Lecture 1: Likelihood Review

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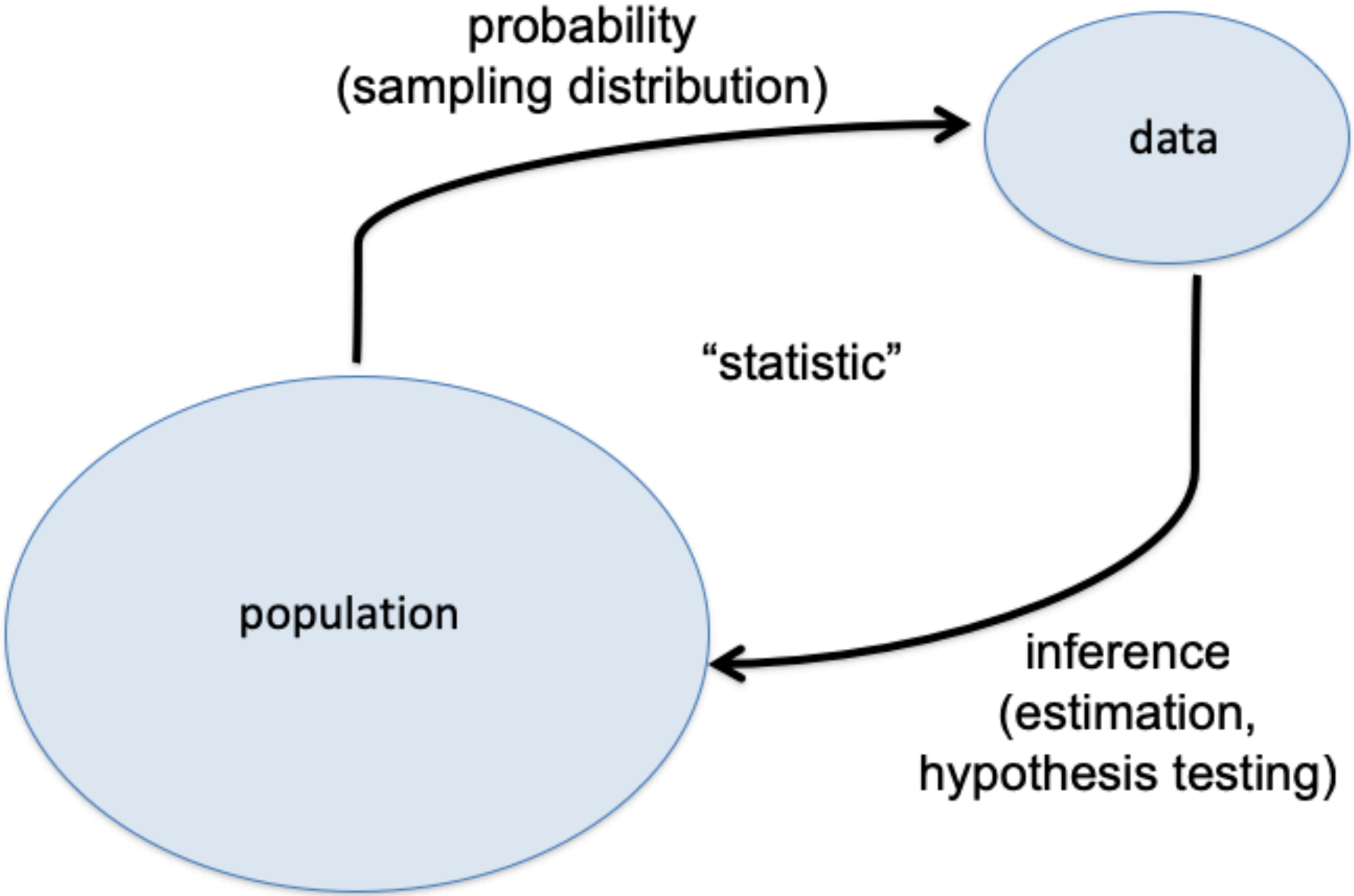
Logistics

- Read: BDA Chapters 1-2
- Sync content using link on course website: https://bit.ly/3NXxk9H
- Annotated lecture slides appear after class

Population and Sample



Independent Random Variables

- $Y_1, \ldots, Y_n$ are random variables
- We say that $Y_1, \ldots, Y_n$ are *conditionally* independent given θ if $P(y_1, \ldots, y_n \mid \theta) = \prod_i P(y_i \mid \theta)$
- Conditional independence means that Y_i gives no additional information about Y_j beyond that in knowing θ

The Likelihood Function

- The likelihood function is the probability density function of the observed data expressed as a function of the unknown parameter (conditional on observed data):
- A function of the unknown constant θ.
- Depends on the observed data $y = (y_1, y_2, \ldots, y_n)$
- Two likelihood functions are equivalent if one is a scalar multiple of the other

Sufficient Statistics

A statistic s(Y) is sufficient for underlying parameter θ if the conditional probability distribution of the Y, given the statistic s(Y), does not depend on θ.

