CSE 152: Computer Vision

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Lecture 12: Multiview Geometry

Some slides in this lecture are courtesy to Prof. Silvio Savarese

Agenda

- Why is stereo useful?
- Epipolar constraints
- Fundamental matrix
- Estimating F

Recovering structure from a single view

Intrinsic ambiguity of the mapping from 3D to image (2D)

Courtesy slide S. Lazebnik

Two eyes help!

Two eyes help!

$$P = l_1 \times l_2$$

[Eq. 1]

This is called triangulation

Triangulation

- Find P* that minimizes

$$d(p_1, M_1P^*) + d(p_2, M_2P^*) \quad \text{[Eq. 2]}$$

Multi (stereo)-view geometry

- **Camera geometry:** Given corresponding points in two images, find camera matrices, position and pose.
- **Scene geometry:** Find coordinates of 3D point from its projection into 2 or multiple images.
- **Correspondence:** Given a point p in one image, how can I find the corresponding point p’ in another one?

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Epipolar geometry

- Epipolar Plane
- Baseline
- Epipolar Lines

- Epipoles e_1, e_2

  = intersections of baseline with image planes

  = projections of the other camera center

Example of epipolar lines

Example: Parallel image planes

- Baseline intersects the image plane at infinity
- Epipoles are at infinity
- Epipolar lines are parallel to the baseline (O_1O_2)

Example: Parallel Image Planes

Epipolar Constraint

- Two views of the same object
- Given a point on left image, how can I find the corresponding point on right image?

Epipolar geometry

Epipolar Constraint

Epipolar Constraint

$P_{O_1} = RP_{O_2} + T$ implies that $P_{O_2} = [R^T, -R^T T]\begin{bmatrix} P_{O_1} \\ 1 \end{bmatrix}$

$$M_1 = K_1[\, I \quad 0 \,] \qquad M_2 = K_2[\, R^T \quad -R^T\, T \,]$$

$$M_1 P = [u_1\ v_1\ 1]^T \qquad M_2 P = [u_2\ v_2\ 1]^T$$

[Eq. 3] [Eq. 4]

P means P_{O_1} in these equations

Epipolar Constraint

$$K_{canonical} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$K_1 = K_2$ are known (canonical cameras)

$$M_1 = K_1[\, I \quad 0 \,] \qquad M_2 = K_2[\, R^T \quad -R^T T \,]$$

$$M_1 = [\, I \quad 0 \,] \qquad M_2 = [\, R^T \quad -R^T T \,]$$

[Eq. 5] [Eq. 6]

Epipolar Constraint

(actually, convert retina plane metric coordinate $\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$ in 2D as $\mathrm{p}_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix}$ in 3D with unit distance to O_2)

p_2 in first camera reference system is = $Rp_2 + T$

$T \times [(Rp_2) + T] = T \times (Rp_2)$ is perpendicular to epipolar plane

$$\rightarrow p_1^T \cdot [T \times (Rp_2)] = 0 \quad \text{[Eq. 7]}$$

Cross product as matrix multiplication

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

$$[a_\times] = -[a_\times]^T$$

"skew-symmetric matrix"

Epipolar Constraint

$$p_1^{\,T} \cdot [T \times (R\, p_2)] = 0 \rightarrow$$

[Eq. 8]

$$p_1^T \boxed{([T_\times]R)} p_2 = 0$$

[Eq. 9]

E = Essential matrix

(Longuet-Higgins, 1981)

Epipolar Constraint

$$p_1^T \cdot E p_2 = 0$$

[Eq. 10]

- $m_1 = E\, p_2$ is the epipolar line associated with p_2
- $m_2 = E^T p_1$ is the epipolar line associated with p_1
- $E\, e_2 = 0$ and $E^T e_1 = 0$
- E is 3x3 matrix; 5 DOF
- E is singular (rank two)

Agenda

- Why is stereo useful?
- Epipolar constraints
- **Fundamental matrix**
- Estimating F

Epipolar Constraint

$$M_1 = K_1[\, I \quad 0 \,] \qquad M_2 = K_2[\, R^T \quad -R^T\, T \,]$$

"ideal" camera

$$\tilde{p}_{1c} = K_1^{-1} p_1 \text{ [Eq. 11]} \qquad \tilde{p}_{2c} = K_2^{-1} p_2 \text{ [Eq. 12]}$$

Goal: Cancel the adjustment made by K which was to accommodate the device

Epipolar Constraint

$$p_{1c} = K_1^{-1} p_1 \quad \text{[Eq. 11]}$$

$$p_{2c} = K_2^{-1} p_2 \quad \text{[Eq. 12]}$$

[Eq.9]

$$p_{1c}^T \cdot [T_\times] \cdot R\, p_{2c} = 0 \rightarrow (K_1^{-1} p_1)^T \cdot [T_\times] \cdot R\, K_2^{-1} p_2 = 0$$

$$p_1^T \boxed{K_1^{-T} \cdot [T_\times] \cdot R\, K_2^{-1}}\, p_2 = 0 \rightarrow p_1^T \boxed{F}\, p_2 = 0 \quad \text{[Eq. 13]}$$

Epipolar Constraint

[Eq. 13] $$\boxed{p_1^T \, F \, p_2 = 0}$$

$$F = K_1^{-T} \cdot [T_\times] \cdot R \, K_2^{-1}$$

[Eq. 14]

F = Fundamental Matrix

(Faugeras and Luong, 1992)

Epipolar Constraint

$$p_1^T \cdot F p_2 = 0$$

- $m_1 = F\ p_2$ is the epipolar line associated with p_2
- $m_2 = F^T p_1$ is the epipolar line associated with p_1
- $F\ e_2 = 0$ and $F^T e_1 = 0$
- F is 3x3 matrix; 7 DOF
- F is singular (rank two)

Why F is useful?

