Lecture 2: One Parameter Models

Professor Alexander Franks

Announcements

- Reading: Chapter 2 (these lecture notes)
- Chapter3 (next lecture notes)
- Homework 1 out.
  - due 2 weeks
- Thursdays 2-3, zoom.

$$P(\theta \mid y) \underset{\propto}{=} \frac{P(y \mid \theta)\, P(\theta)}{\cancel{P(y)}}$$

Data Generate Process (DGP)

"likelihood"

prior

Cromwell's Rule

The use of priors placing a probability of 0 or 1 on events should be avoided except where those events are excluded by logical impossibility.

> I beseech you, in the bowels of Christ, think it possible that you may be mistaken.
> — Oliver Cromwell

If a prior places probabilities of 0 or 1 on an event, then no amount of data can update that prior.

Cromwell's Rule

> Leave a little probability for the moon being made of green cheese; it can be as small as 1 in a million, but have it there since otherwise an army of astronauts returning with samples of the said cheese will leave you unmoved.
> — Dennis Lindley (1991)

If $p(\theta = a) = 0$ for a value of a, then the posterior distribution is always zero, regardless of what the data says

$$p(\theta = a|y) \propto p(y|\theta = a)p(\theta = a) = 0$$

Example: estimating shooting skill in basketball

- On November 18, 2017, an NBA basketball player, Robert Covington, had made 49 out of 100 three point shot attempts.
- At that time, his three point field goal percentage, 0.49, was the best in the league and would have ranked in the top ten all time
- How can we estimate his true shooting skill?
  - Think of "true shooting skill" as the fraction he would make if he took infinitely many shots

$Y \sim Bin(100, \theta)$

- independence
- fatigue
- difficulty

) identically distr.

Example: estimating shooting skill in basketball

- Assume every shot is independent (reasonable) and identically distributed (less reasonble?)
- Let $Y \sim \mathrm{Bin}(n, \theta)$ where θ corresponds to his true skill
- Frequentist inference tells us that the maximum likelihood estimate is simply $\frac{y}{n} = 49/100 = 0.49$
- What would our estimates be if we use Bayesian inference?
  - If our prior reflects "complete ignorance" about basketball?
  - What if we want to incorporate prior domain knowledge?

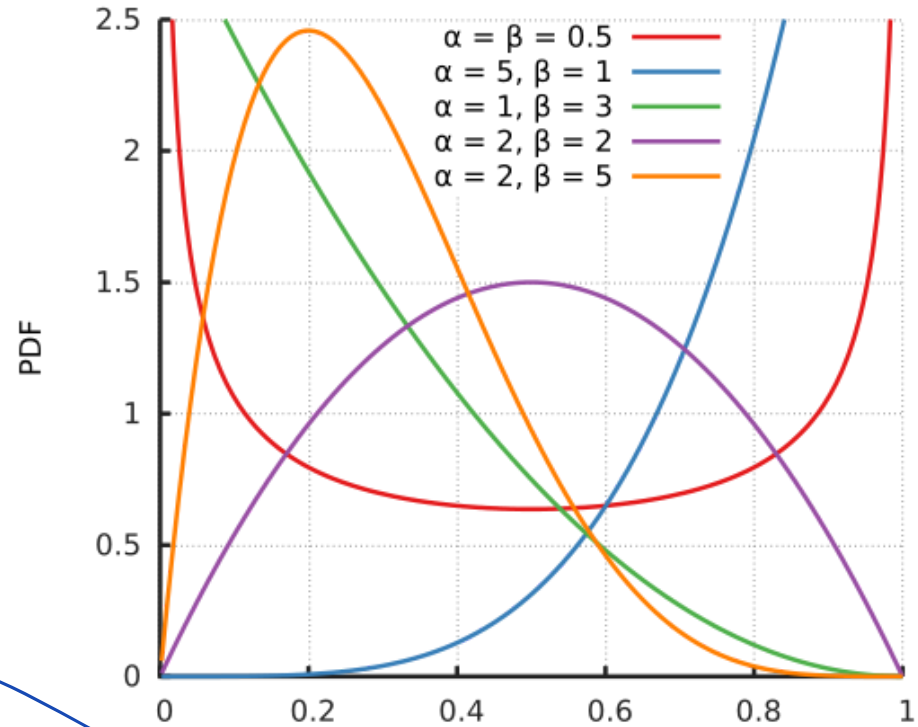
The Binomial Model

- The uniform prior: $p(\theta) = \mathrm{Unif}(0, 1) = \mathbf{1}\{\theta \in [0, 1]\}$
  - A "non-informative" prior
- Posterior: $p(\theta \mid y) \propto \underbrace{\theta^y (1-\theta)^{n-y}}_{\text{likelihood}} \times \underbrace{\mathbf{1}\{\theta \in [0, 1]\}}_{\text{prior}}$
- The above posterior density is is a density over θ.

