

Linear Algebra Primer (cont')

Note: the slides are based on CS131 (Juan Carlos et al) and EE263 (by Stephen Boyd et al) at Stanford. Reorganized, revised, and typed by Hao Su

Outline

- ▶ Geometry of Linear Algebra
 - ▶ Vector spaces
 - ▶ Basis, dimension
 - ▶ Nullspace, range
- ▶ Spectral Decomposition
 - ▶ Eigenpairs
 - ▶ Spectral theory
- ▶ Singular Value Decomposition
 - ▶ Geometry of linear maps
 - ▶ Singular values, singular vectors
 - ▶ Pseudo-inverse
- ▶ Matrix Calculus
 - ▶ Gradient

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Vector Spaces

a **vector space** or **linear space** (over the reals) consists of

- ▶ a set \mathcal{V}
- ▶ a vector sum $+: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$
- ▶ a scalar multiplication: $\mathbb{R} \times \mathcal{V} \rightarrow \mathcal{V}$
- ▶ a distinguished element $0 \in \mathcal{V}$

which satisfy a list of properties

Vector Space Axioms

- ▶ $x + y = y + x, \forall x, y \in \mathcal{V}$
- ▶ $(x + y) + z = x + (y + z), \forall x, y, z \in \mathcal{V}$
- ▶ $0 + x = x, x \in \mathcal{V}$
- ▶ $\forall x \in \mathcal{V} \quad \exists(-x) \in \mathcal{V} \text{ s.t. } x + (-x) = 0$
- ▶ $(\alpha\beta)x = \alpha(\beta x), \quad \forall \alpha, \beta \in \mathbb{R} \quad \forall x \in \mathcal{V}$
- ▶ $\alpha(x + y) = \alpha x + \alpha y, \quad \forall \alpha \in \mathbb{R} \quad \forall x, y \in \mathcal{V}$
- ▶ $(\alpha + \beta)x = \alpha x + \beta x, \quad \forall \alpha, \beta \in \mathbb{R} \quad \forall x \in \mathcal{V}$
- ▶ $1x = x, \quad \forall x \in \mathcal{V}$

$+$ is commutative

$+$ is associative

0 is additive identity

existence of additive inverse

scalar mult. is associative

right distributive rule

left distributive rule

1 is mult. identity

Examples

- ▶ $\mathcal{V}_1 = \mathbb{R}^n$, with standard (componentwise) vector addition and scalar multiplication
- ▶ $\mathcal{V}_2 = \{0\}$ (where $0 \in \mathbb{R}^n$)
- ▶ $\mathcal{V}_3 = \text{span}(v_1, v_2, \dots, v_k)$ where
$$\text{span}(v_1, v_2, \dots, v_k) = \{\alpha_1 v_1 + \dots + \alpha_k v_k \mid \alpha_i \in \mathbb{R}\}$$
and $v_1, \dots, v_k \in \mathbb{R}^n$

Subspaces

- ▶ a **subspace** of a vector space is a **subset** of a vector space which is itself a vector space
- ▶ roughly speaking, a subspace is closed under vector addition and scalar multiplication
- ▶ examples $\mathcal{V}_1, \mathcal{V}_3, \mathcal{V}_3$ above are subspaces of \mathbb{R}^n

Vector Spaces of Functions

- ▶ $\mathcal{V}_4 = \{x : \mathbb{R}_+ \rightarrow \mathbb{R}^n \mid x \text{ is differentiable}\}$, where vector sum is sum of functions:

$$(x + z)(t) = x(t) + z(t)$$

and scalar multiplication is defined by

$$(\alpha x)(t) = \alpha x(t)$$

(a **point** in \mathcal{V}_4 is a **trajectory** in \mathbb{R}^n)

- ▶ $\mathcal{V}_5 = \{x \in \mathcal{V}_4 \mid \dot{x} = Ax\}$
(**points** in \mathcal{V}_5 are **trajectories** of the linear system $\dot{x} = Ax$)
- ▶ \mathcal{V}_5 is a subspace of \mathcal{V}_4

Basis and Dimension

set of vectors $\{v_1, v_2, \dots, v_k\}$ is called a **basis** for a vector space \mathcal{V} if

$$\mathcal{V} = \text{span}(v_1, v_2, \dots, v_k) \\ \text{and} \\ \{v_1, v_2, \dots, v_k\} \text{ is independent}$$

- ▶ equivalently, every $v \in \mathcal{V}$ **can be uniquely** expressed as

$$v = \alpha_1 v_1 + \dots + \alpha_k v_k$$

- ▶ for a given vector space \mathcal{V} , the number of vectors in any basis is the same
- ▶ number of vectors in any basis is called the **dimension** of \mathcal{V} , denoted **$\dim \mathcal{V}$**

