Algorithm Problem Solving (APS): Dynamic Programming

Niema Moshiri UC San Diego SPIS 2019

Imagine you're in a heist crew en route to your next job



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- Your target is a truck carrying electronic goods
- The modified Honda Civic you're driving has a weight limit
 - You know how much each item costs and weighs
 - Which items do you steal to maximize your profits?

• Input: A list of n items, where the i-th item has weight w_i and value v_i ,

and a capacity x

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Output: A set of items such that the sum of their weights is below x
 and the sum of their values is maximized

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V:	\$5	\$5	\$13	\$6

$$x = 20$$

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Can we use a Greedy Algorithm?

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$$x = 20$$

Nope! Not guaranteed to be optimal!

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0, 1, 1, 2

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0, 1, 1, 2, 3

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0, 1, 1, 2, 3, 5

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0, 1, 1, 2, 3, 5

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0, 1, 1, 2, 3, 5, 8

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0, 1, 1, 2, 3, 5, 8

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0, 1, 1, 2, 3, 5, 8, **13**

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0, 1, 1, 2, 3, 5, 8, 13, ...

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Write a recursive Python function!

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Recursive Fibonacci

```
def fib rec(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib_rec(n-1) + fib_rec(n-2)
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                                   Yay! 😄
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Recursive Fibonacci: SLOW!!!

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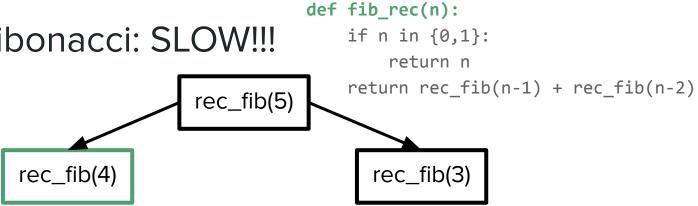
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return n

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rec_fib(2)

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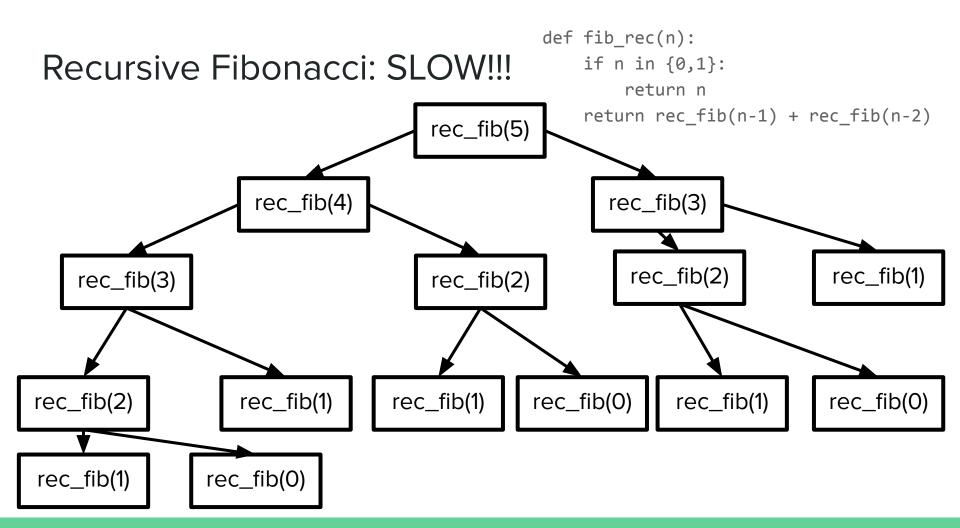
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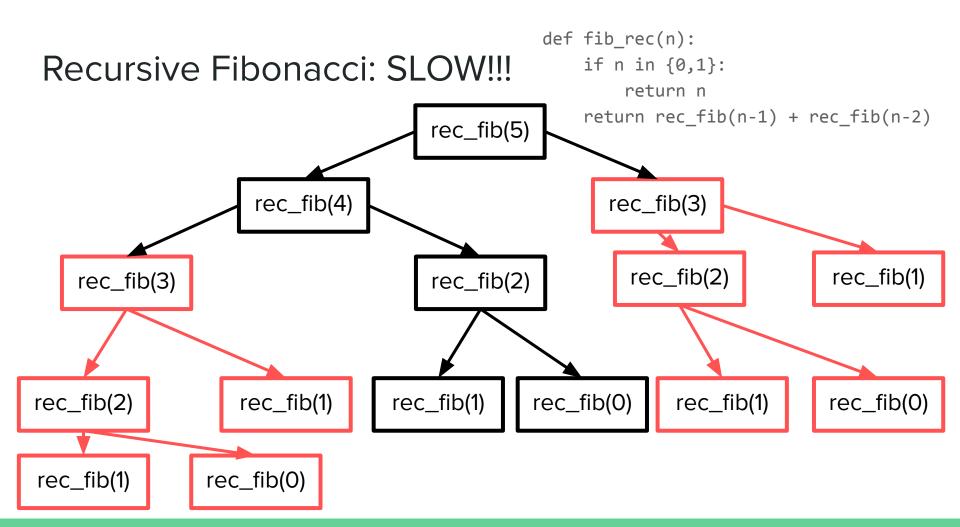
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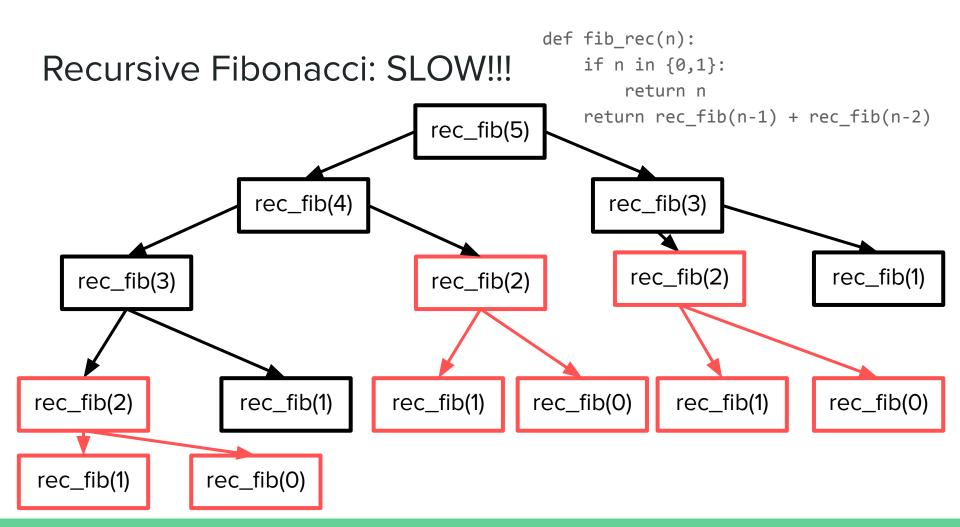
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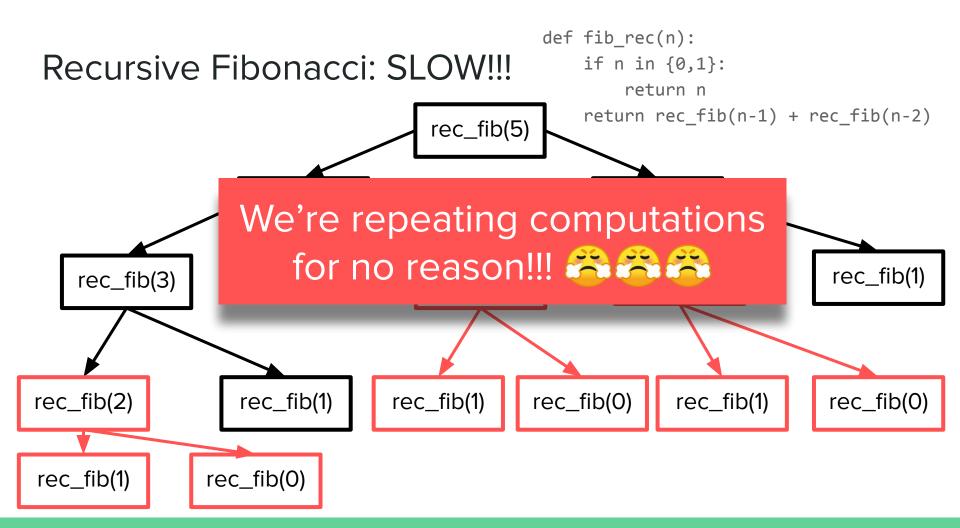
