math108B hw4

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Problem 1

> Find the dual basis of the standard basis on $\mathbb{F}^{3,1}$

Solution

The standard basis $\{e_i\}_{i=1}^3$ on the set of 3×1 matrices (with real values) is $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $e_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The dual basis for the dual space of $\mathbb{F}^{3,1}$ can be defined as

$$\varphi_j\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_j$$

for $j \in \{1, 2, 3\}$

This satisfies the definition of dual basis, that $\varphi_j(e_i) = 1$ if $i = j$ and 0 otherwise

Problem 2

> Let $V = P(\mathbb{R})$ and let $\{v_k\}_{k=0}^{n} = \{x^k\}_{k=0}^{n}$ be the standard basis for V, and let $\{\varphi_j\}_{j=0}^{n}$ be the the corresponding dual basis for V'. Prove that
>
> $$\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$$
>
> for every $p \in V$ and $j = 0, 1, \ldots, n$

Solution

Note that $p^{(j)}$ denotes the jth derivative of p

Note that for any polynomial p that the taylor expansion of p is p itself

Since the kth derivative for a degree n polynomial is 0 for all $k > n$ we have:

$$p = \sum_{k=0}^{n} \frac{p^{(k)}(0)}{k!} x^k$$

And using LADR 3.114 we know that

$p = \varphi_0(p)x^0 + \ldots + \varphi_n(p)x^n$

It is then natural to define the dual basis of V to be

$$\varphi_j(p) = \frac{p^{(j)}(0)}{j!}$$

source: "Taylor's Series of a Polynomial | MIT 18.01SC Single Variable Calculus, Fall 2010" at link: https://www.youtube.com/watch?v=19x213y_uk4

Problem 3

> In the textbook we define $\{\phi_1, ..., \phi_n\}$ to be the dual basis of V where the basis of V is $\{v_1, ..., v_n\}$ satisfying the following:
>
> $$\phi_i(v_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$
>
> Given a constant $\lambda \neq 0$ we define another set $\{T_1, ..., T_n\}$ of elements in V' satisfying that
>
> $$T_i(v_j) = \begin{cases} \lambda \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$
>
> Is the set $\{T_1, ..., T_n\}$ a (nonstandard) dual basis? What is the relationship between T_i and ϕ_i ?

Solution

Note that $T_i \in \{T_1, ..., T_n\}$ is a scalar multiple of $\phi_i \in \{\phi_1, ..., \phi_n\}$ where the scalar multiple is λ

In general, if we have a basis $v_1, ..., v_n$ for a n-dimensional vector space V then for any $\lambda \neq 0 \in \mathbb{F}$ the list $\lambda v_1, ..., \lambda v_n$ is also a basis

To show this we can show that $\lambda v_1, ..., \lambda v_n$ is linearly independent. (this list is of the "right length" n)

Consider $\alpha_1, \lambda v_1 + ... + \alpha_n \lambda v_n = \lambda(\alpha_1 v_1 + ... + \alpha_n v_n) = 0 \in V$

Since $\lambda \neq 0$ then it must be true that $\alpha_1 v_1 + ... + \alpha_n v_n = 0$ Since $v_1, ..., v_n$ are a basis and therefore linearly independent then this implies that $\alpha_1 = ... = \alpha_n = 0 \in \mathbb{F}$

□

Problem 4

> Is $L(V, W)$ isomorphic to $L(W', V')$?
>
> Suppose that V and W are finite dimensional vector spaces over $\mathbb{F}$. Show that the map $D : L(V, W) \to L(W', V')$ defined by $D(T) = T'$ is an isomorphism

Solution

We can try to show that $L(V, W)$ and $L(W', V')$ have the same dimension.

Note that $\dim(W') = \dim(L(W, \mathbb{R})) = \dim(W) \dim(\mathbb{R}) = \dim(W)$

Similarly, $\dim(V') = \dim(V)$ (this result is also shown in LADR 3.111)

Then $\dim L(V, W) = \dim(V) \dim(W) = \dim(W') \dim(V') = \dim(W', V')$ implies that $L(V, W)$ and $L(V', W')$ are isomorphic.

