PSTAT 115 - Section One

Winter 2023

120AB Review

Definition of a Random Variable

notation ← **Definition** (Random Variable). A random variable X represents an outcome from a *random experiment*. Let Ω represent the collection of all outcomes from an experiment. Then, $X : \Omega \to \mathbb{R}$ (is a function from sample space to reals)

Examples.

- Experiment: number of heads after tossing a fair coin twice. $\Omega =$ {HH, HT, TH, TT} and $X = 1 \Rightarrow$ outcome was either TH or HT.
- Experiment: Time it takes for a buss to arrive to bus stop. $\Omega = [0, \infty)$ and $X = 5$ mins means time it took was 5 minutes.

Notation. Random variables are always represented with capital letters (i.e. $X, Y \ldots$). Their observed values are represented with lowercase letters (i.e. $x, y, \ldots$).

Population vs Sample

Definition (Population). A population represents our entire set of interest from which we wish to draw inference on!

Definition. Our sample is the data points from our population of interest. A sequence (or sample) is given by $(X_1, \ldots, X_n)$.

Examples.

(i) Research Question: Most popular 2020 song. Population: ppl who listen to music; Sample $(X_1, \ldots, X_n)$: n=1000; ppl in US

(ii) Research Question: Average height of college male. Population: all college males; Sample $(X_1, \ldots, X_n)$: n=100; UCSB males

Estimator vs Estimate

Definition (Statistic). Given a sequence of random variables $(X_1, X_2, \ldots, X_n)$ a **statistic** is any function h of those random variables.

Special Case: an estimator is a statistic ~~used is~~ used to *estimate* a parameter from the unknown distribution of the RVs.

Note. Estimators are RVs..

Example. Suppose we have a sample $(X_1, X_2, \ldots X_n)$ from an unknown distribution $\sim N(\mu, 9)$. Then, an estimator for μ equals $\frac{1}{n} \cdot \sum_{i=1}^{n} X_i$, avg

Definition. The values that the estimator can take is the estimate. For example, $h(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$ then each X_i has obs. x_i ← little x gives the estimate

$h(x) = \frac{1}{n} \sum_{i=1}^{n} x_i$

Likelihood Function

Problem. Let $X_1, \dots, X_n \sim N(\mu, 1)$. The dataset obtained is $x_1, \dots, x_n$. Compute the likelihood function, log-likelihood and proportional simplification. What is the MLE?

Notes

Problem. Let $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$. The dataset obtained is $x_1, x_2, \dots, x_n$.

(i) Find the MLE of μ, σ^2.

(ii) Find the MLE of μ^3.

Notes

Problem. Let $X_1, \dots, X_n \sim N(\mu, 1)$. The dataset obtained is $x_1, \dots, x_n$. Compute the likelihood function, log-likelihood and proportional simplification. ~~What is the MLE~~?

Def: $L(\theta, x) = \prod_{i=1}^{N} P_\theta(x_i)$; theta unknown parameter.

- Given $X_i \sim N(\mu, 1) \Rightarrow f_{\underset{RV}{X_i}}(\underset{obs}{x_i}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2}}$
- $L(\mu) = \prod_{i=1}^{N} f_{X_i}(x_i) = \prod_{i=1}^{N} 1/\sqrt{2\pi}\, e^{-(x_i-\mu)^2/2}$

$$= \frac{1}{(2\pi)^{N/2}} e^{-\sum_{i=1}^{N} (x_i-\mu)^2/2}$$ (*) Likelihood

(*) Log-Like : $\log(L\mu) = \log(2\pi)^{N/2} + \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{2} = N/2 \log(2\pi) + \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{2}$

- Prop. simplification.

Notation $L(\mu) \propto f(x_1, \dots, x_n) \iff L(\mu) = c \cdot f(x_1, \dots, x_n)$, $c \in \mathbb{R}$.

$L(\mu) \propto e^{-\sum_{i=1}^{N} (x_i-\mu)^2/2}$

Problem. Let $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$. The dataset obtained is $x_1, x_2, \dots, x_n$.

(i) Find the MLE of μ, σ^2.

(ii) Find the MLE of μ^3.

Def: MLE equals the θ which maximizes $L(\theta, x)$.

How to find MLE: 1) Solve $\frac{dL(\theta, x)}{\partial\theta} = 0$ & $\frac{\partial^2 L(\theta, x)}{\partial\theta^2} < 0$

Prop log is monotone & $\Rightarrow$ maximizing $L(\theta, x) \equiv \max_\theta \log L(\theta, x)$.

(i) Let $X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2)$. As a result,

$$L(\mu, \sigma^2; \underset{\sim}{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-1/2\sigma^2 \sum_{i=1}^{n} (x_i - \mu)^2}$$

(set $\log(L) \equiv \ell$)

$$\Rightarrow \log(L) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

To find the MLE of $\hat{\mu}$ & $\hat{\sigma}^2$, we need to solve:

$$\frac{d\ell}{d\mu} = \frac{1}{\sigma^2} \left(\sum_i x_i - n\mu \right) = 0 \qquad (\ell := \log L)$$

$$\frac{d\ell}{d\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \left(\sum_i (x_i - \mu)^2 \right) = 0$$

As a result,

$$\hat{\mu} = \frac{\sum_i x_i}{n} = \bar{x}$$

$$\hat{\sigma}^2 = \frac{1}{n} \left(\sum_i (x_i - \bar{x})^2 \right)$$

(ii) Invariance prop. of MLE For any fn g, $g(\hat{\theta}_{ML})$ is the MLE for $g(\theta)$.

$$\bar{x} \overset{MLE}{=} \mu \Rightarrow (\bar{x})^3 \overset{MLE}{=} \mu^3$$