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- We saw that, in computing rec_fib(5), we ended up calling rec_fib(3)
 twice and rec_fib(2) two times
 - The values of rec_fib(3) and rec_fib(2) never change, so once we compute them the first time, why not just save them somewhere?
- Memoization: Saving the results of expensive functions somewhere and returning the saved result when the same inputs occur again

$$F(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F(n-1) + F(n-2) & \text{if } n > 1 \end{cases}$$

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- Negative weights (j < 0) are impossible, so they should be disallowed
 - $P(i, j) = -\infty$ for all j < 0

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 - Don't add item i to our pack, so our profit would be P(i-1, j)
 - No value added, so it's just our best scenario of looking at the first *i*-1 items with the same (*j*) total weight
 - Take the maximum of these two possible options

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j - w_i), P(i-1, j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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Example: The O-1 Knapsack Problem
$$v$$
: $\begin{vmatrix} x = 20 & w \\ & & \end{vmatrix}$

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

$$j \text{ (total weight)}$$

≘		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	0	0	0	0	0	0	0										0	0	0	0	0	0
	1	0								Ac	ld:	\$!	5 +	0	0							
	2																					
	3								Г)oı	า't	Δα	hh	٠ \$	\cap							
	4										1 (au	. Ψ								

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

$$j \text{ (total weight)}$$

										<i>)</i> (ι	.otai	weig	nt)									
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1		0																				
2	2	0																				
3	3	0																				
4	ļ.	0																				

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ & \text{ j (total weight)} \end{cases}$$

										<i>)</i> (ι	otai	weig	nt)									
Ē		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
items seen)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0																			
er of	2	0	0																			
i (number of	3	0	0																			
<i>i</i> (r	4	0	0																			

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \\ & \text{ j (total weight)} \end{cases}$$

										J) (T	otai	weig	nt)									
Ē		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
items seen)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0																		
er of	2	0	0	0																		
i (number of	3	0	0	0																		
<i>j</i> (r	4	0	0	0																		

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