Beta Density

$\mathrm{Beta}(y+1, n-y+1)$

Beta Distributions



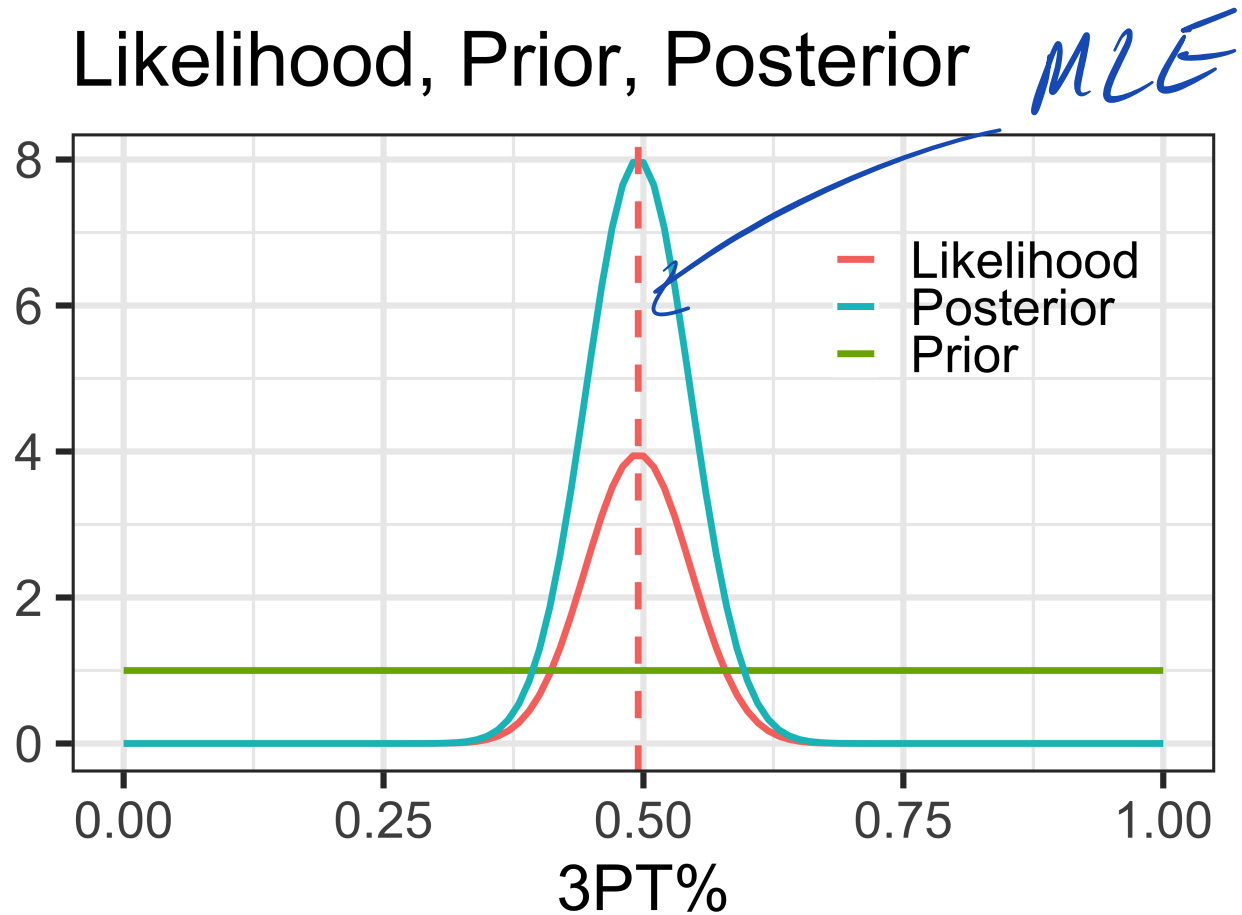
$\alpha = y+1$

$\beta = n-y+1$

normalizing const.

$$\mathrm{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

Example: estimating shooting skill in basketball



Summarizing Posterior Results

Point estimates

- $E[\theta|y] = \int \theta P(\theta|y) d\theta$ "post. mean"
- $\text{argmax } P(\theta|y)$ MAP estimate
- Post. Var: $Var(\theta|y)$
- Credible Intervals: $P(\theta \in R(y)) = 95\%$ for any region

(not "confidence")

Summarizing Posterior Results

- $\text{Beta}(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$
- The mean of a $\text{Beta}(\alpha, \beta)$ distribution r.v. $\frac{\alpha}{\alpha+\beta}$
- The mode of a $\text{Beta}(\alpha, \beta)$ distributed r.v. is $\frac{\alpha-1}{\alpha+\beta-2}$
- The variance of a $\text{Beta}(\alpha, \beta)$ r.v. is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- In R: `dbeta`, `rbeta`, `pbeta`, `qbeta`

The Bayesian Advantage

- Particularly effective with small samples
  - With a true shooting skill of 0.35, there is about a 1/4 chance of making 5 or more out of the first 10 shots
- Ability to incorporate real prior knowledge
- Can easily “share information” across related observations

Informative prior distributions

- At that point in November, Covington's three point field goal percentage, 0.49, was the best in the league and would have ranked in the top ten all time
- It seems very unlikely that this level of skill would continue for an entire season of play.
- A uniform prior distribution doesn't reflect our known beliefs. We need to choose a more *informative* prior distribution (specific players)

- Other players shooting ability.
- R.C's previous play.

Informative prior distributions

- When $p(\theta) \sim U(0,1)$ then the posterior was a Beta distribution
- Remember: the binomial likelihood is $L(\theta) \propto \theta^y (1-\theta)^{n-y}$
- Choose a prior with a similar looking form: $p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$ (Beta(α, β))

$$P(\theta \mid y) \propto \theta^y (1-\theta)^{n-y} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

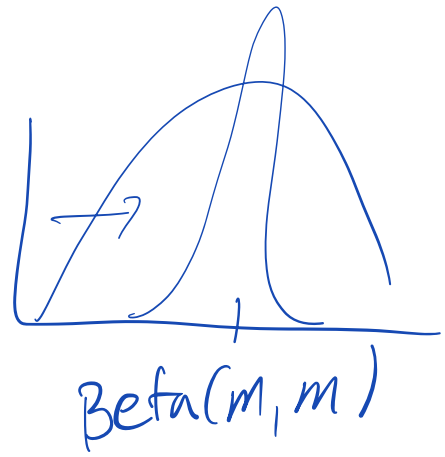
$$\propto \theta^{y+\alpha-1} (1-\theta)^{n-y+\beta-1}$$

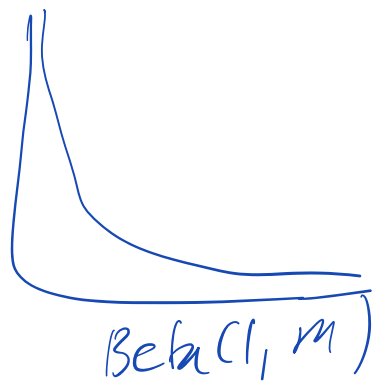
Beta($y+\alpha$, $n-y+\beta$)

"pseudo-counts" interpretation:

α: "prior" makes

β: "prior" misses.





Conjugate Prior Distributions

Definition: A class of prior distributions, $\mathcal{P}$ for θ is called *conjugate* for a sampling model $p(Y|\theta)$ if

$$p(\theta) \in \mathcal{P} \implies p(\theta|y) \in \mathcal{P}$$

e.g. Beta prior $\rightarrow$ Beta post.

- The prior distribution and the posterior distribution are in the same family
- Conjugate priors are very convenient because they make calculations easy
- The parameters for conjugate prior distribution have nice interpretations

Pseudo-Counts Interpretation

- Observe y successes, $n - y$ failures Binomial
- If $p(\theta) \sim \text{Beta}(\alpha, \beta)$ then
  $p(\theta \mid y) = \text{Beta}(y + \alpha, n - y + \beta)$
- What is $E[\theta \mid y]$? (Posterior Mean)

$\tilde{\alpha} = \alpha + y$

$\tilde{\beta} = n - y + \beta.$

$$\frac{\tilde{\alpha}}{\tilde{\alpha} + \tilde{\beta}}$$

$$E[\theta \mid y] = \frac{y + \alpha}{n + \alpha + \beta}$$

$$= \cancel{\frac{y}{n+\alpha+\beta}\left(\frac{n}{n}\right)} + \cancel{\frac{\alpha}{n+\alpha+\beta}\left(\frac{\alpha+\beta}{\alpha+\beta}\right)}$$

$$= \left(\frac{n}{n+\alpha+\beta}\right)\frac{y}{n} + \left(\frac{\alpha+\beta}{n+\alpha+\beta}\right)\frac{\alpha}{\alpha+\beta}$$

