math115A hw5

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Problem 1

> Suppose f and g are multiplicative functions. Prove that the function F defined by
>
> $$F(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$
>
> is also multiplicative.

Solution

Let $n = ab$ then

$$\begin{aligned} F(ab) &= \sum_{d|ab} f(d) g\left(\frac{ab}{d}\right) \\ &= \sum_{d_1|a, d_2|b} f(d_1 d_2) g\left(\frac{ab}{d_1 d_2}\right) \\ &= \sum_{d_1|a, d_2|b} f(d_1) f(d_2) g\left(\frac{a}{d_1}\right) g\left(\frac{b}{d_2}\right) \\ &= \sum_{d_1|a} f(d_1) g\left(\frac{a}{d_1}\right) \cdot \sum_{d_2|b} f(d_2) g\left(\frac{b}{d_2}\right) = F(a)F(b) \end{aligned}$$

By definition, “double for loop” to generate divisors, multiplicativity of f and g, comutativity, and definition respectively. $\square$

Problem 2

> (i) For each positive integer n, show that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$ [Hint: What can you say about the four consecutive intgers $n, n+1, n_2, n_3$ modulo 4 ? If you find yourself doing lots of algebraic manipulations to solve this problem, then you are almost certainly on the wrong track.]
> (ii) For any integer $n \geq 3$, show that
>
> $$\sum_{k=1}^{n} \mu(k!) = 1$$

Solution

(i) Looking at the table of values for μ, we can see that $\mu(n) = 0$ if $4 \mid n$ (this is true because $4 = 2^2$, a power of a prime)
This means that for any positive integer n that at least one of $\mu(n), \mu(n+1), \mu(n+2), \mu(n+3)$ will be 0. Since at least one of $n, n+1, n+2, n+3$ will be divisible by 4

And the claim follows.

(ii)

$\mu(1!) = 1; \mu(2!) = -1; \mu(3!) = 1;$

Since for $n = 4$ we have that $n! = 3! \cdot 4 \Rightarrow \mu(n!) = 0$

And for $n \geq 5$ we have that $n! = 3! \cdot 4 \cdot \prod_{i=5}^{n} i \Rightarrow \mu(n!) = 0$ (recalling the finding in the previous part)

Therefore for $n \geq 3$ we have that $\sum_{k=1}^{n} \mu(k!) = 1 - 1 + 1 + 0 + ... + 0 = 1$ $\square$

Problem 3

> The von Mangoldt function Λ is defined by
>
> $$\Lambda(n) = \begin{cases} \log(p) \text{ if } n = p^k \text{ where } p \text{ is prime and } k \geq 1 \\ 0 \text{ otherwise} \end{cases}$$
>
> Prove that
>
> $$\Lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log(d) = -\sum_{d|n} \mu(d) \log(d)$$
>
> [Hint: First show that $\sum_{d|n} \Lambda(d) = \log(n)$, and then apply the Mobius inversion formula.]

Solution

Let $n = p^k$ then the divisors of n are $1, p, p^2, ..., p^k$ and $\sum_{d|n} \Lambda(d) = \Lambda(1) + \Lambda(p) + ... + \Lambda(p^k) = 0 + \underbrace{\log(p) + ... + \log(p)}_{\text{k times}} = \log(p^k)$

In general, if the prime factorization of n is $n = p_1^{k_1} p_2^{k_2} ... p_s^{k_s}$ then for each $p_i^{k_i}$ we have $\sum_{d|p_i^{k_i}} \Lambda(d) = 0 + \underbrace{\log(p_i) + ... + \log(p_i)}_{\text{k\_i times}} = \log\left(\underbrace{p_i ... p_i}_{\text{k\_i times}}\right) = \log\left(p_i^{k_i}\right)$. Then taking the sum over $i = 1, ..., s$ we have

$$\sum_{d|n} \Lambda(d) = \log(n) = \sum_{i=1}^{s} \sum_{d|p_i^{k_i}} \Lambda(d)$$

which proves the first equality by the definition of the mobius inversion formula

If we try to apply the Mobius inversion formula we find that

$$\Lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log(d) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d)(\log n - \log d) =$$
$$\sum_{d|n} \mu(d)(\log(n)) - \sum_{d|n} \mu(d) \log(d) = 0 - \sum_{d|n} \mu(d) \log(d)$$

where the last equality holds since $\sum_{d|n} \mu(d) = 0$ if $n > 1$ (from class)

If $n = 1$ then $\Lambda(1) = 0$ and since $\log(1) = 0$ (and 1 is the only divisor of 1) then the equality still holds.

