Variation of Existence of Eigenvalue Proof

from linear algebra done right edition 4

polynomials that take an operator as input

Notation

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T^0 is defined to be the identity operator I on V

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Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ where $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_m z_m$ for all $z \in \mathbb{F}$ and $a_0 ... a_m \in \mathbb{F}$

Then $p(T) \in \mathcal{L}(V)$ is defined by $p(T) = a_0 I + a_1 T + a_2 T^2 + ... + a_m T^m$

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Remark

suppose $p, q \in \mathcal{P}(F)$ then

$(pq)(T) = p(T)q(T)$ and $p(T)q(T) = q(T)p(T)$

No matter if $z \in \mathbb{F}$ or $T \in \mathcal{L}(V)$ is the input to p and q, properties will hold

facts about polynomials

Theorem: Fundamental Theorem of Algebra

Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$

In other words for every $p \in \mathcal{P}(\mathbb{C})$ there exists λ such that $p(\lambda) = 0$

facts about polynomials

***Theorem:* Fundamental Theorem of Algebra**

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***Theorem:* each zero of a polynomial corresponds to a degree-one factor**

Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial of degree m

Then for $\lambda \in \mathbb{F}$, we have $p(\lambda) = 0 \Leftrightarrow \exists q \in \mathcal{P}(\mathbb{F})$ of degree $m - 1$ such that $p(z) = (z - \lambda)q(z)$ for every $z \in \mathbb{F}$

From before, we also know that $p(T)(v) = (T - \lambda I)(q(T)v)$ for $T \in \mathcal{L}(V)$ and $v \in V$

existence of eigenvalues

Theorem

> Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

Proof:

Suppose V is a finite-dimensional complex vector space with dimension n

Let $n > 0$ and $T \in L(V)$, and choose $v \in V$ with $v \neq 0$

$\square$

existence of eigenvalues (ii)

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Let $n > 0$ and $T \in L(V)$, and choose $v \in V$ with $v \neq 0$

The list $v, Tv, T^2 v, ..., T^n v$ is not linearly independent (since it has $n + 1$ elements)

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existence of eigenvalues (iii)

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The list $v, Tv, T^2 v, ..., T^n v$ is not linearly independent (since it has $n + 1$ elements)

let $p(T)v = (a_0 I + a_1 T + a_2 T^2 + ... + a_n T^n)(v)$

Then some non-constant p exists exists such that $p(T)v = 0$

$\square$

existence of eigenvalues (iv)

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> Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

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The list $v, Tv, T^2 v, ..., T^n v$ is not linearly independent (since it has $n + 1$ elements)

let $p(T)v = (a_0 I + a_1 T + a_2 T^2 + ... + a_n T^n)(v)$

Then some non-constant p exists exists such that $p(T)v = 0$

Additionally, there exists $\lambda \in \mathbb{C}$ such that $p(\lambda) = 0$ by the fundamental theorem of algebra, and therefore there exists $q \in \mathcal{P}(\mathbb{C})$ such that $p(z) = (z - \lambda)q(z)$

□

existence of eigenvalues (v)

Theorem

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Additionally, there exists $\lambda \in \mathbb{C}$ such that $p(\lambda) = 0$ by the fundamental theorem of algebra, and therefore there exists $q \in \mathcal{P}(\mathbb{C})$ such that $p(z) = (z - \lambda)q(z)$

Then combining the above 2 results gives us $0 = p(T)v = (T - \lambda I)(q(T)v)$

existence of eigenvalues continued

Theorem:

> Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

Proof:

Previously we saw that $0 = p(T)v = (T - \lambda I)(q(T)v)$

Since q has smaller degree than p, then we cannot have $q(T)v \neq 0$

This is because if $T = \lambda I$ then $p(T)v = \left(a_0 I + a_1 T + a_2 T^2 + \ldots + a_n T^n\right)(v)$ is a constant, which is a contradiction

Therefore it must be that $T \neq \lambda I$ which implies $q(T)v \neq 0$ (recall that we chose $v \neq 0$)

□

existence of eigenvalues continued

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Previously we saw that $0 = p(T)v = (T - \lambda I)(q(T)v)$

Since q has smaller degree than p, then we cannot have $q(T)v \neq 0$

Therefore $T - \lambda I$ is not injective, which implies that λ is an eigenvalue of T with eigenvector $q(T)v$

$\square$

Remark

Recall that if $T - \lambda I$ is not injective then λ is an eigenvalue of T