$$j \text{ (total weight)}$$

									<i>J</i> (t	oτal	weig	ht)									
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	-5																	
2	0	0	0																		
3	0	0	0																		
4	0	0	0																		

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1, j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

$$j \text{ (total weight)}$$

										<i>J</i> (1	otal	weig	nt)									
	(0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	(0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	(0	0	0	5																	
2	(0	0	0	5																	
3	(0	0	0																		
4	(0	0	0																		

W:	3	8	10	6
V:	\$5	\$5	\$13	\$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

$$j \text{ (total weight)}$$

										<i>j</i> (1	otal	weig	ht)									
<u>(</u>		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
items seen)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	5																	
er of	2	0	0	0	1 ₅																	
<i>i</i> (number of	3	0	0	0	5																	
<i>i</i> (r	4	0	0	0	T ₅																	

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ \text{ j (total weight)} \end{cases}$$

									_	<i>j</i> (†	total	weig	ht)									
<u>5</u>		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	√ 5															
er of	2	0	0	0	1 5	5	1 ₅															
i (number of	3	0	0	0	5	5 •	5															
) (r	4	0	0	0	T ₅	T ₅	T ₅															

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ & \text{ j (total weight)} \end{cases}$$

									<i>J</i> (1	totai	weig	nt)									
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	1 5	1 5														
2	0	0	0	 	 5	 4 5	1 ₅														
3	0	0	0	₅	 5	 4 5	1 ₅														
4	0	0	0	15	5	1 ₅															

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ & \text{ j (total weight)} \end{cases}$$

									<i>)</i> (1	total	weig	ht)									
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5														
2	0	0	0	1 5	1 5 1 1 1 1 1 1 1 1 1 1	1 ₅	1 ₅														
3	0 -	0	0	5	5	5	1 ₅														
4	0	0	0	5	T ₅	T ₅	6														

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \\ & \text{ j (total weight)} \end{cases}$$

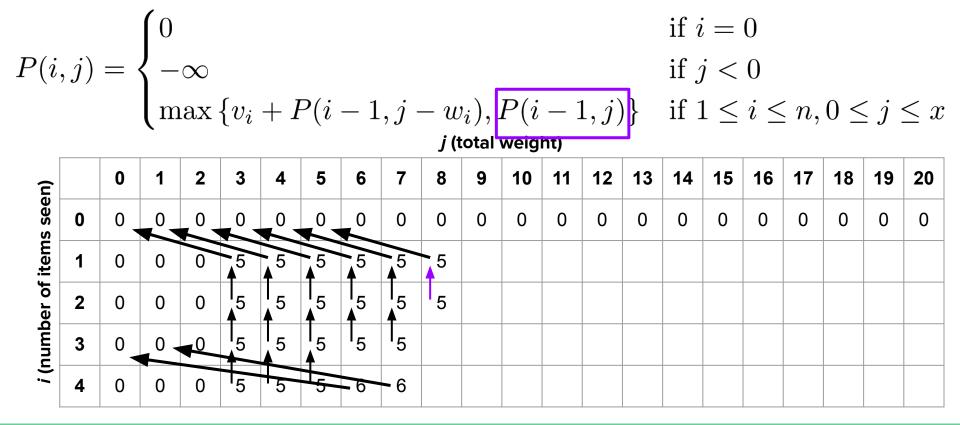
J (total weight)																					
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	1 5	5	√ 5	1 5													
2	0	0	0	5 •	 4	 4 5	 4 5	 4 5													
3	0	0 -	0	5	 5	 4	1 ₅	¹ 5													
4	0	0	0	5	5	5	- 6	- 6													

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

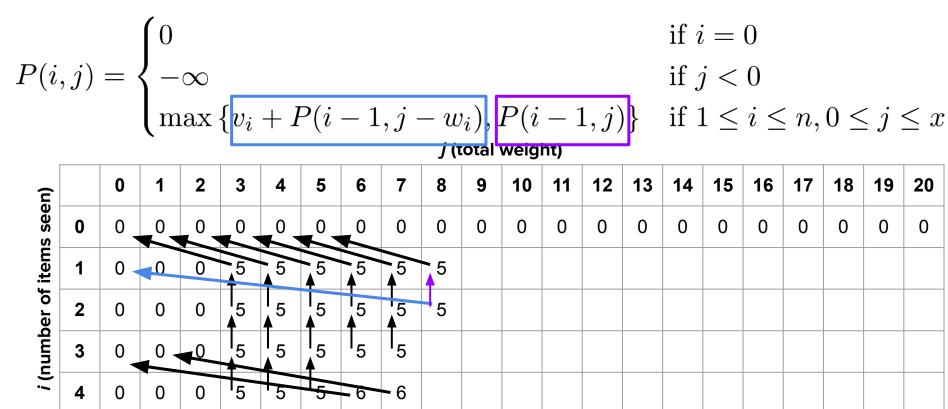
$$j \text{ (total weight)}$$

