Lecture 3: Multiparameter Models

Professor Alexander Franks

1/22/24

- Gradescope → Canvas
- Hw 2 out tonight, due next Sun. 2/4
- Hierarchical (Ch. 5)

Bayesian inference in the normal model

- Assume $y_1, \ldots y_n \sim N(\mu, \sigma^2)$ with σ^2 a known constant
- Let's start with a Jeffreys' (improper prior): $p(\mu) \propto \text{const}$
- What is the posterior distribution $p(\mu \mid y_1, \ldots y_n, \sigma^2)$?

$$\sqrt{I(\mu)} \propto n/\sigma^2 \propto \text{const}$$

$$p(\mu \mid y_1 \ldots y_n) \propto \underbrace{e^{\frac{-\sum (y_i - \mu)^2}{2\sigma^2}}}_{L(\mu)} \times \text{const}$$

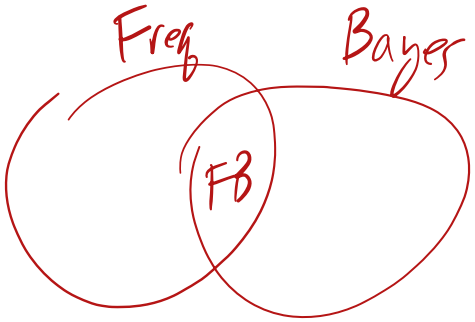
$$\propto e^{\frac{-n\mu^2 - 2\sum y_i \mu + \text{const}}{2\sigma^2}} \quad 1/n \quad 1/n$$

$$\propto e^{\frac{-(\mu-\bar{y})^2}{2\sigma^2/n}}$$

$$P(\mu \mid y_1, \ldots y_n, \sigma^2) \sim N\left(\bar{y}, \frac{\sigma^2}{n}\right)$$

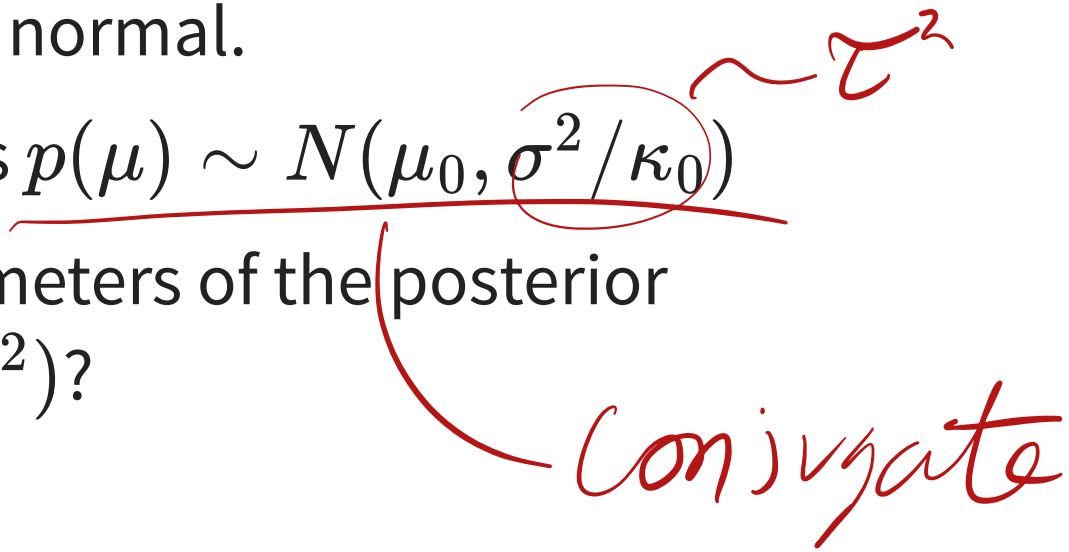
$$\hat{\mu}_{MLE} = \bar{y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\sqrt{\sigma^2/n}\,(\mu - \bar{y}) \sim N(0,1)$$



Bayesian inference in the normal model

- Assume $y_1, \ldots y_n \sim N(\mu, \sigma^2)$ with σ^2 a known constant
- The normal prior distribution is conjugate for μ in the normal sampling model
- Sampling distribution, prior distribution and posterior distribution are all normal.
- Assume the prior is $p(\mu) \sim N(\mu_0, \sigma^2/\kappa_0)$
- What are the parameters of the posterior $p(\mu \mid y_1, \ldots y_n, \sigma^2)$?



$$P(\mu \mid y_1 \dots y_n) \propto L(\mu) P(\mu)$$

$$\propto e^{-\frac{(\mu-\bar{y})^2}{2\sigma^2/n}} e^{-\frac{(\mu-\mu_0)^2}{2\sigma^2/k_0}}$$

$$\propto \exp\left[-\frac{1}{2}\frac{n+k_0}{\sigma^2}\mu^2 - 2\left(\frac{n\bar{y}}{\sigma^2} + \frac{k_0}{\sigma^2}\mu_0\right)\mu + \text{const}\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(\frac{n+k_0}{\sigma^2}\right)\left(\mu - \frac{\frac{n\bar{y}}{\sigma^2} + \frac{k_0}{\sigma^2}\mu_0}{\frac{(n+k_0)}{\sigma^2}}\right)^2\right]$$

Mean: $\frac{n/\sigma^2}{\frac{(n+k_0)}{\sigma^2}}\bar{y} + \frac{\frac{k_0}{\sigma^2}}{\frac{n+k_0}{\sigma^2}}\mu_0$

$$= \frac{n}{n+k_0}\bar{y} + \frac{k_0}{n+k_0}\mu_0$$

$$= w\hat{\mu}_{MLE} + (1-w)\mu_0$$

$$P(\mu \mid y_1, \dots y_n) \sim N\left(w\bar{y} + (1-w)\mu_0, \frac{\sigma^2}{n+k_0}\right)$$

A conjugate prior for the normal likelihood

- The normal distribution is conjugate for the normal likelihood
  - Often called the “normal-normal model”
- $Y_i \sim N(\mu, \sigma^2)$ and $\mu \sim N(\mu_0, \sigma^2/\kappa_0)$ implies that the posterior distribution $p(\mu \mid y)$ is also normally distributed:

$$\mu \mid Y \sim N(\mu_n, \tau_n^2)$$

where $\mu_n = \frac{\frac{\kappa_0}{\sigma^2}\mu_0 + \frac{n}{\sigma^2}\overline{y}}{\frac{\kappa_0}{\sigma^2} + \frac{n}{\sigma^2}}$ and $\tau_n^2 = \frac{1}{\frac{\kappa_0}{\sigma^2} + \frac{n}{\sigma^2}}$

The posterior mean and pseudo-counts

$\tau^2 = \frac{\sigma^2}{K_0}$

$$\mu_n = \frac{\frac{1}{\tau^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\mu_0 + \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\bar{y}$$
$$= (1 - w)\mu_0 + w\bar{y}$$

where $w = \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}$ — Fisher Weight.