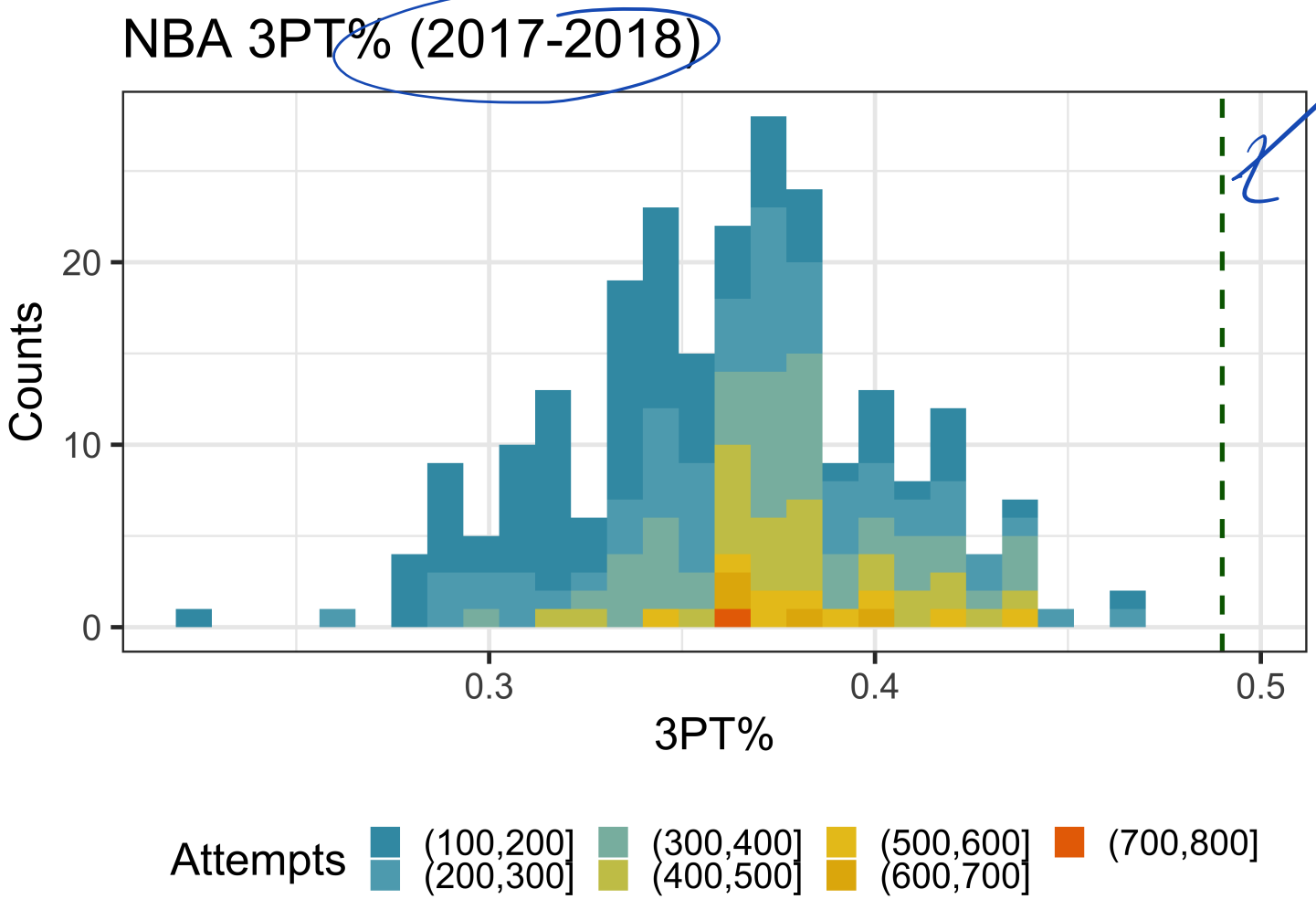
$$= w\,\hat{\theta}_{MLE} + (1-w)\,\hat{\theta}_{\substack{\text{prior}\\ \text{mean}}}$$

$\alpha+\beta$: "pseudo" - sample size.

Example: estimating shooting skill in basketball

- On November 18, 2017, an NBA basketball player, Robert Covington, had made 49 out of 100 three point shot attempts.
- At that time, his three point field goal percentage, 0.49, was the best in the league and would have ranked in the 10 ten all time
- Prior knowledge tells us it is unlikely this will continue!
- How can we use Bayesian inference to better estimate his true skill?

Three point shooting in 2017-2018



MLE for R.C. (1st month)

Regression Toward the Mean

What is a reasonable model?

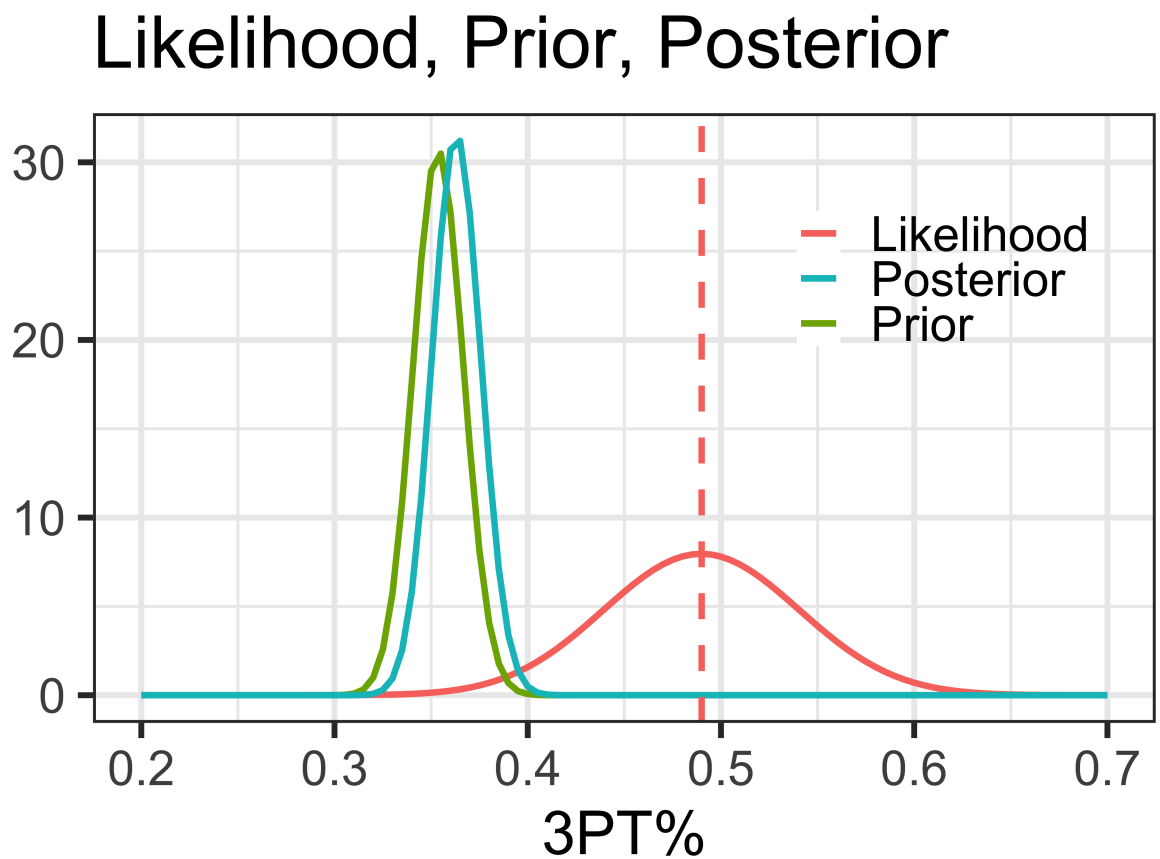
- If we believe that his skill doesn't change much year to year, use past data to inform prior
- In his first 4 seasons combined Robert Covington made a total of 478 out of 1351 three point shots (0.35%, just below average).
- Choose a Beta(478, 873) prior (pseudo-count interpretation)

θ is constant in time!