										<i>j</i> (t	total	weig	ht)									
Ē		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	1 5 1	5	5	5												
er of	2	0	0	0	5	5	1 5	5	5													
(number of	3	0	0 -	0	5	5	1 ₅	1 ₅	1 ₅													
) (r	4	0	0	0	5	5		-6	- 6													

8 10 6 W: x = 20\$5 \$13 \$6







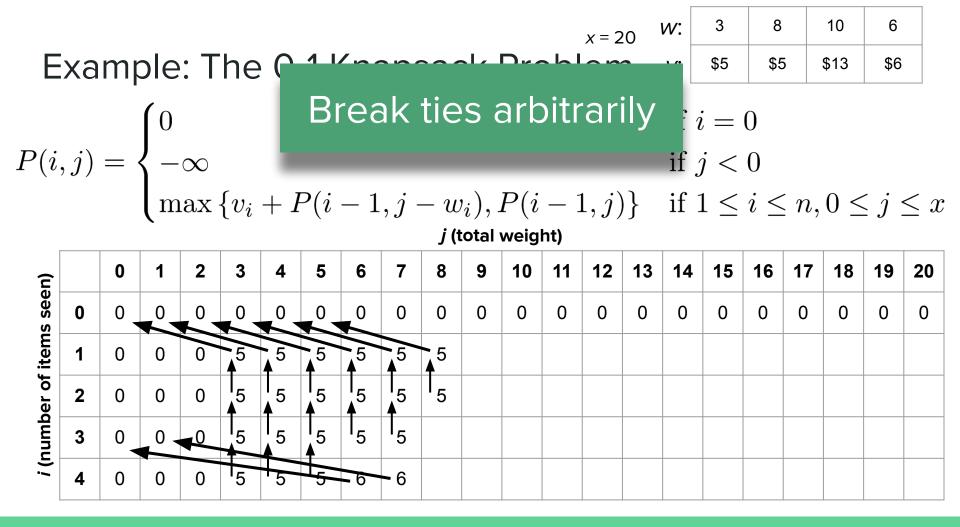
\$13

\$5

W:

6

\$6



$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ & \text{ j (total weight)} \end{cases}$$

			`							j (†	total	weig	ht)									
رة		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	5	5	5	5												
er of	2	0	0	0	5	5	1 5	5	5	5												
(number	3	0	0.	0	5	5	1 ₅	1 ₅	15	5												
; ;	4	0	0	0	5	5	5	- 6	-6	- 6												

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ & \text{j (total weight)} \end{cases}$$

_										<i>J</i> (1	ιοται	weig	nt)									
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	5	5	√ 5	5	5	1 5	√ 5											
	2	0	0	0	1 ₅	5 •	1 ₅	5	1 ₅	1 ₅	5											
	3	0	0 -	0	5	5	 5	1 ₅	1 ₅	15	1 ₅											
	4	0	0	0	1 5	5	1 5	- 6	- 6	- 6												

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

$$j \text{ (total weight)}$$

			`							<i>j</i> (1	total	weig	ht)									
<u>5</u>		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	5	5	5	5	5											
er of	2	0	0	0	 5	 5	1 ₅	5 •	1 ₅	1 5	1 ₅											
(number	3	0	0 -	0	5	5	5	5	1 ₅	15	1 ₅											
, T	4	0	0	0	5	5	5	- б	- 6	- 6	-11											

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \\ & \text{ } j \text{ (total weight)} \end{cases}$$

										<i>j</i> (†	total	weig	ht)									
מפפוו)		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	5	5	5	5	5	5	5	5										
	2	0	0	0	1 ₅	5	1 ₅	5	5	5	5	1 ₅										
	3	0	0 -	0	5	5	5	5	1 ₅	15	15											
	4	0	0	0	5	5	5	- б	-0	- 6	-11											

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ \hline j \text{ (total weight)} \end{cases}$$

Ē		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
items seen)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	5	5	5	5	5	5	5	5										
er of	2	0 -	0	0	5	5	 5	5	5	 5	5	1 ₅										
number	3	0	0 -	0	5	5	5	1 ₅	1 ₅	15	15	13										
<i>i</i> (r	4	0	0	0	5	5	5	- б	- 6	0	-11											

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ \text{ j (total weight)} \end{cases}$$

									<i>j</i> (1	total	weig	ht)									
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	1 5 1 5	5	5	5	1 5 1 5	5										
2	0 -	0	0	5	5	5	1 ₅	1 ₅	5	1 ₅	1 ₅										
3	0	0 -	0	5	5	5	5	5	5	5	13										
4	0	0	0	15	5	5	- 6	- 6	- 6	-11	13										

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

$$j \text{ (total weight)}$$