$y_i \overset{ind}{\sim} N(\mu, \sigma_i^2)$, MLE for μ?

$\hat{\mu}_{MLE} = \sum \frac{1}{\sigma_i^2} y_i \Big/ \sum 1/\sigma_i^2$

Known mean, unknown variance

- Assume we have n mean-zero normal observations with variance σ^2
- Define $d_i = (y_i - \mu)$ for notational convenience
- What is $p(\sigma^2 \mid \mu, d_1, \ldots d_n)$?

$$L\left(\sigma^2\right) \propto \left(\sigma^2\right)^{-n/2} \exp\left\{-\sum_{i=1}^{n} d_i^2 / \left(2\sigma^2\right)\right\}, \ \sigma^2 > 0$$

$$P_{\log\sigma}(\log\sigma) \propto \text{const} \qquad \sigma^2 = \exp(2\log\sigma)$$

$$P_{\sigma^2}(\sigma^2) \propto P_{\log\sigma}(\;) \left| \frac{d\log\sigma}{d\sigma^2} \right|$$

$$\frac{d\,\tfrac{1}{2}\log(\sigma^2)}{d\sigma^2} \propto \frac{1}{\sigma^2}$$

Known mean, unknown variance

- Assume we have n mean-zero normal observations with variance σ^2
- Jeffreys prior $p(\sigma^2) \propto \frac{1}{\sigma^2}$ (careful!)
- The posterior:
  $p(\sigma^2 \mid y) \propto \left(\sigma^2\right)^{-n/2-1} \exp\left\{-\sum_{i=1}^{n} d_i^2 / \left(2\sigma^2\right)\right\}$
- This distribution called an *inverse-Gamma* distribution.
  - If $X \sim \text{Gamma}(a, b)$ then $\frac{1}{X} \sim \text{Inv-Gamma}(a, b)$.

$\rightarrow$ Inv-Gam$\left(n/2, \frac{\sum d_i^2}{2}\right)$

Scaled Inv-$\chi^2\left(n, \frac{\sum d_i^2}{n}\right)$

$\chi^2_k = \sum_k z_i^2$, $z_i \sim N(0,1)$

$\sigma^2 \chi^2_k$ $\qquad$ $\chi^2_k \equiv \text{Gam}(k/2, 1/2)$

A conjugate prior distribution

- In general, the conjugate prior distribution has the form:

$$p(\sigma^2) \propto (\sigma^2)^{-k_1} e^{\frac{k_2}{\sigma^2}}$$

- Inverse-gamma: $p(y \mid a, b) = \frac{b^a}{\Gamma(a)} y^{-a-1} \exp\left(-\frac{b}{y}\right)$
  - The mean of an inverse-Gamma is b/(a-1)

Joint inference for the mean and variance

In the normal model we typically factorize the prior distribution $p(\mu, \sigma^2) = p(\mu \mid \sigma^2)p(\sigma^2)$.

Specifically:

$$\begin{aligned} \sigma^2 &\sim Inv - Gamma(\nu_0/2, \nu_0/2\sigma_0^2) \\ \mu|\sigma^2 &\sim \text{ normal } \left(\mu_0, \sigma^2/\kappa_0\right) \\ Y_1, \ldots, Y_n|\mu, \sigma &\sim \text{ i.i.d. normal } \left(\mu, \sigma^2\right) \end{aligned}$$

- ν_0 is interpreted as the prior sample size
- σ_0^2 is a prior sample variance

$$P(\mu, \sigma^2 | y_1, \dots y_n) \propto \left[\prod^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i-\mu)^2}{\sigma^2}} \right] e^{-\frac{(\mu-\mu_0)^2}{2\sigma^2/\kappa_0}} (\sigma^2)^{-\frac{\nu_0}{2}-1} e^{-\frac{\nu_0\sigma^2}{2\sigma_0^2}}$$

$\kappa_0 \to 0$ implies what about $P(\mu|\sigma^2)$

$\nu_0 \to 0$ implies what about $P(\sigma^2)$

$P(\mu|\sigma^2, y_1 \dots y_n)$ "Conditional post"

$P(\sigma^2 | y_1, \dots y_n)$ "Marginal post."

Joint inference for the mean and variance

- We write down $p(\mu, \sigma^2 \mid y_1, \ldots y_n) \propto L(\mu, \sigma^2) p(\mu, \sigma^2)$. Now what?
- Estimands of potential interest:
  - $E[\mu \mid y_1, \ldots y_n]$
  - $E[\sigma \mid y_1, \ldots y_n]$
  - $E[\sigma/\mu \mid \mid y_1, \ldots y_n]$ (coefficient of variation)

Example: midge wing length

- Modeling wing length of different specifies of midge (small, two-winged flies)
- From prior studies: mean wing length close to 1.9mm.
- Prior mean for μ is $\mu_0 = 1.9$ and Jeffreys prior for σ^2.
- Prior sample sizes: choose $\kappa_0 = 1$
- $(\bar{y}, s^2) = (1.804, 0.0169)$ are the sufficient statistics