θ_{career}. θ_{2016}

Robert Covington 2017-2018 estimates

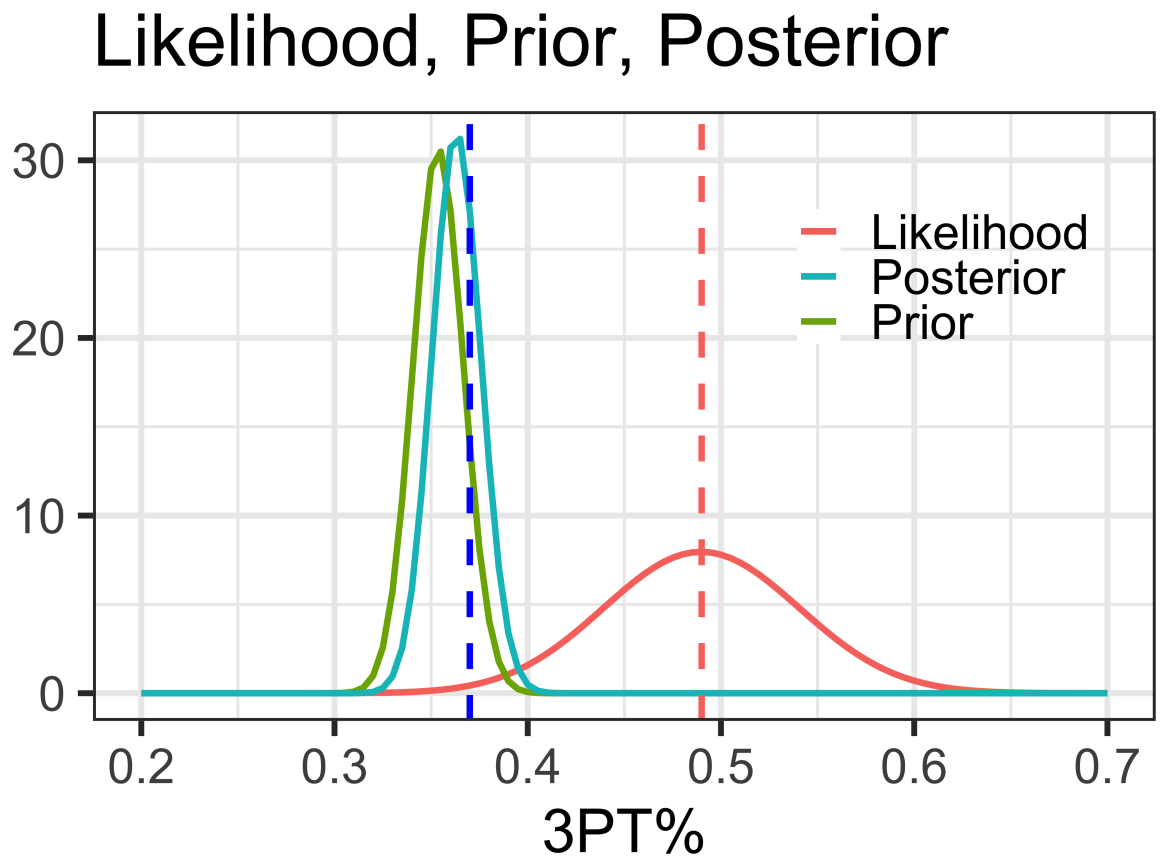
After 100 shots Robert Covington's 3PT% was 0.49



MLE = 0.49, posterior mean = 0.36

How did we do?

Robert Covington's end of season 3PT% was 0.37



MLE = 0.49, posterior mean = 0.36

The Poisson Distribution

- A useful model for count data
- Events occur independently at some rate λ
- Mean = variance = λ.
- Example applications:
  - Epidemiology (disease incidence)
  - Astronomy (e.g. the number of meteorites entering the solar system each year)
  - The number of patients entering the emergency room
  - The number of times a neuron in the brain “fires”

Poisson model with exposure

$$\frac{\lambda^y e^{-\lambda}}{y!}$$

- Assume y_i is Poisson with rate λ and exposure ν_i:

$$p(y_i \mid \nu_i \lambda) = (\nu_i \lambda)^y e^{\nu_i \lambda} / y_i!$$

- How many cars do we expect to pass an intersection in one hour? How many in two hours?
  - If we model the distribution as Poisson, we expect twice as many in two hours as in one hours.

Conjugate Prior for the Poisson Distribution

$$Y_i \overset{indep}{\sim} \text{Pois}(v_i \lambda)$$

$$L(\lambda) \propto \prod_{i=1}^{n} \frac{e^{-v_i \lambda} \lambda^{y_i}}{\cancel{y_i!}} \propto e^{-(\sum v_i)\lambda} \lambda^{\sum y_i}$$

$$P(\lambda) \propto \lambda^{\alpha-1} e^{-\beta\lambda} \longrightarrow \text{Gamma}(a, b)$$

$$P(\lambda \mid y_1, \ldots y_n) \propto \lambda^{\sum y_i + \alpha - 1} e^{-(\sum v_i + \beta)\lambda}$$

$$\text{Gamma}\left(\sum y_i + \alpha,\ \sum v_i + \beta\right)$$

The Gamma distribution

We use the shape-rate parameterization of the Gamma

- $\text{Gamma}(a, b) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$
- $E[\lambda] = a/b$ and $\text{Var}[\lambda] = a/b^2$
- $\text{mode}[\lambda] = (a-1)/b$ if $a > 1, 0$ otherwise
- In R: `dgamma`, `rgamma`, `pgamma`, `qgamma`