										<i>j</i> (t	total	weig	ht)									
<u>C</u>		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0.	0	0	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	5	5	5	5	5	5	5									
er of	2	0 •	0	0	5	5	5	5	1 ₅	5	1 ₅	1 ₅										
(number	3	0	0 -	0	5	5	5	5	1 ₅	5	5	13										
) (r	4	0	0	0	+ ₅	5	5	- б	6	-0	-11	T ₁₃										

Example: The O-1 Knapsack Problem $v: \begin{bmatrix} x = 20 \\ v : \end{bmatrix}$ \$5 \$13 \$6

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \\ & \text{ j (total weight)} \end{cases}$$

			`							<i>j</i> (t	total	weig	ht)									
<u></u>		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0.	0	0	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	5	<u>5</u>	5	5	5	5	5									
er of	2	0 -	10	0	5	5	5	5 •	 5	j 5	5	5	10									
(number	3	0	0 -	0	5	5	5	5	5	5	5	13										
<i>i</i> ,	4	0	0	0	5	5	5	- б	0	- 6	-11	43										

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \\ & \text{ } j \text{ (total weight)} \end{cases}$$

										j (†	total	weig	ht)									
<u>E</u>		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0,	0	0	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	5	5	5	5	1 5 1 5	5	5									
er of	2	0 -	0	10	5	5	5	5	5	5	5	5	10									
(number	3	0	0 -	0	5	5	5	5	5	5	5	13	13									
, T	4	0	0	0	5	5	15	- б	-0	- 6	-11	43	13									

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ & \text{ j (total weight)} \end{cases}$$

0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0.	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5								
2	0 -	-0 -	0-	5	5	5	5	5	1 ₅	5	5	10	10								
3	0	0 -	0	5	5	5	5	5	5	5	13	13	13								
4	0	0	0	15	5	5	- 6	-6	- 6	-11	13	13	13								

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \\ & \text{ j (total weight)} \end{cases}$$

										<i>j</i> (t	otal	weig	ht)									
ت		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0_	0	0-	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	5	5	5	5	5	5	5	5	5							
er of	2	0 -	0	0	5	5	5	5	5	1 5 ▲	1 5	5	10	10	10							
(number	3	0	0 -	0	5	5	5	5	5	5	5	13	13	13								
),(r	4	0	0	0	5	15	5	- б	- 6	- 6	-11	13	13	13								

		$\int 0$													if i	; =	0					
P(i, j	$) = \frac{1}{2}$	{ -	∞												if j	<i>i</i> <	0					
		$\lfloor m \rfloor$	ax	$\{v_i$	+ I	P(i)	- 1	,j	-u	(i)	P(i	_ 2	(1,j))}	if 1	l ≤	$i \leq$	$\leq n,$	0 <	$\leq j$	$\leq x$,
											weig											
<u> </u>	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	

8

\$5

W:

10

\$13

6

\$6

										<i>j</i> (t	otal	weig	ht)									
7		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	0	0	0	0	0	0	0	0	0	0	0.	0-	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	5	5	5	5	5	5	5	5	5	5	5							
