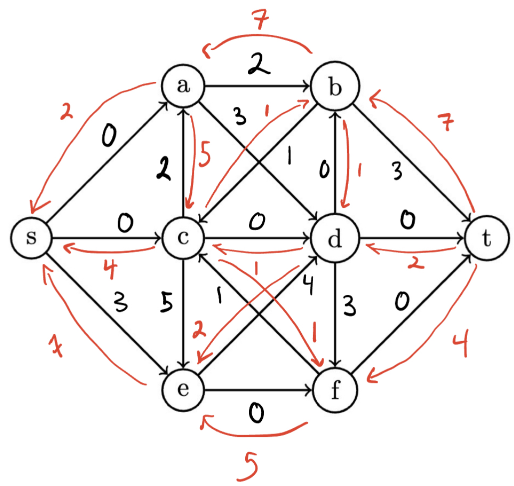
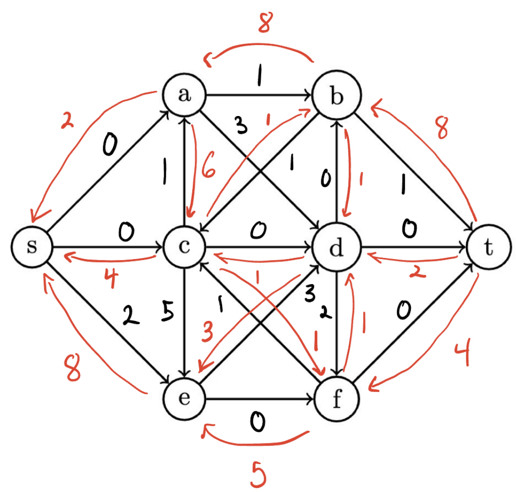
Problem Set 6 Solutions

Calvin Walker

Problem 1: The initial residual graph is shown below:



So we take the path $s \to e \to d \to f \to c \to a \to b \to t$ and route 1 flow along this path. The updated residual graph is:



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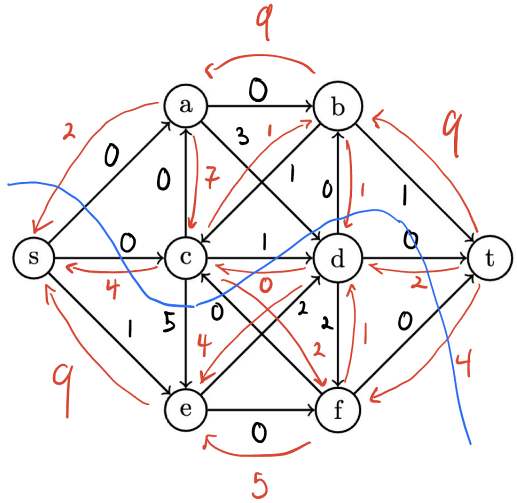

Since only s, e, f, d are reachable from s, the algorithim terminates and we have the min-cut $C = (L, R)$ such that $L = \{s, e, f, d\}$. If we sum the flow of the edges crossing the cut from L to R we get a max-flow of $2 + 4 + 1 + 2 + 4 + 2 = 15$.

Problem 2:

Algorithm: If e was not at full capacity, such that $f(e) < c_e$, then reducing c_e by 1 will have no effect, so F is unchanged. If e is at full capacityy, there must be a path from u to s and a path from t to v in the residual graph. Traverse a path in the residual graph $t \to \cdots \to v$ and a path $u \to \cdots \to s$, and for each reverse edge, undo one unit of flow by decrementing the reverse edges and incrementing the forward edges. Do the same for e unless the new capacity is zero, in which case both the forward and reverse edge $e^{\leftarrow}$ have capacity zero. Then, use Ford-Fulkerson to find the new max-flow in the updated graph.

Correctness: Consider the algorithm in each of the possible scenarios

1. e was not at full capacity, so $f(e) < c_e$ and $f(e) \leq c_e - 1$, therefore the max-flow is unchanged.