$\frac{\text{total counts}}{\text{total time}}$

The Gamma distribution

Mean 1

Mean 2

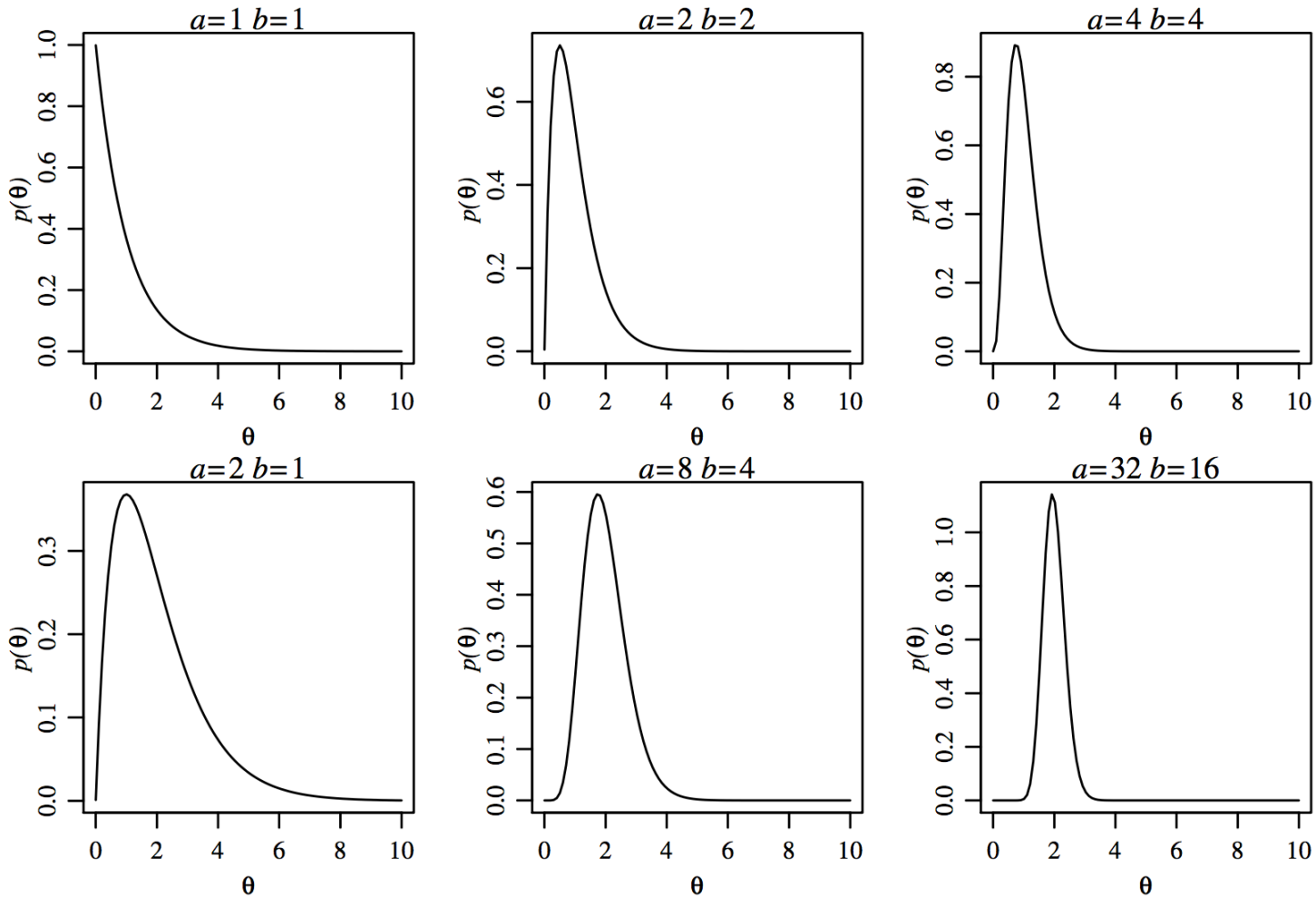


Fig. 3.8. Gamma densities.

The posterior mean

$$\lambda \mid y_1, \dots, y_n \sim \text{Gam}\left(\sum y_i + \alpha, \sum v_i + \beta\right)$$

$$E[\lambda \mid y_1, \dots, y_n] = \frac{\sum y_i + \alpha}{\sum v_i + \beta}$$

$$= \left(\frac{\sum v_i}{\sum v_i}\right)\frac{\sum y_i}{\sum v_i + \beta} + \left(\frac{\beta}{\beta}\right)\frac{\alpha}{\sum v_i + \beta}$$

$$= \frac{\sum v_i}{\sum v_i + \beta}\left(\sum y_i / \sum v_i\right) + \frac{\beta}{\sum v_i + \beta}\left(\frac{\alpha}{\beta}\right)$$

Conjugate Priors and Exponential Families

- An k-dimensional exponential family distribution can be written as $p(y \mid \theta) = h(y)exp(\eta(\theta)T(y) - A(\theta))$
- For any k-parameter exponential family there exists a (k + 1)-parameter conjugate prior.

natural parameter

- log partition

$$L(\theta) \propto \exp\left[\sum^{k} \eta_i(\theta) T(y) - A(\theta)\right]$$

sufficient stat.

$$P_{\alpha}(\theta) \propto$$

$$\exp\left[\sum^{k} \eta_i(\theta)\alpha_i - \alpha_{k+1} A(\theta)\right]$$

$$P(\theta \mid y) \propto \exp\left[\sum \eta_i(\theta)\left[\alpha_i + T_i(y)\right] - (1+\alpha_{k+1})A(\theta)\right]$$

Uncertainty Quantification

Posterior Credible Intervals

- Frequentist interval: $Pr(l(Y) < \theta < u(Y) \mid \theta) = 0.95$
  - Probability that the interval will cover the true value *before* the data are observed.
  - Interval is random since Y is random
- **Bayesian Interval**: $Pr(l(y) < \theta < u(y) \mid Y = y) = 0.95$
  - Information about the the true value of θ *after* observeing $Y = y$.
  - θ is random (because we include a prior), y is observed so interval is non-random.

Posterior Credible Intervals (Quantile-based)

- The easiest way to obtain a credible interval is to use the quantiles of the posterior distribution.

If we want $100 \times (1 - \alpha)$ interval, we find numbers $\theta_{\alpha/2}$ and $\theta_{1-\alpha/2}$ such that:

1. $p(\theta < \theta_{\alpha/2} \mid Y = y) = \alpha/2$
2. $p(\theta > \theta_{1-\alpha/2} \mid Y = y) = \alpha/2$

$$p(\theta \in [\theta_{\alpha/2}, \theta_{1-\alpha/2}] \mid Y = y) = 1 - \alpha$$

Interval for shooting skill in basketball

- The posterior distribution for Covington's shooting percentage is a

$$\text{Beta}(49 + 478, 50 + 873) = \text{Beta}(528, 924)$$

- For a 95% *credible* interval, $\alpha = 0.05$
  - Lower endpoint: `qbeta(0.025, 528, 924)`
  - Upper endpoint: `qbeta(0.975, 528, 924)`
  - $[\theta_{\alpha/2}, \theta_{1-\alpha/2}]$ = [0.34, 0.39]

Interval for shooting skill in basketball

- Bayes credible interval: $\left[\theta_{\alpha/2}, \theta_{1-\alpha/2}\right]$ = [0.34, 0.39]
- Frequentist *confidence* interval: [0.39, 0.59]
- End-of-season percentage was 0.37
- Credible intervals and confidence intervals have different meanings!

Highest Posterior Density (HPD) region

Definition: (HPD region) A $100 \times (1-\alpha)$ HPD region consists of a subset of the parameter space, $R(y) \in \Theta$ such that

1. $\Pr(\theta \in R(y)|Y = y) = 1 - \alpha$
   - The probability that θ is in the HPD region is $1 - \alpha$
2. $\text{If } \theta_a \in R(y), \text{ and } \theta_b \notin R(y), \text{ then } p\left(\theta_a|Y = y\right) > p\left(\theta_b|Y$
   - All points in an HPD region have a higher posterior density tha points out- side the region.

