Hello!

Can you read this in the back???

Environments

2019-05-15

~~Mr.~~ Alex(ander)

Past three weeks

How to *use* essential language constructs?

- Data Types
- Recursion
- Higher-Order Functions

Next two weeks

How to *implement* language constructs?

- Local variables and scope
- Environments and Closures
- Type Inference

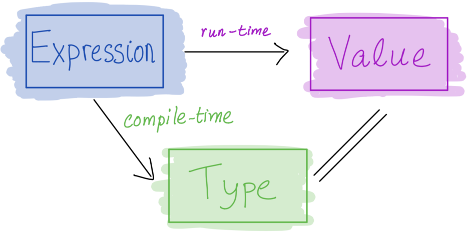
Interpreter

How do we *represent* and *evaluate* a program?

Roadmap: The Nano Language

Features of Nano:

1. **Arithmetic**
2. Variables
3. Let-bindings
4. Functions
5. Recursion



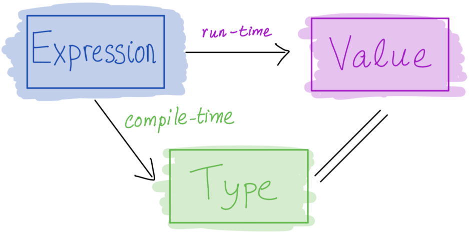
1. Nano: Arithmetic

A “grammar” of arithmetic expressions:

```
e ::= n
    | e1 + e2
    | e1 - e2
    | e1 * e2
```

Expressions		Values
`4`	`==>`	`4`
`4 + 12`	`==>`	`16`
`(4+12) - 5`	`==>`	`11`

Representing Arithmetic Expressions and Values



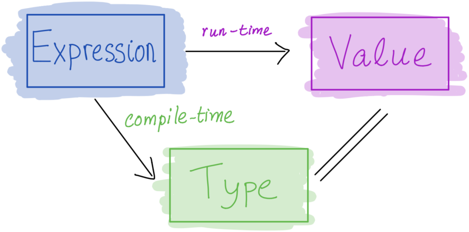
Lets *represent* arithmetic expressions as type

```
data Expr
  = ENum Int             -- ^ n
  | EAdd Expr Expr       -- ^ e1 + e2
  | ESub Expr Expr       -- ^ e1 - e2
  | EMul Expr Expr       -- ^ e1 * e2
```

Lets *represent* arithmetic values as a type

```
type Value = Int
```

Evaluating Arithmetic Expressions



We can now write a Haskell function to *evaluate* an expression:

```
eval :: Expr -> Value
eval (ENum n)      = n
eval (EAdd e1 e2) = eval e1 + eval e2
eval (ESub e1 e2) = eval e1 - eval e2
eval (EMul e1 e2) = eval e1 * eval e2
```

Alternative representation

Lets pull the *operators* into a separate type

```haskell
data Binop = Add                  -- ^ `+`
           | Sub                  -- ^ `-`
           | Mul                  -- ^ `*`

data Expr  = ENum Int              -- ^ n
           | EBin Binop Expr Expr  -- ^ e1 `op` e2
```

QUIZ

Evaluator for alternative representation

```
eval :: Expr -> Value
eval (ENum n)        = n
eval (EBin op e1 e2) = evalOp op (eval e1) (eval e2)
```

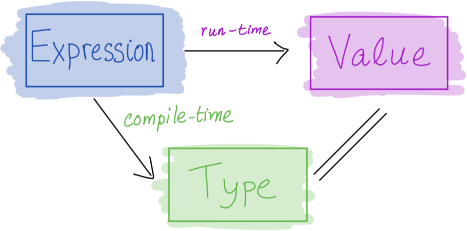
What is a suitable type for `evalOp`?

```
{- 1 -} evalOp :: BinOp -> Value


{- 2 -} evalOp :: BinOp -> Value -> Value -> Value
{- 3 -} evalOp :: BinOp -> Expr  -> Expr -> Value


{- 4 -} evalOp :: BinOp -> Expr -> Expr -> Expr
{- 5 -} evalOp :: BinOp -> Expr -> Value
```