Working with the log posterior

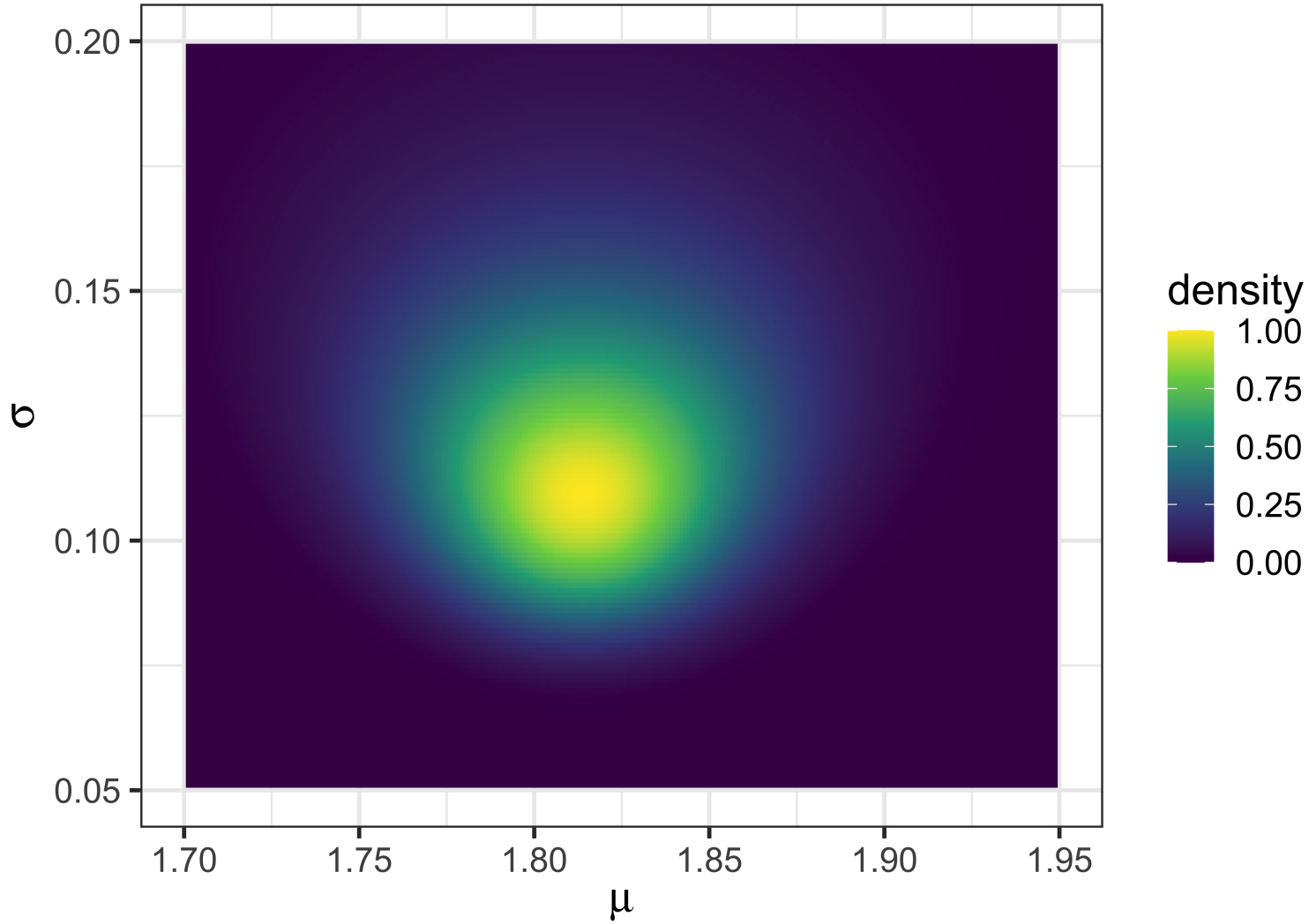
- As always, we will write down $p(\theta \mid y) \propto p(y \mid \theta)p(\theta)$
- In code, we always work with the **log-posterior** for numerical reasons
  - Mathematically it makes no difference, but computationally it is important
  - $L(\theta) \propto \prod p(y_i \mid \theta)$ is very small for moderate sample size (underflow)
  - $\ell(\theta) = \sum log(p(y_i \mid \theta))$ is numerically stable
- Monte Carlo methods only require that we can evaluate the log posterior

Grid approximation

```
log_normal_posterior <- Vectorize(function(mu, sigma) {

  ### log likelihood
  sum(dnorm(y, mu, sigma, log=TRUE)) +
  ## plus log prior
  dnorm(mu, mu0, sigma/sqrt(k0), log=TRUE) +
  -2*log(sigma)

})


post_grid <- as_tibble(
  expand.grid(seq(1.6, 2.0, by=0.001),
              seq(0.01, 0.25, by=0.001)))
colnames(post_grid) <- c("mu", "s")
```

