math108B hw2

Jonas Chen

January 08, 2024

Problem 1

> Construct an example that for 1-norm $\| \cdot \|_1$ in $\mathbb{R}^2$, the parallelogram equality does not hold, that is, $\|u+v\|_1^2 + \|u-v\|_1^2 \neq 2(\|u\|_1^2 + \|v\|_1^2)$
> As a result, 1-norm $\| \cdot \|_1$ cannot be induced from any inner-product.

Solution

The 1-norm $\|u\|_1$ for some $v = (v_1, v_2) \in \mathbb{R}^2$ is defined as

$$|v_1| + |v_2|$$

Addition and subtraction of two vectors is defined as adding or subtracting component wise in $\mathbb{R}^2$

Let $u = (1, 0), v = (0, 1)$

Then $\|u+v\|_1^2 = \|(1,1)\|_1^2 = 4$
Then $\|u-v\|_1^2 = \|(1,-1)\|_1^2 = 4$
And $\|u\|_1^2 = 1$
And $\|v\|_1^2 = 1$

But $8 \neq 4$

Problem 2

> Some exercises about the Cauchy-Schwarz
>
> a. Prove
>
> $$\|\frac{\langle u, v\rangle}{\|v\|^2} v\|^2 = \frac{|\langle u, v\rangle\ |^2}{\|v\|^2}$$
>
> (This is a step in the proof to the Cauchy-Schwarz Inequality)
>
> b. In question (8) of HW1, we proved the equivalent norm in $\mathbb{R}^d$. Can you use the Cauchy-Schwarz inequality to find a constant C such that for any $v \in \mathbb{R}^d$,
>
> $$\|v\|_1 \leq C\ \|v\|_2$$

Solution

a.

$$\|\frac{\langle u, v\rangle}{\|v\|^2} v\ \|^2 = \langle \frac{\langle u, v\rangle}{\|v\|^2} v, \frac{\langle u, v\rangle}{\|v\|^2} v\rangle = \left(|\frac{\langle u, v\rangle}{\|v\|^2}| \right)^2 \ \|v\|^2 = \frac{|\langle u, v\rangle|^2}{\|v\|^2}$$

abusing the fact that $\langle cv, cv\rangle = c\bar{c}\langle v, v\rangle = |c|^2\ \langle v, v\rangle$

and that $\|\cdot\|^2 \geq 0$ hence $|\ \|\cdot\|^2\ | = \|\cdot\|^2$

b. Set $n := d$. Using the cauchy schwarz inquality $|\langle u, v\rangle| \leq \|u\|\ \|v\|$ and using the standard inner (dot) product in $\mathbb{R}^n$:
$|v \cdot w| \leq \|u\|\ \|v\|$

We have

$$\|v\|_1 = \sum_{i=1}^{n} |v_i| \cdot \vec{1} \leq \sqrt{\sum_{i=1}^{n} v_i^2} \sqrt{\sum_{i=1}^{n} 1^2} = \sqrt{n}\ \|v\|_2$$

which implies that $\frac{1}{\sqrt{n}}\ \|v\|_1 \leq \|v\|_2$ or $\|v\|_1 \leq \sqrt{n}\ \|v\|_2 \Rightarrow C = \sqrt{n}$
where $\vec{1} \in \mathbb{R}^d$ is a vector of $1'$s and $v = (v_1, ..., v_n)$

Proposition

> Orthogonal decomposition

Proof
Given $u, v \in V, v \neq 0$ then $u = cv + w$ where cv is parallel to v and w is normal to v i.e. $\langle w, v\rangle = 0$

In fact we can explicitly compute c and w by $c = \frac{\langle u, v\rangle}{\|v\|^2}$ and $w = u - \frac{\langle u, v\rangle}{\|v\|^2} v$

$\square$

Problem 3

> Consider the space of polynomials $P([-1,1])$ on $[-1,1]$. Define its inner product by
>
> $$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x$$
>
> a. Is the polynomial $f(x) = x$ orthogonal to the polynomial $g(x) = 1$
> b. Decompose the polynomial $h(x) = x^2$ into a part that is parallel to the polynomial $g(x) = 1$, and a part that is orthogonal to the polynomial $g(x)$