Nullspace of a Matrix

the **nullspace** of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\mathbf{null}(A) = \{x \in \mathbb{R}^n | Ax = 0\}$$

- ▶ $\mathbf{null}(A)$ is set of vectors mapped to zero by $y = Ax$
- ▶ $\mathbf{null}(A)$ is set of vectors orthogonal to all rows of A

$\mathbf{null}(A)$ gives **ambiguity** in x given $y = Ax$:

- ▶ if $y = Ax$ and $z \in \mathbf{null}(A)$, then $y = A(x + z)$
- ▶ conversely, if $y = Ax$ and $y = A\tilde{x}$, then $\tilde{x} = x + z$ for some $z \in \mathbf{null}(A)$

$\mathbf{null}(A)$ is also written $\mathcal{N}(A)$

Zero Nullspace

A is called **one-to-one** if 0 is the only element of its null space

$$\text{null}(A) = \{0\}$$

Equivalently,

- ▶ x can always be uniquely determined from $y = Ax$ (i.e., the linear transformation $y = Ax$ doesn't 'lose' information)
- ▶ mapping from x to Ax is one-to-one: different x 's map to different y 's
- ▶ columns of A are independent (hence, a basis for their span)
- ▶ A has a **left inverse**, i.e., there is a matrix $B \in \mathbb{R}^{n \times m}$ s.t. $BA = I$
- ▶ $A^T A$ is invertible

Range of a Matrix

the **range** of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\text{range}(A) = \{Ax | x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

range(A) can be interpreted as

- ▶ the set of vectors that can be 'hit' by linear mapping $y = Ax$
- ▶ the span of columns of A
- ▶ the set of vectors y for which $Ax = y$ has a solution

range(A) is also written $\mathcal{R}(A)$

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 - ▶ Jacobian
 - ▶ Hessian

Eigenvector and Eigenvalue

- ▶ an eigenvector x of a linear transformation A is a non-zero vector that, when A is applied to it, does not change direction

$$Ax = \lambda x, \quad x \neq 0.$$

- ▶ applying A to the vector only scales the vector by the scalar value λ , called an **eigenvalue**.

Eigenvector and Eigenvalue

- ▶ we want to find all the eigenvalues of A :

$$Ax = \lambda x, \quad x \neq 0.$$

- ▶ which can be written as:

$$Ax = (\lambda I)x, \quad x \neq 0.$$

- ▶ therefore:

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

Eigenvector and Eigenvalue

- ▶ we can solve for eigenvalues by solving :

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

- ▶ above means that $\lambda I - A$ is not full rank, thus we can instead solve the above equation as:

$$|(\lambda I - A)| = 0.$$

- ▶ this is called **characteristic polynomial** of an $n \times n$ matrix

Properties of Eigenvalues

- ▶ the trace of A is equal to the sum of its eigenvalues:

$$\text{tr}(A) = \sum_{i=1}^n \lambda_i$$

- ▶ the determinant of A is equal to the product of its eigenvalues

$$|A| = \prod_{i=1}^n \lambda_i$$

- ▶ the rank of A is equal to the number of non-zero eigenvalues of A
- ▶ for general A , it can be proved by Schur Decomposition easily (omitted)
- ▶ for diagonalizable A , the proof is straightforward

Diagonalization

- ▶ if matrix A can be diagonalized, that is,

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- ▶ then:

$$AP = P \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

- ▶ write $P = [\vec{\alpha}_1, \dots, \vec{\alpha}_n]$, the above equation can be rewritten as

$$A\vec{\alpha}_i = \lambda_i\vec{\alpha}_i$$

Diagonalization by Spectral Decomposition

- ▶ here is a sufficient (but not necessary) condition
- ▶ assuming all λ_i 's are unique, by eigenvalue equation:

$$AV = VD$$

$$A = VDV^{-1}$$

- ▶ why?
 - ▶ eigenvectors associated with different eigenvalues are linearly independent, thus A invertible
 - ▶ in fact, if A is symmetric, V would be orthonormal and $A = VDV^T$

Diagonalization (Summary)

- ▶ an $n \times n$ matrix A is diagonalizable if it has n linearly independent (in fact, orthogonal) eigenvectors.
- ▶ matrices with n distinct eigenvalues are diagonalizable

