PSTAT 115 - Section Week 7

Winter 2023

1 Normal-Normal Set-Up

Theorem. *Let μ represent the* unknown mean *parameter where* $\boxed{\mu \sim N(\theta,\tau^2)}$ *& the vector, $(Y_1,\ldots,Y_n)$ represent an **independent** normal $N(\mu,\sigma^2)$* sample *(σ^2 known).*

After observing data $y=(y_1,\ldots y_n)$ with mean $\overline{y}=\frac{1}{n}\sum_{i=1}^{n} y_i$, the posterior model μ is:

$$\mu|y \sim N\left(\underbrace{\theta\frac{\sigma^2}{n\tau^2+\sigma^2}+\bar{y}\frac{n\tau^2}{n\tau^2+\sigma^2}}_{\nu_1}, \underbrace{\frac{\tau^2\sigma^2}{n\tau^2+\sigma^2}}_{\sigma_0^2}\right)$$

Proof. Notes

□

2 Normal-Normal Problem

(Section 5.11; Bayes Rules). Prof. Abebe and Prof. Morales both recently finished their PhDs and are teaching their first statistics classes at Bayesian University. Their colleagues told them that the average final exam score across all students, μ, varies **Normally** from year to year, with a mean of 80 points and standard deviation of 4. Further, individual students' scores Y vary **Normally** around μ with a known standard deviation of 3 points.

Problem. Prof. Abebe conducts the final exam and observe that his 32 students scored an average of 86 points. Calculate the posterior mean and variance of μ *using data* from Prof. Abebe's course.

Notes

Problem. Prof. Morales conducts the final exam and observes that her 32 students scored an average of 82 points. Calculate the posterior mean and variance of μ using the data from Prof. Morales' class.

Notes

Problem. Next, use Prob. Abebe and Prof. Morales' *combined* exams to calculate the posterior mean and variance of μ.

Notes

1) Prob.

→ If $Y \sim N(\mu, \sigma^2)$, then $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2} \propto e^{-(y-\mu)^2/2\sigma^2}$

→ Given, $Y_i|\mu \sim N(\mu, \sigma^2)$, $\mu \sim N(\theta, \tau^2)$

- $p(\mu|\vec{y}) \propto p(\mu) \cdot L(\mu|y)$

$$= p(\mu) \cdot p(y|\mu)$$

i) $p(\mu) \propto e^{-(\mu-\theta)^2/2\tau^2}$

ii) $p(y|\mu) = \prod_{i=1}^{n} p(y_i|\mu)$

$$\propto \prod_{i=1}^{n} e^{-(y_i-\mu)^2/2\sigma^2}$$

As a result,

$$p(\mu|\vec{y}) \propto e^{-\sum_{i=1}^{n}(y_i-\mu)^2/2\sigma^2} \cdot e^{-(\mu-\theta)^2/2\tau^2}$$

- keep only terms involving μ.

- $\sum_{i=1}^{n}(y_i-\mu)^2 = \sum_{i=1}^{n}(y_i^2+\mu^2-2y_i\mu) \equiv n\mu^2 - 2\mu\sum_{i=1}^{n} y_i$

- $(\mu-\theta)^2 \equiv \mu^2 - 2\mu\theta^2$

Answer

$$\Rightarrow p(\mu|\vec{y}) \propto e^{-\left(\frac{n\mu^2}{2\sigma^2} - \frac{2\mu}{2\sigma^2}\sum_i y_i\right) - \frac{1}{2}\tau^2(\mu^2 - 2\mu\theta)}$$

$$= e^{-n\mu^2/2\sigma^2 + \mu/\sigma^2 \sum_i y_i - \mu^2/2\tau^2 + \mu\theta^2/\tau^2}$$

$$= e^{\mu\left(1/\sigma^2 \sum_i y_i + \theta^2/\tau^2\right) - \frac{\mu^2}{2}\underbrace{\left(1/\tau^2 + n/\sigma^2\right)}_{=: 1/\sigma_0^2}}$$

$$= e^{\left(-\mu^2/2\sigma_0^2 + \mu \cdot \frac{1}{\sigma_0^2} \cdot \sigma_0^2 \cdot \underbrace{\left(1/\sigma^2 \sum_i y_i + \theta^2/\tau^2\right)}_{\text{equals } \mu_1}\right)}$$

$$= e^{-\mu^2/2\sigma_0^2 + \mu/\sigma_0^2 \cdot \mu_1}$$

$$\propto \frac{1}{\sqrt{2\pi}\sigma_0} \cdot e^{-(\mu - \mu_1/2\sigma_0)^2}$$

$$\sim N(\mu_1, \sigma_0^2) \checkmark.$$

Given: $\mu \sim N(\theta = 80, \tau^2 = 4^2)$, $Y_i \sim N(\mu, \sigma^2 = 3^2)$

(Section 5.11; Bayes Rules). Prof. Abebe and Prof. Morales both recently finished their PhDs and are teaching their first statistics classes at Bayesian University. Their colleagues told them that the average final exam score across all students, μ, varies **Normally** from year to year, with a mean of 80 points and standard deviation of 4. Further, individual students' scores Y vary **Normally** around μ with a known standard deviation of 3 points.

Problem. Prof. Abebe conducts the final exam and observe that his 32 students scored an average of 86 points. Calculate the posterior mean and variance of μ *using data* from Prof. Abebe's course.

We know, $p(\mu | \underset{\sim}{y}) \sim N\Big(\underbrace{\theta \frac{\sigma^2}{n\tau^2+\sigma^2} + \bar{y}\frac{n\tau^2}{n\tau^2+\sigma^2}}_{\mu_1}, \underbrace{\frac{\tau^2\sigma^2}{n\tau^2+\sigma^2}}_{\sigma_0^2}\Big)$

$\theta = 80$ $\quad\quad \tau^2 = 4^2$

$\sigma^2 = 3^2$

$n_A = 32$

$\bar{y}_A = 86$

Answer, $p(\mu | \underset{\sim}{y}) \sim N(85.8964, 0.2764)$.

Problem. Prof. Morales conducts the final exam and observes that her 32 students scored an average of 82 points. Calculate the posterior mean and variance of μ using the data from Prof. Morales' class.

$n_M = 32$

$\bar{y}_M = 82$

Answer $\sim N(81.9655, .2764)$

Problem. Next, use Prob. Abebe and Prof. Morales' *combined* exams to calculate the posterior mean and variance of μ.

$n_{com} = 32 + 32 = 64$

$$\bar{y}_{com} = \frac{\sum_i y_{A,i} + \sum_j y_{M,j}}{32+32}$$

$$= \frac{32/32 \sum_i y_{A,i} + 32/32 \sum_j y_{M,j}}{64}$$

$$= \frac{32\bar{y}_A + 32\bar{y}_M}{64}$$