Algorithm Problem Solving (APS):
Greedy Method

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Example: The Change Problem (USA Currency)

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  - **C** = {1¢ (penny), 5¢ (nickel), 10¢ (dime), 25¢ (quarter)}

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- **The issue: your problem formulation was not specific!**

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- A *correct* solution optimizing the *objective function* is **optimal**

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  - Imagine if $\boldsymbol{C} = \{1¢, 2¢, 3¢, 4¢\}$ and $\boldsymbol{x} = 5¢$
  - [1¢, 4¢] and [2¢, 3¢] are equally-optimal solutions
- You should be happy receiving any such solution
  - If not, you need to fix your objective function!

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- **C** = {1¢ (penny), 5¢ (nickel), 10¢ (dime), 25¢ (quarter)}

- Imagine I owe you 42¢. How should I choose the coins to give you?

Let's solve the problem!

Revisiting the Change Problem (USA Currency)

```
Algorithm change_USA(x,C):
    change ← empty list
    For each coin c in C (descending order):
        While x >= c:
            Add c to change
            x ← x - c
    Return change
```

Revisiting the Change Problem (USA Currency)

```
Algorithm change_USA(x,C):
    change ← empty list
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Does this work for *any arbitrary* currency?

```
            Add c to change
            x ← x - c
    Return change
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  - **Exhaustive:** Simply looking at every possible solution
- When we try to cleverly search for an optimal solution more quickly:
  - **Global:** We can look at entire solutions at a time
  - **Local:** We can break solutions into parts and optimize part-by-part

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Local Search: The Greedy Method

- **Greedy Method:** Selecting the best possible choice at each step
- **Note that this does not always work!!!**
    - We often skip what's immediately best to improve in the long-run
    - <u>Example</u>: Buying vs. leasing a car
- Thus, it's important to prove the correctness of a Greedy Algorithm

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- Imagine I owe you 6¢. How should I choose the coins to give you?
  - The greedy algorithm would return [4¢, 1¢, 1¢]
  - The optimal solution is [3¢, 3¢]
  - **Our greedy algorithm doesn’t work for all possible currencies!!!**

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- Greedy: Take the best immediate benefit and ignore opportunity costs
    - Optimal when immediate benefit outweighs opportunity costs

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    - You charge a flat rate, regardless of the length of the event
    - Thus, you want to schedule as many events as possible
    - However, events cannot overlap

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- **Input:** All $\boldsymbol{n}$ possible events $\boldsymbol{E} = [(start_1, end_1), \ldots, (start_{\boldsymbol{n}}, end_{\boldsymbol{n}})]$

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- **Input:** All n possible events $E = [(start_1, end_1), \ldots, (start_n, end_n)]$
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- If we wanted to design a greedy algorithm, what would we optimize?

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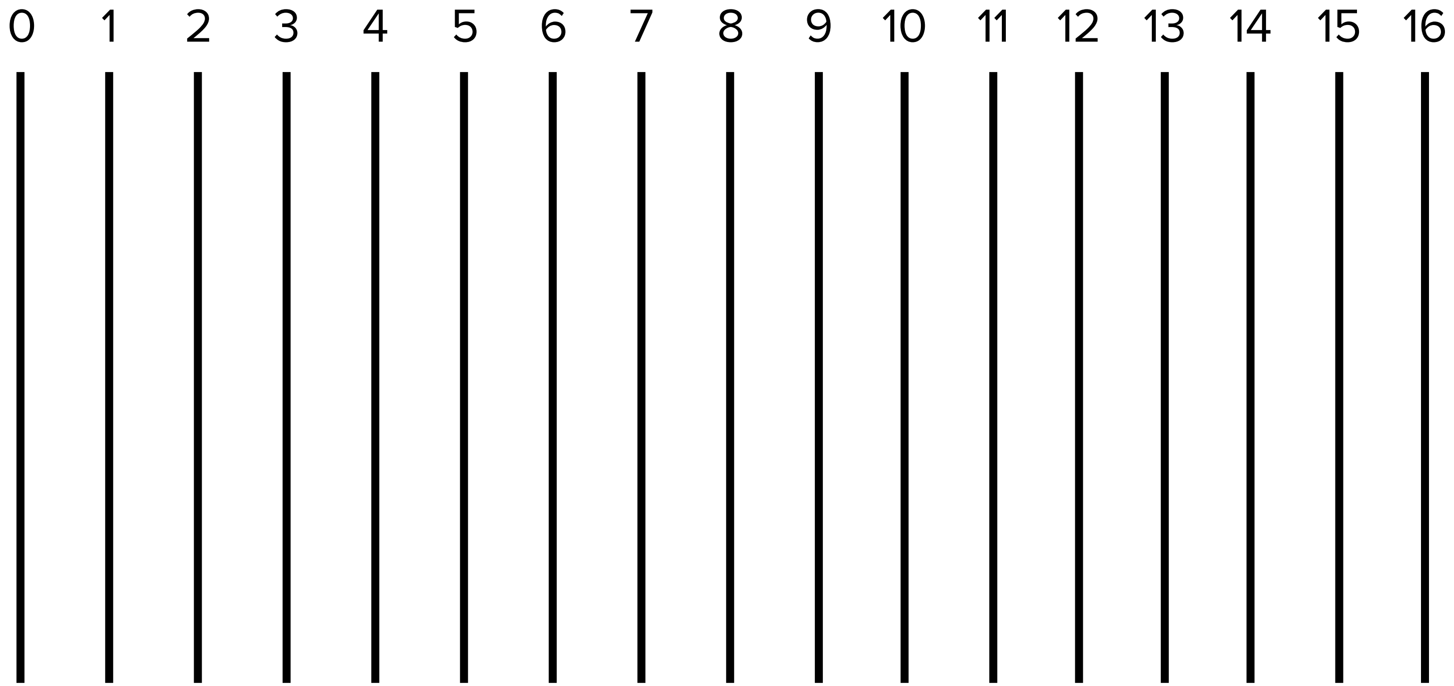
- **Input:** All $\boldsymbol{n}$ possible events $\boldsymbol{E} = [(start_1, end_1), \ldots, (start_{\boldsymbol{n}}, end_{\boldsymbol{n}})]$
- **Output:** A non-overlapping subset of $\boldsymbol{E}$ maximizing its size
- If we wanted to design a greedy algorithm, what would we optimize?
    - Shortest duration?
    - Earliest start time?
    - Fewest conflicts?
    - Earliest end time?

Counterexample: Shortest Duration



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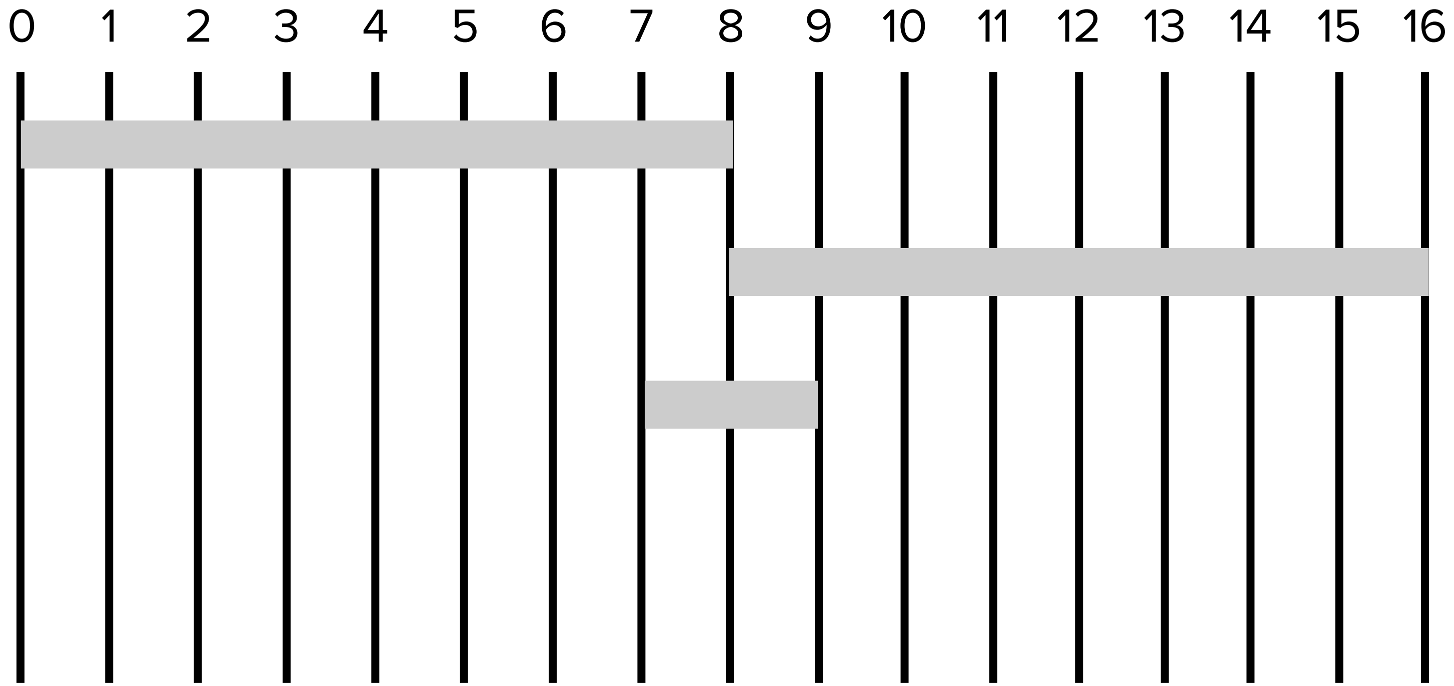