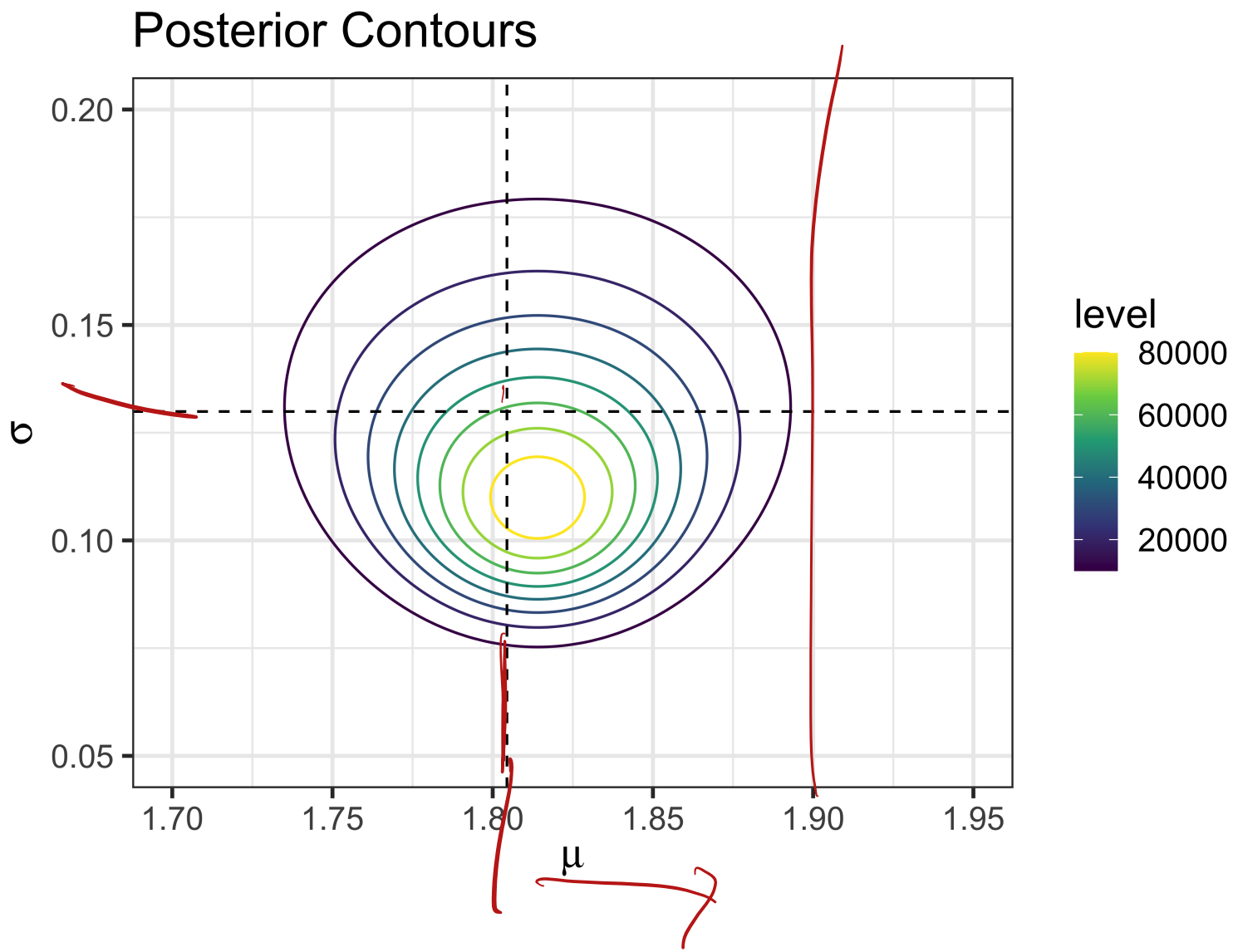
Grid approximation

```
post_grid %>%
  mutate(log_density = log_normal_posterior(mu, s)) %>%
  mutate(density = exp(log_density - max(log_density))) %>%
  ggplot() +
  geom_raster(aes(mu, s, fill=density)) +
  xlim(c(1.7, 1.95)) + ylim(c(0.05, 0.2)) +
  xlab(expression(mu)) +
  ylab(expression(sigma)) +
  theme_bw(base_size=16
           ) +
  scale_fill_continuous(type="viridis")
```


0.20
0.15
0.10
0.05
σ
1.70
1.75
1.80
1.85
1.90
1.95
μ
density
1.00
0.75
0.50
0.25
0.00

Contour Plot (Standard Deviation)

```
post_grid %>%
  mutate(density = exp(log_normal_posterior(mu, s))) %>%
  ggplot() +
  geom_contour(aes(mu, s, z=density, colour=stat(level))) +
  xlim(c(1.7, 1.95)) + ylim(c(0.05, 0.2)) +
  xlab(expression(mu)) + ylab(expression(sigma)) +
  ggtitle("Posterior Contours") +
  theme_bw(base_size=16) +
  scale_color_continuous(type="viridis") +
  geom_hline(yintercept=s, linetype="dashed") +
  geom_vline(xintercept=ybar, linetype="dashed")
```

