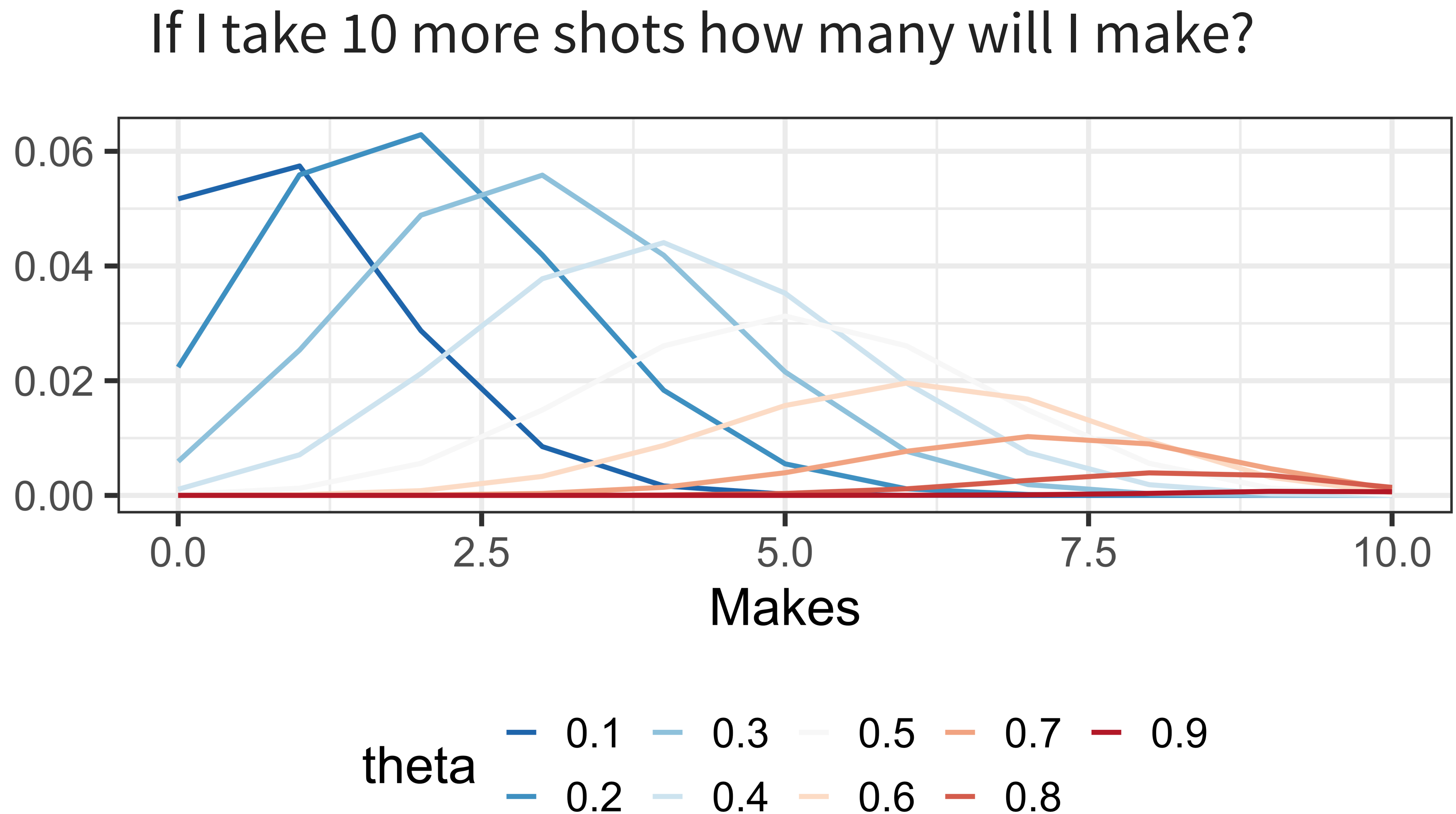
If you know θ, then we know the distribution over future attempts:

$$\tilde{Y} \sim \mathrm{Bin}(10, \theta)$$

Posterior Prediction

- We already observed 1 make out of 2 tries.
- Assume a Beta(1, 3) prior distribution
  - e.g. a priori you think I'm more likely to make 25% of my shots
- Then $p(\theta \mid Y = 1, n = 2)$ is a $\mathrm{Beta}(2, 4)$
- Intuition: weight $\tilde{Y} \sim \mathrm{Bin}(10, \theta)$ by $p(\theta \mid Y = 1, n = 2)$

