math115A hw1

Jonas Chen

January 08, 2024

Problem 1

> (i) Suppose that $n > 1$ is a composite integer, with $n = rs$, say. Show that $2^n - 1$ is divisible by $2^r - 1$. (This shows that $2^n - 1$ is prime only if n is prime. Primes of the form $2^n - 1$ are called Mersenne primes.)
>
> (ii) Show that if $2^k + 1$ is prime, then k must be a power of 2. (This explains why Fermat only had to consider numbers of the form $f_n = 2^{2^n} + 1$ (using the notation described in class) when he was hunting for primes, rather than more general numbers of the form $2^k + 1$ for an arbitrary positive integer k.

Solution

(i) Want to show that $2^{rs} - 1$ is divisible by $2^r - 1$ where $n = rs$

To do this, we can see that

$$2^{rs} - 1 = 2^{r(s)} - 1^1$$

this construction allows us to use the difference of powers of formula:

$$2^{r(s)} - 1 = (2^r - 1)\left[(2^r)^{s-1} + ... + 2^r + 1\right]$$

therefore $2^n - 1$ is divisible by $2^r - 1$ $\square$

(ii) We want to show that if $2^k + 1$ is prime then k must be a power of 2

suppose that k cannot be divisible be an odd prime p (and is therefore not a power of 2) so that $k = mp$

then it holds that $2^k + 1 = 2^{mp} + 1 = (2m + 1)\left(1 - 2^m + 2^{2m} - ... + 2^{(p-1)m}\right)$ (geometric series expansion)

which means that $2^k + 1$ is divible by some (2m + 1) and is therefore composite, leading to a contradiction.

therefore in order for $2^k + 1$ to be prime k must be a power of 2. $\square$

Problem 2

> The Fibbonaci sequence $(F_n)_{n\geq 1}$ is defined recursively by the equations $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.
> Let α be any real number larger than $\beta = \frac{1}{2}\left(1 + \sqrt{5}\right)$. Prove by induction or otherwise that, for $n \geq 1$, $F_n < \alpha^n$. [Hint: You may find it helpful to note that β is a solution of the equation $x^2 = x + 1$]

Solution

For the base cases n=1 and n=2 we can observe that

$$F_n = F_1 = f_2 = 1 < \beta^1 = \frac{1}{2}\left(1 + \sqrt{5}\right) \approx 1.6 < \alpha^1$$

For the inductive hypothesis, assume that $F_n < \alpha^n$ for some $n \leq k$

Then $F_{k+1} = F_k + F_{k-1} < \alpha^k + \alpha^{k-1} < \alpha^{k-1}(\alpha + 1) < \alpha^{k+1}$ where the last statement comes from the fact that

$$\beta^2 = \beta + 1 \Rightarrow \alpha^2 > \alpha + 1 \Rightarrow \alpha^2\alpha^{k-1} > \alpha^{k-1}(\alpha + 1) \equiv \alpha^{k+1} > \alpha^{k-1}(\alpha + 1)$$

Therefore the claim holds (by induction). Since we proved the base case $k = 1$ as well as all $k + 1$

$\square$

Problem 3

> Prove that for $n \geq 1$, $\sum_1^n m = \frac{1}{2}n(n+1)$

Solution

Since m is not defined, we can reformulate the problem and get a solution using induction.

Reformulate the problem to the following:

Prove that for $n \geq 1$,

$$\sum_{i=1}^{n} m_i = \frac{n(n+1)}{2}$$

where $m_i = i$

For the base case $n = 1$ we can observe that 1 = (1(1+1))/2 = 1

For the inductive hypothesis assume that the claim holds for some $n = k$.

Then

$$\sum_{i=1}^{k+1} m_i = \sum_{i=1}^{k} m_i + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

$\square$

Problem 4

> Suppose that n pairs of gloves of different sizes are mixed together in a drawer. How many individual gloves must you take out if you are to be sure of having at least one complete pair. (justify your answer)

Solution

$n + 1$ pairs is the minimum to ensure having at least one complete pair.
If we draw n gloves, they may be all distinct in the worst case (one glove from each of the n pairs). However, the next glove must form a complete pair,

Problem 5

> prove that the integer $n = 2^{10}(2^{11} - 1)$ is not a perfect number by showing that $\sigma(n) \neq 2n$. [Hint: $2^{11} - 1 = 23 \times 89$]

Solution

Firstly, we know that if $2^n - 1$ is prime then $2^{n-1}(2^n - 1)$ is a perfect number. It is known that there exists a one-to-one mapping between mersenne primes of the form $2^n - 1$ and the corresponding perfect number $2^{n-1}(2^n - 1)$ by the Euclid–Euler theorem. Since $2^{11} - 1$ is not a mersenne prime (it's not prime), then it follows that n is not perfect.

Alternatively, it is known that $\sigma(mn) = \sigma(m)\sigma(n)$ so long as m and n are coprime.

Since 2^{10} is a power of 2 this means that all of its divisors must also be powers of 2 or 1 (since $2^{10} = \underbrace{2 \cdot 2}_{10 \text{ times}}$).