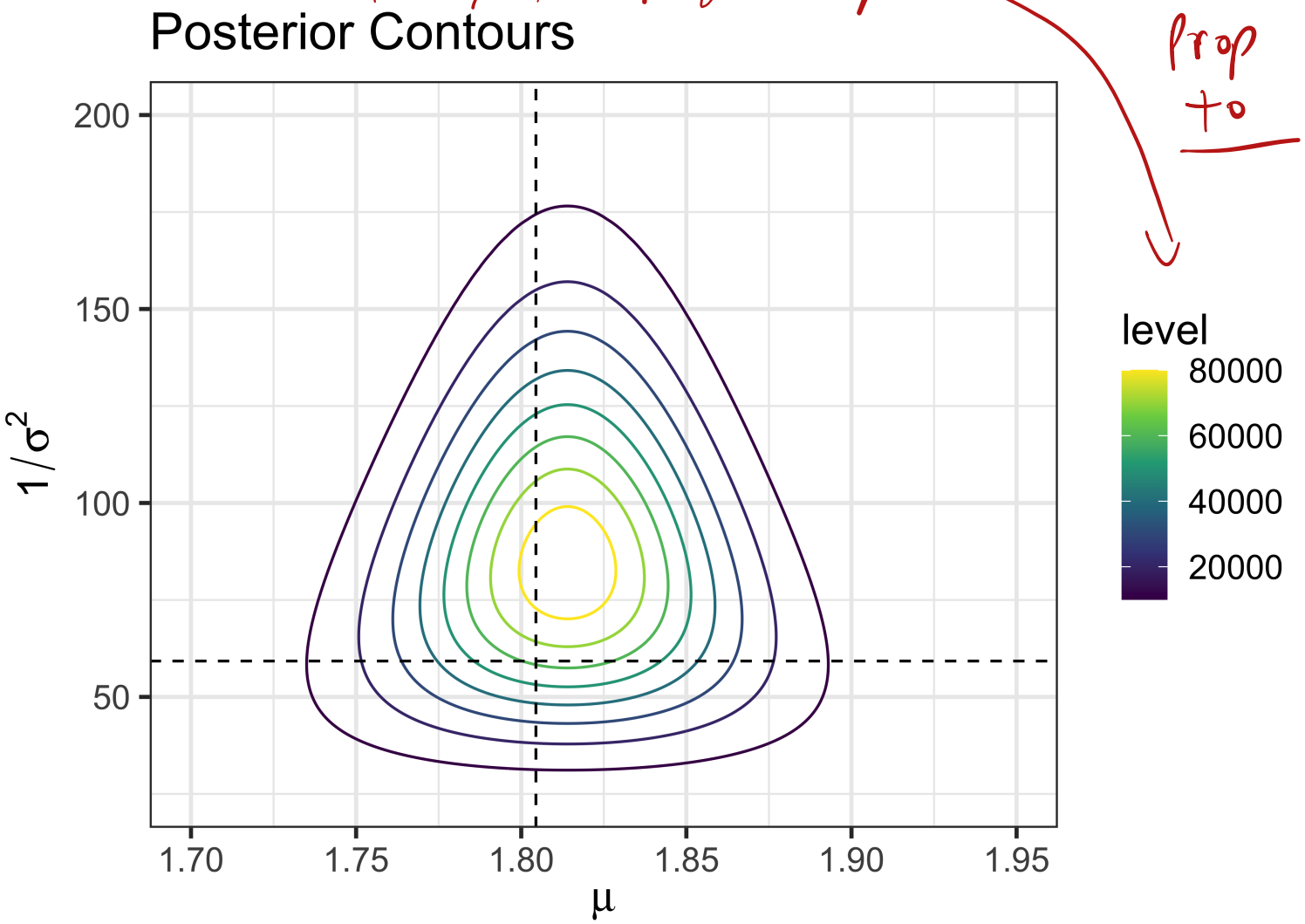

Posterior Contours
0.20
0.15
0.10
0.05
σ
1.70
1.75
1.80
1.85
1.90
1.95
μ
level
80000
60000
40000
20000

Contour Plot (Precision)

```r
post_grid$prec <- 1/post_grid$s^2

post_grid %>%
  mutate(density = exp(log_normal_posterior(mu, sqrt(1/prec)))) %>%
  ggplot() +
  geom_contour(aes(mu, prec, z=density, colour=stat(level))) +
  xlim(c(1.7, 1.95)) + ylim(c(25, 200)) +
  xlab(expression(mu)) + ylab(expression(1/sigma^2)) +
  ggtitle("Posterior Contours") +
  theme_bw(base_size=16) +
  scale_color_continuous(type="viridis") +
  geom_hline(yintercept=1/s^2, linetype="dashed") +
  geom_vline(xintercept=ybar, linetype="dashed")
```

$P(\mu, 1/\sigma^2 \mid \text{data})$

Prop to

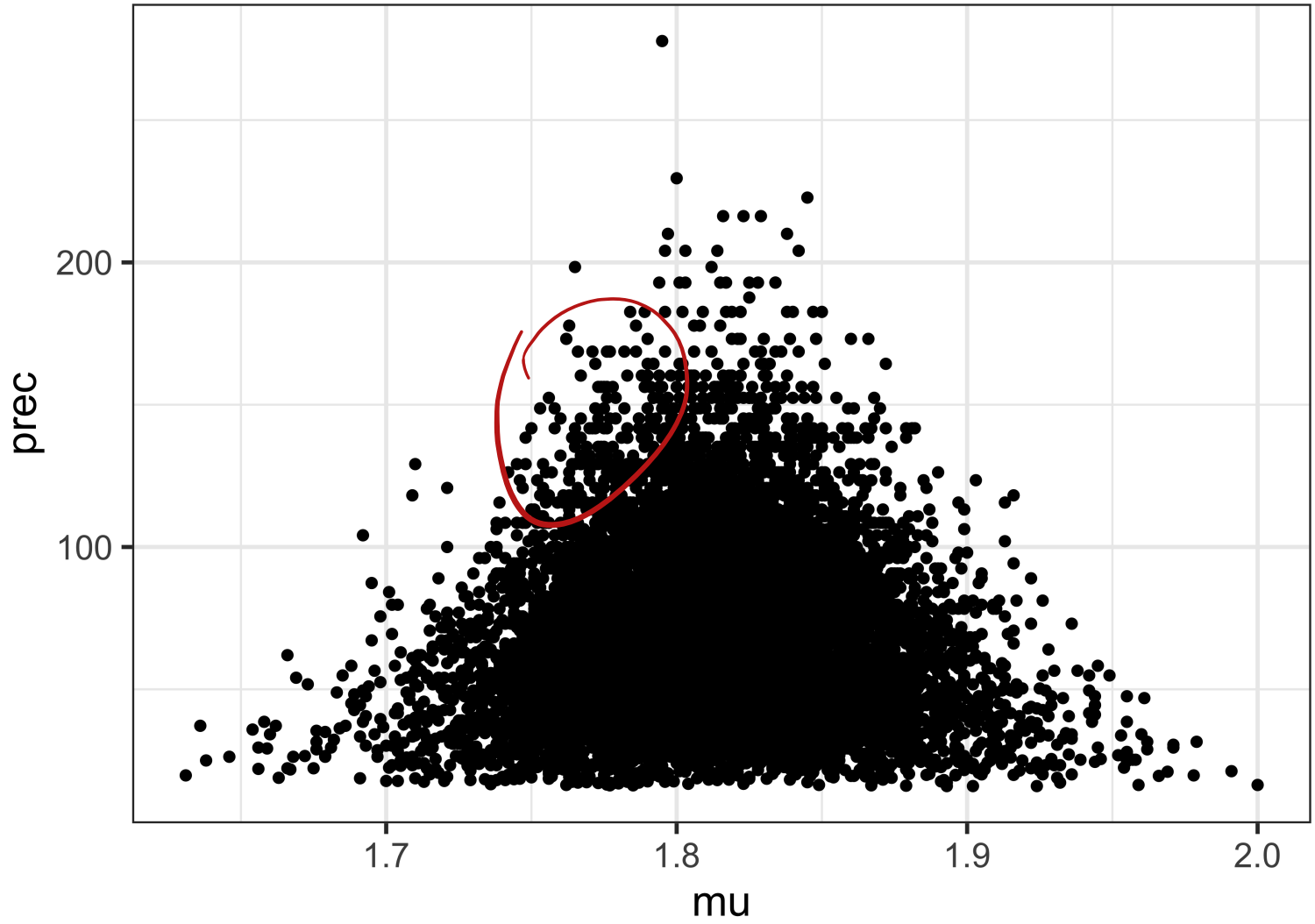


Sampling from the joint posterior

- Contour and raster plots allow us to visualize the posterior (in two dimensions)
  - Grid sampling approximates posterior samples
  - Need to know approximately where the high posterior density is (not easy)
- When we have more than 2 parameters visualization isn't feasible
- How do we summarize the posterior?
  - e.g. posterior means, posterior probabilities, intervals, etc..

