Problem Set 1 Solutions

Calvin Walker

1. Probability Review

(a) (i)

$$P(A|B,E)P(B|E) = \frac{P(A,B,E)}{P(B,E)}P(B|E) = \frac{P(A,B,E)}{P(E)P(B|E)}P(B|E)$$
$$= \frac{P(A,B,E)}{P(E)} = P(A,B|E)$$

So $P(A,B|E) = P(A|B,E)P(B|E)$ as needed.

(ii)

(b) .

(c) .

(d) (i)

$$P(H|E_1,E_2) = \frac{P(H,E_1,E_2)}{P(E_1,E_2)} = \frac{P(H)P(E_1|H)P(E_2|E_1,H)}{P(E_1,E_2)}$$
$$= \frac{P(H)P(E_1,E_2|H)}{P(E_1,E_2)}$$

So option (b) is sufficient to compute $P(H|E_1,E_2)$

(ii) If $(E_1 \perp E_2|H)$, then:

$$\frac{P(H)P(E_1,E_2|H)}{P(E_1,E_2)} = \frac{P(H)P(E_1|H)P(E_2|H)}{\sum_i P(E_1|H_i)P(E_2|H_i)}$$

So option (c) is sufficient to compute $P(H|E_1,E_2)$

(e) (i) .

(ii)

$$-E_p[-\log P(X)] = -\sum_X P(X)(-\log P(X))$$
$$= \sum_X P(X) \sum_X \log P(X)$$
$$\leq \log \sum_X P(X) = 0$$

So $E_p[-\log P(X)] \geq 0$ as needed

(iii)

$$-E_p[\log \frac{P(X)}{Q(X)}] = -\sum_X P(X) \log \frac{P(X)}{Q(X)}$$
$$= \sum_X P(X) \log \frac{Q(X)}{P(X)}$$
$$\leq \log \sum_X Q(X) = 0$$

So $E_p[\log \frac{P(X)}{Q(X)}] \geq 0$ as needed

(f) (i)

$$\begin{aligned}
H_p(X|Y) - H_p(X) &= E_p[-\log P(X|Y)] - E_p[-\log P(X)] \\
&= E_p\left[\log \frac{P(X)}{P(X|Y)}\right] \\
&= \sum_{X,Y} P(X,Y) \log \frac{P(X)}{P(X|Y)} = \sum_{X,Y} P(X,Y) \log \frac{P(X)}{P(X,Y)/P(Y)} \\
&\leq \log \sum_{X,Y} P(X)P(Y) = 0
\end{aligned}$$

So $H_p(X|Y) - H_p(X) \leq 0$ and $H_p(X|Y) \leq H_p(X)$ as needed

(ii)

$$\begin{aligned}
-I(X;Y) &= -E_p\left[\log \frac{P(X|Y)}{P(X)}\right] \\
&\leq -\log \sum_{X,Y} P(X,Y) \frac{P(X|Y)}{P(X)} = -\log \sum_{X,Y} P(X,Y) \frac{P(X,Y)/P(Y)}{P(X)} \\
&= \log \sum_{X,Y} P(X)P(Y) = 0
\end{aligned}$$

So $I(X;Y) \geq 0$ as needed

(g) (i) .

(ii)