math108B hw1

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Problem 1

> For the space of real-valued polynomials $P([0, 1])$ over $\mathbb{R}$ we define:
>
> $$\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x$$
>
> Prove that it is indeed an inner product. (Hint, you might want to reformulate "conjugate symmetry" specifcally for the real-nuber case).

Solution

Let the inner product be defined as in Linear Algebra done Right (LADR) 6.2

Note that $f(x)f(x) = (f(x))^2$ is always non negative for all $x \in \mathbb{R}$, therefore its integral must be non negative

Suppose $\langle f, f\rangle = \int_0^1 f(x)f(x) = 0$. This implies $f(x) = 0 \in P([0, 1])$ (see the below link)
Additionally, if $f(x) = 0$ then it's clear that $\langle f, f\rangle = 0$. (due to definite integrating 0) Thus the inner product definition satisifes "definiteness"

Next, recall that $P([0, 1])$ over $\mathbb{R}$ is a vector space. Therefore, distributivity holds i.e. for $f_1, f_2, g \in P([0, 1])$ it is true that $(f_1 + f_2)g = f_1 g + f_2 g$

Then

$$\langle f_1 + f_2, g\rangle = \int_0^1 (f_1 + f_2)(x)g(x) = \int_0^1 f_1(x)g(x) + f_2(x)g(x)$$
$$= \int_0^1 f_1(x)g(x) + \int_0^1 f_2(x)g(x) = \langle f_1, g\rangle + \langle f_2, g\rangle$$

Additionally, let $\lambda \in \mathbb{R}$ then

$$\langle \lambda f, g\rangle = \int_0^1 \lambda f(x)g(x) = \lambda \int_0^1 f(x)g(x) = \lambda\langle f, g\rangle$$

For conjugate symmetry, recall from lecture that for any $x \in \mathbb{R}$ that $x = \overline{x}$. Note that $f(x) \in \mathbb{R}$ and $g(x) \in \mathbb{R}$ (after we evaluate the polynomial) therefore $\langle f, g\rangle = \overline{\langle f, g\rangle}$ and $\langle g, f\rangle = \overline{\langle g, f\rangle}$.

Additionally, recall that elements in vector spaces are commutative, therefore $\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x = \int_0^1 g(x)f(x)\mathrm{d}x = \langle g, f\rangle = \overline{\langle g, f\rangle}$ as desired. $\square$

references:
https://math.stackexchange.com/questions/1889443/prove-that-if-integral-of-a-squared-function-is-zero-then-function-is-zero-func

Problem 2

> Propose an inner product for the space $P_2([0,1])$ of real-valued polynomials up to the second order.
> $$P_2([0,1]) = \{a_0 + a_1x + a_2x^2\} \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

Solution

Let $p(t), q(t) \in P_2$ then we propose the following inner product:

$$\langle p(t), q(t) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

Verifying the inner product properties:

a) $\langle p(t), q(t) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) = q(0)p(0) + q(1)p(1) + q(2)p(2) = \langle q(t), p(t) \rangle$ (recall from problem 1 this is sufficient for conjugate symmetry since we are dealing with real vector space)

b) next, let

$$r(t) \in P_2$$

then $\langle p(t) + r(t), q(t) \rangle =$
$(p(0) + r(0))q(0) + (p(1) + r(1))q(1) + (p(2) + r(2))q(2) =$
$(p(0)q(0) + p(1)q(1) + p(2)q(2)) + (r(0)q(0) + r(1)q(1) + r(2)q(2)) = \langle p(t), q(t) \rangle + \langle r(t), q(t) \rangle$

c) next let $c \in \mathbb{R}$ then

$$\langle cp(t), q(t) = (cp(0))q(0) + (cp(1))q(1) + (cp(2))q(2) =$$

$$c(p(0)q(0) + p(1)q(1) + p(2)q(2)) =$$

$$c\langle p(t), q(t) \rangle$$

d) next, $\langle p(t), p(t) \rangle = p(0)p(0) + p(1)p(1) + p(2)p(2) = p(0)^2 + p(1)^2 + p(2)^2 \geq 0$.