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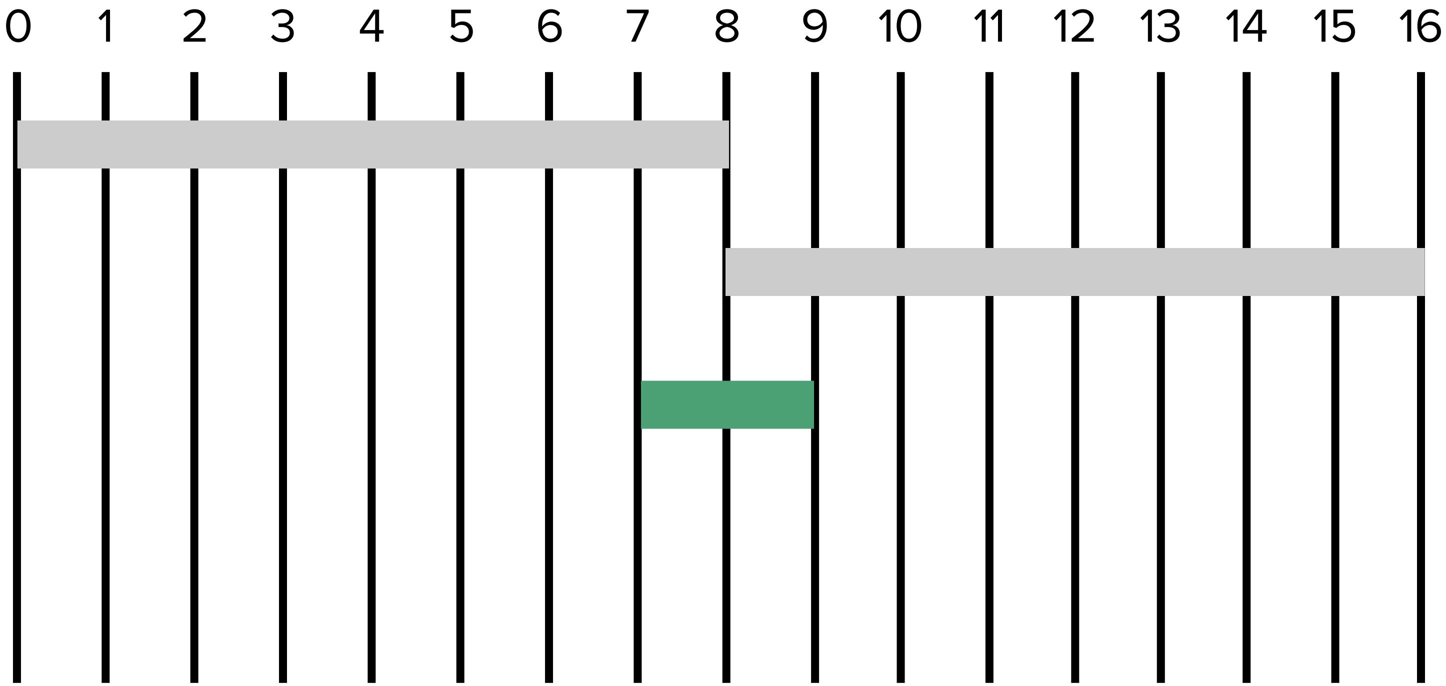

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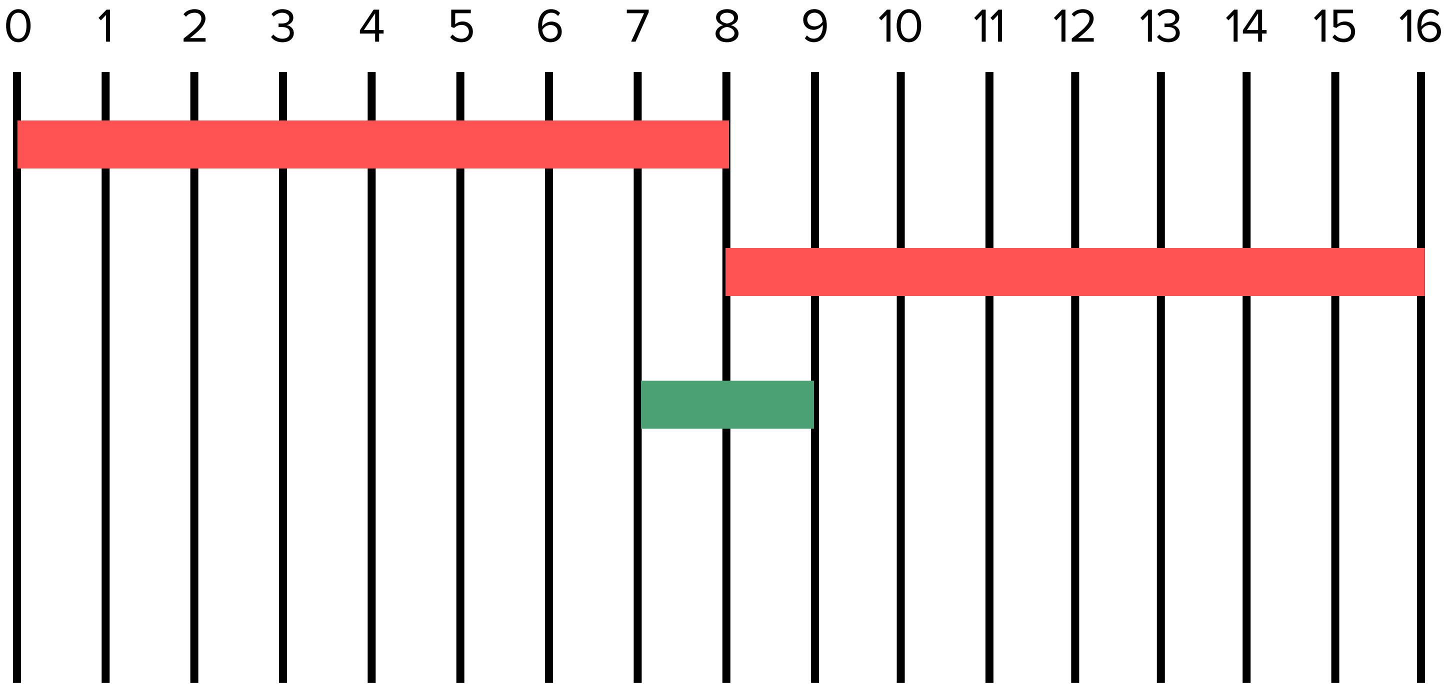

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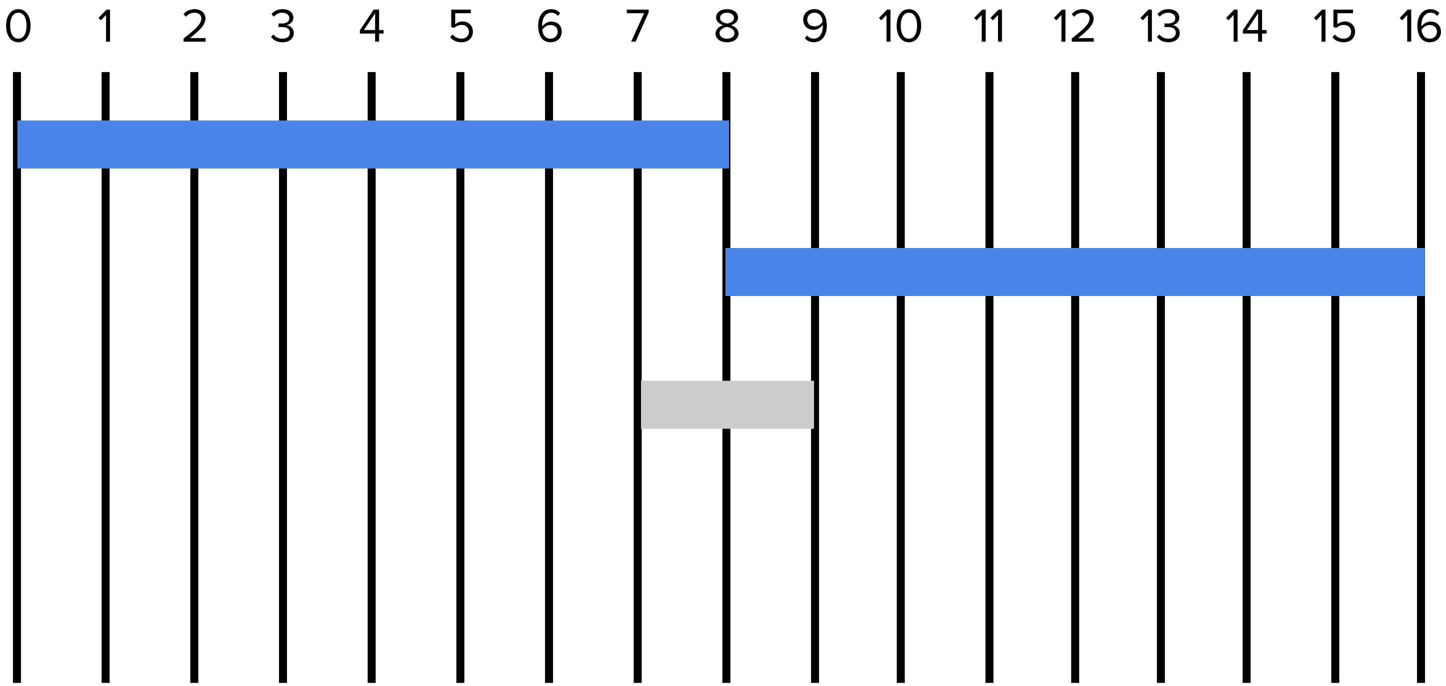