- Suppose F is known
- No additional information about the scene and camera is given
- Given a point on left image, we can compute the corresponding epipolar line in the second image

Why F is useful?

- F captures information about the epipolar geometry of 2 views + camera parameters

- MORE IMPORTANTLY: F gives constraints on how the scene changes under view point transformation (without reconstructing the scene!)

- Powerful tool in:
  - 3D reconstruction
  - Multi-view object/scene matching

Agenda

- Why is stereo useful?
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- Fundamental matrix
- **Estimating F**

Estimating F

The Eight-Point Algorithm

(Longuet-Higgins, 1981)

(Hartley, 1995)

$$p_1^T F p_2 = 0$$

Estimating F

Suppose we have a pair of corresponding points:

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

[Eq. 13] $\mathrm{p}^{\mathrm{T}}\,\mathrm{F}\,\mathrm{p}' = 0$ ➡

$$(u, v, 1)\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = 0$$

➡

$$(uu', uv', u, vu', vv', v, u', v', 1)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

[Eq. 14]

Let's take 8 corresponding points

Estimating F

Estimating F

$$\left(u_i u'_i, u_i v'_i, u_i, v_i u'_i, v_i v'_i, v_i, u'_i, v_i', 1\right)\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \qquad \text{[Eq. 14]}$$

Estimating F

$$\mathbf{W}\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix}\begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0 \quad \text{[Eqs. 15]}$$

$\mathbf{f}$

- Homogeneous system $\mathbf{W}\,\mathbf{f} = 0$
- Rank 8 ⟶ A non-zero solution exists (unique)
- If N>8 ⟶ Lsq. solution by SVD! ⟶ $\hat{\mathrm{F}}$

$$\|\mathbf{f}\| = 1$$

$\hat{F}$ satisfies: $p^T \hat{F} p' = 0$

and estimated $\hat{F}$ may have full rank ($\det(\hat{F}) \neq 0$)

But remember: fundamental matrix is Rank2

Find F that minimizes $\left\| F - \hat{F} \right\|$

Frobenius norm (*)

Subject to rank(F)=2

SVD (again!) can be used to solve this problem

(*) Sq. root of the sum of squares of all entries

Find F that minimizes $\left\| F - \hat{F} \right\|$

Frobenius norm (*)

Subject to det(F)=0

$$F = U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

Where:

$$U \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} V^T = SVD(\hat{F})$$

[HZ] pag 281, chapter 11, "Computation of F"

Problems with the 8-Point Algorithm

$$\mathbf{W}\,\mathbf{f} = 0, \quad \|\mathbf{f}\| = 1 \xrightarrow[\text{by SVD}]{\text{Lsq solution}} \mathrm{F}$$

- Recall the structure of W:
  - do we see any potential (numerical) issue?

Problems with the 8-Point Algorithm

$$\mathbf{Wf} = 0$$

$$\begin{pmatrix} u_1u'_1 & u_1v'_1 & u_1 & v_1u'_1 & v_1v'_1 & v_1 & u'_1 & v'_1 & 1 \\ u_2u'_2 & u_2v'_2 & u_2 & v_2u'_2 & v_2v'_2 & v_2 & u'_2 & v'_2 & 1 \\ u_3u'_3 & u_3v'_3 & u_3 & v_3u'_3 & v_3v'_3 & v_3 & u'_3 & v'_3 & 1 \\ u_4u'_4 & u_4v'_4 & u_4 & v_4u'_4 & v_4v'_4 & v_4 & u'_4 & v'_4 & 1 \\ u_5u'_5 & u_5v'_5 & u_5 & v_5u'_5 & v_5v'_5 & v_5 & u'_5 & v'_5 & 1 \\ u_6u'_6 & u_6v'_6 & u_6 & v_6u'_6 & v_6v'_6 & v_6 & u'_6 & v'_6 & 1 \\ u_7u'_7 & u_7v'_7 & u_7 & v_7u'_7 & v_7v'_7 & v_7 & u'_7 & v'_7 & 1 \\ u_8u'_8 & u_8v'_8 & u_8 & v_8u'_8 & v_8v'_8 & v_8 & u'_8 & v'_8 & 1 \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} = 0$$

- Highly un-balanced (not well conditioned)
- Values of W must have similar magnitude
- This creates problems during the SVD decomposition

HZ pag 108

Normalization

- IDEA: Transform image coordinates such that the matrix W becomes better conditioned (pre-conditioning)

- For each image, apply a transformation T (translation and scaling) acting on image coordinates such that:

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels

Example of normalization

Coordinate system of the image before applying T

Coordinate system of the image after applying T

- Origin = centroid of image points
- Mean square distance of the image points from origin is ~2 pixels

Normalization

$$q_i = T\ p_i$$

$$q'_i = T'\ p'_i$$

The Normalized Eight-Point Algorithm

Compute T and T' for image 1 and 2, respectively

Normalize coordinates in images 1 and 2:

$$q_i = T\ p_i \qquad q_i' = T'\ p_i'$$

Use the eight-point algorithm to compute $\hat{F}_q$ from the corresponding points q_i and q_i' .

Enforce the rank-2 constraint: $\rightarrow$ $\mathrm{F_q}$ such that:

$$\begin{cases} \mathrm{q}^{\mathrm{T}} \mathrm{F_q}\ \mathrm{q}' = 0 \\ \det(\mathrm{F_q}) = 0 \end{cases}$$

De-normalize F_q:

$$F = T^T F_q\ T'$$

Without normalization

Mean errors:
10.0pixel
9.1pixel

With normalization

Mean errors:
1.0pixel
0.9pixel