Problem Set 2 Solutions

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1.

(a)

$$\begin{aligned}
P(E) &= \sum_{r,m,h,s,b,a,l} P(r,m,h,s,b,a,l,E) \\
&= \sum_{r,m,h,s,b,a,l} P(r)P(m)P(h|r,m)P(s|h)P(a|h)P(b|h)P(E|s,b)P(l|b,a) \\
&= \sum_b \sum_s P(E|s,b) \sum_h P(s|h)P(b|h) \sum_a P(a|h) \sum_l P(l|b,a) \sum_r P(r) \sum_m P(h|r,m)P(m)
\end{aligned}$$

(b) The corresponding elimination ordering is $\prec = M, R, L, A, H, S, B$

(c)

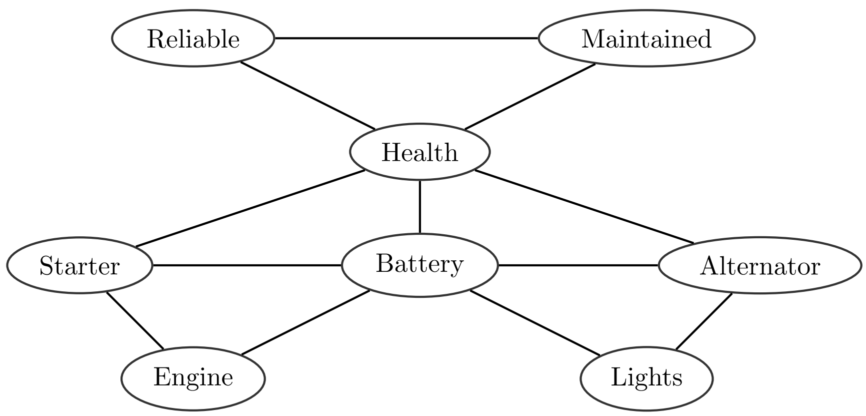

Figure 1: Moralized graph for the Bayesian Network

(d) The initial set of factors is $\Phi = \{\phi(H,M,R), \phi(S,B,H), \phi(H,B,A), \phi(S,B,E), \phi(B,A,L)\}$

Step	Variable Eliminated	Intermediate Factor	Variables Involved	New Factor
1	M	$\psi_1(H,M,R) = \phi(H,M,R)$	H,M,R	$\tau_1(H,R) = \sum_m \psi_1(H,m,R)$
2	R	$\psi_2(H,R) = \tau_1(H,R)$	H,R	$\tau_2(H) = \sum_r \psi_2(H,r)$
3	L	$\psi_3(B,A,L) = \phi(B,A,L)$	B,A,L	$\tau_3(B,A) = \sum_l \psi_3(B,A,l)$
4	A	$\psi_4(A,B,H) = \tau_3(B,A)\phi(H,B,A)$	H,B,A	$\tau_4(B,H) = \sum_a \psi_4(a,B,H)$
5	H	$\psi_5(S,B,H) = \tau_2(H)\tau_4(B,H)\phi(S,B,H)$	S,B,H	$\tau_5(S,B) = \sum_h \psi_5(S,B,h)$
6	B	$\psi_6(S,B,E) = \tau_5(S,B)\phi(S,B,E)$	S,B,E	$\tau_6(S,E) = \sum_l \psi_6(S,b,E)$
7	S	$\psi_6(S,E) = \tau_6(S,E)$	S,E	$\tau_7(E) = \sum_s \psi_7(s,E)$

Table 1: Variable Elimination Procedure for $P(E)$

(e) The largest induced scope is 2, and assuming each variable can take k possible values, the computational complexity would be $O(nk^2)$, since the factor has k^2 entries.

(f) The induced graph is the same as the moralized netowrk in part (c), since there are already edges between all variables appearing in some ψ generated by the variable elimination.