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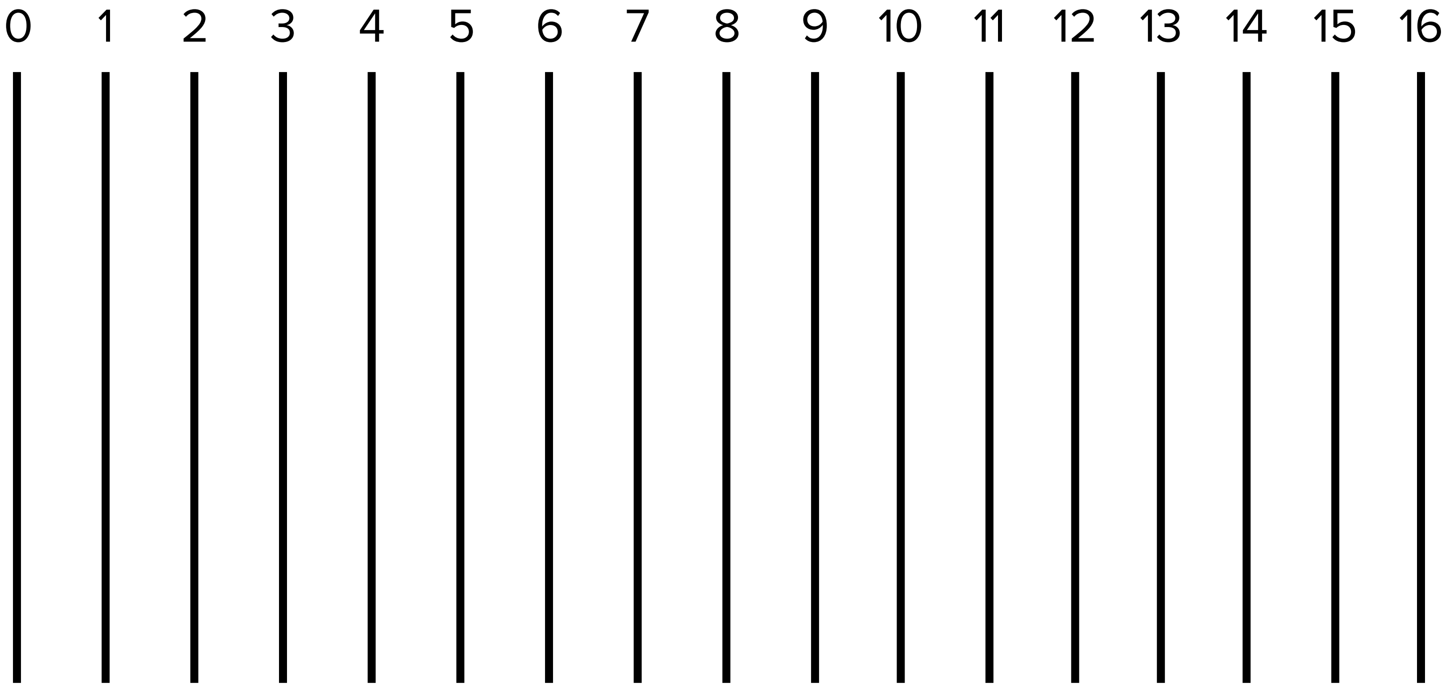

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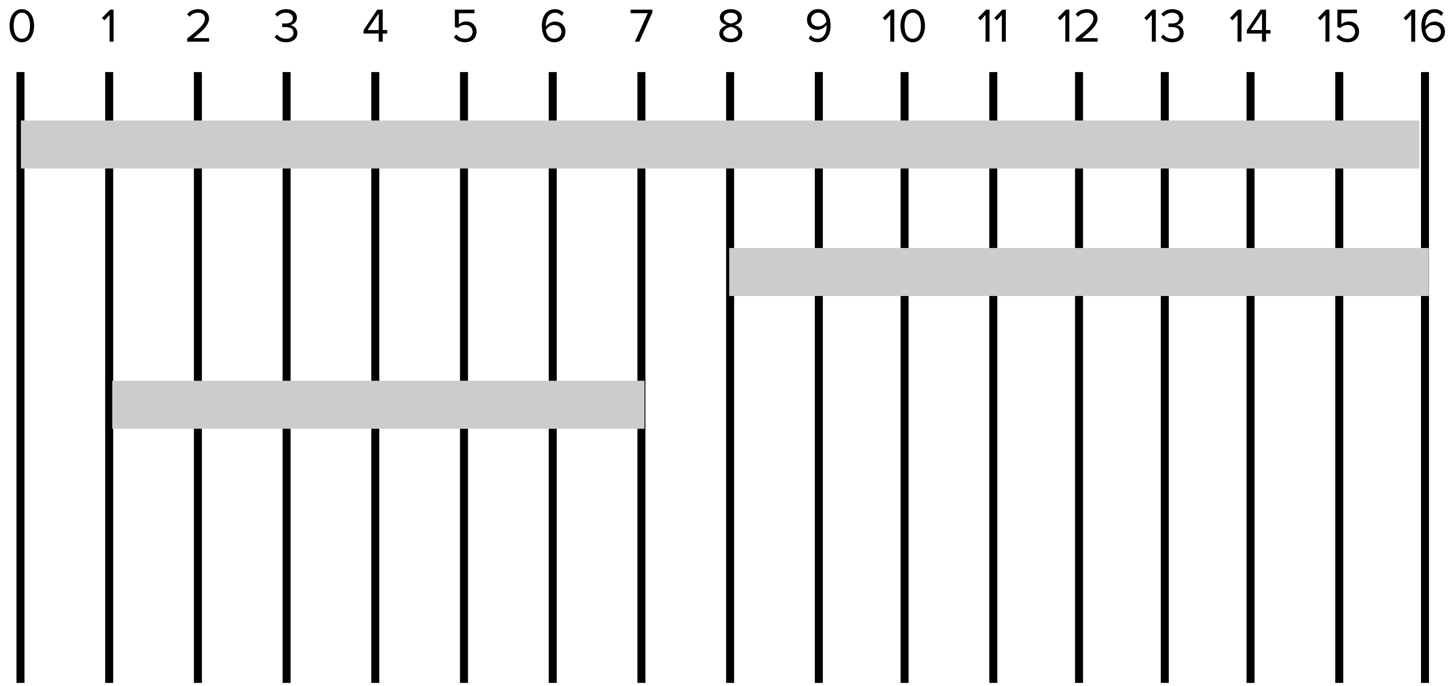

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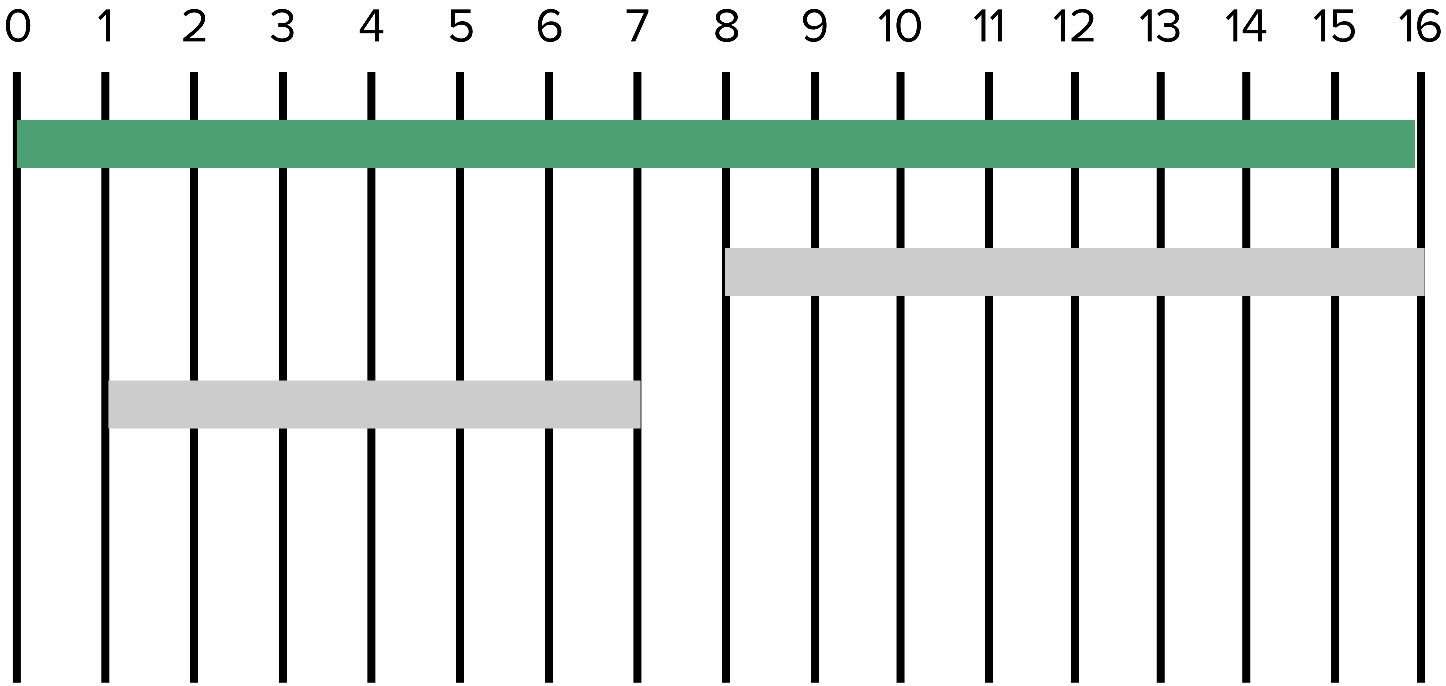