Symmetric Matrices

Properties

- ▶ for a real symmetric matrix A , all the eigenvalues are real
- ▶ A is diagonalizable
- ▶ the eigenvectors of A are orthonormal

$$A = VDV^T$$

Symmetric Matrices

- ▶ therefore

$$x^T A x = x^T V D V^T x = y^T D y = \sum_{i=1}^n \lambda_i y_i^2$$

where $y = V^T x$

- ▶ so, if we wanted to find the vector x that

$$\max_{x \in \mathbb{R}^n} x^T A x \quad \text{subject to } \|x\|_2^2 = 1$$

Symmetric Matrices

- ▶ therefore

$$x^T A x = x^T V D V^T x = y^T D y = \sum_{i=1}^n \lambda_i y_i^2$$

where $y = V^T x$

- ▶ so, if we wanted to find the vector x that

$$\max_{x \in \mathbb{R}^n} x^T A x \quad \text{subject to } \|x\|_2^2 = 1$$

is the same as finding the eigenvector that corresponds to the largest eigenvalue.

Spectral Theory

- ▶ we call an eigenvalue λ and an associated eigenvector an **eigenpair**
- ▶ the space of vectors where $(A - \lambda I)x = 0$ is often called the **eigenspace** of A associated with the eigenvalue λ
- ▶ the set of all eigenvalues of A is called its **spectrum**:

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is singular}\}$$

Spectral Theory

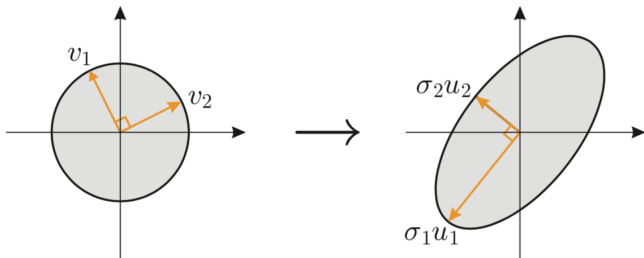
- ▶ the magnitude of the largest eigenvalue (in magnitude) is called the spectral radius

$$\rho(A) = \max\{|\lambda_1|, \dots, |\lambda_n|\}$$

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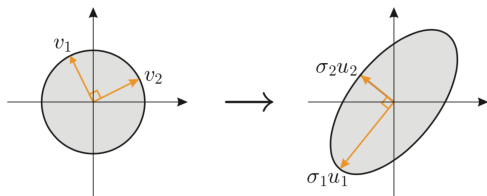
Geometry of Linear Maps



every matrix $A \in \mathbb{R}^{m \times n}$ maps the unit ball in \mathbb{R}^n to an ellipsoid in \mathbb{R}^m

$$S = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \quad AS = \{Ax \mid x \in S\}$$

Singular Values and Singular Vectors



- ▶ first, assume $A \in \mathbb{R}^{m \times n}$ is skinny and full rank
- ▶ the numbers $\sigma_1, \dots, \sigma_n > 0$ are called the **singular values** of A
- ▶ the vectors u_1, \dots, u_n are called the **left** or **output singular vectors** of A . These are **unit vectors** along the principal semi-axes of AS
- ▶ the vectors v_1, \dots, v_n are called the **right** or **input singular vectors** of A . These map to the principal semi-axes, so that

$$Av_i = \sigma_i u_i$$

Thin Singular Value Decomposition

$$Av_i = \sigma_i u_i \text{ for } 1 \leq i \leq n$$

For $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = n$, let

$$U = [u_1 \ u_2 \ \dots \ u_n] \quad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \quad V = [v_1 \ v_2 \ \dots \ v_n]$$

the above equation is $AV = U\Sigma$ and since V is orthogonal

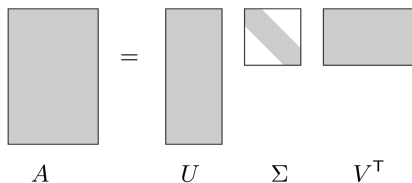
$$A = U\Sigma V^T$$

called the **thin SVD** of A

Thin SVD

For $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$, the **thin SVD** is

$$A = U \Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$$



here

- ▶ $U \in \mathbb{R}^{m \times r}$ has orthonormal columns,
- ▶ $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$, where $\sigma_1 \geq \dots \geq \sigma_r > 0$
- ▶ $V \in \mathbb{R}^{n \times r}$ has orthonormal columns

SVD and Eigenvectors

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T) = V \Sigma^2 V^T$$

hence:

- ▶ v_i are eigenvectors of $A^T A$ (corresponding to nonzero eigenvalues)
- ▶ $\sigma_i = \sqrt{\lambda_i(A^T A)}$ (and $\lambda_i(A^T A) = 0$ for $i > r$)
- ▶ $\|A\| = \sigma_1$