The other number, $2^{11} - 1 = 23 \times 89$. Therefore 2^{10} and $2^{11} - 1$ are coprime (23 and 89 do not have 2 as a divisor.)

Next,

$$\sigma(2^{10}(2^{11} - 1)) = \sigma(2^{10})\sigma(2^{11} - 1) = (2^{10+1} - 1)\sigma(23 \times 89) = (2^{11} - 1)(24)(90)$$

where we use the fact that $\sigma(2^n) = 2^{n+1} - 1$. Also, since 23 and 89 are prime they are also coprime.

On the other hand, $2n = 2 \times 2^{10}(2^{11} - 1) = 2^{11}(2^{11} - 1) \neq (24 \times 90)(2^{11} - 1) = \sigma(n)$

$\square$

Problem 6

> Verify each of the following statements:
>
> a) No power of a prime can be a perfect number
>
> b) The product of two odd primes is never a perfect number

Solution

a) Let p be a prime. We want to see if $p^k = \sigma(p^k) = 1 + p + p^2 + \ldots + p^{k-1} = 1 + p(1 + p + \ldots + p^{k-2})$.

But since p^k is divisible by p but $1 + p(1 + p + \ldots + p^{k-2})$ is not, $p^k \neq 1 + p + p^2 + \ldots + p^{k-1}$ implies p^k is not perfect.

Note that the definition of $\sigma(n)$ for this subproblem does not include n itself in the sum, and that prime numbers must be greater than 1.
$\square$ b. By the euclid-euler theorem, a perfect number must be en even number multiplied by an odd prime number (2 is not a mersenne prime), which proves the claim.

Alternatively, let p_1, p_2 be both odd and prime. Then we want to see (or show) that $2p_1p_2 = \sigma(p_1p_2)$, supposing this is true for the sake of contradiction.

Note that p_1, p_2 are coprime therefore

$$\sigma(p_1p_2) = \sigma(p_1)\sigma(p_2) = (p_1 + 1)(p_2 + 1) = p_1p_2 + p_1 + p_2 + 1$$

Equivalently, if we exclude p_1p_2 from the sum of divisors we can see if

$$p_1 + p_2 + 1 = p_1p_2$$

But if this is true then $p_1p_2 - p_1 - p_2 + 1 = 2 = (p_1 - 1)(p_2 - 1)$

But since $p_1, p_2 \geq 3 \Rightarrow (p_1 - 1)(p_2 - 1) \geq 4 > 2$ which is a contradiction. $\square$

Problem 7

> a) If n is a perfect number, prove that $\sum_{d|n} \left(\frac{1}{d}\right) = 2$
>
> b) Show that no proper divisor of a perfect number can be perfect.

Solution

a. Note that $\sigma(n) = 2n \Rightarrow \frac{\sigma(n)}{n} = 2$

and that $\frac{\sigma(n)}{n} = \sum_{d|n} \frac{d}{n} = \sum_{d|n} \left(\frac{1}{d}\right)$

b) Note that all known perfect numbers are composite

Therefore if we have some proper divisor i of a perfect number n then using part (a):

$2 = \sum_{d|n} \left(\frac{1}{d}\right) > \sum_{d|i} \left(\frac{1}{d}\right)$ which implies that i is not perfect.

Problem 8

> If $\sigma(n) = kn$ where $k \geq 3$, then the positive integer n is called a k-perfect number. Establish the following assertions concerning k-perfect numbers:
> (a) If n is a 3-perfect number, and $3 \nmid$ n, then 3n is 4-perfect.
> (b) If n is a 5-perfect number, and $5 \nmid$ n, then 5n is 6-perfect.
> (c) If 3n is a 4k-perfect number, and $3 \nmid$ n, then n is 3k-perfect.

Solution

a) since n is 3-perfect, then we have $\sigma(3n) = \sigma(3)\sigma(n) = 4\sigma(n) = 4(3n)$ therefore $3n$ is 4-perfect and the multiplicative properties hold for coprime numbers 3 and n (note that $3 \nmid n$)

b) similar to part (a) we can compute $\sigma(5n) = \sigma(5)\sigma(n) = 6(5n)$ therefore $5n$ is 6 perfect.

c) since $3n$ is $4k$ perfect, we know that $\sigma(3n) = (4k)(3n)$. since $3 \nmid n$ like in part (a) we also know that $\sigma(3n) = 4(3n) = 4\sigma(n)$.

This implies that $4\sigma(n) = (4k)(3n)$. Dividing 4 on both sides gives $\sigma(n) = 3kn \Rightarrow n$ is $3k$ perfect.

Resources:

https://en.wikipedia.org/wiki/Perfect_number

https://penguinmaths.blogspot.com/2019/07/proof-sum-of-divisors-function-is.html

https://people.math.harvard.edu/~knill/seminars/perfect/handout.pdf

https://math.stackexchange.com/questions/157419/show-that-sum-nolimits-dn-frac1d-frac-sigma-nn-for-every-pos

https://math.stackexchange.com/questions/4051082/prove-that-sigman-n-sum-d-mid-n-1-d