The HPD region is the *smallest* region with probability $(1-\alpha)$

Frequentist behavior of Bayesian intervals

- Bayesian credible intervals usually won't have exactly correct frequentist coverage
- If our prior was well-calibrated and the sampling model was correct, we'd have well-calibrated credible intervals
- And: asymptotically, a central posterior interval will cover the true value 95% of the time under repeated sampling

Frequentist-Bayes Unification

Under regularity conditions:

$$p(\theta \mid y_1, \ldots, y_n) \approx \mathrm{N}\left(\hat{\theta}, [I(\hat{\theta})]^{-1}\right)$$

Classical result:

$$\hat{\theta} \approx N(\theta, [I(\theta)]^{-1})$$

Sequential Bayesian Updating

$$\underbrace{p(\theta)}_{\text{prior}} \rightarrow \underbrace{p\left(\theta \mid y_1\right)}_{\text{reveal the first observation}} \rightarrow \underbrace{p\left(\theta \mid y_1, y_2\right)}_{\text{reveal the second observation}} \cdots$$

$$\rightarrow \cdots \rightarrow \underbrace{p\left(\theta \mid y_1, \ldots, y_n\right)}_{\text{reveal all } n \text{ observations}}$$

When data are i.i.d., final posterior is the same, regardless of whether we analyze data sequentially or as a single batch.

Improper prior distributions

- For the Beta distribution we chose a uniform prior (e.g. $p(\theta) \propto \text{const}$). This was ok because
  - $\int_0^1 p(\theta) d\theta = \text{const} < \infty$
  - We say this prior distribution is *proper* because it is integrable
- For the Poisson distribution, try the same thing: $p(\lambda) \propto \text{const}$
  - $\int_0^\infty p(\lambda) d\lambda = \infty$
  - In this case we say $p(\lambda)$ is an *improper* prior

Improper prior distributions

- Sometimes there is an absence of precise prior information
- The prior distribution does not have to be proper but the posterior does!
  - A proper distribution is one with an integrable density
  - If you use an improper prior distribution, you need to check that the posterior distribution is also proper

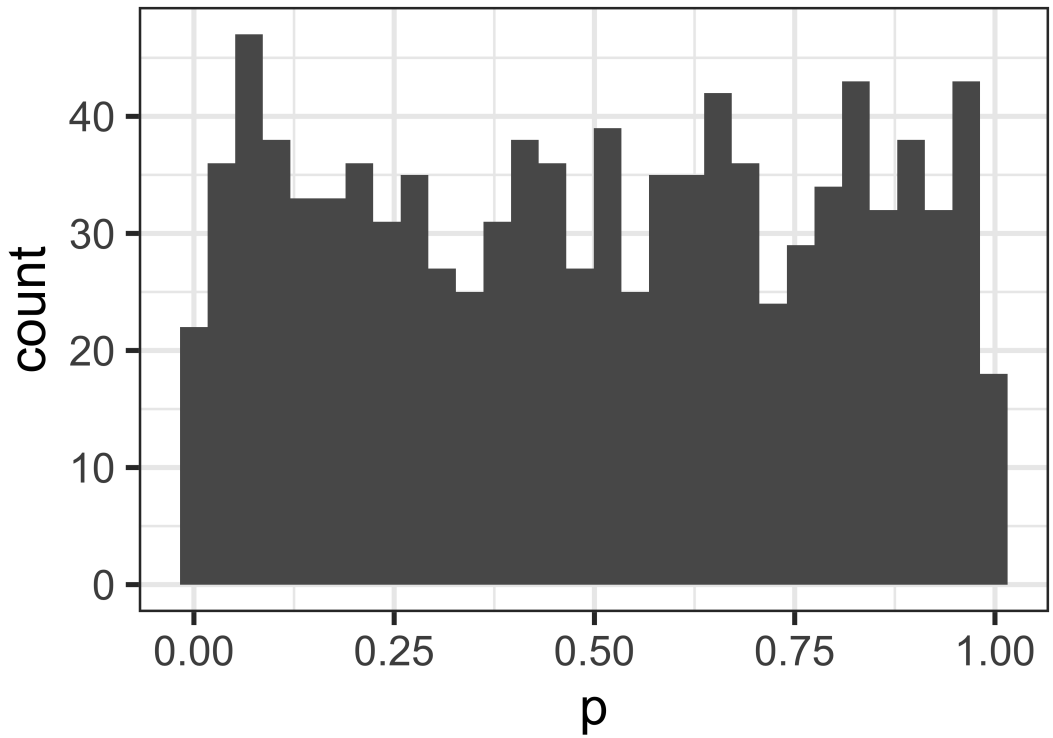
Objective Bayes

- Also called “default”, “reference”, “non-informative” prior distributions
- Laplace’s principle of insufficient reason
- Principle of maximum entropy (MAXENT).
- Matching prior distributions
  - Find prior distributions which lead to posterior intervals with approximate frequentist coverage
- Invariant priors

Laplace's principle of insufficient reason

Uniform distribution for p

```
p <- runif(1000)
tibble(p=p) %>% ggplot() +
  geom_histogram(aes(x=p), bins=30) +
  theme_bw(base_size=24)
```



Laplace's principle of insufficient reason

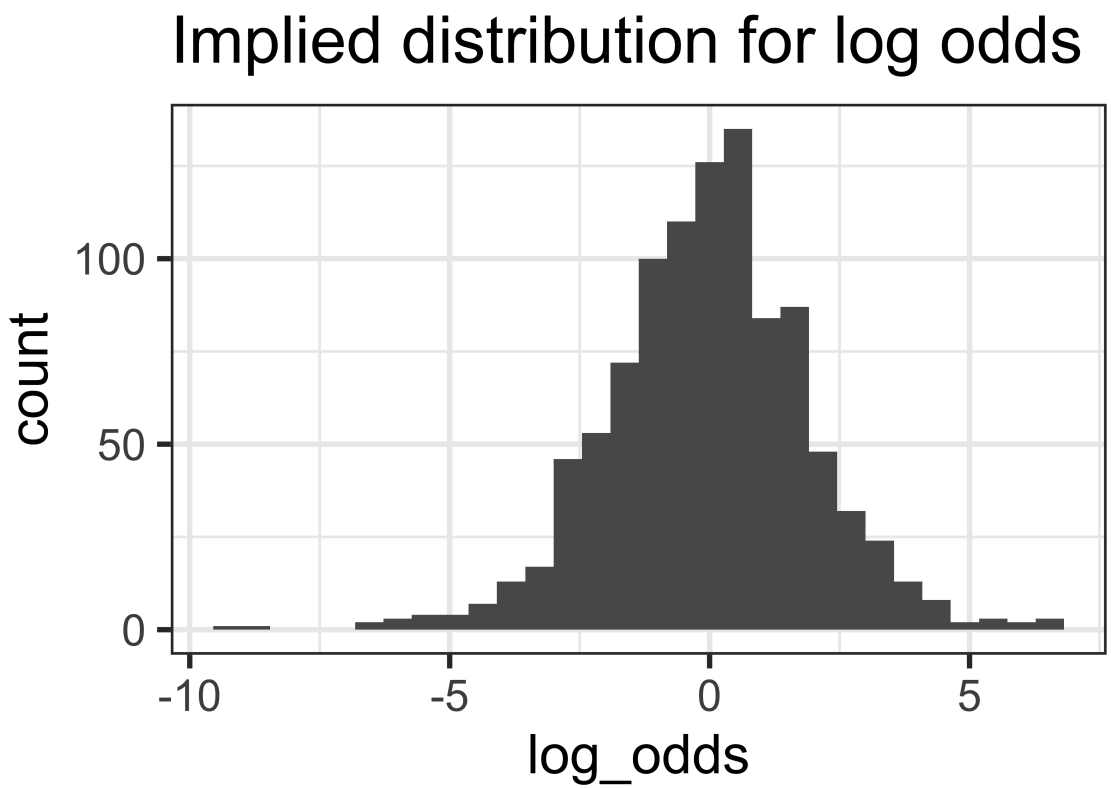
- Assume that $Y \sim \mathrm{Bin}(n, \theta)$ but that we're most interested the log odds:

$$\gamma = \log \mathrm{odds}(\theta) = \log \frac{\theta}{1-\theta}$$

- What prior should we use if we want to be "noninformative"?