The Nano Language

Features of Nano:

1. Arithmetic *[done]*
2. **Variables**
3. Let-bindings
4. Functions
5. Recursion



2. Nano: Variables

Let's add variables and `let` bindings!

```
e ::= n                   -- OLD
    | e1 + e2
    | e1 - e2
    | e1 * e2
                          -- NEW
    | x                   -- variables
```

Lets extend our datatype

```
type Id = String

data Expr
  = ENum Int              -- OLD
  | EBin Binop Expr Expr
                         -- NEW
  | EVar Id              -- variables
```

QUIZ

What should the following expression evaluate to?

```
x + 1
```

(A) 0

(B) 1

(C) Error

Environment

An expression is evaluated in an **environment**

- A **phone book** which maps *variables* to *values*

```
[ "x" := 0, "y" := 12, ...]
```

A type for *environments*

```
type Env = [(Id, Value)]
```

Evaluation in an Environment

We write

```
(eval env expr) ==> value
```

to mean

When `expr` is **evaluated in environment** `env` the result is `value**`

That is, when we have variables, we modify our `evaluator` to take an input environment `env` in which `expr` must be evaluated.

```
eval :: Env -> Expr -> Value
eval env expr = ... value-of-expr-in-env...
```

First, lets update the evaluator for the arithmetic cases `ENum` and `EBin`

```
eval :: Env -> Expr -> Value
eval env (ENum n)        = ???
eval env (EBin op e1 e2) = ???
```

QUIZ

What is a suitable `?value` such that

```
eval [ "x" := 0, "y" := 12, ...] (x + 1)  ==>  ?value
```

(A) `0`

(B) `1`

(C) Error

QUIZ

What is a suitable `env` such that

```
eval env (x + 1)    ==>    10
```

(A) `[]`

(B) `[x := 0, y := 9]`

(C) `[x := 9, y := 0]`

(D) `[x := 9, y := 10, z := 666]`

(E) `[y := 10, z := 666, x := 9]`

Evaluating Variables

Using the above intuition, lets update our evaluator to handle variables i.e. the `EVar` case:

```
eval env (EVar x)        = ???
```

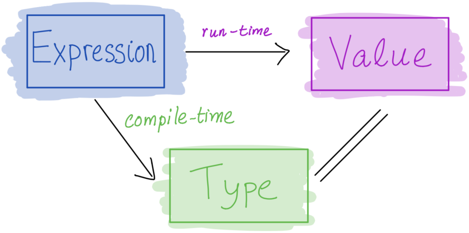
Lets confirm that our `eval` is ok!

```
envA = []
envB = ["x" := 0 , "y" := 9]
envC = ["x" := 9 , "y" := 0]
envD = ["x" := 9 , "y" := 10 , "z" := 666]
envE = ["y" := 10, "z" := 666, "x" := 9  ]

-- >>> eval envA (EBin Add (EVar "x") (ENum 1))
-- >>> eval envB (EBin Add (EVar "x") (ENum 1))
-- >>> eval envC (EBin Add (EVar "x") (ENum 1))
-- >>> eval envD (EBin Add (EVar "x") (ENum 1))
-- >>> eval envE (EBin Add (EVar "x") (ENum 1))
```

The Nano Language

Features of Nano:

1. Arithmetic expressions *[done]*
2. Variables *[done]*
3. **Let-bindings**
4. Functions
5. Recursion



2. Nano: Variables

Let's add variables and `let` bindings!

```
e ::= n                   -- OLD
    | e1 + e2
    | e1 - e2
    | e1 * e2
    | x
                          -- NEW
    | let x = e1 in e2
```

Lets extend our datatype

```
type Id = String

data Expr
  = ENum Int              -- OLD
  | EBin Binop Expr Expr
  | EVar Id
                         -- NEW
  | ELet Id Expr Expr
```

QUIZ

What *should* the following expression evaluate to?

```
let x = 0
in
  x + 1
```

(A) Error

(B) 1

(C) 0

QUIZ

What *should* the following expression evaluate to?

```
let x = 0
in
  let y = 100
  in
    x + y
```