$p(0) = p(1) = p(2) = 0 \Rightarrow \langle p(t), p(t) \rangle = 0$

And if $\langle p(t), p(t) \rangle = 0$ then $p(t)$ has three distinct zeroes, but this can only happen if $p(t) = 0$ for all $t \in [0,1]$. Therefore $\langle p(t), p(t) \rangle = 0 \Leftrightarrow p(t) = 0$ $\square$

Took directly from https://youtu.be/RqEOv38uv1I?si=NtHxuedW2jxia4II

Problem 3

> Note that the inner product $\langle \cdot, \cdot \rangle$ defined in 6.3 in the textbook only restricts that the first slot if linear, given a fixed second slot. That is, $\langle \cdot, u \rangle$ is linear for a fixed $u \in V$.
>
> For a vector space V over the real numbers $\mathbb{R}$, and associated with an inner product $\langle \cdot, \cdot, \rangle : V \times V \to \mathbb{R}$. Prove that the second slot is also linaer, given a fixed first slot $v \in V$. That is, $\langle v, \cdot \rangle$ is a linear map.

Solution

We know from https://math.libretexts.org/Bookshelves/Linear_Algebra/A_First_Course_in_Linear_Algebra_(Kuttler)/06%3A_Complex_Numbers/6.01%3A_Complex_Numbers (and previous classes) that the following two properties hold for all complex numbers z_1, z_2:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$
$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

We would like to show the following: Let $v \in V$ be fixed and let $w, w_1, w_2 \in V$ and $\lambda \in \mathbb{R}$. Then $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$ and $\langle v, \lambda w \rangle = \lambda \langle v, w \rangle$

Firstly,

$$\langle v, w_1 + w_2 \rangle = \overline{\langle v, w_1 + w_2 \rangle} = \langle w_1 + w_2, v \rangle = \langle w_1, v \rangle + \langle w_2, v \rangle = \overline{\langle w_1, v \rangle} + \overline{\langle w_2, v \rangle} = \langle v, w_1 \rangle + \langle v, w_2 \rangle$$

Secondly, $\langle v, \lambda w \rangle = \overline{\langle v, \lambda w \rangle} = \langle \lambda w, v \rangle = \lambda \langle w, v \rangle = \lambda \overline{\langle w, v \rangle} = \lambda \langle v, w \rangle$

In both parts we exploit the fact that if v, w have all real entries then $\langle v, w \rangle = \overline{\langle v, w \rangle}$ similar to problem 1 $\square$

Problem 4

> For an induced norm $\| \cdot \|$ associated with a vector space V over the complex numbers $\mathbb{C}$, prove that $\|\lambda v\| = |\lambda| \; \|v\|$ for any $\lambda \in \mathbb{C}$ and $v \in V$. Recall that if $\lambda = a + bi \in \mathbb{C}$, then $|\lambda| = \sqrt{a^2 + b^2}$

Solution

$\|\lambda v\|^2 = \langle \lambda v, \lambda v \rangle = \lambda \langle v, \lambda v \rangle = \lambda \overline{\lambda} \langle v, v \rangle = |\lambda|^2 \; \|v\|^2$, then take the square root of both sides to obtain the desired result. This is taken from LADR 6.9 and works for both $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$ $\square$

The square root operation should work out here since the norm $\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{\overline{v} \cdot v} \in \mathbb{R}$ for complex vector spaces, and for completeness $|\lambda| \in \mathbb{R}$. (since $a, b \in \mathbb{R}$)

Alternatively, for $v \in V(\mathbb{C})$ we have $\|v\| = \sqrt{|v_1|^2 + \ldots + |v_n|^2} \in \mathbb{R}$ where $v = (v_1, \ldots, v_n)$ and $|v_i| = \sqrt{a_i^2 + b_i^2}$ where $v_i = a_i + b_i$ and $a_i, b_i \in \mathbb{R}$

https://math.stackexchange.com/questions/1670156/norm-of-complex-vector

Problem 5

> Explain why the following are not inner products for the given spaces.
>
> a) $\langle (a,b),(c,d)\rangle = ac - bd$ for $\mathbb{R}^2$
>
> b) $\langle A, B\rangle = \operatorname{tr}(A+B)$ on the space of n by n matrices $\mathbb{R}^{n\times n}$