Difficulties with non-informative priors

```
log_odds <- log(p/(1-p))
tibble(log_odds=log_odds) %>% ggplot() +
  geom_histogram(aes(x=log_odds)) +
  theme_bw(base_size=24) +
  ggtitle("Implied distribution for log odds")
```



Method of transformations

Assume the prior density, $p(\theta)$. What is the implied prior density for the transformed parameter, $\gamma = g(\theta)$?

1. Find the inverse, $\theta = g^{-1}(\gamma)$
2. Compute $\frac{dg^{-1}(\gamma)}{d\gamma}$
3. Find $p_\gamma(\gamma) = \left| \frac{dg^{-1}(\gamma)}{d\gamma} \right| \times p_\theta(g^{-1}(\gamma))$

Jeffreys prior

- Idea: find a parameterization that is invariant under transformations
- Derivation:

Weakly Informative Priors

- A proper prior distribution, but intentionally include less information than is actually available a priori
- Construction:
  - Start with a noninformative prior and add then add enough information to constrain inferences to be more reasonable
  - Or: start with an informative prior and remove information
- Example: coefficients in a logistic regression should have a magnitude less than 10, in general

Prediction

Posterior predictive distribution

- An important feature of Bayesian inference is the existence of a predictive distribution for new observations.
  - Let $\tilde{y}$ be a new (unseen) observation, and $y_1, \ldots y_n$ the observed data.
  - The Posterior predictive distribution is $p(\tilde{y} \mid y_1, \ldots y_n)$

Posterior predictive distribution

- The predictive distribution does not depend on unknown parameters
- The predictive distribution only depends on observed data
- Asks: what is the probability distribution for new data given observations of old data?

Another Basketball Example

- I take free throw shots and make 1 out of 2. How many do you think I will make if I take 10 more?
- If my true "skill" was 50%, then $\tilde{Y} \sim \mathrm{Bin}(10, 0.50)$
- Is this the correct way to calculate the predictive distribution?

Posterior Prediction

If you know θ, then we know the distribution over future attempts:

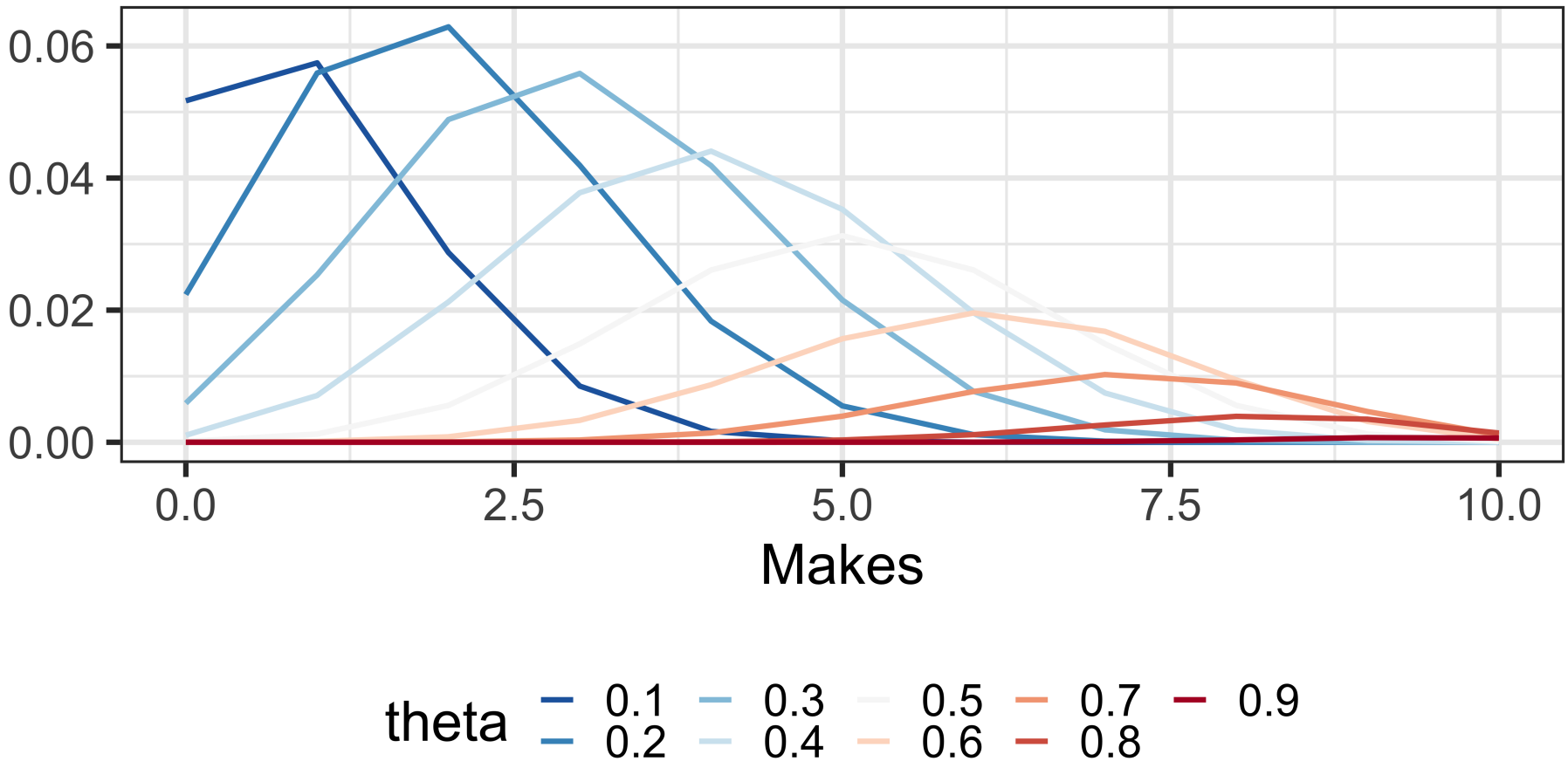
$$\tilde{Y} \sim \mathrm{Bin}(10, \theta)$$

Posterior Prediction

- We already observed 1 make out of 2 tries.
- Assume a Beta(1, 3) prior distribution
  - e.g. a priori you think I'm more likely to make 25% of my shots
- Then $p(\theta \mid Y = 1, n = 2)$ is a $\mathrm{Beta}(2, 4)$
- Intuition: weight $\tilde{Y} \sim \mathrm{Bin}(10, \theta)$ by $p(\theta \mid Y = 1, n = 2)$