	2	0 🔻	0	0	5	5	5	5	5	5	5 A	5	10	10	10							
	3	0	0 -	0	5	5	5	5	5	5	5	13	13	13	18							
	4	0	0	0	5	15	1 5	- 6	-0	- 6	- 11	13	13	13								

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ & \text{ j (total weight)} \end{cases}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0.	0-	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5	5	5	5	5	5	5	5							
2	0 -	0	0	5	5	5	5	5	5	5 •	5	10	10	10							
3	0	0 -	0	5	5	5	5	5	5	15	13	13	13	1 8							
4	0	0	0	5	5	$\frac{1}{5}$	- б	- б	6	-11	կ ₃	13	կ ₃	18							

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

$$\underbrace{\int_{j \text{ (total weight)}}}$$

									<i>)</i> ('	Ulai	weig	111,									
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0.	0-	0	0	0	0	0	0	0	0	0	0
1	0	0	0	5	5	5	5		5	5	5	5	5	5	5	5					
2	0 -	0	0	5	5=	5	5	5	5	5	5	10	10	10	10	10					
3	0	0 -	0	5	5	5	5	5	5	5	13	13 •	13 •	18	18	18					
4	0	0	0	15	5	5	- б	- 6	- 6	-11	43	13	43	18	1 ₁₈	18					

<i>x</i> = 20						
Example: The 0-1 Knapsack Problem	V:	\$5	\$5	\$13	\$6	
·						

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ \hline j \text{(total weight)} \end{cases}$$

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ j \text{ (total weight)} \end{cases}$$

_			_							<i>j</i> (t	total	weig	ht)									
<u>ت</u>		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0.	0.	0-	0	0	0	0	0	0	0	0	0	0
items	1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5				
er of	2	0 -	10-	0	5	5		54	5	5	5	5	10	10	10	10	10	10				
(number	3	0	0.	0	5	5	5	5	5	5	5	13-	13	13	18	18	18	18				
, (r	4	0	0	0	5	5	1 5	- 6	- 0	-6	-11	13	13	13	18	18	16	- 19				

<i>x</i> = 20						
Example: The 0-1 Knapsack Problem	V:	\$5	\$5	\$13	\$6	
·						

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1,j-w_i), P(i-1,j)\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ \hline j \text{ (total weight)} \end{cases}$$

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Example:

<i>x</i> = 20	L					
The 0-1 Knapsack Problem	V :	\$5	\$5	\$13	\$6	

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \\ & \text{ j (total weight)} \end{cases}$$

									<i>j</i> (t	total	weig	ht)									
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0.	0.	0-	0	0	0	0	0	0	0 -	0	0	0
1	0	0	0	5	5	5	5-		5	5	5	5	5	5	5	5	5	5	5	5	5
2	0 -	0-	0	5	54	5	54	5-		5	5 <	110	10	10	10	10	10	10	10	10	10
3	0	0 -	0	5	5	5	5	5	5	5	13-	13	13.	18 .	18	18	18	18	18	19	18
4	0	0	0	5	5	15	- б	б	-6	-11	13	13	13	18	18	lδ	-19	-19	19	19	19

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\left\{v_i + P(i-1,j-w_i), P(i-1,j)\right\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