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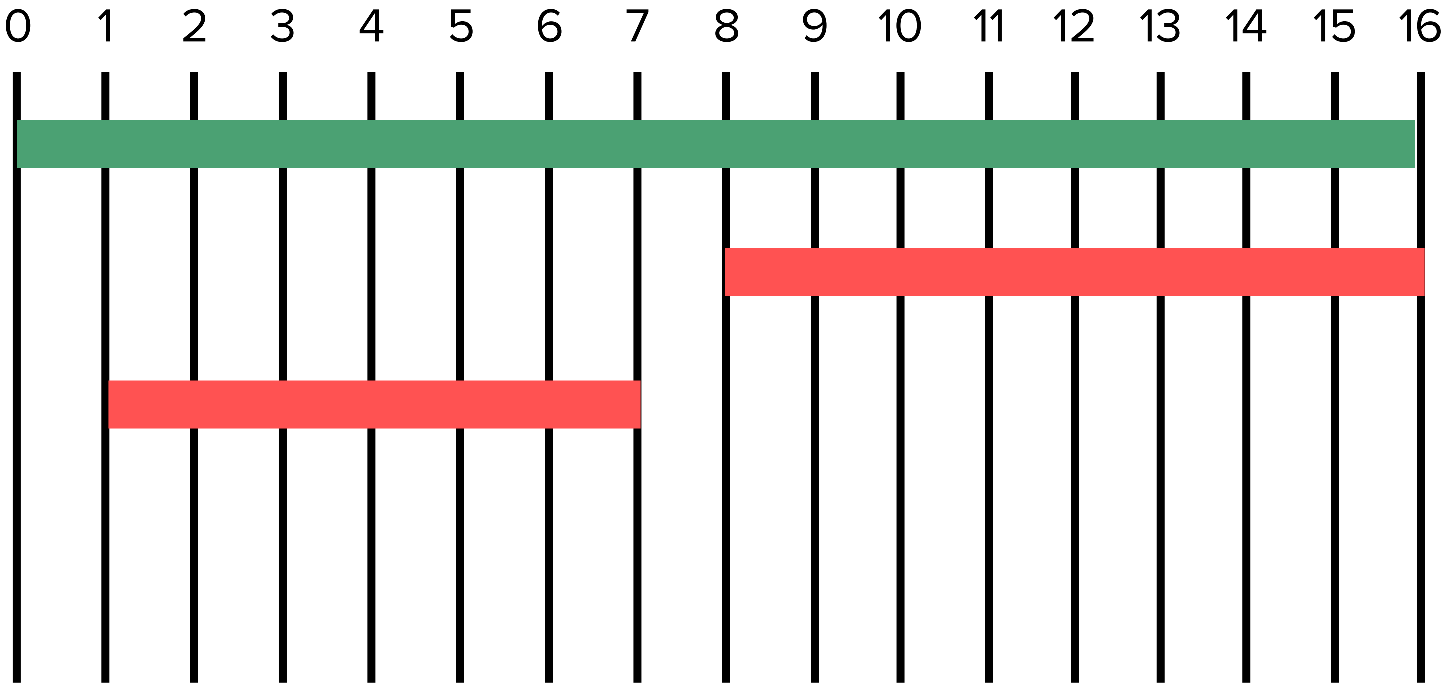

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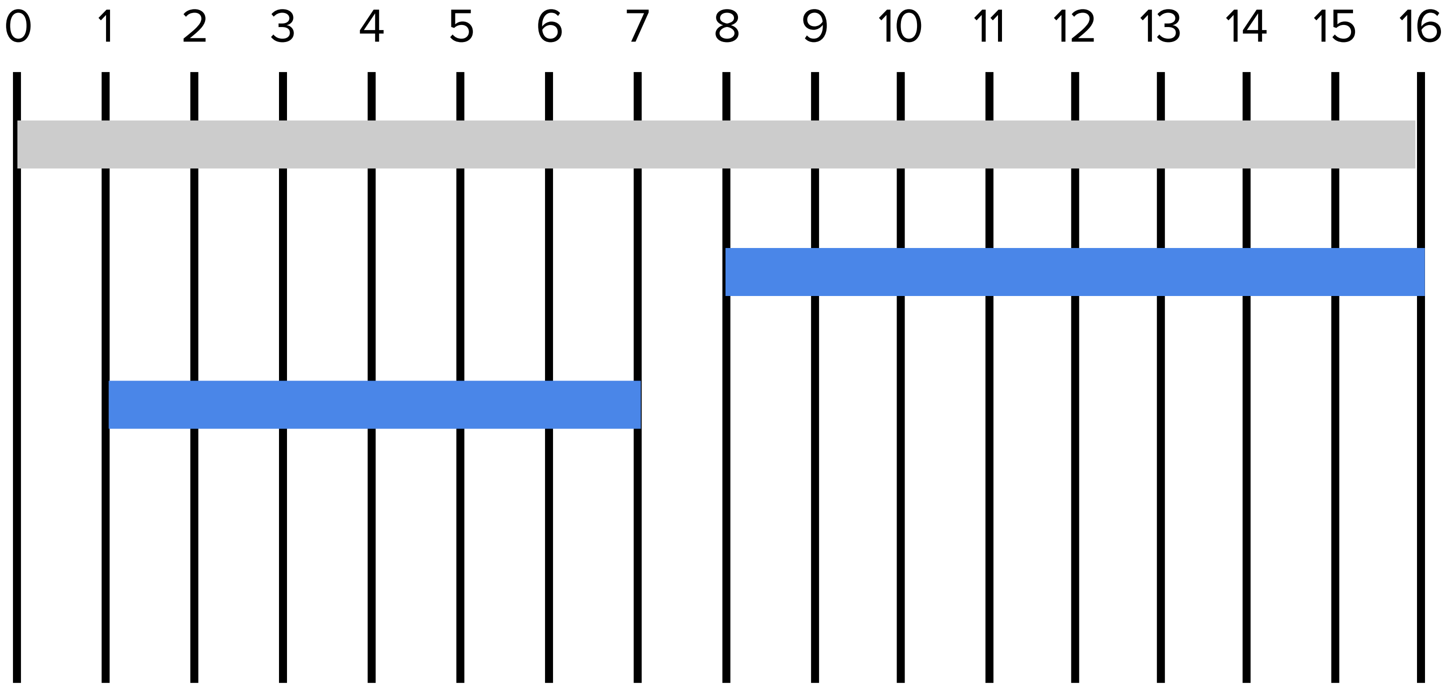

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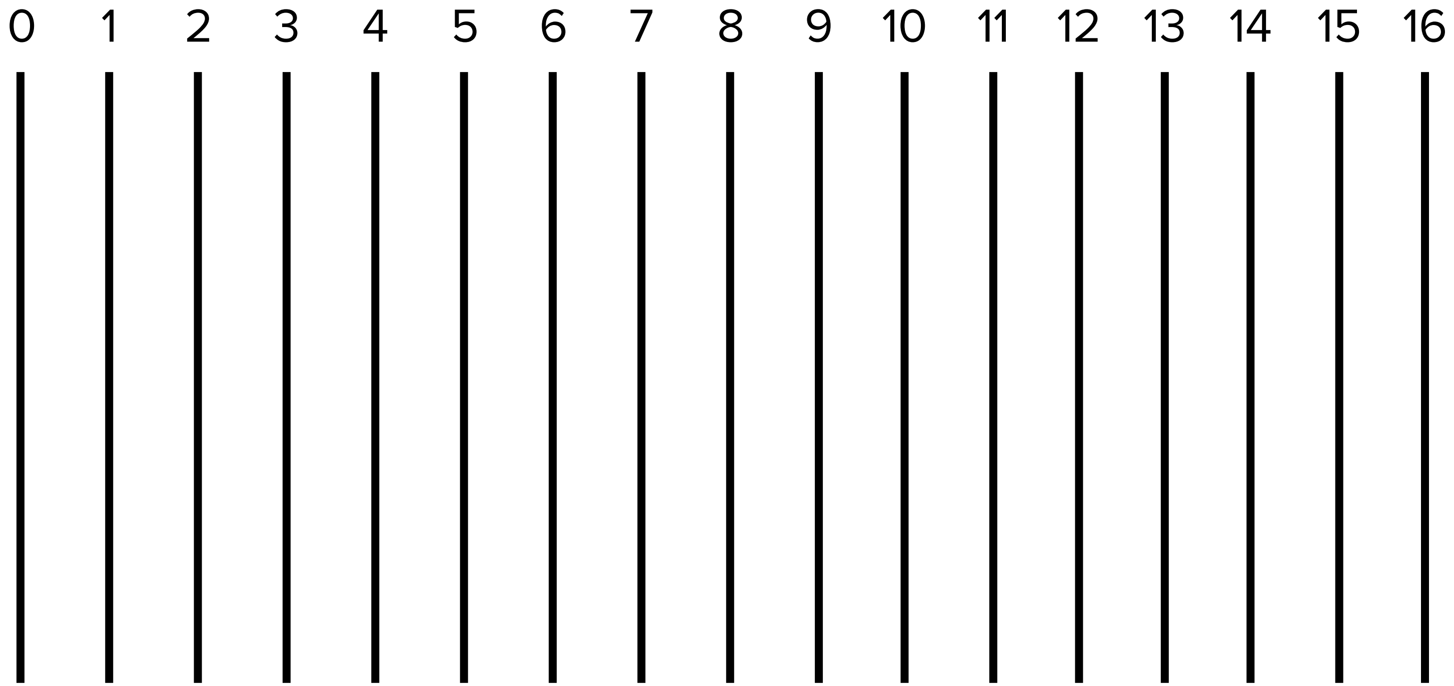