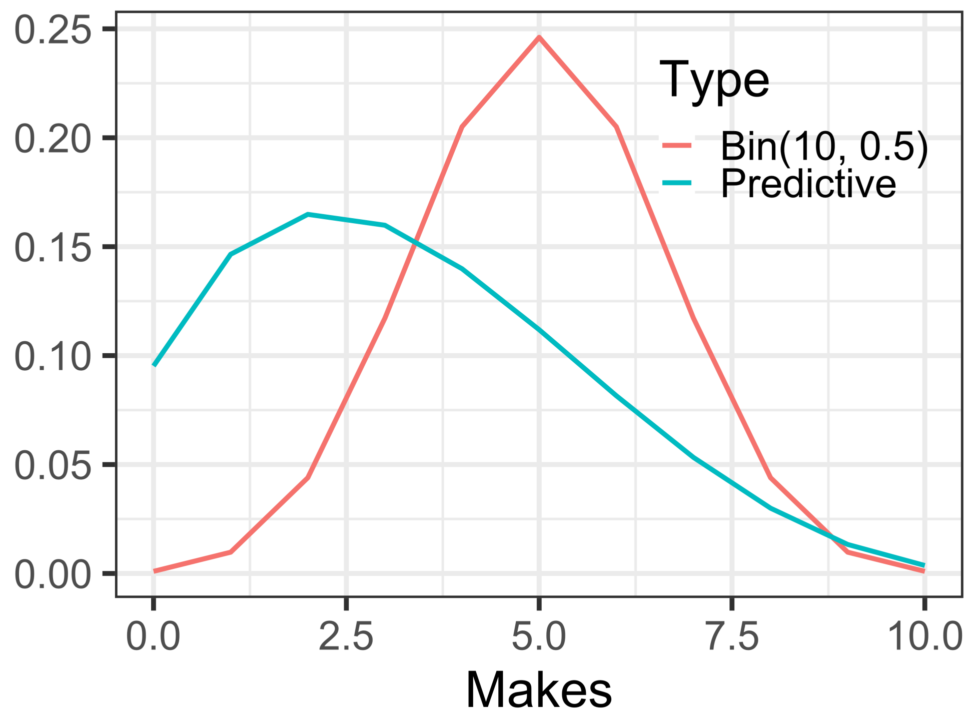
Posterior Prediction



Posterior predictive distribution

Posterior predictive distribution

$$p(\theta) = \text{Beta}(1,3), p(\theta \mid y) = \text{Beta}(2,4)$$



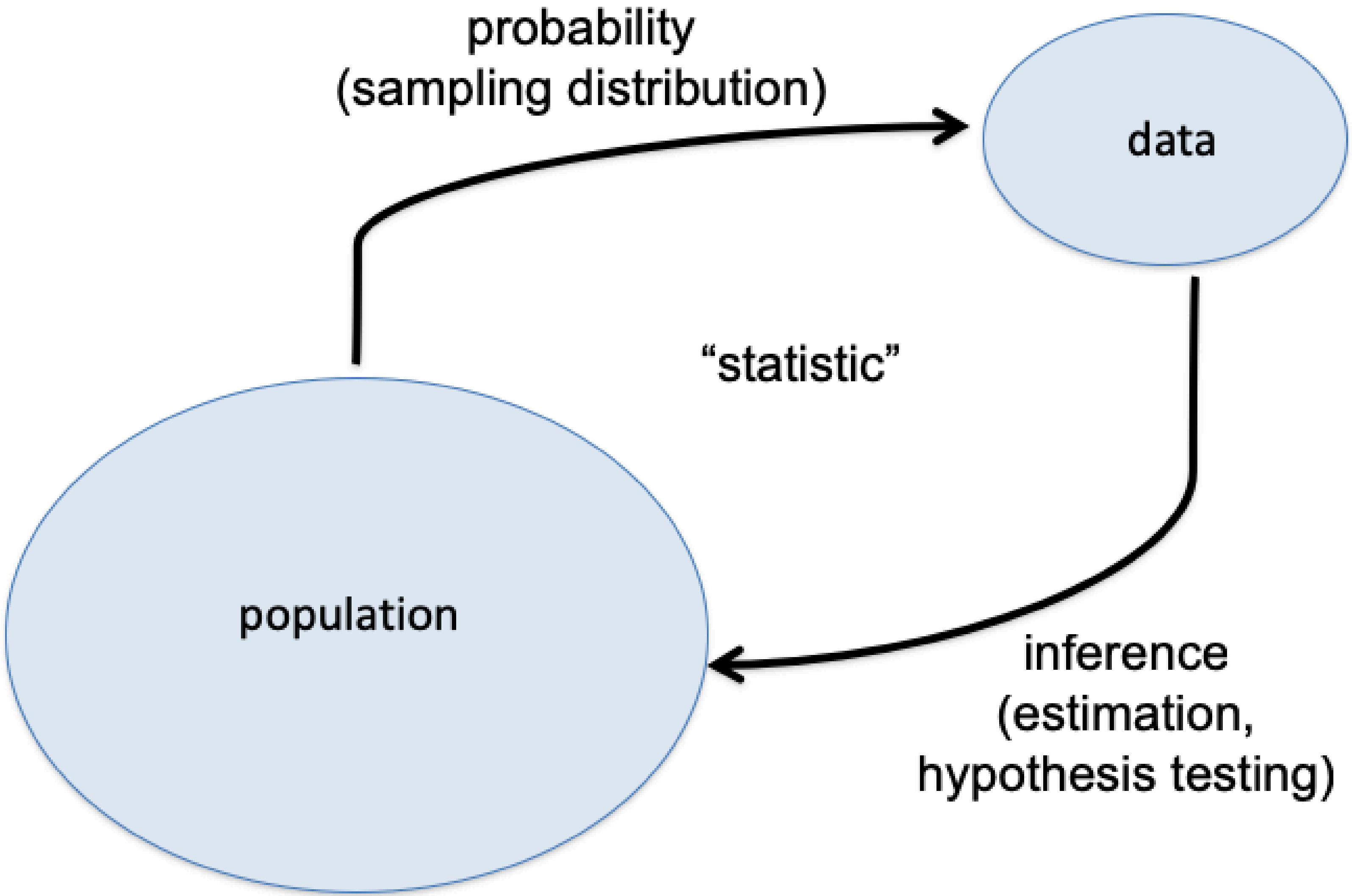
The predictive density, $p(\tilde{y} \mid y)$, answers the question “if I take 10 more shots how many will I make, given that I already made

The posterior predictive distribution

$$\begin{aligned} p(\tilde{y} \mid y_1, \ldots y_n) &= \int p(\tilde{y}, \theta \mid y_1, \ldots y_n) d\theta \\ &= \int p(\tilde{y} \mid \theta) p(\theta \mid y_1, \ldots y_n) d\theta \end{aligned}$$

- The posterior predictive distribution describes our uncertainty about a new observation after seeing n observations
- It incorporates uncertainty due to the sampling in a model $p(\tilde{y} \mid \theta)$ *and* our posterior uncertainty about the data generating parameter, $p(\theta \mid y_1, \ldots y_n)$

Posterior Predictive Density



The prior predictive distribution

$$
\begin{aligned}
p(\tilde{y}) &= \int p(\tilde{y}, \theta) d\theta \\
&= \int p(\tilde{y} \mid \theta) p(\theta) d\theta
\end{aligned}
$$

- The prior predictive distribution describes our uncertainty about a new observation before seeing data
- It incorporates uncertainty due to the sampling in a model $p(\tilde{y} \mid \theta)$ *and* our prior uncertainty about the data generating parameter, $p(\theta)$