Visualizing Posterior Samples

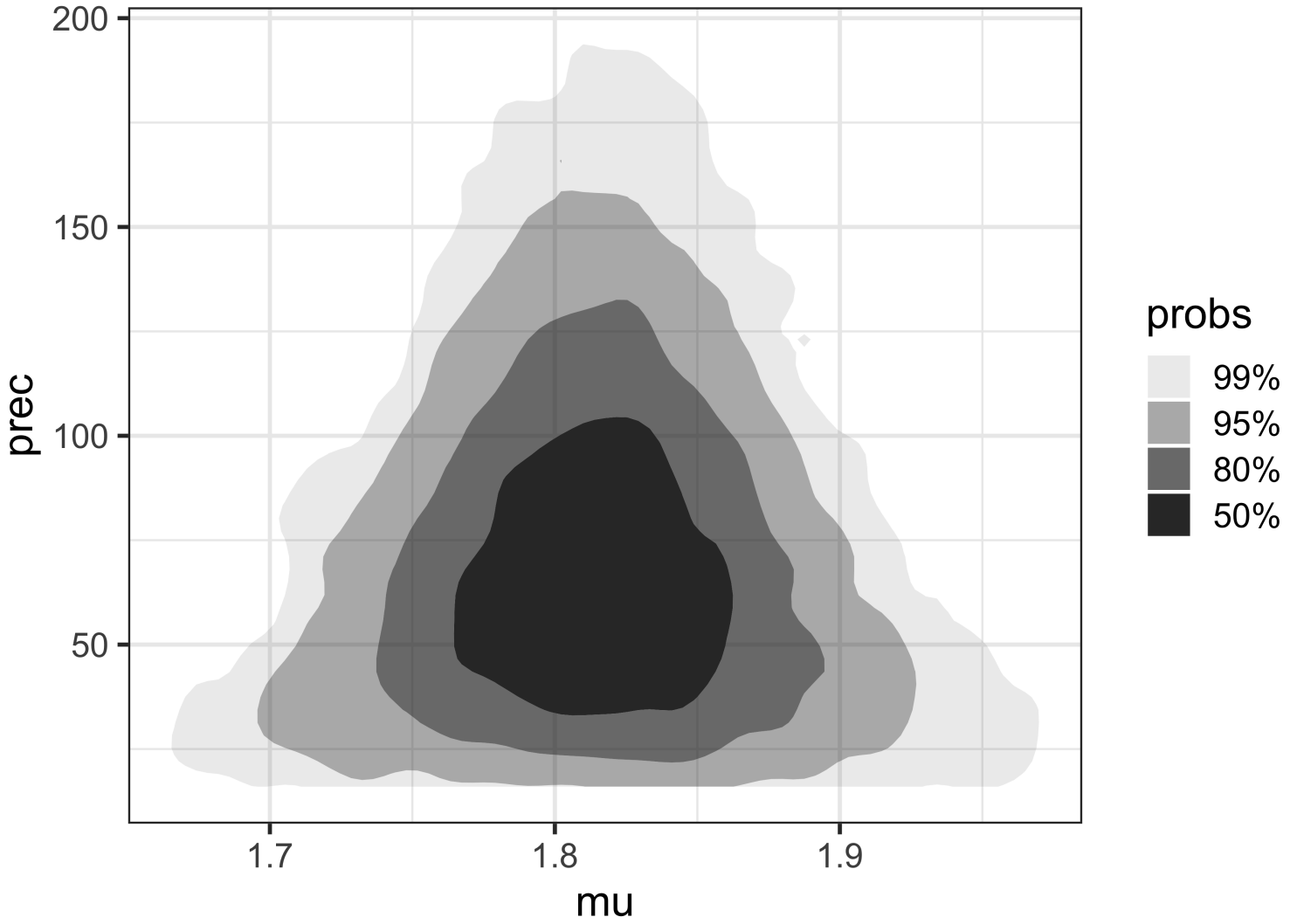
```r
post_grid %>%
  mutate(density = exp(log_normal_posterior(mu, sqrt(1/prec)))) %>%
  mutate(density = density / sum(density)) -> post_grid

sample_indices <- sample(1:nrow(post_grid), size=10000, replace=TRUE, prob=

post_grid[sample_indices, ] %>%
  mutate(prec = 1/s^2) %>%
  ggplot() +
  geom_point(aes(x=mu, y=prec)) +
  theme_bw(base_size=16)
```


200
100
prec
1.7
1.8
1.9
2.0
mu

Visualizing Posterior Samples

```
post_grid %>%
  mutate(density = exp(log_normal_posterior(mu, sqrt(1/prec)))) %>%
  mutate(density = density / sum(density)) -> post_grid

sample_indices <- sample(1:nrow(post_grid), size=10000, replace=TRUE, prob=

post_grid[sample_indices, ] %>%
  mutate(prec = 1/s^2) %>%
  ggplot() +
  ggdensity::geom_hdr(aes(x=mu, y=prec)) +
  theme_bw(base_size=16)
```

Visualizing Posterior Samples



Direct Sampling

- $p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y)p(\sigma^2 \mid y)$
- Need to know how to sample from
  $p(\sigma^2 \mid y) = \int p(\mu, \sigma^2 \mid y) d\mu$ and $p(\mu \mid \sigma^2, y)$
- With the proposed conjugate priors, this integral is tractable

$$P(\sigma^2 / y_1 \dots y_n) \sim \text{Inv-Gamma}\left(\frac{\nu_n}{2}, \frac{\nu_n \sigma_n^2}{2}\right)$$

$$\nu_n = \nu_0 + n,$$

$$\sigma_n^2 = \frac{1}{\nu_n}\left[\nu_0 \sigma_0^2 + (n-1)s^2 + \frac{k_0 n}{n+k_0}(\bar{y} - \mu_0)^2\right]$$

$$P(\mu \mid y_1, \dots y_n) = \int P(\mu \mid \sigma^2, y) P(\sigma^2 \mid y) d\sigma^2$$

Normal

$\mu = cZ, \quad Z \sim N(0,1)$

$c \sim$ Inv-Gamma

t-distn.