SVD and Eigenvectors

similarly,

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma^2 U^T$$

hence:

- ▶ u_i are eigenvectors of AA^T (corresponding to nonzero eigenvalues)
- ▶ $\sigma_i = \sqrt{\lambda_i(AA^T)}$ (and $\lambda_i(AA^T) = 0$ for $i > r$)

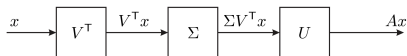
SVD and Range

$$A = U\Sigma V^T$$

- ▶ u_1, \dots, u_r are orthonormal basis for **range**(A)
- ▶ v_1, \dots, v_r are orthonormal basis for **null**(A)[⊥]

Interpretations

$$A = U\Sigma V^T = \sum_{i=1}^r \sigma_i u_i v_i^T$$



linear mapping $y = Ax$ can be decomposed as

- ▶ compute coefficients of x along input directions v_1, \dots, v_r
- ▶ scale coefficients by σ_i
- ▶ reconstitute along output directions u_1, \dots, u_r

difference with eigenvalue decomposition for symmetric A : input and output directions are **different**

General Pseudo-inverse

if $A \neq 0$ has SVD $A = U\Sigma V^T$, the **pseudo-inverse** or **Moore-Penrose inverse** of A is

$$A^\dagger = V\Sigma^{-1}U^T$$

- ▶ if A is skinny and full rank,

$$A^\dagger = (A^T A)^{-1} A^T$$

gives the least-squares approximate solution $x_{ls} = A^\dagger y$

- ▶ if A is fat and full rank,

$$A^\dagger = A^T (A A^T)^{-1}$$

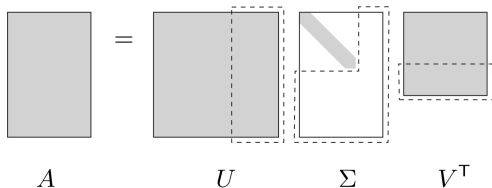
gives the least-norm solution $x_{ln} = A^\dagger y$

Full SVD

SVD of $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A)=r$

$$A = U_1 \Sigma_1 V_1^T = [u_1 \ \cdots \ u_r] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}$$

Add extra columns to U and V , and add zero rows/cols to Σ_1



Full SVD

- ▶ find $U_2 \in \mathbb{R}^{m \times (m-r)}$ such that $U = [U_1 \ U_2] \in \mathbb{R}^{m \times m}$ is orthogonal
- ▶ find $V_2 \in \mathbb{R}^{n \times (n-r)}$ such that $V = [V_1 \ V_2] \in \mathbb{R}^{n \times n}$ is orthogonal
- ▶ add zero rows/cols to Σ_1 to form $\Sigma \in \mathbb{R}^{m \times n}$

$$\Sigma = \left[\begin{array}{c|c} \Sigma_i & 0_{r \times (n-r)} \\ \hline 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{array} \right]$$

then the full SVD is

$$A = U_1 \Sigma_1 V_1^T = \left[\begin{array}{c|c} U_1 & U_2 \end{array} \right] \left[\begin{array}{c|c} \Sigma_i & 0_{r \times (n-r)} \\ \hline 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{array} \right] \left[\begin{array}{c} V_1^T \\ V_2^T \end{array} \right]$$

which is $A = U \Sigma V^T$

Image of Unit Ball under Linear Transformation

full SVD:

$$A = U\Sigma V^T$$

gives interpretation of $y = Ax$

- ▶ rotate (by V^T)
- ▶ stretch along axes by σ_i ($\sigma_i = 0$ for $i > r$)
- ▶ zero-pad (if $m > n$) or truncate (if $m < n$) to get m -vector
- ▶ rotate (by U)

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Matrix Calculus – the Gradient

- ▶ let a function $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ take as input a matrix A of size $m \times n$ and returns a real value
- ▶ then the **gradient** of f :

$$\nabla_A f(A) \in \mathbb{R}^{m \times n} = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \frac{\partial f(A)}{\partial A_{12}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \frac{\partial f(A)}{\partial A_{21}} & \frac{\partial f(A)}{\partial A_{22}} & \cdots & \frac{\partial f(A)}{\partial A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial A_{m1}} & \frac{\partial f(A)}{\partial A_{m2}} & \cdots & \frac{\partial f(A)}{\partial A_{mn}} \end{bmatrix}$$

Matrix Calculus – the Gradient

Properties

- ▶ $\nabla_x(f(x) + g(x)) = \nabla_x f(x) + \nabla_x g(x)$
- ▶ For $t \in \mathbb{R}$, $\nabla_x(tf(x)) = t\nabla_x f(x)$