(A) Error

(B) 0

(C) 1

(D) 100

(E) 101

QUIZ

What *should* the following expression evaluate to?

```
let x = 0
in
  let x = 100
  in
    x + 1
```

(A) Error

(B) 0

(C) 1

(D) 100

(E) 101

QUIZ

What *should* the following expression evaluate to?

```
let x = 0
in
  (let x = 100 in
   in
     x + 1
  )
  +
  x
```

(A) Error

(B) 1

(C) 101

(D) 102

(E) 2

Principle: Static/Lexical Scoping

Every variable *use* gets its value from a unique *definition*:

- "Nearest" `let`-binder in program *text*

"Static" means you can tell *without running the program*

Great for readability and debugging

1. Define *local* variables
2. Be sure *where* each variable got its value

Don't have to scratch head to figure where a variable got "assigned"

How to **implement** static scoping?

QUIZ

Lets re-evaluate the quizzes!

let y=1 in

```
                -- env
let x = 0
in              -- ??? what env to use for `x + 1`?
  x + 1
```

(A) env } don't mention 'x' at all

(B) [ ]

(C) [("x" := 0)] ← loses 'env'

[✓ **(D)**] ("x" := 0) : env ←

env = "cat"; 3

(E) env ++ ["x" := 0]

(x=1):(x=0):env

QUIZ

input

```
                         -- env
let x = 0
in                          -- (x := 0) : env
  let y = 100
  in                        -- ??? what env to use for `x + y` ?
    x + y
```

(A) `("x" := 0) : env`

(B) `("y" := 100) : env`

[✓ **(C)**] `("y" := 100) : ("x" := 0) : env`

2nd

outer let

(D) `("x" := 0) : ("y" := 100) : env`

(E) `[("y" := 100), ("x" := 0)]`

QUIZ

Lets re-evaluate the quizzes!

```
                     -- env
let x = 0
in                   -- ("x" := 0) : env
  let x = 100
  in                 -- ??? what env to use for `x + 1`?
    x + 1
```

(A) `("x" := 0) : env`

(B) `("x" := 100) : env`

(C) `("x" := 100) : ("x" := 0) : env`

(D) `("x" := 0) : ("x" := 100) : env`

(E) `[("x" := 100)]`

Extending Environments

Lets fill in `eval` for the `let x = e1 in e2` case!

```
eval env (ELet x e1 e2) = ???
```

1. **Evaluate** `e1` in `env` to get a value `v1`
2. **Extend** environment with value for `x` i.e. to `(x := v1) : env`
3. **Evaluate** `e2` using *extended* environment.

Lets make sure our tests pass!

Run-time Errors

Haskell function to *evaluate* an expression:

```
eval :: Env -> Expr -> Value

eval env (Num n)          = n

eval env (Var x)          = lookup x env       -- (A)

eval env (Bin op e1 e2) = evalOp op v1 v2    -- (B)
  where
    v1                    = eval env  e1       -   (C)
    v2                    = eval env  e2       -- (C)

eval env (Let x e1 e2)  = eval env1 e2
  where
    v1                    = eval env e1
    env1                  = extend env x v1    -- (D)
```

QUIZ

Will `eval env expr` always return a `value` ? Or, can it *crash*?

[✓ **(A)**] operation at `A` may fail

(B) operation at `B` may fail

(C) operation at `C` may fail

(D) operation at `D` may fail

(E) nah, its all good. . . , always returns a `Value`

Undefined Variables

How do we make sure `lookup` doesn't cause a run-time error?

Bound Variables

Consider an expression `let x = e1 in e2`

- An occurrence of `x` is **bound** in `e2`
- i.e. when occurrence of form `let x = ... in ... x ...`
- i.e. when `x` occurs "under" a `let` binding for `x`.