Sufficient Statistics

- Let $L(\theta) = p(y_1, \ldots y_n \mid \theta)$ be the likelihood and $s(y_1, \ldots y_n)$ be a statistic
- *Factorization theorem*: $s(y)$ is a sufficient statistic if we can write:

$$L(\theta) = h(y_1, \ldots y_n) g(s(y), \theta)$$

  - g is only a function of s(y) and θ only
  - h is *not* a function of θ
- $L(\theta) \propto g(s(y), \theta)$

The Likelihood Principle

- **The likelihood principle**: All information from the data that is relevant to inferences about the value of the model parameters is in the equivalence class to which the likelihood function belongs
- Two likelihood functions are equivalent if one is a scalar multiple of the other
- Frequentist testing and some design based estimators violate the likelihood principle

Binomial vs Negative Binomial

Score and Fisher Information

- **The score function:** $\frac{d\ell(\theta;y)}{d\theta}$
  - $E[\frac{d\ell(\theta;Y)}{d\theta} \mid \theta] = 0$ (under certain regularity conditions)
- **Fisher information** is a measure of the amount of information a random variable carries about the parameter
  - $I(\theta) = E\left[\frac{(d\ell(\theta;Y)}{d\theta})^2 \mid \theta\right]$ (variance of the score)
  - Equivalently: $I(\theta) = -E\left[\frac{d^2\ell(\theta;Y)}{d^2\theta}\right]$

Fisher Information

Data Generating Process

Data Generating Process (DGP)

- I select 100 random students at UCSB to 10 free throw shots at the basketball court
- Assume there are two groups: experienced and inexperienced players
- Skill is identical conditional on experience level

Data Generating Process (DGP)

- Tell a plausible story: some students play basketball and some don’t.
- Before you take your shots we record whether or not you have played before.

```
assume theta_1 > theta_0
for (i in 1:100)
  - Generate z_i from Bin(1, phi)
    - p_i = theta_0 if z_i=0
    - p_i = theta_1 if z_i=1
  - Generate y_i from a Binom(10, p_i)
return y = (y_1, ... y_100) and z = (z_1, ..., z_100)
```

