Linear Algebra Primer

Note: the slides are based on CS131 (Juan Carlos et al) and EE263 (by Stephen Boyd et al) at Stanford. Reorganized, revised, and typed by Hao Su

Outline

- Vectors and Matrices
  - Basic matrix operations
  - Determinants, norms, trace
  - Special matrices
- Transformation Matrices
  - Homogeneous matrices
  - Translation
- Matrix inverse
- Matrix rank

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Vector

- A column vector $v \in \mathbb{R}^{n \times 1}$ where

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A row vector $v^T \in \mathbb{R}^{1 \times n}$ where

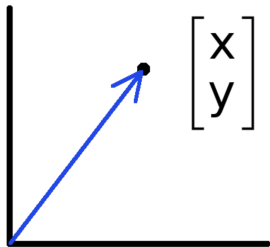$$v^T = [v_1 v_2 \ldots v_n]$$

T denotes the **transpose** operation

Vector

▶ We'll default to column vectors in this class

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

▶ You'll want to keep track of the orientation of your vectors when programming in Python

Vectors have two main uses

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

▶ Data (pixels, gradients at an image keypoint, etc) can also be treated as a vector

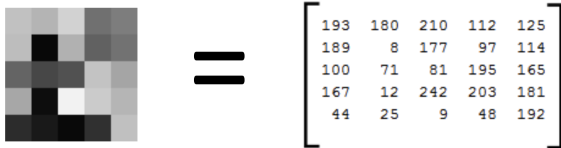▶ Such vectors do not have a geometric interpretation, but calculations like "distance" can still have value

▶ Vectors can represent an offset in 2D or 3D space

▶ Points are just vectors from the origin

Matrix

- A matrix $A \in \mathbb{R}^{m \times n}$ is an array of numbers with size m by n, i.e., m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

- if $m = n$, we say that A is square.

Images

$$\begin{bmatrix} 193 & 180 & 210 & 112 & 125 \\ 189 & 8 & 177 & 97 & 114 \\ 100 & 71 & 81 & 195 & 165 \\ 167 & 12 & 242 & 203 & 181 \\ 44 & 25 & 9 & 48 & 192 \end{bmatrix}$$

▶ Python represents an image as a matrix of pixel brightness
▶ Note that the upper left corner is $(y, x) = [0, 0]$

Color Images

- Grayscale images have one number per pixel, and are stored as an $m \times n$ matrix
- Color images have 3 numbers per pixel – red, green, and blue brightness (RGB)
- stored as an $m \times n \times 3$ matrix
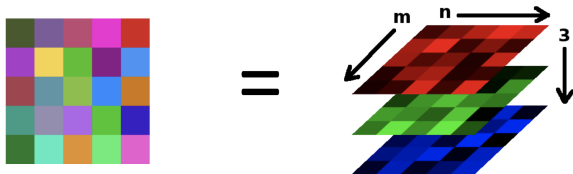
Basic Matrix Operations

We will discuss:

- Addition
- Scaling
- Dot product
- Multiplication
- Transpose
- Inverse/pseudo-inverse
- Determinant/trace

Matrix Operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+4 \end{bmatrix}$$

  - Can only add a matrix with matching dimensions or a scalar

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 7 = \begin{bmatrix} a+7 & b+7 \\ c+7 & d+7 \end{bmatrix}$$

- Scaling

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 3 = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

Vectors

- Norm: $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
- More formally, a norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 4 proerties:
  - Non-Negativity: For all $x \in \mathbb{R}^n$, $f(x) \geq 0$
  - Definiteness: $f(x) = 0$ if and only if $x = 0$
  - Homogeneity: For all $x\mathbb{R}^n$, $t \in \mathbb{R}$, $f(tx) = |t| f(x)$
  - Triangle inequality: For all $x, y \in \mathbb{R}^n$, $f(x + y) \leq f(x) + f(y)$

Matrix Operations

- Example norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|_\infty \qquad \|x\|_\infty = \max_i |x_i|$$