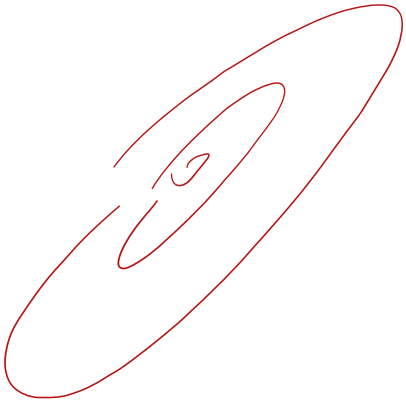
$y_p = AZ, \quad Z \sim N(0, I)$

$y \sim N(0, \Sigma)$

$\Sigma = AA^T$

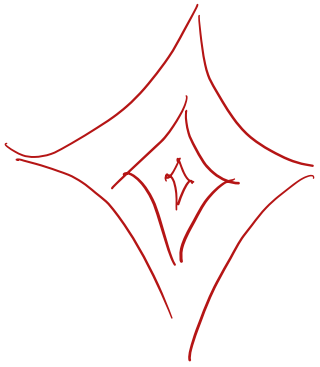
$y = \sqrt{\frac{\nu}{c}} AZ \sim MVT.$

$c \sim$ Inv-Gamma



Still elliptical

$$y_2 = \begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix} z$$



Direct Sampling

- $p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$
- Need to know how to sample from $p(\sigma^2 \mid y) = \int p(\mu, \sigma^2 \mid y) d\mu$ and $p(\mu \mid \sigma^2, y)$
- With the proposed conjugate priors, this integral is tractable

$$\left\{1/\sigma^2 \mid y_1, \ldots, y_n\right\} \sim \text{gamma}\left(\nu_n/2, \nu_n \sigma_n^2/2\right), \text{ where}$$

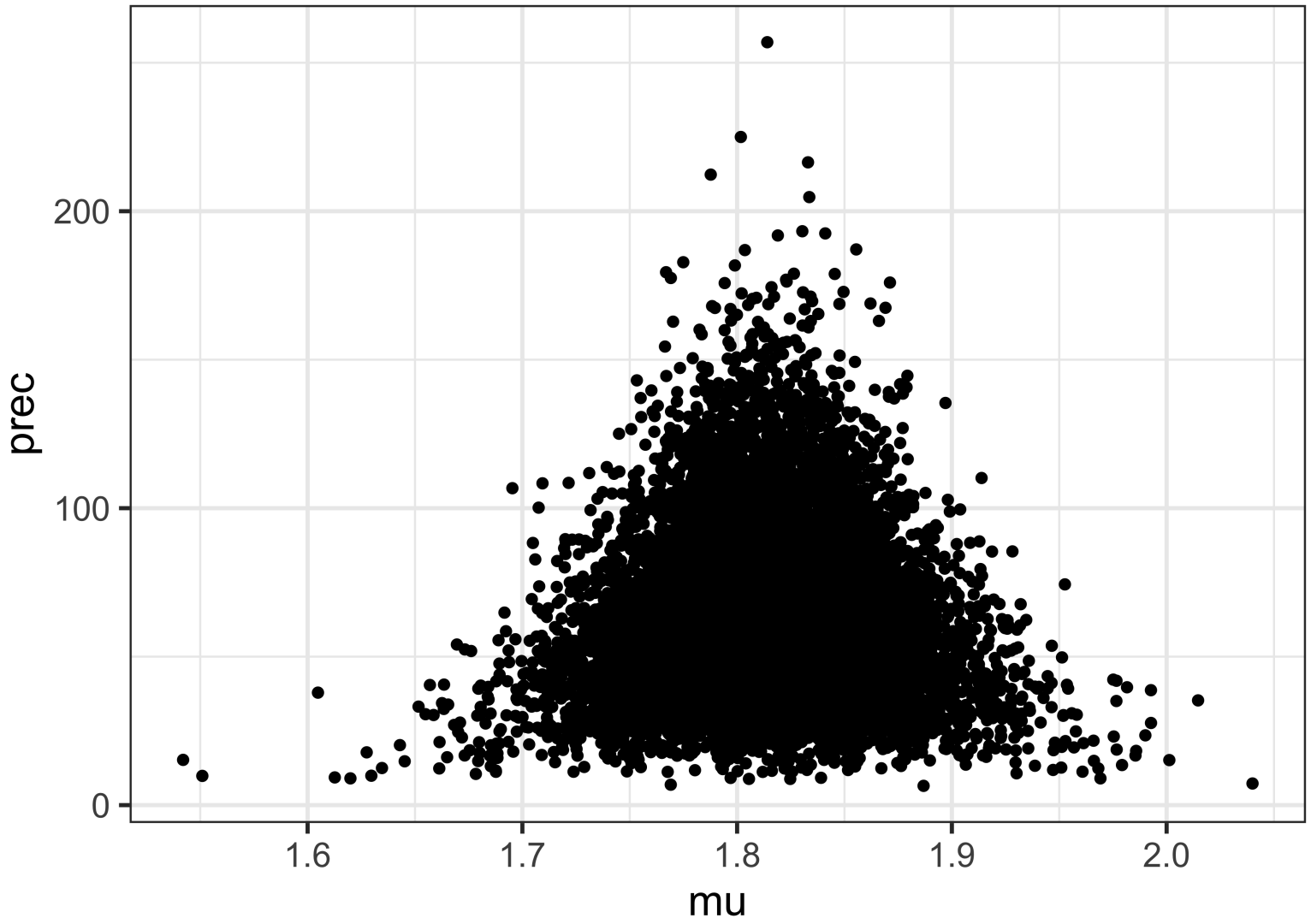
$$\nu_n = \nu_0 + n$$

$$\sigma_n^2 = \frac{1}{\nu_n}\left[\nu_0 \sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_n}(\bar{y} - \mu_0\right.$$

