math108B hw4

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Problem 1

> On textbook, we actually never define the dimension for a subset which is not a subspace. In addition, the isomorphism to identify two subspaces is an invertible linear map. Now, we relax these definitions.
>
> We say the dimension of ausbset S is the dimension of a subspace U if there is a subspace $U \subset V$ and an injective and surjective map between S and U.
>
> Use this definition, to prove that the dimension of the subet $S = \{(x, y, 2) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$ is two.

Solution

Let $V = \mathbb{R}^3$ and $U \subset V = \mathbb{R}^2$. Let $s = (x, y, 2) \in S$ and define the consider map $M : S \to U$ by $M(s) \to (x, y)$

Injective: let $s, s' \in S$ and $s = (x_1, y_1, 2), s' = (x_2, y_2, 2)$ then if $M(s) = M(s') \Rightarrow (x_1, y_1) = (x_2, y_2) \Rightarrow x_1 = x_2, y_1 = y_2 \Rightarrow s = s'$

Surjective: let s be an element in the range of M and $(x, y, 2) \in S$. Then $s = (x, y) = M((x, y, 2))$

Since we have an injective and surjective map between S and U then the dimesnions of S is the dimension of U i.e. 2

Problem 2

> Construct an example of a subset S to conject that whether $S \subset \left(S^{\perp}\right)^{\perp}$ is true or $\left(S^{\perp}\right)^{\perp} \subset S$ is true. And prove your findings for arbitrary subset S in a vector space V.

Solution

consider $S, S^{\perp}, \left(S^{\perp}\right)^{\perp}$ where $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ and $S^{\perp} = \left\{ \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}$ and $\left(S^{\perp}\right)^{\perp} = \left\{ \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} : z \in \mathbb{R} \right\}$

then clearly, $S \subset \left(S^{\perp}\right)^{\perp} \land \left(S^{\perp}\right)^{\perp} \not\subset S$ (Note that $S, S^{\perp}, \left(S^{\perp}\right)^{\perp}$ is a subset of $\mathbb{R}^3$)

In general:

a) let $u \in S$ then for all $w \in S^{\perp}$ we have $\langle u, w \rangle = 0$. Since each u is orthogonal to any element in $S^{\perp}$, it follows that $u \in \left(S^{\perp}\right)^{\perp}$ and therefore $S \subset \left(S^{\perp}\right)^{\perp}$

b) Second, let $v \in \left(S^{\perp}\right)^{\perp}$

Since S is a subset and not a subspace, we do not know if $S + S^{\perp}$ form a direct sum of V and it is also not guaranteed that $S \cup S^{\perp} = V$. Although it is clear that $v \notin S^{\perp}$ it is possible that $v \notin S$ since V may be a union of $S \cup S^{\perp} \cup S_0$ for some $S_0 \subset V$

We can conclude that in general, $\left(S^{\perp}\right)^{\perp} \not\subset S$

Definition: Orthogonal projection of V onto a subspace U

Define the orthogonal projection of V onto a subspace U by $P_U : V \to V$ which maps $v \in V \to u \in U$ where $v = u + w$ such that $u \in U, w \in U^\perp$

Problem 3

> Let P_U be the orthogonal projection operator onto the subspace U in a vector space V. Prove that $P_U(u) = u$ for any $u \in U$ and $P_U(w) = 0$ for any $w \in U^\perp$

Solution

If $u \in U$ then $u = u + 0$ where $0 \in U^\perp$. Then by definition $P_U(u) = u$

If $w \in U^\perp$ then $w = 0 + w$ where $0 \in U$. Then by definition $P_U(w) = 0$

Definition: Projection

Consider $P : V \to V$, then P is a projection if $P(P(v)) = P(v)$ i.e. $P^2 = P$ ("two consecutive operations equivalent to one operation")

Problem 4

> Define a new projection operator from $\mathbb{R}^2$ onto the subspace $U = \{(x, y) \in \mathbb{R}^2 : y = x\}$ that is different from the ones in class. In class, we define the horizontal projection as $P_U^h((x, y)) = (x, x)$ and (also define) the orthogonal projection. You need to express it in an explicit formula $P_U((x, y))$ and prove it is a projection by definition.

Solution

Note that U can be thought of as $\mathtt{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

The projection of one vector onto another is defined in $\mathbb{R}^2$ with standard dot product as $\frac{a \cdot b}{b \cdot b} b$ where a is projected onto b. Here we are trying to project $a = (x, y)$ onto $b = (1, 1)$ so $(x, y) \cdot (1, 1) = x + y$ and $(1, 1) \cdot (1, 1) = 2$. Therefore,

we can define $P_U((x, y)) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$ since $P_U(P_U(x, y)) = P_U\left(\frac{x+y}{2}, \frac{x+y}{2}\right) = \left(\frac{2(x+y)}{4}, \frac{2(x+y)}{4}\right) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right) = P_U(x, y)$

Problem 5

> Construct an example of a vector $u, v \in \mathbb{R}^2$ such that $\langle v, P_U^h(u)\rangle \neq \langle P_U^h(v), u\rangle$, where $P_U^h(v)$ is the horizontal projection defined in class and in (4).