- General ℓ_p norms:

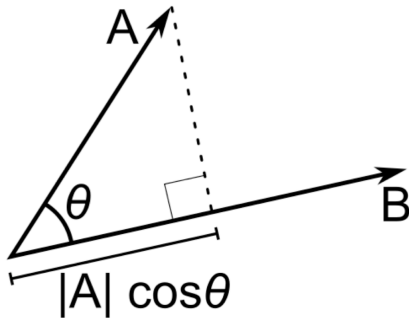$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

Matrix Operations

- Inner product (dot product) of vectors
  - Multiply corresponding entries of two vectors and add up the result
  - $x \cdot y$ is also $|x||y| \cos$(the angel between x and y)

$$x^T y = [x_1 \dots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \qquad \text{(scalar)}$$

Matrix Operations

- Inner product (dot product) of vectors
  - If B is a unit vector, then $A \cdots B$ gives the length of A, which lies in the direction of B

Matrix Operations

- The product of two matrices

$$A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}$$

$$C = AB \in \mathbb{R}^{m \times p}$$

$$C_{ij} = \sum_{i=1}^{n} A_{ik} B_{kj}$$

$$C = AB = \begin{bmatrix} -a_1^T- \\ -a_2^T- \\ \vdots \\ -a_m^T- \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}$$

Matrix Operations

Multiplication example:

A x B

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

Each entry of the matrix product is made by taking the dot product of the corresponding row in the left matrix, with the corresponding column in the right one.

$$\begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix} \quad \begin{bmatrix} 10 & 14 \\ 34 & 54 \end{bmatrix}$$

$$0 \cdot 3 + 2 \cdot 7 = 14$$

Matrix Operations

- The product of two matrices

  Matrix multiplication is associative: (AB)C=A(BC)

  Matrix multiplication is distributive: A(B+C)=AB+AC

  Matrix multiplication is, in general, *not* commutative; that is, it can be the case that $AB \neq BA$ (For example, if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times q}$, the matrix product BA does not even exist if m and q are not equal!)
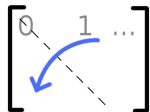
Matrix Operations

- Powers
    - By convention, we can refer to the matrix product AA as A^2, and AAA as A^3, etc.
    - Obviously only square matrices can be multiplied that way

Matrix Operations

- Transpose – flip matrix, so row 1 becomes column 1

$$\begin{bmatrix} 0 & 1 & \dots \\ & & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$

Matrix Operations

- Determinant
    - $\det(A)$ returns a scalar
    - Represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix
    - For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \det(A) = ad - bc$
    - Properties:

$$\det(AB) = \det(A)\det(B)$$
$$\det(AB) = \det(BA)$$
$$\det(A^{-1}) = \frac{1}{\det(A)}$$
$$\det(A^T) = \det(A)$$
$$\det(A) = 0 \iff A \text{ is singular}$$

Matrix Operations

- Trace
  - trace(A) = sum of diagonal elements

$$\text{tr}(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}) = 1 + 7 = 8$$

  - Invariant to a lot of transformations, so it's used sometimes in proofs. (Rarely used in this class, though)
  - Properties:

$$\text{tr}(AB) = \text{tr}(BA)$$
$$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$$

Matrix Operations

- Vector norms

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \qquad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

- Matrix norms: Norms can also be defined for matrices, such as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2} = \sqrt{\mathrm{tr}(A^T A)}$$

Special Matrices

- Identity matrix I

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Special Matrices

- Symmetric matrix: $A^T = A$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

- Skew-symmetric matrix: $A^T = -A$

$$\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$$
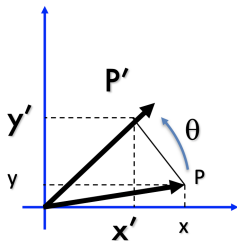
Outline

Transformation

- Matrices can be used to transform vectors in useful ways, through multiplication: $x' = Ax$
- Simplest is scaling:

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

(Verify by yourself that the matrix multiplication works out this way)

Rotation (2D case)

Counter-clockwise rotation by an angle θ

$$x' = \cos\theta x - \sin\theta y$$
$$y' = \cos\theta y + \sin\theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = RP$$

Transformation Matrices

- Multiple transformation matrices can be used to transform a point:

$$p' = R_2 R_1 S p$$

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$$(R_2(R_1(Sp)))$$

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- In the example above, the result is

$$(R_2(R_1(Sp)))$$

- The result is exactly the same if we multiply the matrices first, to form a single transformation matrix:

$$p' = (R_2 R_1 S)p$$

Homogeneous System

- In general, a matrix multiplication lets us linearly combine components of a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

  - This is sufficient for scale, rotate, skew transformations
  - But notice, we cannot add a constant! :(

Homogeneous System

- The (somewhat hacky) solution? Stick a "1" at the end of every vector:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Now we can rotate, scale, and skew like before, AND translate (note how the multiplication works out, above)
- This is called "homogeneous coordinates"

Homogeneous System

▶ In homogeneous coordinates, the multiplication works out so the rightmost column of the matrix is a vector that gets added

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

▶ Generally, a homogeneous transformation matrix will have a bottom row of [001], so that the result has a "1" at the bottom, too.

Homogeneous System

- One more thing we might want: to divide the result by something:
  - Matrix multiplication cannot actually divide
  - So, **by convention**, in homogeneous coordinates, we'll divide the result by its last coordinate after doing a matrix multiplication

$$\begin{bmatrix} x \\ y \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} x/7 \\ y/7 \\ 1 \end{bmatrix}$$

2D Transformation using Homogeneous Coordinates
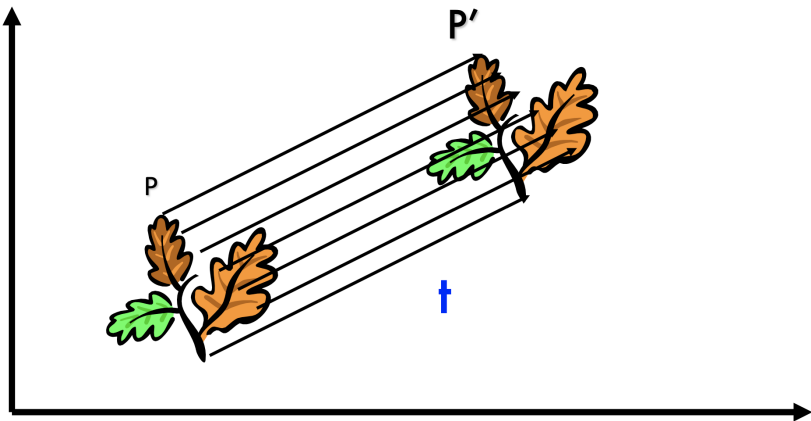
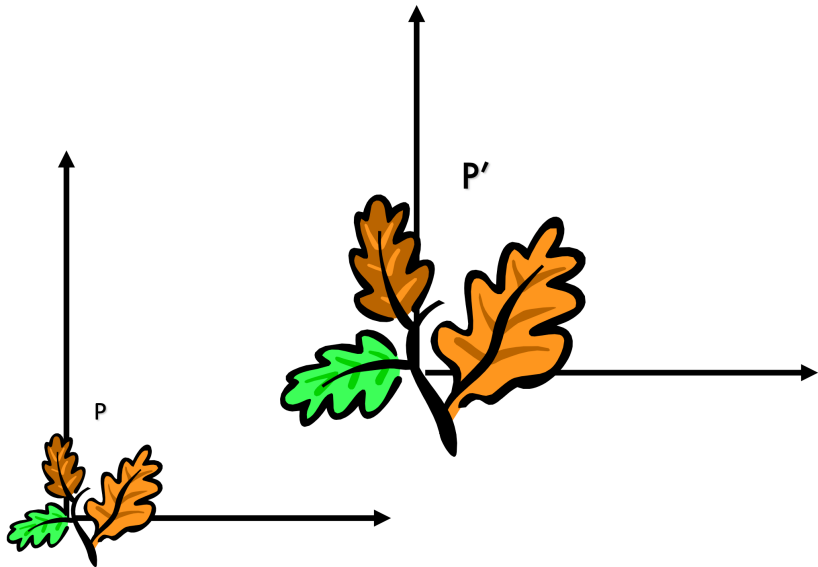# 2D Transformation using Homogeneous Coordinates
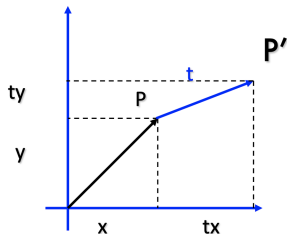
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$
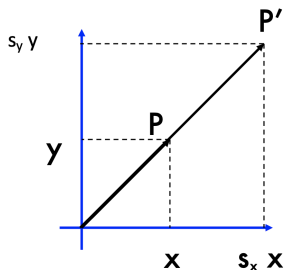
$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling