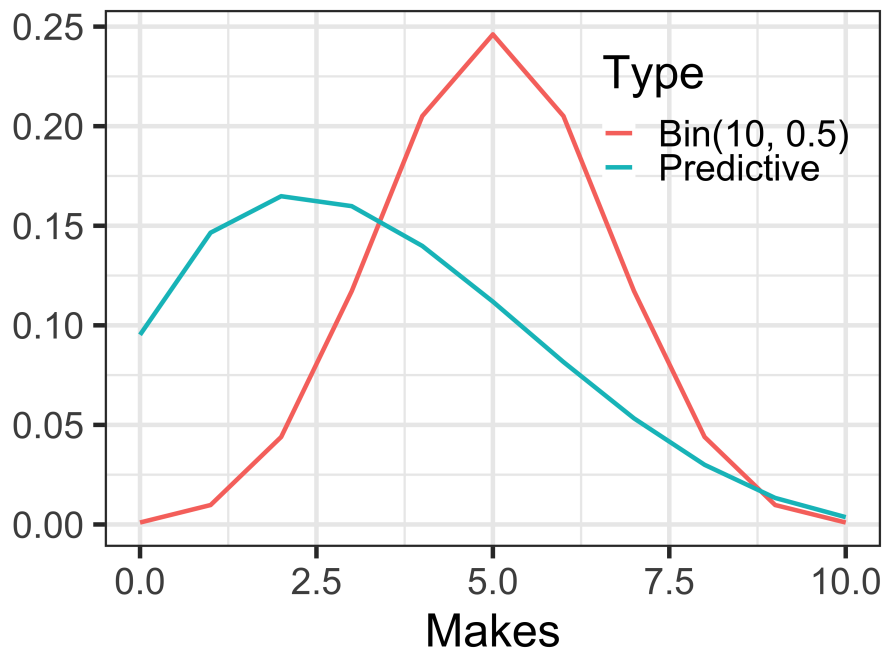
Posterior Prediction

If I take 10 more shots how many will I make?



Posterior predictive distribution

$$p(\theta) = \text{Beta}(1, 3), p(\theta \mid y) = \text{Beta}(2, 4)$$



The predictive density, $p(\tilde{y} \mid y)$, answers the question "if I take 10 more shots how many will I make, given that I already made

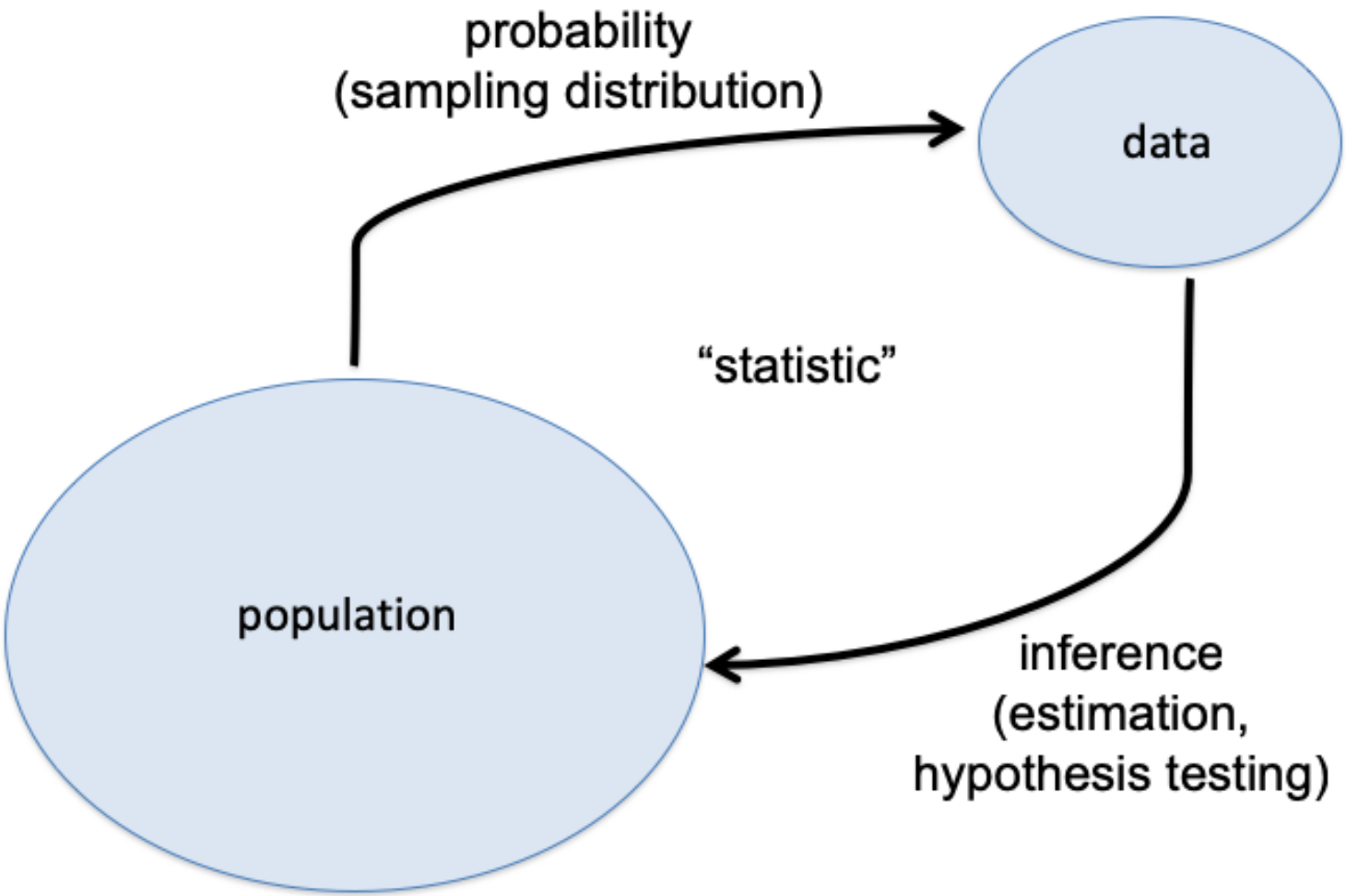
The posterior predictive distribution

$$
\begin{aligned}
p(\tilde{y} \mid y_1, \ldots y_n) &= \int p(\tilde{y}, \theta \mid y_1, \ldots y_n) d\theta \\
&= \int p(\tilde{y} \mid \theta) p(\theta \mid y_1, \ldots y_n) d\theta
\end{aligned}
$$

- The posterior predictive distribution describes our uncertainty about a new observation after seeing n observations
- It incorporates uncertainty due to the sampling in a model $p(\tilde{y} \mid \theta)$ *and* our posterior uncertainty about the data generating parameter, $p(\theta \mid y_1, \ldots y_n)$

The Beta-Binomial Distribution

Posterior predictive density



The prior predictive distribution

$$
\begin{aligned}
p(\tilde{y}) &= \int p(\tilde{y}, \theta) d\theta \\
&= \int p(\tilde{y} \mid \theta) p(\theta) d\theta
\end{aligned}
$$

- The prior predictive distribution describes our uncertainty about a new observation before seeing data
- It incorporates uncertainty due to the sampling in a model $p(\tilde{y} \mid \theta)$ *and* our prior uncertainty about the data generating parameter, $p(\theta)$

Summary

- Conjugate priors
- Credible intervals
- Noninformative priors
- Posterior (or prior) predictive distribution