Solution

Consider the following counterexamples for each part:

a. let $a = 2$ $b = 3$ $c = 2$ $d = 3$ then $\langle (2,3)(2,3)\rangle = 4 - 9 = -5 < 0$ which violates positivity

b. let $A \in \mathbb{R}^{n\times n}$ have all of its diagonal entries be −1. Then $\langle A, A\rangle = \operatorname{tr}(A + A)$ but $A + A$ will now have −2 in all its diagonal entries. Therefore the trace is $(-2)n < 0$ which also violates positivity

Problem 6

> In class, for $\mathbb{R}^2$ with the standard inner product, we show that the induced norm $\|v\|^2 = \langle v, v\rangle$ satisfies $\|u + v\| \leq \|u\| + \|v\|$, by showing that $\langle u, v\rangle \leq |u\| \ \|v\|$ for any $u, v \in \mathbb{R}^2$.
>
> In the recorded video, we will show that the above holds for any induced norm $\|v\|^2 = \langle v, v\rangle$ for a general inner product space $(V, \langle\cdot,\cdot\rangle)$
>
> State your reason that one does not have to show $\|u + v\| \leq \|u\| + \|v\|$ if the norm $\|\cdot\|$ is the general norm defined via axioms.

Solution

The property that $\|u + v\| \leq \|u\| + \|v\|$ (triangle equality) must hold given any general norm $\|\cdot\|$.

However, if we define an arbitrary inner product which then induces a norm then we must show that the norm has all the properties of a general norm including the triangle equality

Problem 7

> For the vector space $(\mathbb{R}^2, \|\cdot\|)$ with an unspecified norm $\|\cdot\|$, its unit ball (which is a subset of $\mathbb{R}^2$) is defined by:
>
> $$B_{\|\ \|} = \{v \in \mathbb{R}^2 : \|v\| \leq 1\}$$
>
> Draw a picture for $B_1, B_2, B_{\max}$ when we pick the norm $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_{\max}$ respectively.

Solution

For any arbitrary point $p \in \mathbb{R}^2$

B_1 is a “unit diamond” centered at p

B_2 is the unit circle centered at p

$B_{\max}$ is the unit square (lines are parallel to axes) centered at p

see attached (the attached uses (0,0) as origin)

Problem 8

> Prove that $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent in $\mathbb{R}^d$, that is, for any $v \in \mathbb{R}^d$, htere exists some constants, C_1, C_2 sich that
> $$C_1 \ \|v\|_1 \leq \|v\|_2 \leq C_2 \|v\|_1$$

Solution

Pick $C_1 \leq \frac{\|v\|_2}{\|v\|_1}$ and $C_2 \geq \frac{\|v\|_2}{\|v\|_1}$ then since $\|v\|_1$ and $\|v\|_2$ are explicitly defined C_1, C_2 must exist.

However, if the constants C_1, C_2 cannot depend on v. Then set $n := d$ we can try to prove the inequality

$$\frac{1}{\sqrt{n}} \ \|v\|_1 \leq \|v\|_2 \leq \sqrt{n} \ \|v\|_1$$

using the cauchy schwarz inquality $|\langle u, v \rangle| \leq \|u\| \ \|v\|$ or using the standard inner (dot) product in $\mathbb{R}^n$: $|v \cdot w| \leq \|u\| \ \|v\|$

Firstly, note that

$$\|v\|_1 = \sum_{i=1}^{n} |v_i| \cdot 1 \leq \sqrt{\sum_{i=1}^{n} v_i^2} \sqrt{\sum_{i=1}^{n} 1^2} = \sqrt{n} \ \|v\|_2$$

which implies that $\frac{1}{\sqrt{n}} \ \|v\|_1 \leq \|v_2\|$

Next note that if

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2} \leq \sqrt{n} \sum_{i=1}^{n} |v_i| = \sqrt{n} \ \|v\|_1$$

and taking the square of both sides gives

$$\sum_{i=1}^{n} v_i^2 \leq n \sum_{i=1}^{n} v_i^2$$

which implies that $\|v\|_2 \leq \sqrt{n} \ \|v\|_1$

Therefore the claim holds if we set $C_1 = \frac{1}{\sqrt{n}}, C_2 = \sqrt{n}$ where $n = d$ $\square$

The following was helpful: https://math.stackexchange.com/questions/1426471/in-the-proof-that-l1-norm-and-l2-norm-are-equivalent