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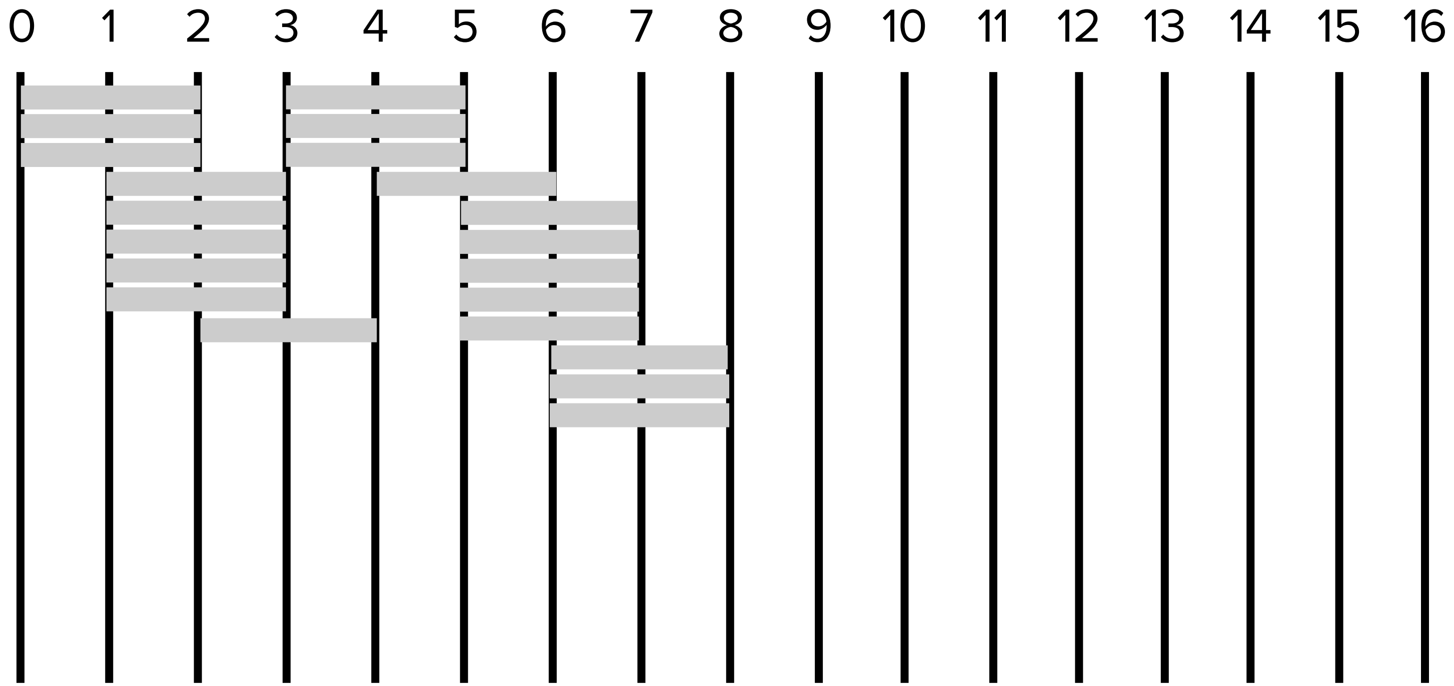

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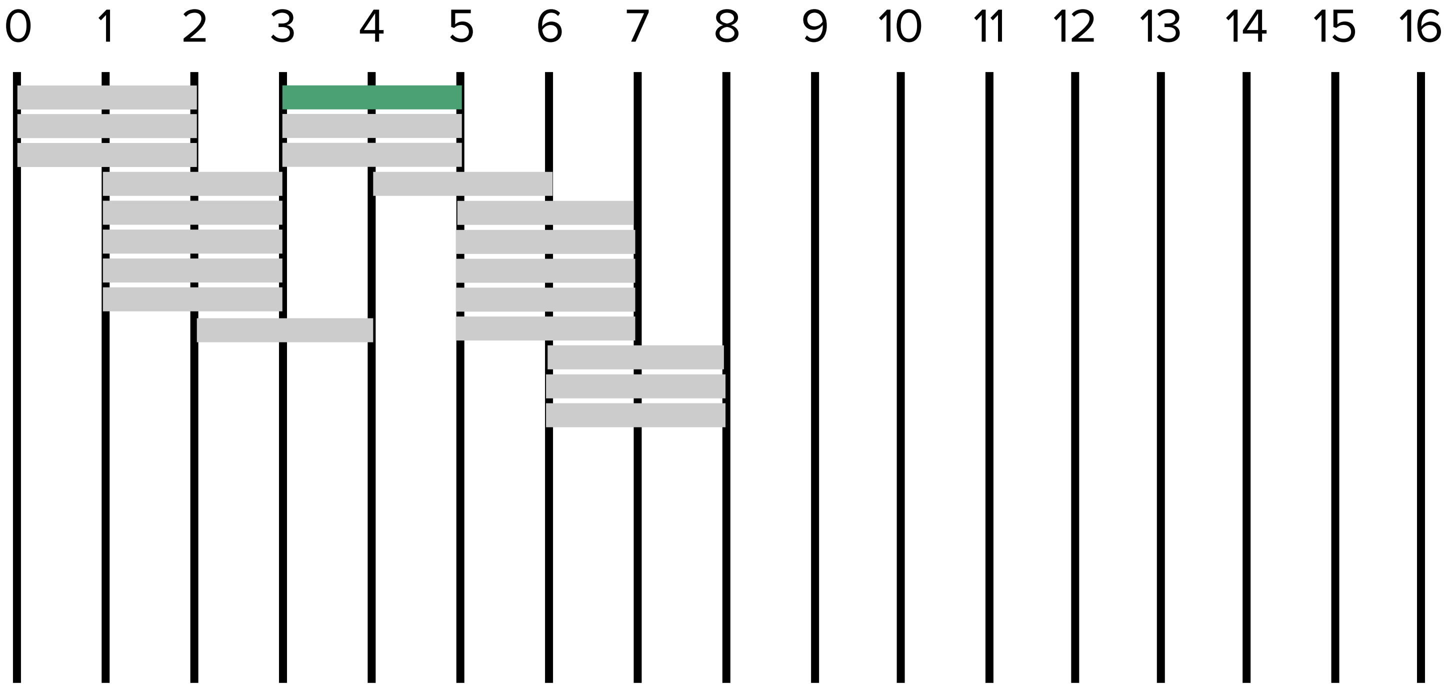

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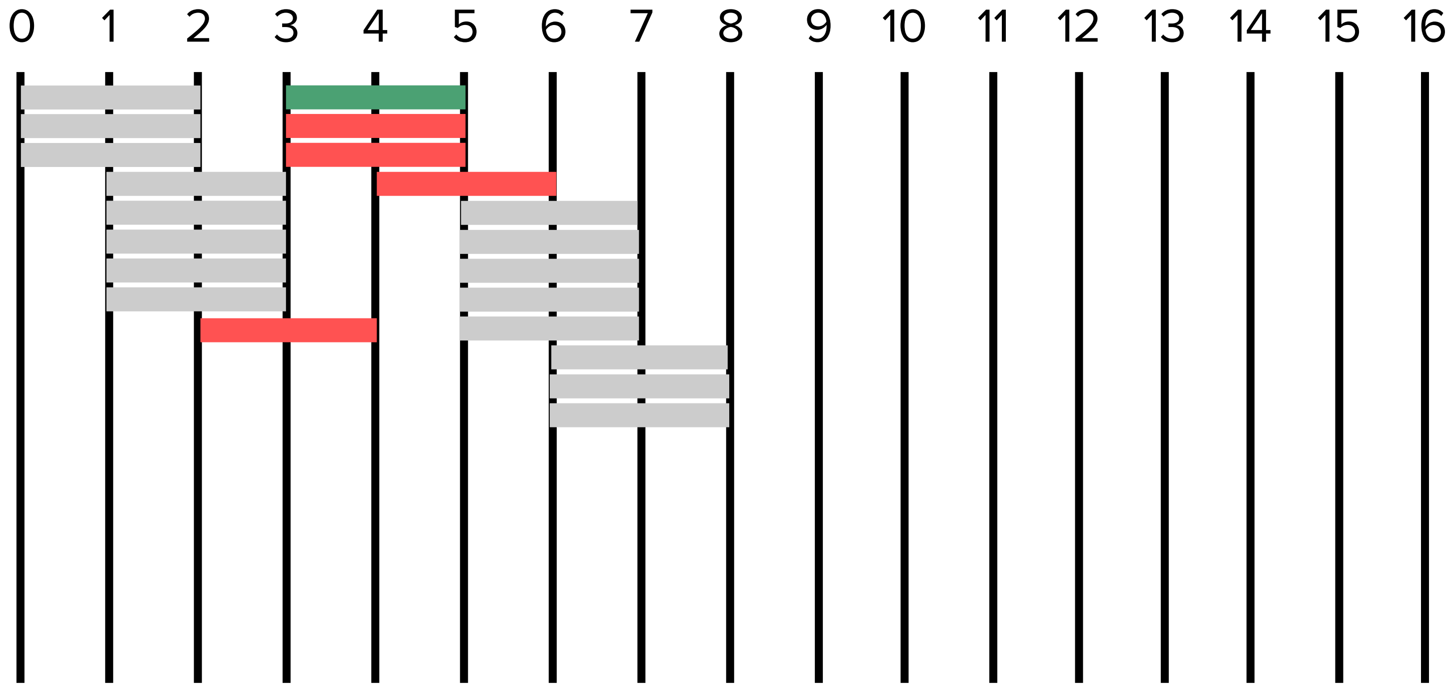