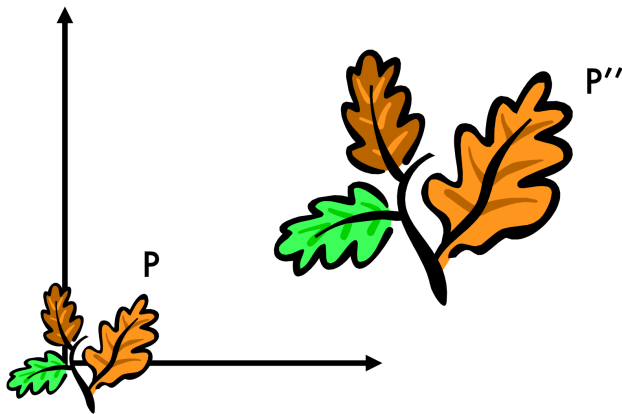

P

P'

Scaling Equation

$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating

$$P'' = T \cdot P' = T \cdot (S \cdot P) = T \cdot S \cdot P$$

Scaling & Translating

$$P'' = T \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation & Scaling versus Scaling & Translating

$$P''' = T \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

Translation & Scaling $\neq$ Scaling & Translating

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$$= \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$P''' = S \cdot T \cdot P = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

Translation & Scaling $\neq$ Scaling & Translating

$$P''' = T \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
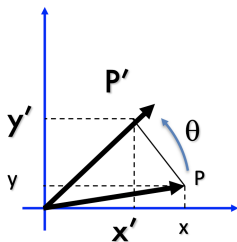
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Rotation

P

P'
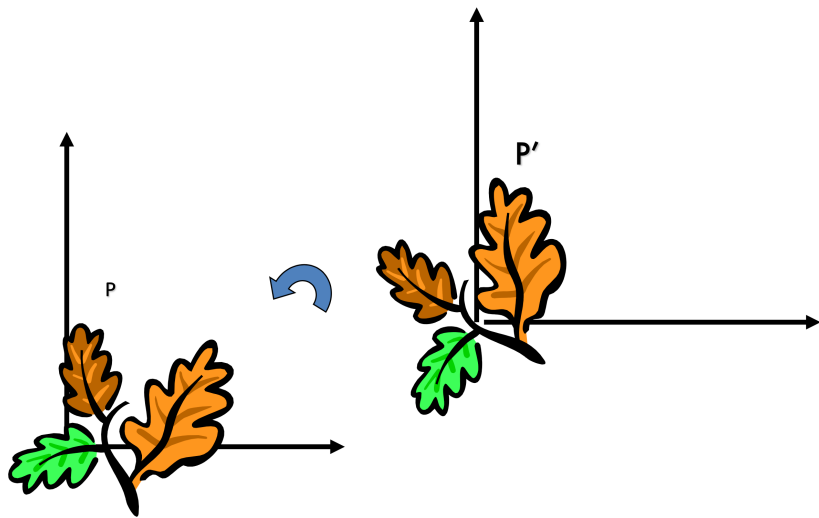
Rotation

Counter-clockwise rotation by an angle θ

$$x' = \cos\theta x - \sin\theta y$$
$$y' = \cos\theta y + \sin\theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = RP$$