It is not possible that $n < 0$ since Λ is defined over domain of positive integers

$\square$

Problem 4

> Let $n = p_1^{k_1} p_2^{k_2} ... p_r^{k_r}$ be the prime factorisation of an integer $n > 1$. If f is a multiplicative function that is not identically zero, prove that
>
> $$\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2))...(1 - f(p_r))$$
>
> [Hint: Use the fact that the function F defined by $F(n) = \sum_{d|n} \mu(d) f(d)$ is multiplicative (why is this so?), and is therefore determined by its values on powers of primes.]

Solution

Let F be defined as in the problem statement

The result from problem 1 implies that F is multiplicative. Therefore $F\left(p_1^{k_1} p_2^{k_2} ... p_r^{k_r}\right) = F\left(p_1^{k_1}\right) F\left(p_2^{k_2}\right) ... F\left(p_r^{k_r}\right)$ Also note that:

$$\begin{aligned} F\left(p_i^{k_i}\right) = \sum_{d|p_i^{k_i}} \mu(d) f(d) &= \mu(1) f(1) + \mu(p_i) f(p_i) + \mu(p_i^2) f(p_i^2) + ... + \mu\left(p_i^{k_i}\right) f\left(p_i^{k_i}\right) \\ &= 1 - f(p_i) \end{aligned}$$

and the claim follows if $f(1) = 1$

Which is true since $f(1 \cdot x) = f(1) f(x) \Rightarrow f(1) = 1$ (where 1 is multiplicative identity for $\mathbb{Z}$, $x \in \mathbb{Z}$)

□

Problem 5

> Let $n = p_1^{k_1} p_2^{k_2} ... p_r^{k_r}$ be the prime factorisation of an integer $n > 1$. Use the result of Problem 4 above to establish the following:
> (a) $\sum_{m|n} \mu(m) d(m) = (-1)^r$
> (b) $\sum_{d|n} \mu(d) \sigma(d) = (-1)^r p_1 p_2 ... p_r$
> (c) $\sum_{d|n} \frac{\mu(d)}{d} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)...\left(1 - \frac{1}{p_r}\right)$
> (d) $\sum_{d|n} d\mu(d) = (1 - p_1)(1 - p_2)...(1 - p_r)$

Solution

(a) $\sum_{m|n} \mu(m) d(m) = (1 - d(p_1))(1 - d(p_2))...(1 - d(p_r)) = \underbrace{(-1)...(-1)}_{\text{r times}} = (-1)^r$

(b) $\sum_{d|n} \mu(d) \sigma(d) = (1 - \sigma(p_1))(1 - \sigma(p_2)...(1 - \sigma(p_r)) = (1 - (1 + p_1))(1 - (1 + p_2))...(1 - (1 + p_r)) = (-p_1)(-p_2)...(-p_r) = (-1)^r p_1 p_2 ... p_r$

(c) follows if we let $f(d) = \frac{1}{d}$ (note that f is multiplicative since $f(d_1 d_2) = \frac{1}{d_1 d_2} = \frac{1}{d_1}\left(\frac{1}{d_2}\right) = f(d_1) f(d_2)$ in particular $\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))...(1 - f(p_r)) = \left(1 - \frac{1}{p_1}\right)...\left(1 - \frac{1}{p_r}\right)$

(d) follows if we let $f(d) = d$ since $\sum_{d|n} \mu(d) f(d) = (1 - f(p_1))...(1 - f(p_r)) = (1 - p_1)...(1 - p_r)$

Problem 6

> Let $S(n)$ denote the number of square-free divisors of n. Show that
>
> $$S(n) = \sum_{d|n} |\mu(d)| = 2^{\omega(n)}$$
>
> where $\omega(n)$ is the number of distinct prime factors of n

Solution

Let $n = p_1^{k_1} p_2^{k_2} ... p_r^{k_r}$ be the prime factorization of n

Then

$$S(n) = \sum_{d_1 | p_1^{k_1}, ..., d_r | p_r^{k_r}} |\mu(d_1 ... d_r)|$$

Since $\mu(n) = 0$ if $p^2 \mid n$ for some prime p, then the only non zero terms in the above sum are when $d_1 ... d_r$ are distinct primes and/or any of $d_1 ... d_r$ are 1. If $\mu(d_1 ... d_r) \neq 0$ if follows by definition that $|\mu(d_1 ... d_r)| = 1$. Therefore we are counting the number of combinations of $d_1 ... d_r$ such that $\mu(d_1 ... d_r)$ is non-zero

In other words, $\mu(d_1 ... d_r) \neq 0$ if $d_1 ... d_r = (d_1') ... (d_r')$ where each $d_i' = 1$ or d_i

Therefore there are 2^r such combinations of choices for $d_1' ... d_r'$ where r is the number of distinct prime factors of n $\square$