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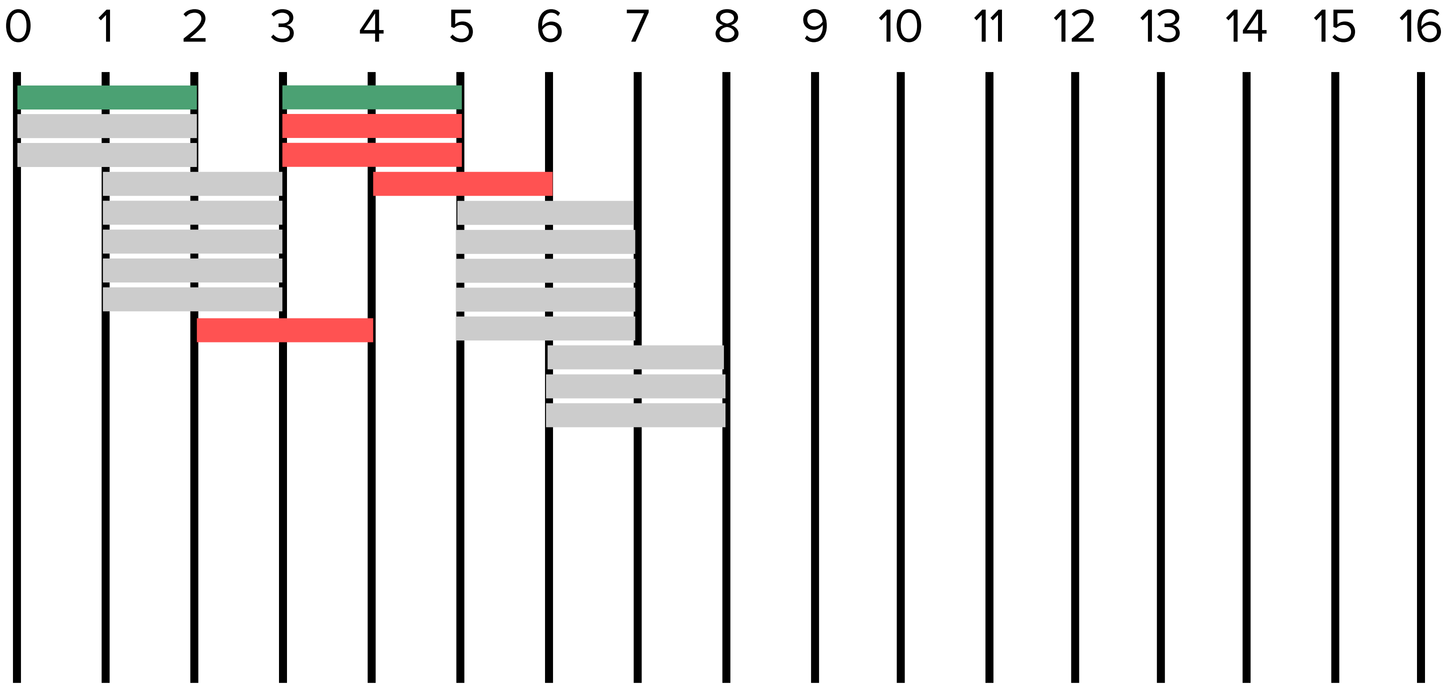

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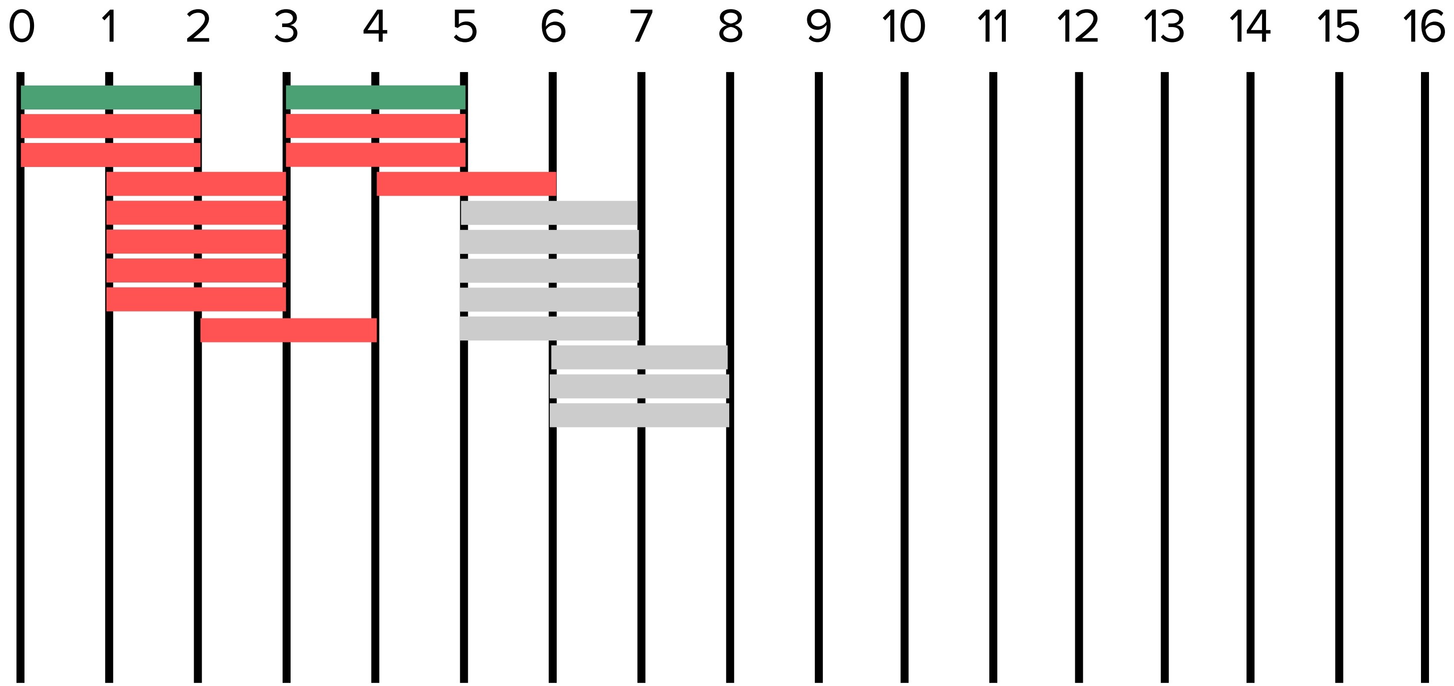