Free Variables

An occurrence of `x` is **free** in `e` if it is **not bound** in `e`

free(x) ⇒ no-bound(x)

X

```
let x = e
in .
   :
   .
   x
   .
   :
```

Undefined Variables (continued)

env := ('x', 3)

x ('y', 5)

x

Closed Expressions

An expression e is **closed** in environment env:

- If all **free** variables of e are defined in env

Successful Evaluation

lookup will never fail

- If eval env e is only called on e that is closed in env

lookup env x

QUIZ

Which variables occur free in the expression?

```
let y = (let x = 2
          in x   ) + x
in
  let x = 3
  in
    x + y
```

??

x is free b.c. no definition

(A) None

[✓ (B) x]

(C) y

(D) x and y

Exercise

- free variables

is it safe

check e to see if it has f.v.'s !!

Consider the function

```
evaluate :: Expr -> Value
evaluate e
  | isOk e    = eval emptyEnv e
  | otherwise = error "Sorry! bad expression, it will crash `eval`!"
  where
    emptyEnv  = []                     -- has NO bindings
```

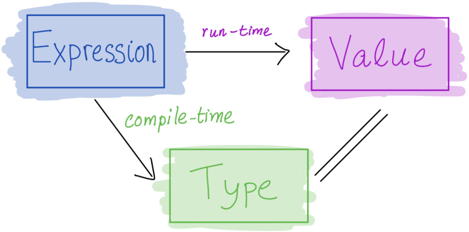
What should isOk check for? (Try to implement it for nano...)

if e has fv then eval will crash

The Nano Language

Features of Nano:

1. Arithmetic expressions *[done]* $e_1 + e_2$
2. Variables *[done]* x
3. Let-bindings *[done]* $x = e$
4. **Functions**
5. Recursion



$\textcircled{x}$

$\text{env} :: [(\text{Id}, \text{Value})]$

$\text{eval env } x$

$\text{let } x = e \Rightarrow v$

$\text{in } e'$

$\textcircled{x} + 1$

Nano: Functions

Let's add

- *lambda abstraction* (aka function definitions)
- *application* (aka function calls)

```
e ::= n                         -- OLD
    | e1 `op` e2
    | x
    | let x = e1 in e2
                                -- NEW
    | \x -> e                   -- abstraction
    | e1 e2                     -- application
```

function

Example

```
let incr = [\x -> x + 1]
in
  (incr 10) ⟶ 11
```

Representation

$\lambda x.e \rightsquigarrow \text{Expr} \longrightarrow \text{Value}$

```
data Expr
  = ENum Int                -- OLD
  | EBin Binop Expr Expr
  | EVar Id
  | ELet Id Expr Expr
                           -- NEW
  | ???                    -- abstraction \x -> e
  | ???                    -- application (e1 e2)
```

Representation

```
data Expr
  = ENum Int                  -- OLD
  | EBin Binop Expr Expr
  | EVar Id
  | ELet Id Expr Expr
                              -- NEW
  | ELam Id Expr              -- abstraction \x -> e
  | EApp Expr Expr            -- application (e1 e2)
```

Example

```
let incr = \x -> x + 1
in
   incr 10
```

is represented as

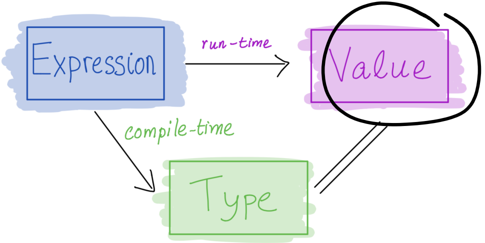
```
                                         ??
                                  e
              λ    x
ELet "incr" (ELam "x" (EBin Add (EVar "x") (ENum 1)))
  (
    EApp (EVar "incr") (ENum 10)
  )
            e1            e2
```

Functions are Values

✓ syntax / feature

✓ Expr

Recall the trinity



But... what is the *value* of a function?

Lets build some intuition with examples.

QUIZ

What does the following expression evaluate to?

```
let incr = \x -> x + 1       -- abstraction ("definition")
in
   incr 10                   -- application ("call")
```

(A) Error/Undefined

(B) 10

[✓ **(C)** 11]

(D) 0

(E) 1

$$(\lambda x \to x + 1)\ 10$$

$$\downarrow$$

$$10 + 1$$

$$\downarrow$$

$$11$$

What is the Value of `incr`?

mtv env (λx → x+1)

let incr = ℓ

in (incr 10)

... (v 10)

'code'

- Is it an `Int` ?
- Is it a `Bool` ?
- Is it a ???