Problem 5

> Is the invertible operator equivalent to the invertible matrix that represents that operator?
>
> Suppose that V is a finite dimensional vector space over $\mathbb{F}$. Let $\{v_k\}_{k=1}^n$ be a basis for V, let $T \in L(V)$ nand let A be the matrix of T relative to $\{v_k\}_{k=1}^n$
>
> Prove that T is an invertible operator if and only if A is an invertible matrix.

Solution

Let $A = [T]_\beta^\beta$ be the matrix for T relative to $\beta = \{v_k\}_{k=1}^n$ then

If T is an invertible operator, then there exists some T^{-1} such that $TT^{-1} = T^{-1}T = I$

The matrix for $T^{-1}$is $[T^{-1}]_\beta^\beta$ and $[T \circ T^{-1}]_\beta^\beta = [T]_\beta^\beta [T^{-1}]_\beta^\beta = [I]_\beta = [T^{-1}]_\beta^\beta [T]_\beta^\beta = [T^{-1} \circ T]_\beta^\beta$

Therefore $[T]_\beta^\beta$ is invertible.

Conversely, suppose that $[T]_\beta^\beta$ is invertible then there exists some matrix $[T^{-1}]_\beta^\beta$ such that $[T]_\beta^\beta [T^{-1}]_\beta^\beta = [I]_\beta = [T^{-1}]_\beta^\beta [T]_\beta^\beta$ which implies that T is invertible since if $[T^{-1}]_\beta^\beta$ exists then so does T^{-1}

Note that in the above we use the fact that there is an isomoprhism between linear maps and matrices which represent those linear maps i.e. $T \to [T]_\beta^\beta$ for each $T \in L(V)$ is an isomophism

Problem 6

> Prove that an operator $T \in L(V)$ on a finite dimensional vector space V is invertible if and only if 0 is not an eigenvalue of T

Solution

$\Rightarrow$ Suppose that 0 is an eigenvalue of T, then consider the equation $Tv = 0v = \mathbf{0}$. Then any $v \in V$ will satisfy this equation, which implies that the null space of T is not only $\{\mathbf{0} \in V\}$ and therefore T cannot be injective and therefore is not invertible. (this is the contrapositive statement)

$\Leftarrow$ Suppose that T is not invertible then T is no injective. Then $\exists v \in V$ such that $v \neq \mathbf{0}$ and $T(v) = \mathbf{0} = 0v$ so that v is an eigenvector with zero eigenvalue. (this is also the contrapositive statement)

Problem 7

> Prove that the sum of two invariant subspaces is invariant

Solution

Suppose that U_1, U_2 are invariant subspaces under $T \in L(V)$

Then $U_1 + U_2$ is invariant if $T(z \in U_1 + U_2) \in U_1 + U_2$ for all $z \in U_1 + U_2$

let $z = u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$ (by the definition of sums of subspaces)

Then $T(u_1 + u_2) = T(u_1) + T(u_2)$ by linearity and

Then we can conclude that $T(u_1) + T(u_2)$ is an element of $U_1 + U_2$ since $T(u_1) \in U_1$ and $T(u_2) \in U_2$ by the assumption that U_1, U_2 are invariant subspaces under T

Problem 8

> Let V be a vector space over $\mathbb{F}$, let $T \in L(V)$ and let $W \subset V$ be a subspace invariant under T. Prove that $\text{null}(T|_W) = (\text{null}T) \cap W$

Solution

Let $T|_W : W \to W$

Suppose that $v \in \text{null}(T|_W)$ then $T|_W \ (v) = 0$ Clearly $v \in W$ and $v \in V$ and since $0 \in W \Rightarrow 0 \in V$ we have that v is in the null space of T. Then $v \in \text{null}T \cap W$

Suppose that $v \in \text{null}T \cap W$ then $v \in \text{null}T \wedge v \in W$
Note that W and V share the same 0 element since W is a subspace of V. Then $T(v) = 0 \in W$ which is the same condition for v being in the null space of $T|_W$ (that a vector in W must map to $0 \in W$)