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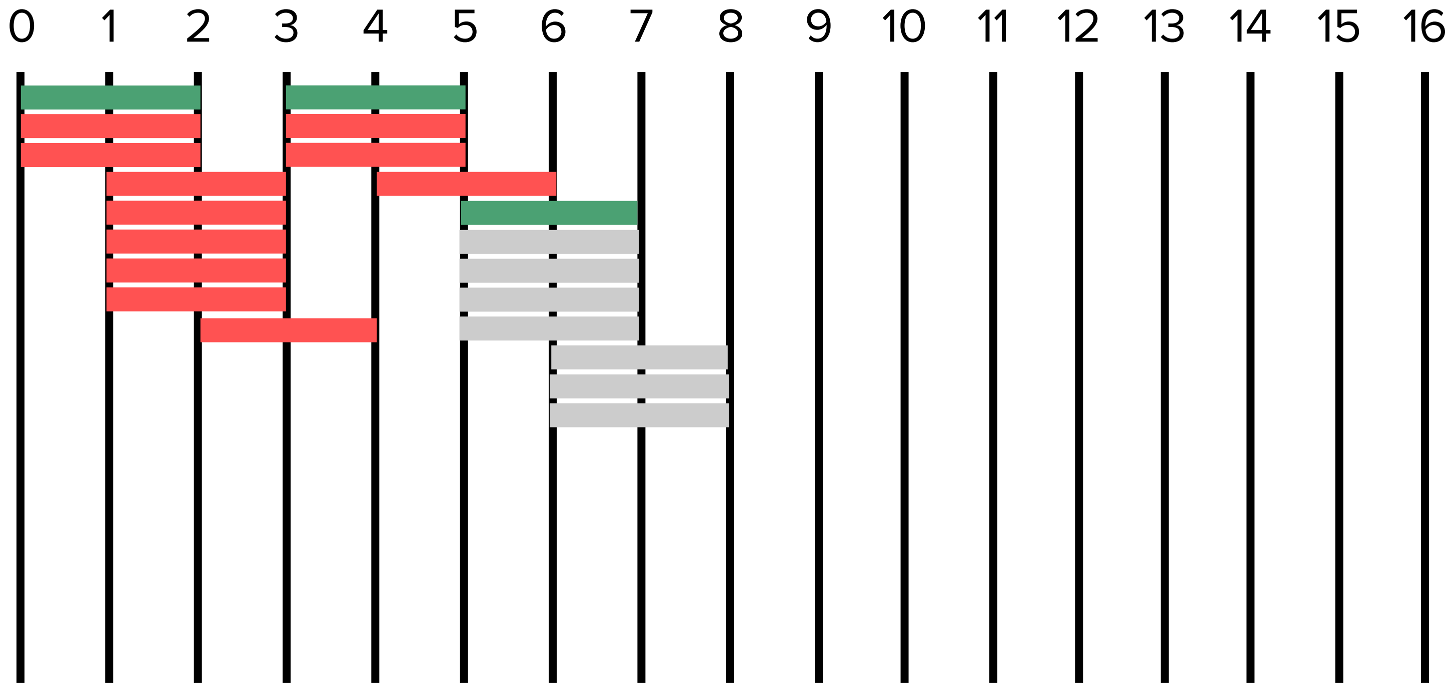

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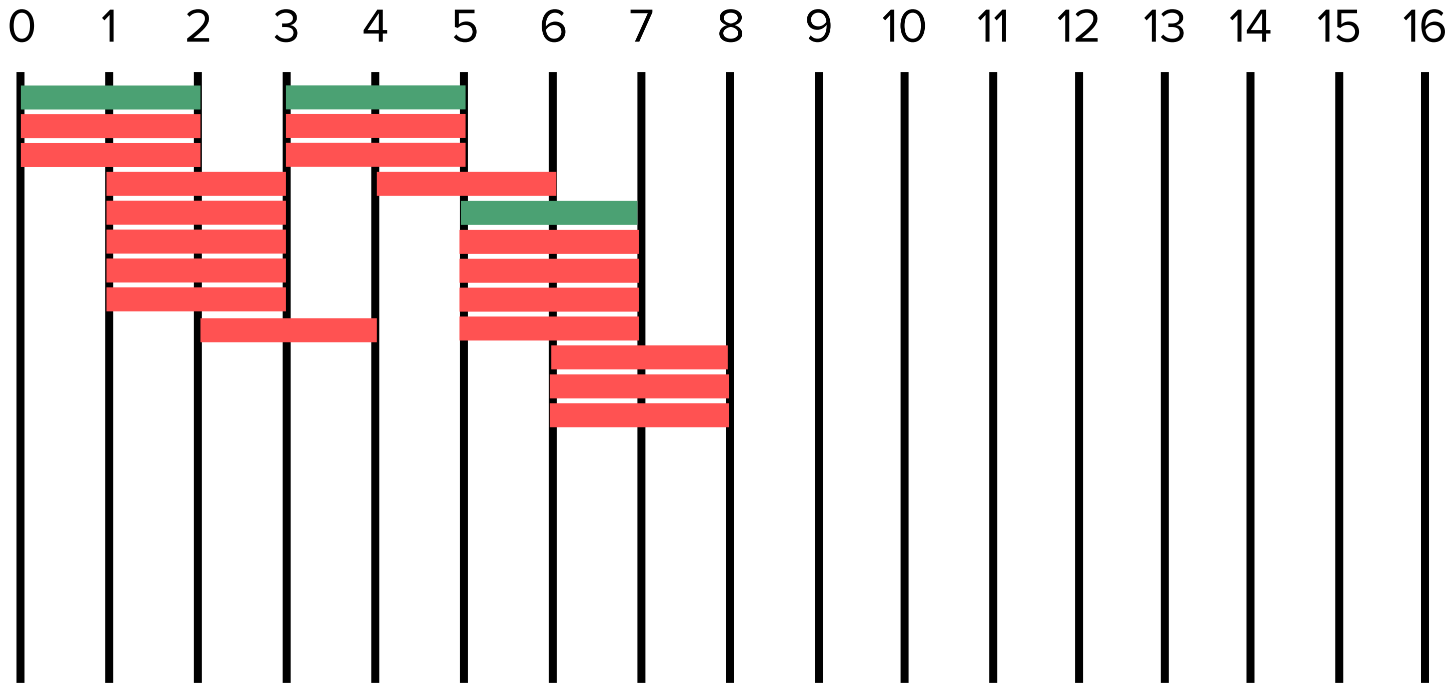

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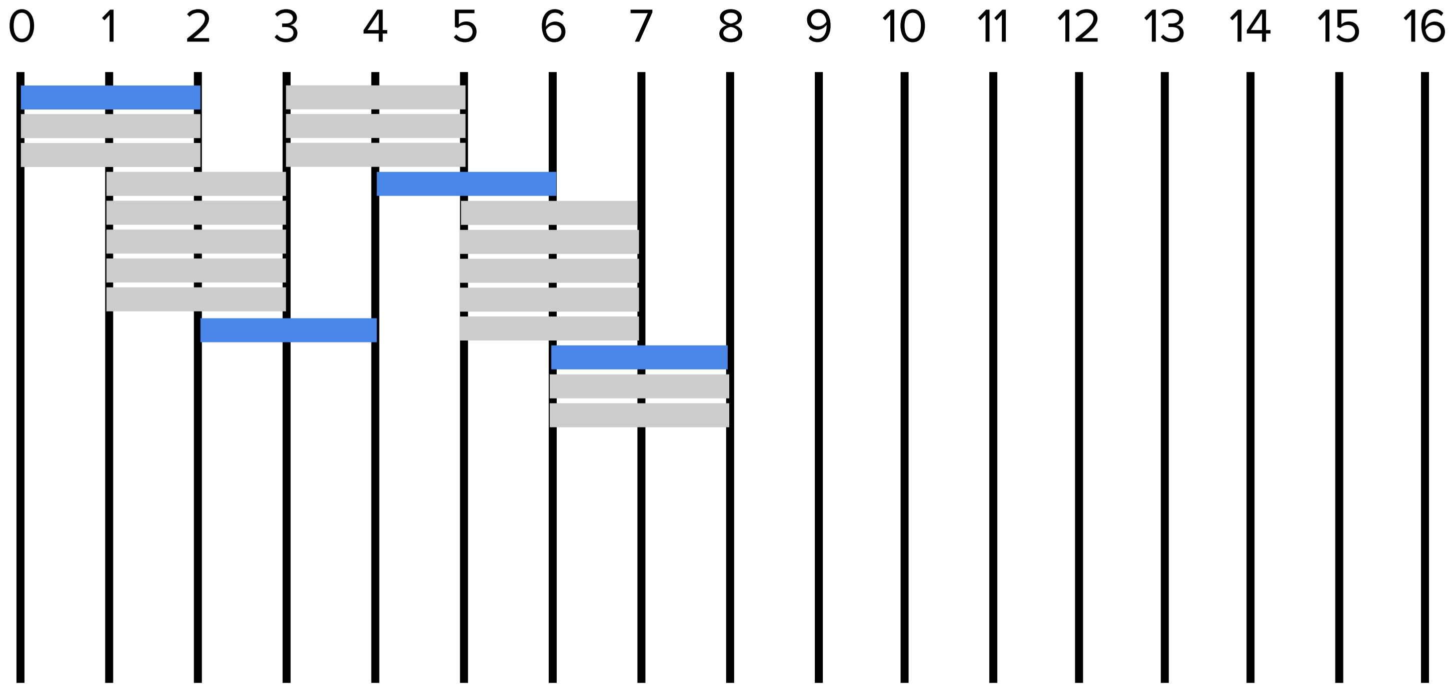