Solution

Let the inner product be the dot product (standard for $\mathbb{R}^n$)

Let $v = (1, 0), u = (0, 1)$ Then $P_U^h(v) = (1, 1)$ and $P_U^h(u) = (0, 0)$ however $(1, 1) \cdot (0, 0) \neq (1, 1) \cdot (0, 1)$

Problem 6

> Suppose that V_1, V_2, V_3 are subspaces of V. Define a linear map $T : V_1 \times V_2 \times V_3 \to V_1 + V_2 + V_3$ (recall $V_1 + V_2 + V_3$) is still a subspace of V by
>
> $$T(v_1, v_2, v_3) = v_1 + v_2 + v_3$$
>
> for $v_i \in V_i$
>
> (i) prove T is surjective
> (ii) Prove that if $V_1 + V_2 + V_3$ is indeed a direct sum, then T is injective.

Solution

(i) Let x be an element in the range of T i.e. $x \in V_1 + V_2 + V_3$. By the definition of sums of subspaces (LADR 1.36) we know that $x = v_1 + v_2 + v_3$ where $v_i \in V_i$. Take $z = (v_1, v_2, v_3)$ then $T(z) = x$ for all x. Therefore any x can be generated by $z = (v_1, v_2, v_3)$ through T

(ii) Suppose that $V_1 + V_2 + V_3$ is a direct sum. Then $V_1 \cap V_2 \cap V_3 = \{0\}$ (LADR 1.46). Although we could use this fact, it may be more concise to argue that since each element of $V_1 + V_2 + V_3$ can be written in exactly one way as $v_1 + v_2 + v_3$. Therefore each $(v_1, v_2, v_3) \in V_1 \times V_2 \times V_3$ will map to a unique $v_1 + v_2 + v_3 \in V_1 + V_2 + V_3$, hence T is injective. $\square$

Alternatively, assume $x \neq 0 \in V_1 \times V_2 \times V_3$ and try to show that $T(x) = 0 = v_1 + v_2 + v_3$. Therefore at least one $v_i \neq 0$, and it follows that the other v_i must add to its additive inverse. However, the additive inverse of $v_i \neq 0$ is also in V_i which contradicts $V_1 \cap V_2 \cap V_3 = \{0\}$. Therefore $x = 0$ in order for $T(x) = 0 \therefore$ $\text{null}(T) = \{0\}$ and T is injective.

Note

In class, given a closed convex set C in a vector space V equipped with an induced norm $\|\cdot\|$, we define the orthogonal projection $P_C : V \to V$ that maps a vector $x \in V$ to a vector $y^* \in C$ by

$$P_C(x) = y^* = \arg\min_{y \in C} \|x - y\|$$

***Theorem:* orthogonal projection characterization**

> Given a closed convex subset C of V, for every $x \in V$ we have
>
> $$z = P_C(x) \Leftrightarrow \langle y - z, x - z \rangle \leq 0 \forall y \in C$$

Problem 7

> Use the characterization theorem for the orthogonal projection to show that for a given cloased convex set C, for any $x, y \in V$ we have
>
> $$\|P_C(x) - P_C(y)\| \leq \|x - y\|$$

Solution

By the projection characterization theorem we have that

$\langle x - P_C(x), P_C(y) - P_C(x) \rangle \leq 0$ since $P_C(y) \in C$ and additionally,

$\langle y - P_C(y), P_C(x) - P_C(y) \rangle \leq 0$ since $P_C(x) \in C$

Factoring -1 from the second inequality gives

$\langle P_C(y) - y, P_C(y) - P_C(x) \rangle \leq 0$ and then we can use linearity (first slot) of inner products to obtain from the first equation:

$\langle x - y - P_C(x) + P_C(y), P_C(y) - P_C(x) \rangle \leq 0$ and apply linearity again to obtain

$\langle x - y, P_C(y) - P_C(x) \rangle + \|P_C(y) - P_C(x)\|^2 \leq 0$

$\|P_C(y) - P_C(x)\|^2 \leq -\langle x - y, P_C(y) - P_C(x) \rangle$

$\|P_C(y) - P_C(x)\|^2 \leq \langle -x + y, P_C(y) - P_C(x) \rangle$ and by Cauchy-Schwarz

$\|P_C(y) - P_C(x)\|^2 \leq \langle -x + y, P_C(y) - P_C(x) \rangle \leq \| - x + y\| \, \|P_C(y) - P_C(x)\|$

Then dividing both sides by $\|P_C(y) - P_C(x)\|$ and multiplying by −1 proves the desired result.

Problem 8

> If one interprets the norm $\|x - y\|$ as the distance between x and y, interpret the meaning of the above inequality.

Solution

The orthogonal projection map is a contraction. (of the vector connecting x and y)