Homework 1

Subjective Bayesianism

- So far we have focused on defining priors using domain expertise
- “Subjective” Bayes
  - Essentially what we have discussed so far
  - Priors usually represent subjective judgements can’t always be rigorously justified
- Alternative: “objective” Bayes

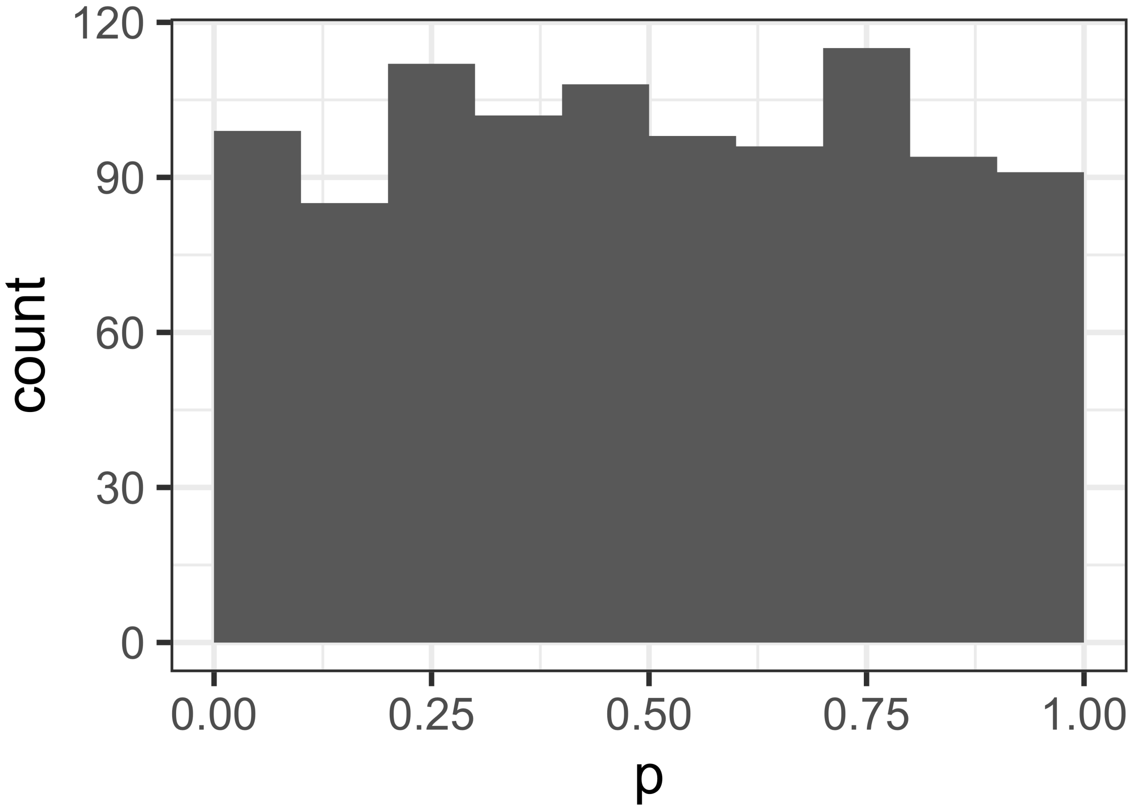
Objective Bayesianism

- Is there a way to define “objective” prior distributions?
  - Good default prior distributions for some problems?
  - “Non-informative” prior distributions?
- Also called “reference” or “default” priors
- Can we find prior distributions that lead to (approximately) correct frequentist calibration?
- Can we find prior distributions which minimize the amount of information contained in the distribution?
  - Principle of maximum entropy (MAXENT).