Rotation Matrix Properties

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A 2D rotation matrix 2×2

Note: R belongs to the category of *normal* matrices and satisfies many interesting properties:

$$R \cdot R^T = R^T \cdot R = I$$
$$\det(R) = 1$$

Rotation Matrix Properties

- Transpose of a rotation matrix produces a rotation in the opposite direction

$$R \cdot R^T = R^T \cdot R = I$$
$$\det(R) = 1$$

- The rows of a rotation matrix are always mutually perpendicular (a.k.a. orthogonal) unit vectors
  - (and so are its columns)

Scaling+Rotation+Translation

$$P' = (T \; R \; S) \; P$$

$$P' = T \cdot R \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} RS & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Outline

Inverse

- Given a matrix A, its inverse A^{-1} is a matrix such that $AA^{-1} = A^{-1}A = I$

- e.g.,

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

- Inverse does not always exist. If A^{-1} exists, A is *invertible* or *non-singular*. Otherwise, it is *singular*.

- Useful identities, for matrices that are invertible:

$$(A^{-1})^{-1} = A$$
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$A^{-T} \triangleq (A^T)^{-1} = (A^{-1})^T$$

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  - Homogeneous matrices
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- Matrix inverse
- **Matrix rank**

Linear Independence

- Suppose we have a set of vectors $v_1, \ldots, v_n$
- If we can express v_1 as a linear combination of the other vectors $v_2, \ldots, v_n$, then v_1 is linearly *dependent* on the other vectors
  - The direction v_1 can be expressed as a combination of the directions $v_2, \ldots, v_n$ (e.g., $v_1 = 0.7v_2 - 0.7v_4$)
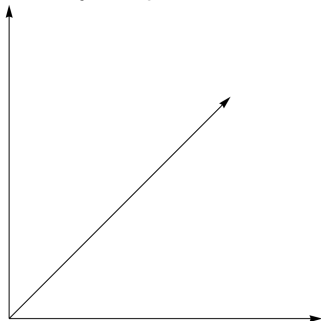
Linear Independence

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- If no vector is linearly dependent on the rest of the set, the set is linearly *independent*.
  - Common case: a set of vectors $v_1, \ldots, v_n$ is always linearly independent if each vector is perpendicular to every other vector (and non-zero).

Linear Independence

Linearly independent set
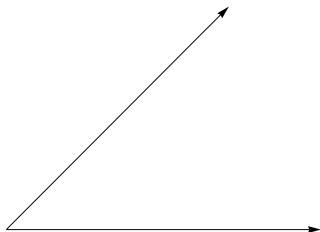
Not linearly independent

Matrix Rank

▶ Column/row rank

col-rank(A) = the maximum number of linearly independent column vectors of A

row-rank(A) = the maximum number of linearly independent row vectors of A

   ▶ Column rank always equals row rank

▶ Matrix rank

$$\text{rank}(A) \triangleq \text{col-rank}(A) = \text{row-rank}(A)$$

Matrix Rank

▶ For transformation matrices, the rank tells you the dimensions of the output

▶ e.g. if rank of A is 1, then the transformation

$$p' = Ap$$

maps points onto a line.

▶ Here's a matrix with rank 1:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + 2y \end{bmatrix}$$

Matrix Rank

- If an $m \times m$ matrix is rank m, we say it is "full rank"
  - Maps an $m \times 1$ vector uniquely to another $m \times 1$ vector
  - An inverse matrix can be found
- If rank $< m$, we say it is "singular"
  - At least one dimension is getting collapsed. No way to look at the result and tell what the input was
  - Inverse does not exist
- Inverse also does not exist for non-square matrices