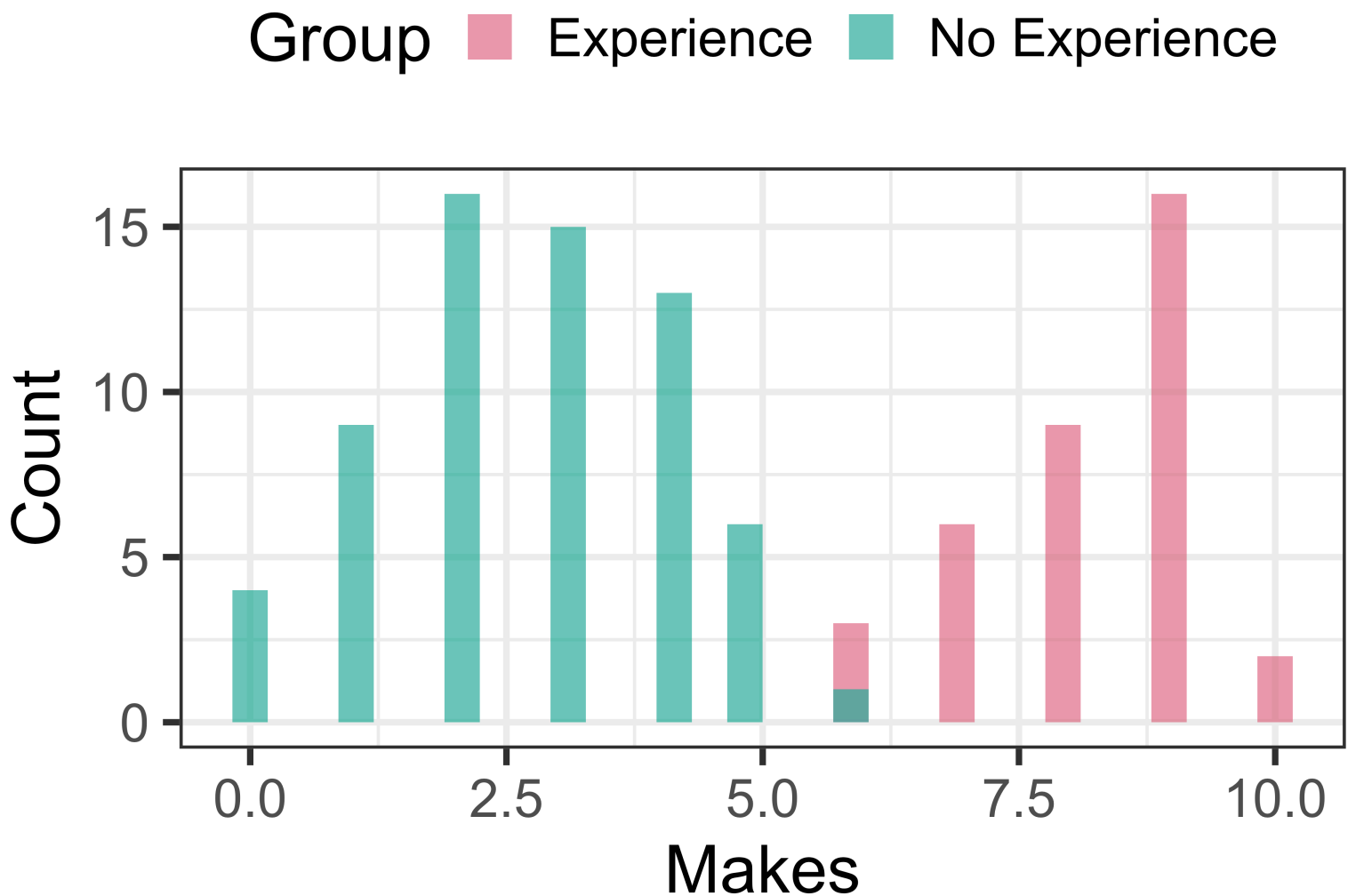
Mixture models

$$Z_i = \begin{cases} 0 & \text{if the } i^{th} \text{ if student doesn't play basketball} \\ 1 & \text{if the } i^{th} \text{ if student does play basketball} \end{cases}$$

$$Z_i \sim Bin(1, \phi)$$

$$Y_i \sim \begin{cases} \text{Bin}(10, \theta_0) & \text{if } Z_i = 0 \\ \text{Bin}(10, \theta_1) & \text{if } Z_i = 1 \end{cases}$$

A Mixture Model



Note: z is observed

Sufficient statistics When Z_i is observed

Together, the following quantities are sufficient for $(\theta_0, \theta_1, \phi)$

- $\sum y_i z_i$ (total number of shots made by experienced players)
- $\sum y_i(1 - z_i)$ (total number of shots made by inexperienced players)
- $\sum z_i$ (total number experienced players)

