PSTAT 115 - Section Three

Winter 2023

1 Summary of Model

Review: What are the 4 ingredients common to each and every Bayesian model?

i) *Prior Model:* We have a variable of interest π. The prior model specifies two pieces of info, 1. The values that π takes on and 2. the plausibility of each value.

ii) Data Dependence: to learn about π, we take collect data Y (another random variable). Summarize the dependence of the data on π with **conditional pmf**. $f(y|\pi)$

iii) *Likelihood function:* after observing the data (i.e. $Y=y$), we have $L(\pi|y) = f(y|\pi)$. This represents the likelihood of seeing the dataset y given different values (of π).

iv) *Posterior model:* aims to balance the prior & likelihood. We get,

$$\boxed{f(\pi|y)} = \frac{f(\pi)L(\pi|y)}{f(y)} \propto f(\pi)L(\pi|y) \Rightarrow \text{posterior} = \frac{\text{prior} \cdot \text{likelihood}}{\text{constant}}.$$

2 Gamma Integral

Question 1. *Let $K = \int L(\lambda; y)p(\lambda)d\lambda$ be the integral of the proportional posterior. Then the proper posterior density, i.e. a true density integrates to 1, can be expressed as $p(\lambda|y) = \frac{L(\lambda;y)p(\lambda)}{K}$. Compute this posterior density & express the density as a mixture of two gamma distributions.*

Notes

Question 1. *Let* $K = \int L(\lambda; y)p(\lambda)d\lambda$ *be the integral of the proportional posterior. Then the proper posterior density, i.e. a true density integrates to 1, can be expressed as* $p(\lambda|y) = \frac{L(\lambda;y)p(\lambda)}{K}$. *Compute this posterior density & express the density as a mixture of two gamma distributions.*

Note: Equation above, $p(\lambda|y) = \dfrac{L(\lambda;y)p(\lambda)}{g(y)}$

- What's $g(y)$?

joint dist.

By LTP = $g(y) = \int g(y,\lambda)\, d\lambda$

$P(A,B) = P(B)P(A|B)$ (20BA)

$= \int g(y|\lambda)p(\lambda)d\lambda$

equiv.

$= \int L(\lambda;y)p(\lambda)d\lambda. \quad \equiv K$

- Step 1) Find K.

- Equation. $\Gamma(z) = \int_0^\infty \lambda^{z-1}e^{-\lambda}\,d\lambda.$

$$K = \int e^{-1767\lambda}\cdot\lambda^8\left(\frac{2000^3}{\Gamma(3)}\lambda^2 e^{-2000\lambda} + \frac{1000^7}{\Gamma(7)}\lambda^6 e^{-1000\lambda}\right)d\lambda$$

$$= \int\left(e^{-1767\lambda}\frac{\cdot 2000^3}{\Gamma(3)}\cdot\lambda^{10}e^{-2000\lambda} + \frac{1000^7 e^{-1767\lambda}\lambda^{14}e^{-1000\lambda}}{\Gamma(7)}\right)d\lambda$$

$$= \frac{2000^3}{\Gamma(3)}\int e^{-1767\lambda}\cdot\lambda^{10}\cdot e^{-2000\lambda} + \frac{1000^7}{\Gamma(7)}\int \lambda^{14}e^{-1767\lambda}e^{-1000\lambda}d\lambda$$

$$= \frac{2000^3}{\Gamma(3)}\underbrace{\int \lambda^{10}e^{-3767\lambda}d\lambda}_{T1} + \frac{1000^7}{\Gamma(7)}\underbrace{\int \lambda^{14}e^{-2767\lambda}d\lambda}_{T2}.$$

T1: $\int \lambda^{10}e^{-3767\lambda}d\lambda$

$u = 3767\lambda$

$du = 3767\,d\lambda$

$$= \frac{1}{3767}\int e^{-u}\cdot\left(\frac{1}{3767}\right)^{10}u^{10}\,du$$

$$= \frac{1}{(3767)^{11}} \int e^{-u} u^{10} \, du$$

$$= \frac{1}{(3767)^{11}} \Gamma(11)$$

T2: $\int \lambda^{14} e^{-2767\lambda} \, d\lambda$

$u = 2767\lambda$, $du = 2767 \, d\lambda$

$$= \int e^{-u} \cdot \frac{1}{(2767)^{15}} \cdot u^{14} \, du$$

$$= \frac{\Gamma(15)}{(2767)^{15}}.$$

Combining T1, T2.

$$K = \frac{2000^3}{\Gamma(3)} \cdot \frac{\Gamma(11)}{(3767)^{11}} + \frac{1000^7}{\Gamma(7)} \cdot \frac{\Gamma(15)}{(2767)^{15}}$$

- Step 2) Compute $p(\lambda|y)$

- $p(\lambda|y) = \frac{L(\lambda;y)p(\lambda)}{K}$

where $K = \int L(\lambda;y)p(\lambda) \, d\lambda$.

Answer: $$\frac{\frac{2000^3}{\Gamma(3)} \lambda^{10} e^{-3767\lambda}}{K} + \frac{\frac{1000^7}{\Gamma(7)} \lambda^{14} e^{-2767\lambda}}{K}$$

Simplifies to,

$$\frac{\frac{2000^3}{\Gamma(3)} \cdot \frac{\Gamma(11)}{3767^{11}}}{\underbrace{\frac{2000^3}{\Gamma(3)} \frac{\Gamma(11)}{3767^{11}}}_{:= \omega} + \frac{1000^7}{\Gamma(7)} \frac{\Gamma(15)}{2767^{15}}} \cdot \frac{2767^{11}}{\Gamma(11)} \lambda^{10} e^{-3767\lambda}$$

$$+ (1-\omega) \cdot \frac{2767^{15}}{\Gamma(15)} \lambda^{14} e^{-2767\lambda}$$

$$\equiv K p_U(\lambda) + (1-\omega) p_V(\lambda)$$

$U \sim \text{Gamma}(11, 1/3767)$, $V \sim \text{Gamma}(15, 1/2767)$

i.e. Posterior density mixture of 2 gamma distributions