Direct Sampling

```r
## posterior parameters
kn <- k0 + n
nun <- nu0 + n
mun <- (k0 * mu0 + n * ybar) / kn
s2n <- (nu0*s20 + (n-1)*s2  + k0*n / kn * (ybar - mu0)^2) / nun

nsamps <- 10000
prec_samps <- rgamma(nsamps, nun/2, nun*s2n/2)
mu_samps <- rnorm(nsamps, mun, sqrt((1/prec_samps) / kn))

tibble(mu=mu_samps, prec=prec_samps) %>%
  ggplot() +
  geom_point(aes(x=mu, y=prec)) +
  #ggdensity::geom_hdr(aes(x=mu, y=prec)) +
  theme_bw(base_size=16)
```

Direct Sampling



The Multivariate Normal Distribution

$$Y_{p\times 1} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix} \sim N_p(\mu, \Sigma)$$

$$P(Y=y \mid \mu, \Sigma) = (2\pi)^{-p/2} |\Sigma|^{-1/2} \exp\left(-\tfrac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right)$$

$\mu \in \mathbb{R}^p$

$\Sigma_{p\times p} \in S_p^+$ cone of sym. pos. def matrices.

$a'\Sigma a > 0$

$p(p+1)/2$ free parameters

Assume Σ is known.

$$P(\mu \mid \Sigma, y_1, \dots y_n)$$

$$\mu \sim N_p(\mu_0, \Lambda_0)$$

$$L(\mu) \propto \prod^{n} (2\pi)^{-p/2} |\Sigma|^{-1/2} \exp\left[-\tfrac{1}{2}(y_i-\mu)'\Sigma^{-1}(y_i-\mu)\right]$$

$$\propto \exp(-1/2\,\mu' A \mu + \mu' b)$$

$$A = n\Sigma^{-1}, \quad b = n\Sigma^{-1}\bar{y}$$

$$P(\mu) \propto \exp(-\mu' \Lambda_0^{-1} \mu + \mu' \Lambda_0^{-1} \mu_0)$$

$$P(\mu \mid y_1, \dots y_n) \propto \exp(-\mu' A_n \mu + \mu'(b + \Lambda_0^{-1}\mu_0))$$

$$A_n = \Sigma^{-1} + \Lambda_0^{-1}, \quad n\Sigma^{-1}\bar{y} + \Lambda_0^{-1}\mu_0$$

$$\sim N(\mu_n, \Sigma_n)$$

$$\Sigma_n = (n\Sigma^{-1} + \Lambda_0^{-1})^{-1}$$

$$\mu_n = (n\Sigma^{-1} + \Lambda_0^{-1})^{-1}(n\Sigma^{-1}\bar{y} + \Lambda_0^{-1}\mu_0)$$

Σ unknown, $\mu = 0$

$y_1 \dots y_n \sim N(0, \Sigma)$

$$S = YY' = \begin{pmatrix} \sum_i y_{i1}^2 & \cdots & \left(\sum_{i=1}^n y_{ij} y_{ik}\right) \\ & \ddots & \\ & & \sum y_{ip}^2 \end{pmatrix}$$

j, k — jth row, kth col.

"sum of squares matrix"

$$S \sim \text{Wishart}(n, \Sigma)$$

$$P(S) \sim |S|^{\frac{-(n-p-1)}{2}} e^{-1/2\,\text{tr}(\Sigma^{-1}S)}$$

$n > p$, S is pos. def w.p. 1.

$$E[S] = n\Sigma, \quad \frac{yy^T}{n} \to \Sigma$$

$$X \sim \text{Wish.}, \quad X^{-1} \sim \text{Inv-wish.}$$

Inv-wish is conjugate for Σ

$$L(\Sigma) \propto |\Sigma|^{-n/2} \exp\left(-\tfrac{1}{2}\underbrace{\sum_{i=1}^{n} y_i' \Sigma^{-1} y_i}_{\text{scalar}}\right)$$

$\text{tr}(ABC) = \text{tr}(BCA)$

$\text{tr}(y'\Sigma^{-1}y) = \text{tr}(\Sigma^{-1}yy')$

$$\propto |\Sigma|^{-n/2} \exp\left(-\tfrac{1}{2}\text{tr}(\Sigma^{-1}S)\right)$$

$$\Sigma \sim IW(\nu, \Lambda_0) \to |\Sigma|^{-\frac{\nu+p+1}{2}} e^{-1/2\,\text{tr}(\Sigma^{-1}\Lambda_0)}$$

$$P(\Sigma \mid y_1, \dots y_n) \sim IW(n+\nu, S+\Lambda_0)$$

$$\text{Jeffreys} \propto |\Sigma|^{-(p+1)/2}$$

Dirichlet-Multinomial

Metagenomics example

- Metagenomics is the study of genetic material recovered directly from environmental samples
- Map counts of genetic material to counts of microbial species
- Assume species are sampled with replacement
  - Observed sample is a multinomial distribution
- Total counts isn’t meaningful (hard to control how much total sample)
- Relative counts are meaningful

Multinomial Density

- Let $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ with $\sum_i = 1^k \theta_i = 1$
- If $Y = (Y_1, \ldots, Y_K) \sim Mult(n, \theta)$, then:
  - $Y_i \sim Bin(n, \theta_i)$
  - $Y_i + Y_j \sim Bin(n, \theta_i + \theta_j)$
- What is $\hat{\theta}_{MLE}$?

Dirichlet Distribution