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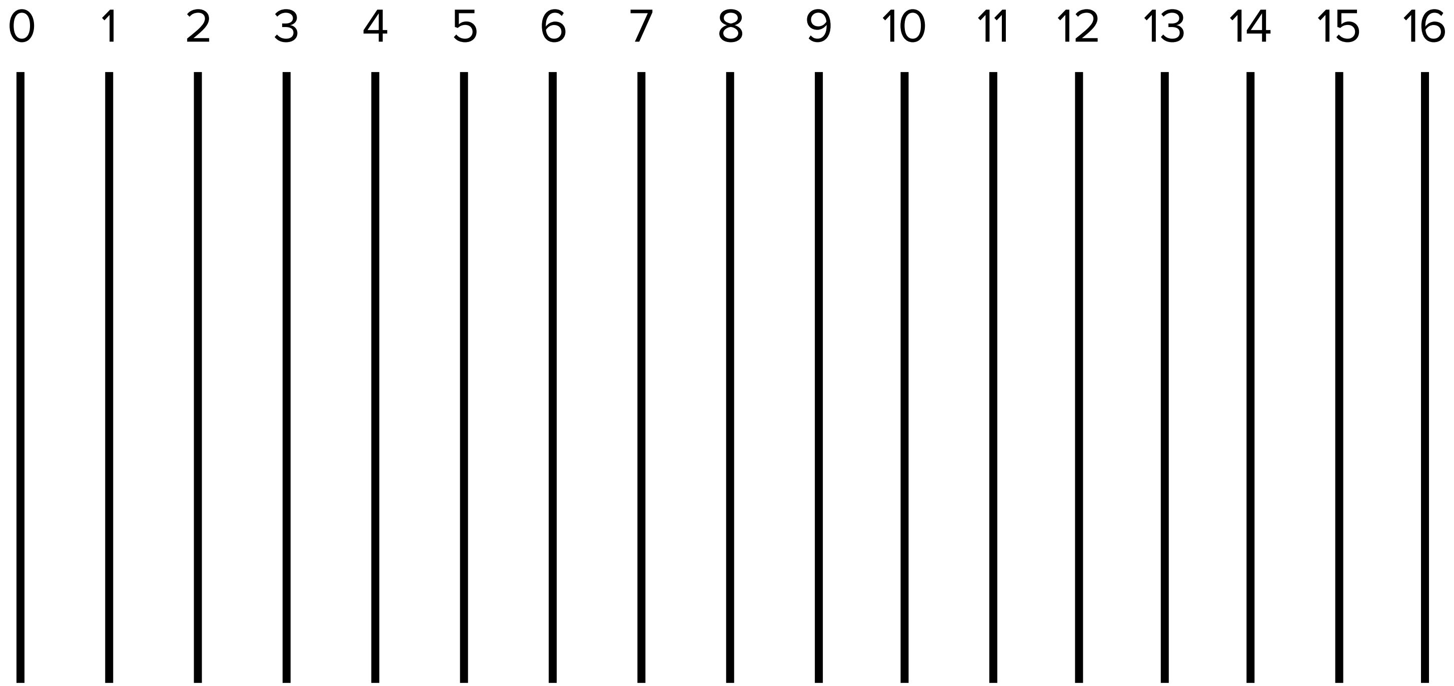

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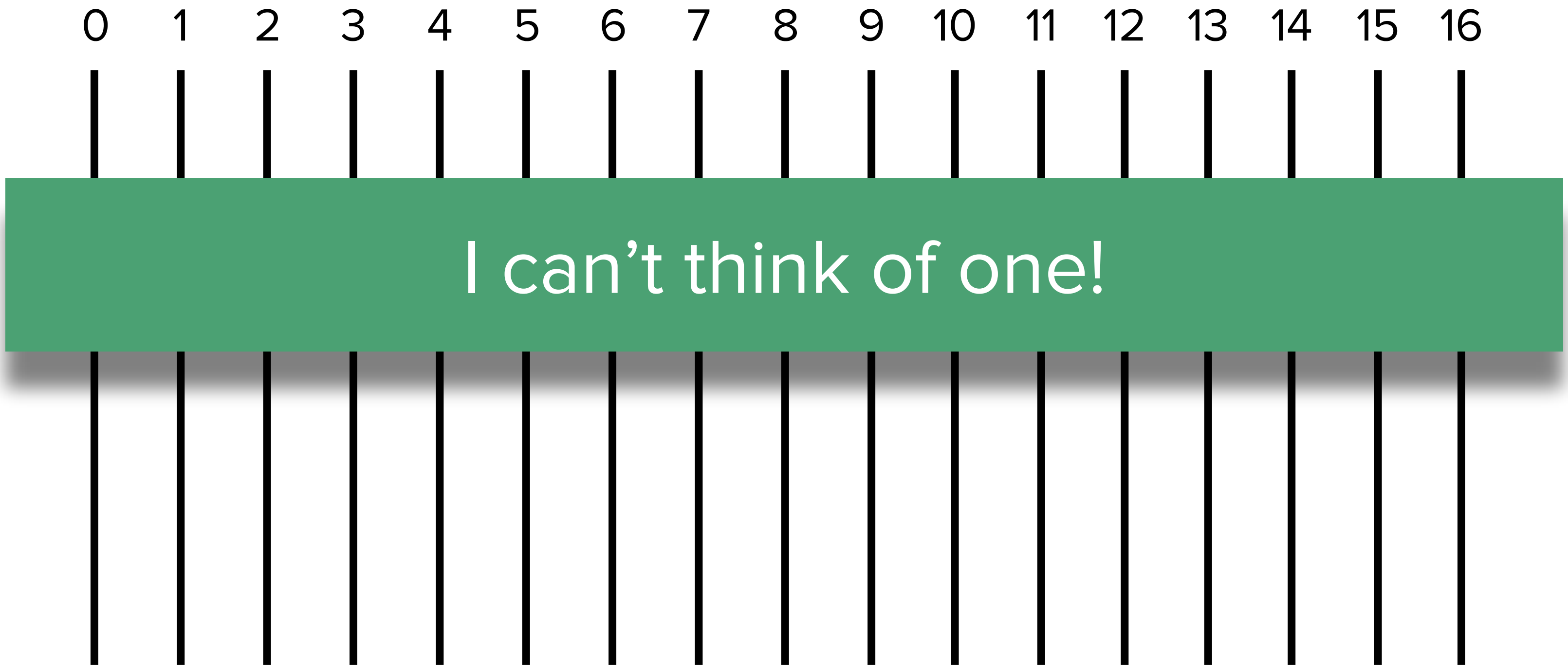

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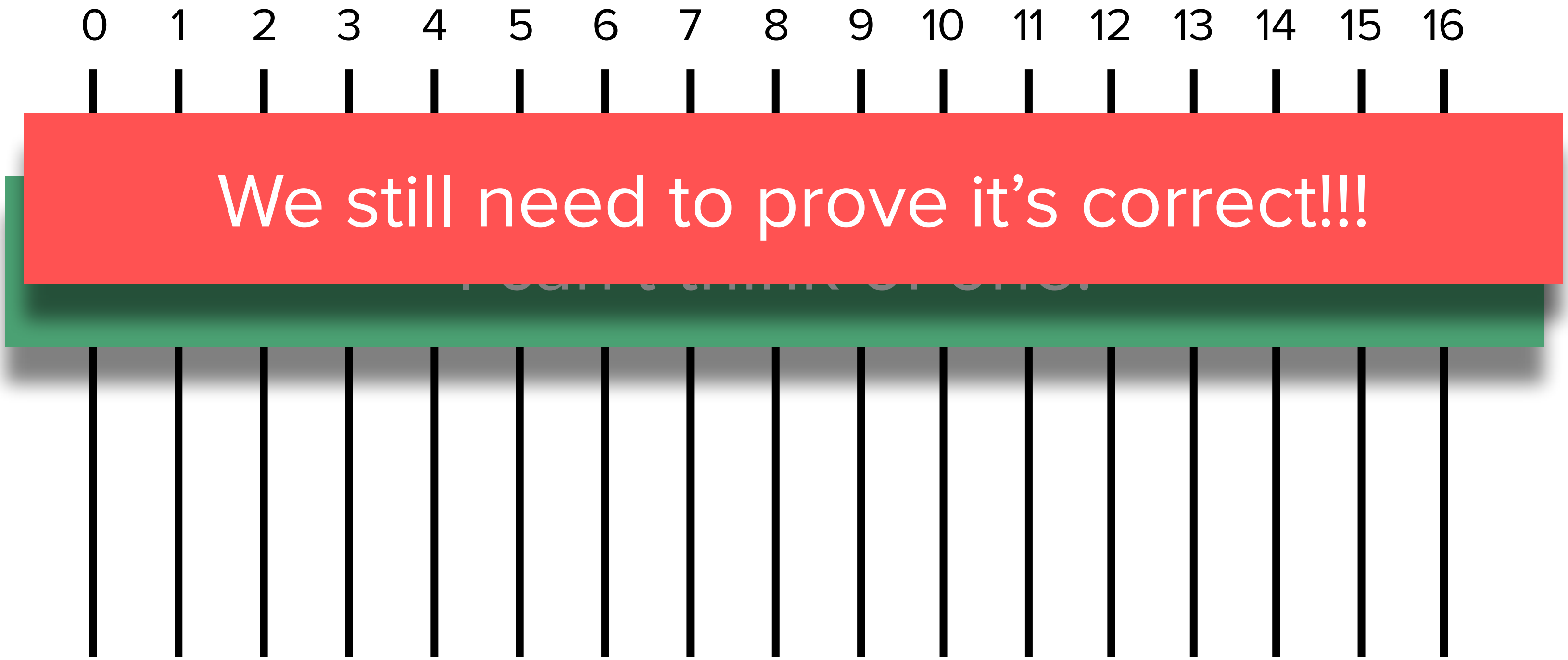

Example: The Event Scheduling Problem

```
Algorithm schedule(E):
    Sort E in ascending order of end time
    curr_time ← negative infinity
    events ← empty list
    For each event (start,end) in E:
        If start ≥ curr_time:
            Add (start,end) to events
            curr_time ← end
    Return events
```

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Let's try to prove our algorithm!

  - Create *S'* by exchanging a choice in *S* with *g* and show that
    - *S'* is a valid solution
    - *S'* is just as good, or better than, *S*