Difficulties with non-informative priors

Uniform distribution for p

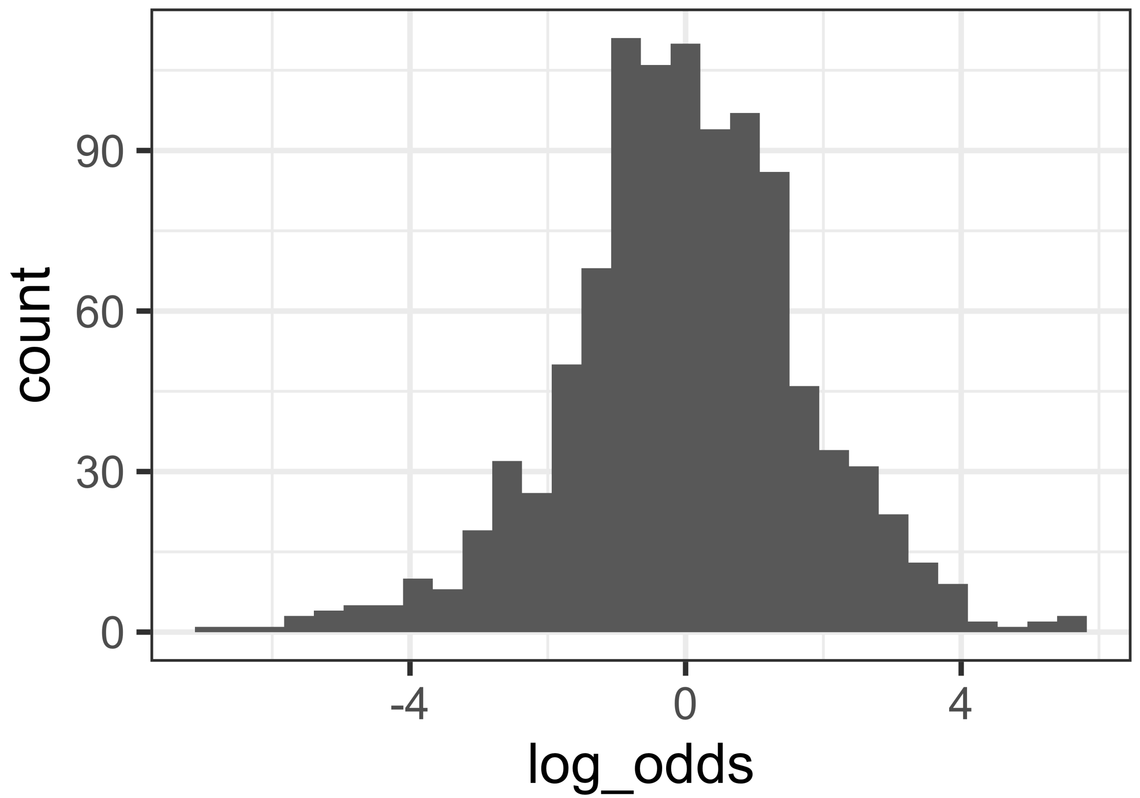
```
p <- runif(1000)
tibble(p=p) %>% ggplot() +
  geom_histogram(aes(x=p), boundary=0.5, binwidth=0.1) +
  theme_bw(base_size=24)
```



Difficulties with non-informative priors

Implied distribution for odds = p/(1-p)

```r
log_odds <- log(p/(1-p))
tibble(log_odds=log_odds) %>% ggplot() +
  geom_histogram(aes(x=log_odds)) +
  theme_bw(base_size=24)
```



Improper prior distributions

- For the Beta distribution we chose a uniform prior, where $p(\theta) \propto \text{const}$. This was ok because:
  - $\int_0^1 p(\theta) d\theta = \text{const} < \infty$
  - We say this prior distribution is *proper* because it is integrable
- For the Poisson distribution, try the same thing: $p(\lambda) \propto \text{const}$
  - $\int_0^\infty p(\lambda) d\lambda = \infty$
  - In this case we say $p(\lambda)$ is an *improper* prior

Improper prior distributions

- Sometimes there is an absence of precise prior information
- The prior distribution does not have to be proper but the posterior does!
  - A proper distribution is one with an integrable density
  - If you use an improper prior distribution, you need to check that the posterior distribution is also proper

Summary

- Bayesian credible intervals
  - Posterior probability that the value falls in the interval
  - Still strive for well-calibrated intervals (in the frequentist sense)
- Posterior predictive distributions
  - Estimated distribution for new data our uncertainty about the parameters
- Non-informative prior distributions