Solution

a. Orthogonal condition: $\langle f, g\rangle = 0$

Note that $f(x)g(x) = x(1) = x$ and that $\langle f, g\rangle = \int_{-1}^{1} x\mathrm{d}x = \frac{x^2}{2} \mid_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$. Hence f is orthgonal to g

b. $\langle h, g\rangle = \int_{-1}^{1} x^2 = \frac{1}{3}x^3 \mid_{-1}^{1} = \frac{2}{3}$

$\|g\|^2 = \langle g, g\rangle = \int_{-1}^{1} 1 = x \mid_{-1}^{1} = 2$

set the constant $c = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$

Then the component parallel to g is $cg = \frac{1}{3}(1) = \frac{1}{3}$

And the component orthogonal to g is $o \coloneqq h - cg = x^2 - \frac{1}{3}$

So that $cg + o = x^2 = h(x)$ and additionally $\langle o, g\rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)(1) = \left(\frac{1}{3}x^3 - \frac{1}{3}x\right) \mid_{-1}^{1} = 0$

Problem 4

> Construct an example such that Cauchy-Schwarz inequality does not hold for 1-norm. That is, you need to find two vectors $u, v \in \mathbb{R}^d$ such that $|\langle u, v\rangle\ | > \|u\|_1\ \|v\|_1$. You can take $d = 2$ for simplicity but are welcome to deal with general d

Solution

No solution for finite dim vector spaces

Problem 5

> Decompose the vector $v = (3, 1)$ into a part that is parallel to the vector $w = (1, 2)$ and a part that is orthogonal to the vector w.

Solution

Assume the inner product is the dot product and norm of a vector to be l2 norm, and
Let cw by the component parallel to w and o be the component orthogonal to w

$\langle v, w\rangle = 3 + 2 = 5$
$\|w\|^2 = \sqrt{1^2 + 2^2}^2 = \sqrt{5}^2 = 5$
$cw = \frac{5}{5}(1, 2) = (1, 2)$
$o = (3, 1) - (1, 2) = (2, -1)$

Then $cw + o = (3, 1)$. Since cw is a scalar multiple of w, cw is parallel to w

Additionally, $\langle o, w\rangle = 2 - 2 = 0$

Theorem: **Gram - Schmidt Procedure (LADR 6.32)**

> Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $f_1 = v_1$. For $k = 2, \ldots, m$, define f_k inductively by
>
> $$f_k = v_k - \frac{\langle v_k, f_1\rangle}{\|f_1\|^2} f_1 - \ldots - \frac{\langle v_k, f_{k-1}\rangle}{\|f_{k-1}\|^2} f_{k-1}$$
>
> For each $k = 1, \ldots, m$ let $e_k = \frac{f_k}{\|f_k\|}$. Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that
>
> $$\texttt{span}\ (v_1, \ldots, v_k) = \texttt{span}\ (e_1, \ldots, e_k)$$
>
> for each $k = 1, \ldots, m$.

Problem 6

Apply Grahm-Schmidt onto a set $S = \{(1,1,1),(0,1,1),(0,0,1)\}$ of vectors in $\mathbb{R}^3$ to obtain a set of orthonormal vectors.

Solution

Define the inner product to be the dot product, then confirm S is a linearly independent set of vectors in $\mathbb{R}^3$ (they are by calculator)