What information do we need to store (in the `Env`) about `incr`?

(incr ⟼ ??)

input / output type

A Function’s Value is its Code

```
                        -- env
let incr = \x -> x + 1
in                      -- ("incr" := <code>) : env
    incr 10             -- evaluate <code> with parameter := 10
```

What information do we store about <code> ?

A Call's Value

code
i.e.
the Expr

How to evaluate the "call" `incr 10` ?

1. Lookup the <code> i.e. `<param, body>` for `incr` (stored in the environment),

   ("x"

2. Evaluate `body` with `param` set to 10!

Two kinds of Values

We now have *two* kinds of Values

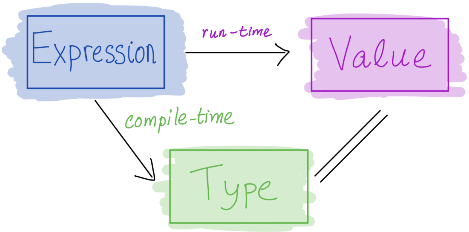
```
v ::= n                    -- OLD
    | <x, e>               -- <param, body>
```

1. Plain `Int` (as before)
2. A function's "code": a pair of "parameter" and "body-expression"

```haskell
data Value
  = VInt  Int              -- OLD
  | VCode Id Expr          -- <x, e>
```

x + 1

'x'



Evaluating Lambdas and Applications

```
eval :: Env -> Expr -> Value
                                -- OLD
eval env (ENum n)        = ???
eval env (EVar x)        = ???
eval env (EBin op e1 e2) = ???
eval env (ELet x  e1 e2) = ???
                                -- NEW
eval env (ELam x e)      = ???
eval env (EApp e1 e2)    = ???
```

Lets make sure our tests work properly!

```
exLam1 = ELet "incr" (ELam "x" (EBin Add (EVar "x") (ENum 1)))
          (
            EApp (EVar "incr") (ENum 10)
          )

-- >>> eval [] exLam1
-- 11
```

QUIZ

What should the following evaluate to?

```
let c = 1
in
   let inc = \x -> x + c)
   in
      inc 10  ~> 11
```

(A) Error/Undefined

(B) 10

[✓ **(C)** 11]

(D) 0

(E) 1

QUIZ

And what should *this* expression evaluate to?

```
let c = 1
in
  let inc = \x -> x + c
  in
    let c = 100
    in
      inc 10
```

c:=1

env

evaluated here

memory
inc ↦ ...
c ↦ 100

c:=1

110

let y = 100
in
inc 10

1

(A) Error/Undefined

(B) 110

[✓ **(C)**] 11

The "Immutability Principle"

A function's behavior should *never change*

- A function must *always* return the same output for a given input

Why?

```
> myFunc 10
0
```

```
> myFunc 10
10
```

Oh no! How to find the bug? Is it

- In `myFunc` or
- In a global variable or
- In a library somewhere else or
- ...

My worst debugging nightmare

The Immutability Principle ?

How does our `eval` work?

```
exLam3 = ELet "c" (ENum 1)
           (
           ELet "incr" (ELam "x" (EBin Add (EVar "x") (EVar "c")))
              (
               ELet "c" (ENum 100)
                 (
                 EApp (EVar "incr") (ENum 10)
                 )
              )
           )

-- >>> eval [] exLam3
-- ???
```

Oops?

```
                          -- []
let c = 1
```

Enforcing Immutability with Closures

How to enforce immutability principle

- `inc 10` **always** returns 11?