Mixture models

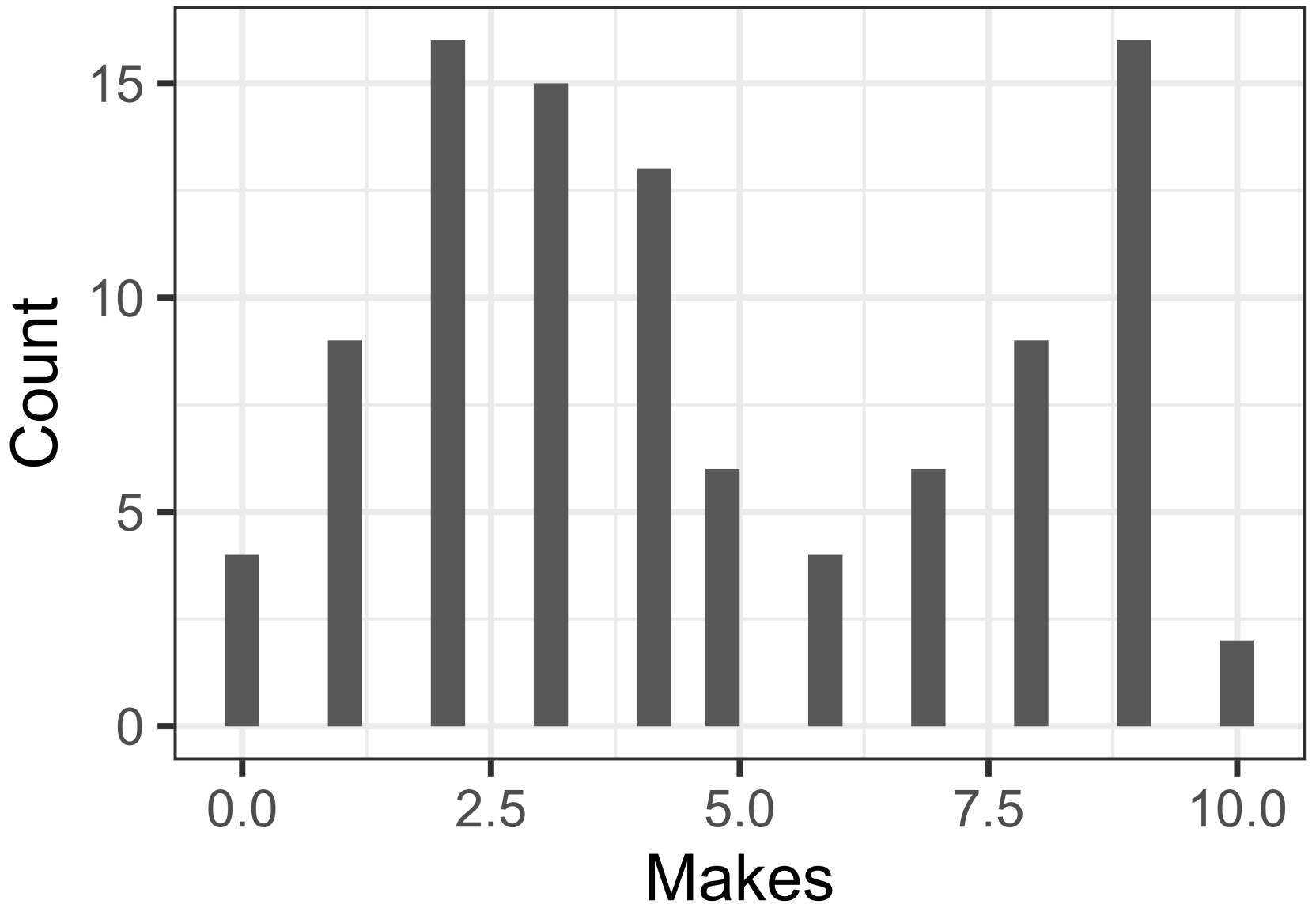
- A mixture model is a probabilistic model for representing the presence of subpopulations
- The subpopoluation to which each individual belongs is not necessarily known
  - e.g. do we ask: “have you played basketball before?”
- When z_i is not observed, we sometimes refer to it as a clustering model
  - *unsupervised* learning

Data Generating Process (DGP)

```
for (i in 1:100)
  - Generate z_i from Bin(1, phi)
    - p_i = theta_1 if z_i=1
    - p_i = theta_0 if z_i=0
  - Generate y_i from a Binom(10, p_i)
return y = (y_1, ... y_100)
```

This time we don’t record who has experience with basketball.

