math115A hw3

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Problem 1

> (Euler, 1770). Divide 100 by two summands such that one is divisible by 7 and the other by 11.

Solution

We can try to solve the linear diophantine equation $7x + 11y = 100$

The gcd of 7,11 is 1 which divides 100 so the equation should be solvable

We can first try to solve $7x_i + 11y_i = 1$ and by inspection we can we that one solution is $x_i = -3, y_i = 2$

Then one solution will be $\left(x_i \cdot \frac{n}{d}, y_i \frac{n}{d}\right) = (-3 \cdot 100, 2 \cdot 100) = (-300, 200)$ where $n = 100$

We can verify the solution by resubstituing into the original equation: (-2100+2200=100)

Thus the two summands are -2100 and 2200 which are divisible by 7 and 11 respectively.

Problem 2

> Find the last two digits of the number 9^{9^9}. [Hint: $9^9 \equiv 9 (\operatorname{mod} 10)$; hence $9^{9^9} = 9^{9+10k}$; now use the fact that $9^9 \equiv 89 (\operatorname{mod} 100)$]

Solution

Using the hint we know that

$$9^{9^9} = 9^{9+10k} = 9^9 \cdot 9^{10k} \equiv 89 \cdot \left(9 \cdot 9^9\right)^k \equiv 89(1)^k \equiv 89 (\operatorname{mod} 100)$$

Thus we know that $9^{9^9} = 100(k) + 89$ and therefore the last two digits are 89.

Additionally note that $9^9 \cdot 9 \equiv 89 \cdot 9 = 801 \equiv 1 (\operatorname{mod} 100)$ which justifies the second to last equivalence.

Problem 3

> Show that 2^n divides an integer N if and only if 2^n divides the number made up of the last n digits of N

Solution

Let

$$N = a_{n+i}10^{n+i} + \ldots + a_n 10^n + a_{n-1}10^{n-1} + \ldots + a_0$$

be the base 10 expansion of N where $i \geq 0, n \geq 1$.

Then, note that $10^n = 2^n \cdot 5^n$

Therefore $N = 10^n(a_{n+i}10^i + \ldots + a_n) + (a_{n-1}10^{n-1} + \ldots + a_0)$ is divisible by 2^n iff the last digits (here $a_{n-1} \ldots a_0$) are divisible by 0

Since, by rules of modular arithmetic if $10^n(a_{n+i}10^i + \ldots + a_n) \equiv 0 (\bmod 2^n)$ and $(a_{n-1}10^{n-1} + \ldots + a_0) \equiv 0 (\bmod 2^n)$ and since N is a sum of these two terms then if their sum is divisible by 2^n then N will also be divisible by 2^n.

Problem 4

> a. Let $N = a_m 10^m + ... + a_2 10^2 + a_1 10 + a_0$, where $0 \leq a_k \leq 9$ be the decimal expansion of a positive integer N. Prove that $7, 11$, and 13 all divide N if and only if $7, 11$, and 13 divide the integer $M = (100a_2 + 100a_1 + a_0) - (100a_5 + 100a_4 + a_3) + (100a_8 + 100a_7 + a_6) - ...$
>
> [Hint: If n is even, then $10^{3n} \equiv 1, 10^{3n+1} \equiv 10, 10^{3n+2} \equiv 100 (\operatorname{mod} 1001)$; If n is odd, then $10^{3n} \equiv -1, 10^{3n+1} \equiv -10, 10^{3n+2} \equiv -100 (\operatorname{mod} 1001)$]
>
> b. Show that the number 29310478561 is divisible by 7 and 13, but not by 11.

Solution

a.

First observe that $7 \times 13 \times 11 = 1001$ and that $7, 11, 13$ are all prime.

Thus, $7, 13, 11 \mid N \Rightarrow N \equiv 0 (\operatorname{mod} 1001)$ and similarly, $7, 13, 11 \mid M \Rightarrow M \equiv 0 (\operatorname{mod} 1001)$

Thus if we can show that $M \equiv N (\operatorname{mod} 1001)$ then the claim will follow

Several students believe that there is a typo, and that M should be defined as

$$M = (100a_2 + 10a_1 + a_0) - (100a_5 + 10a_4 + a_3) + (100a_8 + 10a_7 + a_6) - ... + ...$$

Then the claim will hold by the hints given since

$$\begin{gathered} a_0 \equiv a_0 (\operatorname{mod} 1001) \\ 10a_1 \equiv 10a_1 (\operatorname{mod} 1001) \\ 100a_2 \equiv 100a_2 (\operatorname{mod} 1001) \\ 10^3 a_3 \equiv -1a_3 (\operatorname{mod} 1001) \\ 10^4 a_4 \equiv -10a_4 (\operatorname{mod} 1001) \\ 10^5 a_5 \equiv -100a^5 (\operatorname{mod} 1001) \\ 10^6 \equiv 1a_6 (\operatorname{mod} 1001) \\ \vdots \end{gathered}$$

(Starting from the fourth equivalence we can start apply the hints)

Then since these numbers are pairwise congruent mod 1001 we can the numbers on the left column (which are from N) and the numbers on the right column (numbers from M) and maintain congruence mod 1001. Therefore $M \equiv N (\operatorname{mod} 1001)$ and the claim holds.

b.

From part a, we know that 29310478561 is divisible by $7, 11, 13$ if and only if $561 - 478 + 310 - 29 = 364$ is divisible by $7, 11, 13$. However, a possible factorization of 364 is $364 = 7 \times 13 \times 4$ therefore divisibility by 7 and 13 holds.

In fact, the number does not end in 1 so divisibility by 11 will not hold. Also from class we know that divisibility by 11 holds iff the alternating sum of the digits of a numbers are divisible by 11. But the alternating sum of 29310478561 is $1 - 6 + 5 - 8 + 7 - 4 + 0 - 1 + 3 - 9 + 2 = -10$.

Problem 5

> An old and somewhat illegible invoice show that 72 canned hams were purchased for $\$x67.9y$ Find the missing digits.

Solution

Since $72 = 8 \times 9$ then $9 \mid x679y$

A number is divisible by 9 if the sum of its digits is also divisible by 9, therefore $x + 6 + 7 + 9 + y \equiv 0 (\operatorname{mod} 9)$

If $x + y = 5$ then the result (27) will be divisible by 9.

After trying a few combinations if $x = 3, y = 2$ then we have $367.92/72 = 5.11$ (each can was purchased for $\$5.11$) (the decimal point doesn't matter since everything is in base 10)

Problem 6

> Prove that there exist one million consecutive integers, each of which is divisible by the cube of an integer > 1

Solution

Since there are infinitely many primes, it follows that there are infinitely many cubes of primes that are coprime with eachother.

Pick 1 million consecutive coprime cubes of integers, and assume there exist 1 million consecutive integers that are divisible by each corresponding cube. Then the chinese remainder theorem guarantees that this is true (the unique x is 0) (see below)

□

The chinese remainder theorem states that there exist unique x such the following congruences hold:

$$x \equiv a_1 (\bmod n_1) ... x \equiv a_k (\bmod n_k)$$

as long as $n_1, ..., n_k$ are coprime and $0 \leq a_i \leq n_i$