Then, $f_1 = (1,1,1)$

$$f_2 = v_2 - \frac{\langle v_2, f_1\rangle}{\|f_1\|^2} f_1 = (0,1,1) - \frac{\langle (0,1,1),(1,1,1)\rangle}{\|(1,1,1)\|^2}(1,1,1) = (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$f_3 =$$
$$v_3 - \left(\frac{\langle v_3, f_1\rangle}{\|f_1\|^2} f_1\right) - \left(\frac{\langle v_3, f_2\rangle}{\|f_2\|^2} f_2\right) =$$
$$(0,0,1) - \left(\frac{\langle (0,0,1),(1,1,1)\rangle}{\|(1,1,1)\|^2}\right)(1,1,1) - \left(\frac{\langle (0,0,1), \left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)\rangle}{\|\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)\|^2}\right)\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) =$$
$$(0,0,1) - \left(\frac{1}{3}\right)(1,1,1) - \left(\frac{1}{3}\frac{9}{6} = \frac{1}{2}\right)\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = (0,0,1) - \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) - \left(-\frac{2}{6}, \frac{1}{6}, \frac{1}{6}\right) =$$
$$\left(0,0,\frac{6}{6}\right) - \left(\frac{2}{6}, \frac{2}{6}, \frac{2}{6}\right) - \left(-\frac{2}{6}, \frac{1}{6}, \frac{1}{6}\right) = \left(0, -\frac{3}{6}, \frac{3}{6}\right)$$

Next we can normalize:

$$e_1 = \frac{f_1}{\|f_1\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$
$$e_2 = \frac{f_2}{\|f_2\|} = \frac{\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)}{\sqrt{\frac{2}{3}}} = \left(-\frac{2}{3\sqrt{\frac{2}{3}}}, \frac{1}{3\sqrt{\frac{2}{3}}}, \frac{1}{3\sqrt{\frac{2}{3}}}\right)$$
$$e_3 = \frac{f_3}{\|f_3\|} = \frac{\left(0,-\frac{3}{6},\frac{3}{6}\right)}{\sqrt{\frac{1}{2}}} = \left(0, -\frac{3}{6\sqrt{\frac{1}{2}}}, \frac{3}{6\sqrt{\frac{1}{2}}}\right)$$

We can easily inspect that (e_1, e_2, e_3) and orthonormal and using a calculator we know they are linearly independent, and hence form a basis for $\mathbb{R}^3$

https://www.emathhelp.net/calculators/linear-algebra/linear-independence-calculator/?i=%5B%5B1%2Fsqrt%283%29%2C-2%2F%283sqrt%282%2F3%29%29%2C0%5D%2C%5B1%2Fsqrt%283%29%2C1%2F%283sqrt%282%2F3%29%29%2C-3%2F%286sqrt%281%2F2%29%29%5D%2C%5B1%2Fsqrt%283%29%2C1%2F%283sqrt%282%2F3%29%29%2C3%2F%286sqrt%281%2F2%29%29%5D%5D

We can further simplify the vectors to $e_1 = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ $e_2 = \left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}\right)$ $e_3 = \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Problem 7

Consider the space of polynomials $P((-\infty, \infty))$ on $(-\infty, \infty)$. Define its inner product by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) e^{-\frac{x^2}{2}} \mathrm{d}x$$

Use the following identity

$$\int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} \mathrm{d}x = \begin{cases} \sqrt{2\pi}(n-1)!! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

where $k!! = 1 \cdot 3 \cdot 5 \cdots k$ is the semi-factorial of an odd number k, and the Grahm-Schmidt on the polynomials $\{1, x, x^2\}$, to obtain a set of orthogonal polynomials. You do not need to normalize them.

Solution

after gram-schmidt we obtain $\{1, x, x^2 - 1\}$

see the attached page

Problem 8

> Let T be a linear operator on an inner product space V, that is $T : V \to V$, satisfying
> $$\|T(v)\| = \|v\|$$
> for any $v \in V$. Prove that T is injective.

Solution

Recall that T is injective if and only if the null space of T is $\{0\}$

We can try to show that if $T(v) = 0$ then $v = 0$

Assume $T(v) = 0$ then $\|T(v)\| = \|0\| = 0 = \|v\|$ by $(\star)$ and by definition in problem statement.

And since $0 = \|v\|$ by $(\star)$ we know that $v = 0$ and therefore the null space of T is $\{0\}$, hence T is injective. $\square$

$(\star)$: recall from lecture that $\|v\| = 0$ only if $v = 0$

(T is also an isometry)