$$j \text{ (total weight)}$$

			m	ax	$\{v_i$	+I	P(i)	– 1	,j	-u	(i), i	P(i	_ [(1,j))	if I	1 ≤	$i \leq$	$\leq n,$	0 <	$\leq j$	$\leq x$
-			-	ī			i	i	1	<i>j</i> (1	otal	weig	ht)	ī	ī	i	ī	i	ī	i	ī	
Ē		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0.	0.	0-	0	0	0	0	0	0.	0 -	0	0	0
items	1	0	0	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
oer of	2	0 🔻	0-	0	5	5	5	54	54		5	5	110	10	10	10	10	10	10	10	10	10
(number	3	0	0 -	0	5	5	5	5	5	15	¹ 5	13-	13-	13	18	18	18	18	18	18	19	18
, 	4	0	0	0	5	+5	15	- 6	- 6	- 6	-11	43	13	13	18	18	8	-19	-19	-19 -	19	1 9

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\left\{v_i + P(i-1,j-w_i), P(i-1,j)\right\} & \text{if } 1 \leq i \leq n, 0 \leq j \leq x \end{cases}$$

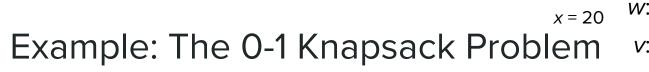
$$j \text{ (total weight)}$$

			$\lfloor m \rfloor$	ax	$\{v_i$	+ I	P(i)	– 1	,j	-u	(i),	P(i	_ [(1,j))}	if	1 ≤	$i \leq$	n	0 <	$\leq j$	$\leq x$
										<i>j</i> (t	otal	weig	ht)									
Ē		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0	0.	0-	0	0	0-	0	0	0	0 -	0	0	0
items	1	0	0	0	5	5	5	5	5	5	5	5	5	5	<u>5</u>	5	5	5	5	5	5	5
er of	2	0 🗸	0	0	5	5-	5	5	5.4		5.	5 -	110	10	10	10	10	10	10	10	10	10
number	3	0	0 -	0	5	5	5	5	5	5	5	13-	13	13	18	18	18	18	18	18	19	18
; =	4	0	0	0	5	5	1 5	- 6	6	0	-11	13	13	43	18	18	8	-19	-19	-19 -	19	-1 9

$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max{\{v_i + P(i-1, j-w_i), P(i-1, j)\}} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

$$j \text{ (total weight)}$$

				ax								weig										
Ĵ.		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0	0.	0-	0	0	0	0	0	0 .	0 -	0	0	0
בון	1	0	0	0	5	5	5	5		5	5	5	5	5	5	5	5	5	5	5	5	5
5 5	2	0 🗸	0	0	5	5-	<u>5</u>	5	5.4		5.	5 -	110	10	10	10	10	10	10	10	10	10
	3	0	0 -	0	5	5	5	5	5	5	15	13-	13-	13.	18 -	18	18	18	18	18	19	18
?	4	0	0	0	5	5	5	_ ნ	- 6	- 6	-11	43	13	13	18	18	18	19	-19	-19 -	19	1 9



$$P(i,j) = \begin{cases} 0 & \text{if } i = 0 \\ -\infty & \text{if } j < 0 \\ \max\{v_i + P(i-1, j-w_i), P(i-1, j)\} & \text{if } 1 \le i \le n, 0 \le j \le x \end{cases}$$

$$j \text{ (total weight)}$$

\$13

\$6

			m	ax	$\{v_i$	+I	P(i)	– 1	,j -	-u	(i), i	P(i	— [(1,j))	if I	1 <	$i \leq$	$\leq n,$	0 <	$\leq j$	$\leq x$
г	j (total weight)																					
جَ		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
s seen)	0	0	0	0	0	0	0	0	0	0.	0.	0-	0	0	0	0	0	0.	0 -	0	0	0
items	1	0	0	0	5	5	5	5	5	5	5	5	5	5	<u>5</u>	5	5	5	5	5	5	5
er of	2	0 🔻	0-	0	5	5-	5	5	- 5.4 - 5.4		5	5	110	10	10	10	10	10	10	10	10	10
(number	3	0	0 -	0	5	5	5	5	5	5	5	13-	13-	13	18	18	18	18	18	18	19	18
,	4	0	0	0	5	+5	15	- 6	-6	-6	-11	43	13	13	18	18	18	-19	-19	-19 -	19	1 9