A Mixture Model



A finite mixture model

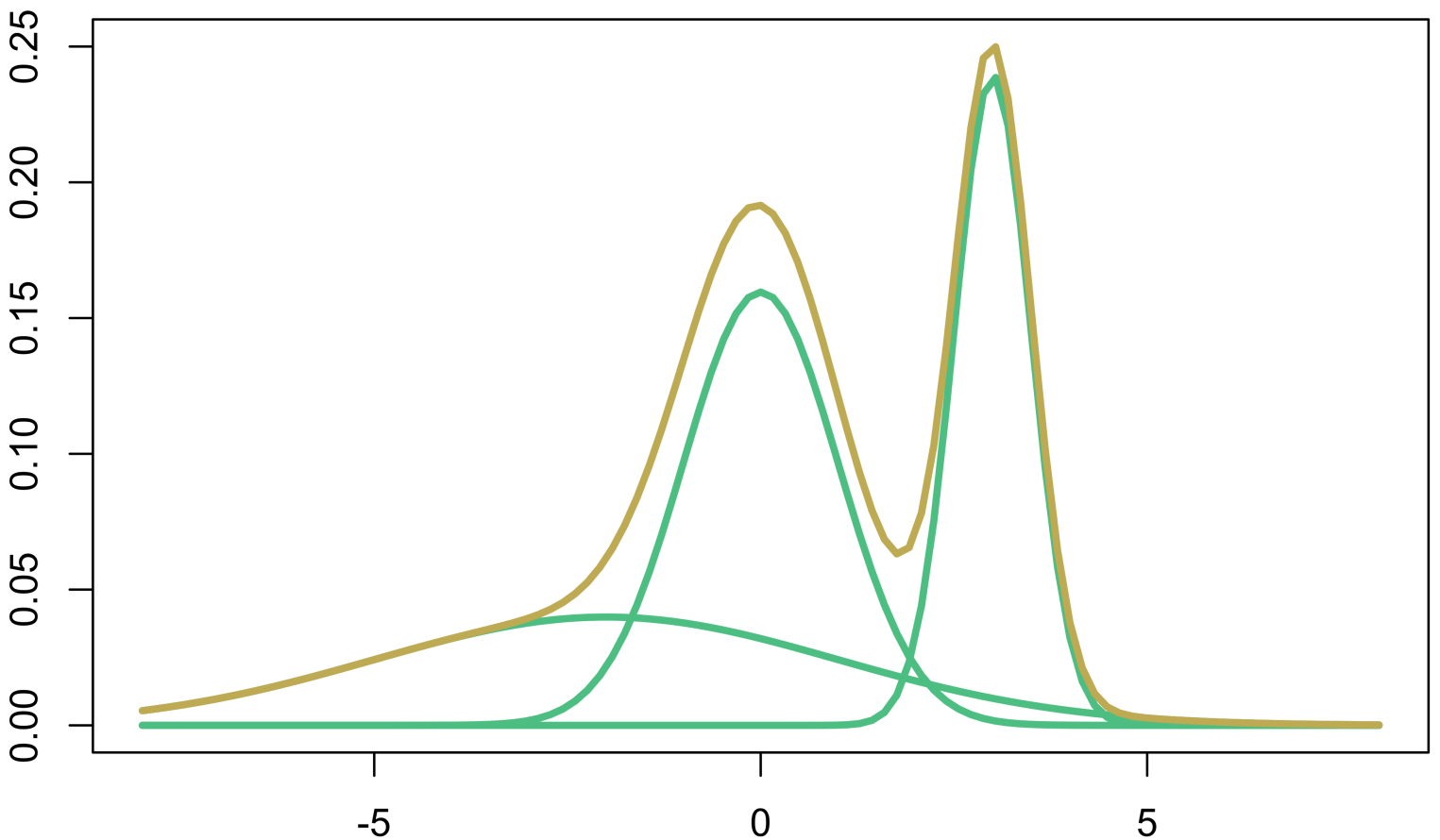
- Often crucial to understand the complete data generating process by introducing *latent* variables
- Write the *observed data likelihood* by integrating out the latent variables from the *complete data likelihood*

$$
\begin{aligned}
p(Y \mid \theta) &= \sum_z p(Y, Z = z \mid \theta) \\
&= \sum_z p(Y \mid Z = z, \theta) p(Z = z \mid \theta)
\end{aligned}
$$

In general we can write a K component mixture model as:

$$
p(Y) = \sum_k^K \pi_k p_k(Y) \text{ with } \sum \pi_k = 1
$$

Finite mixture models



Infinite Mixture Models

- Often helpful to think about infinite mixture models
- Example 1: normal observations with normally distributed mean

$$\mu_i \sim N(0, \tau^2)$$
$$Y_i \sim N(\mu_i, \sigma^2)$$

What is the distribution of Y_i given τ^2 and σ^2 (integrating over μ)?

Infinite Mixture Models

Example 2: Poisson observations with random rates

$$\lambda \sim Gamma(\alpha, \beta)$$
$$Y \sim Pois(\lambda)$$

Infinite Mixture Models

- Example 3: normal observations with exponentially distributed scale

$$\sigma_i^2 \sim Exponential(1/2)$$
$$Y_i \sim N(0, \sigma_i^2)$$

What is the distribution of Y_i?

Summary

- Likelihood, log likehood
- Sufficient statistics
- Fisher information
- Mixture models

Assignments

- Read chapter 1-2 BDA3