Key Idea: Closures

At definition: *Freeze* the environment the function's value
At call: Use the *frozen* environment to evaluate the *body*
Ensures that `inc 10` *always* evaluates to the *same* result!

```
                                    -- []
let c = 1
in                                  -- ["c" := 1]
   let inc = \x -> x + c
   in                               -- ["inc" := <frozenv, x, x+c>, c := 1]  <<< frozenv =
      let c = 100
      in                            -- ["c" := 100, "inc" := <frozenv, x, x+c>, "c" := 1]
        inc 10
```

Now we evaluate
`eval env (inc 10)`

use this in eval

Representing Closures

Lets change the `Value` datatype to also store an `Env`

```
data Value
  = VInt  Int           -- OLD
  | VClos Env Id Expr   -- <frozenv, param, body>
```

Evaluating Function Definitions

How should we fix the definition of `eval` for `ELam`?

```
eval :: Env -> Expr -> Value

eval env (ELam x e) = ???
```

Hint: What value should we *bind* `incr` to in our example above?

(Recall **At definition** *freeze* the environment the function's value)

Evaluating Function Calls

How should we fix the definition of `eval` for `EApp`?

(e1 e2)

v

```
eval :: Env -> Expr -> Value

eval env (EApp e1 e2) = ???
```

<env, x, body>

(Recall **At call:** Use the *frozen* environment to evaluate the *body*)

eval

Hint: What value should we *evaluate* `incr 10` to?

closure

1. Evaluate `incr` to get `<frozenv, "x", x + c>`
2. Evaluate `10` to get `10` ← arg
3. Evaluate `x + c` in `x:=10 : frozenv`

body

(x:=v):env

body

Let's generalize that recipe!

1. Evaluate `e1` to get `<frozenv, param, body>`
2. Evaluate `e2` to get `v2`
3. Evaluate `body` in `param := v2 : frozenv`

Immutability Achieved

Lets put our code to the test!

```
exLam3 =
  ELet "c" (ENum 1)
    (
    ELet "incr" (ELam "x" (EBin Add (EVar "x") (EVar "c")))
       (
        ELet "c" (ENum 100)
          (
          EApp (EVar "incr") (ENum 10)
          )
       )
    )

-- >>> eval [] exLam3
-- ???
```

QUIZ

What should the following evaluate to?

```
let add = \x -> (\y -> x + y)
in
  let add10 = add 10
  in
    let add20 = add 20
    in
      (add10 100) + (add20 1000)
```

(A) Error/Undefined

(B) 1100

(C) 30

[✓ (D) 1130]

env = []

closure

< • → [] , x , λy → x + y >

add

env: add ↦ ,

add ↦ ...

< • , (20 + y) >

y

(10 + 100)

+

(20 + 1000)

(λy → 10 + y) →

< • , y , 10 + y >

QUIZ

What should the following evaluate to?

```
let add = \x -> (\y -> x + y)
in
  let add10 = add 10
  in
    let doTwice = \f -> (\x -> f (f x))
    in
      doTwice add10 100
```

(A) Error/Undefined

(B) 1130

[✓ **(C)** 120]

(D) 110

$add := \langle \cdot, x, \lambda y. x+y \rangle$

$add10 := \langle \cdot, y, x+y \rangle$

$\langle \cdot, f, \lambda x \to f(f x) \rangle$

doTwice $\rightsquigarrow$

$(\langle \cdot, f, \lambda x \to f (f x) \rangle \; add10) \; 100$

$\rightsquigarrow (\lambda x. add10 \; (add10 \; x)) \; 100$

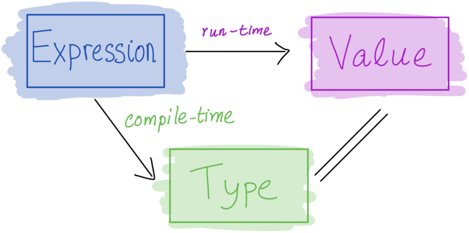
$add10 \; (add10 \; 100) \rightsquigarrow 120$

Functions Accepting Functions Achieved!

```
exLam4 = ...

-- >>> eval [] exLam4
```

TODO

The Nano Language



Features of Nano:

1. Arithmetic expressions *[done]*
2. Variables *[done]*
3. Let-bindings *[done]*
4. Functions *[done]*
5. **Recursion**

. . . You figure it out **Hw4** . . . :-)