(g) Since the ordering in part (b) produced the most efficient variable elimination in part (d), the associated clique tree is that of part (d), shown below:

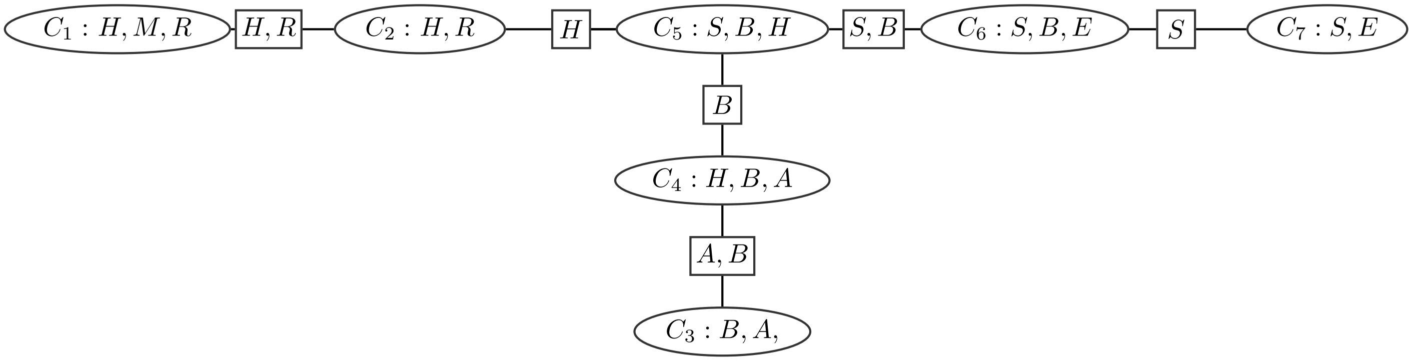

Figure 2: Clique Tree for the Bayesian Network

(h) If we first eliminate H, there is an induced scope of 5, as we create a new factor $\tau_1(R, M, S, B, A)$, next we could eliminate B, to induce a scope of 6, as we create a new factor $\tau_2(R, M, S, A, E, L)$ with the remaining 6 random variables. So any elimination ordering starting $\prec = H, B \ldots$ would lead to a computational complexity of $O(nk^6)$

2.

(a) x

(b) x

(c) x

(d) Either $\mathcal{T}' = \mathcal{T}$ or there are $C_i, C_j \in \mathcal{T}$ such that $C_i \subseteq C_j$. If we eliminate C_i, $\mathcal{T}'$ is family preserving since Scope$[\phi_i] \subseteq C_i \subseteq C_j$. We connect all neighbors of C_i to C_j when removing C_i, so any path through C_i is preserved via C_j since there can be no $X \in C_i$ not in C_j. Thus, if $\mathcal{T}$ satisfies the running intersection property, so does $\mathcal{T}'$

3.

(a) We can follow the algorithim for out of clique inference in a clique tree to solve for $P(X_i, X_j)$, given a calibrated clique tree $\mathcal{T}$. First we define $\mathcal{T}'$ to be the subtree of $\mathcal{T}$ such that $(X_i, X_j) \subseteq \mathit{Scope}[\mathcal{T}']$, then define a new set of factors (letting C_j be the root):

$$\Phi = \{\beta_j\} \cup \{\phi_k = \frac{\beta_k}{\mu_{k,k+1}} \mid k \in \mathcal{V}_{\mathcal{T}'} - \{j\}\}$$

Which can be done in linear time. Then we perform sum-product variable elimination on a ordering $\prec = X_{i+1} \ldots X_{j-1}$ of $\mathbf{Z} = \mathit{Scope}[\mathcal{T}'] - \{X_i, X_j\}$. The running time of the variable elimination is $O(nk)$ since the largest induced width is one.

(b) Naively, we must run the $O(nk)$ process for all $\binom{n}{2}$ combinations of i and j, so the runing time would be $O(n^3k)$

(c) Following the procedure in part (a), perform variable elimination on the ordering $\prec = X_2, X_3$ and set of factors $\Phi = \{\beta_3(X_3, X_4), \frac{\beta_1(X_1,X_2)}{\mu_{1,2}(X_2)}, \frac{\beta_2(X_2,X_3)}{\mu_{2,3}(X_3)}\}$. So

$$\begin{aligned} P(X_1, X_4) &= \sum_{X_3} \frac{\beta_3(X_3, X_4)}{\mu_{2,3}(X_3)} \sum_{X_2} \frac{\beta_1(X_1, X_2)\beta_2(X_2, X_3)}{\mu_{1,2}(X_2)} \\ &= \end{aligned}$$