2. e was at full capacity, and there is a path from u to v in the residual graph, so if capacity c_e is decremented by one, we can route this unit of flow along the alternate path from u to v using Ford-Fulkerson, and the max-flow will be unchanged.

3. e was at full capacity, and there is not a path from u to v in the residual graph.

   Since the flow into u must equal the flow out of u, and e has capacity at least 1, there must be at least one path from s to u having positive flow. Thus, there is a path in the residual graph using reverse edges from u to s. Similarly, the flow into v must equal the flow out of v, and the same can be said for all verticies on a path from v to t, so there must be a path from v to t having positive flow. Thus, there is a path in the residual graph using reverse edges from t to v.

   If we undo one unit of flow along each of these paths and undo one unit of flow through e, then we have a valid flow for the updated graph, since we have reduced the flow into and out of e by one to satisfy its new capacity, and the Ford-Fulkerson algorithm can now find the updated max-flow.

Runtime: Since each edge in the updated graph had at most one unit of flow undone, the Ford-Fulkerson algorithm with either terminate with one less flow, or find the new max-flow in one iteration and then subsequently terminante, yeilding $O(|E|)$ time. We can traverse the residual graph to find and update the paths from t to v and from u to s in $O(|V| + |E|)$ time. So the total runtime is $O(|V| + |E|)$.

Problem 3:

Algorithm: Initialize a directed graph $G = (V, E)$ such that there is a vertex s with edges to all $i \in [m]$ of capacity s_i, each vertex $i \in [m]$ has edges to all $j \in [n]$ with capacity c_{ij}, and all $j \in [n]$ have edges to a vertex t with capacity b_j. Then, compute the max-flow from s to t using Ford-Fulkerson, and return the integer value of the flow as T.

Proof: Since the correctness of Ford-Fulkerson is accepted, we will prove the correctess of the algorithm by proving that this problem is reducible to max flow. Consider a sales plan S with total sales $T = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$. Let (u, v) denote the edge between to verticies in G, where $c_{(u,v)}$ is the capactiy of (u, v). If S is a valid sales plan, then:

- For all $i \in [m]$, $\sum_j^n x_{ij} \leq s_i = c_{(s,i)}$, so the flow $f((s, i)) \leq c_{(s,i)}$
- For all $i \in [m]$ and $j \in [n]$, $x_{ij} \leq c_{ij} = c_{(i,j)}$, so the flow $f((i, j)) \leq c_{(i,j)}$
- For all $j \in [n]$, $\sum_i^m x_{ij} \leq b_j = c_{(j,t)}$, so the flow $f((j, t)) \leq c_{(j,t)}$

So for every edge in G, no capacity is exceeded for sales plan S, so the flow where $f((i, j)) = x_{ij}$ has value T in G. Now, assume there is an integer-valued flow with value T in G. Then we take the sales plan $S = \{x_{ij} = f((i, j))\}$, which has total sales T. So there is a bijection between integer valued flows on G and sales plans. Thus, the problem is reducible to max flow.

Runtime: We can initialize the graph G in linear time. The maximum flow on G is bound by the supply $S = \sum_i^m s_i$ and there are $mn + m + n$ edges in G. So Ford-Fulkerson will run in $O(Smn)$ time, giving the algorithim a runtime of $O(Smn)$.

Problem 4:

Algorithm: Initialize a graph G such that vertex s has edges with capacity $\lceil \sum_{j \in [n]: i \in S_j} \frac{1}{|S_j|} \rceil$ to each vertex $i \in [k]$, each $i \in [k]$ has an edge with capacity 1 to each $j \in [n]$ for which they are carpooling on day j, and each $j \in [n]$ has a n edge of capacity 1 to a vertex t. Run Ford-Fulkerson to obtain an integer valued max-flow F, and return the corresponding driving schedule $S = \{p_{i_j} : f((i, j)) = 1\}$

Proof: As seen in the description of the algorithim, given an integer valued max-flow F, we can take the corresponding driving schedule $S = \{p_{i_j} : f((i, j)) = 1\}$. This satisfies the constraints since $f((s, i)) \leq c_{(s,i)} = \lceil \sum_{j \in [n]: i \in S_j} \frac{1}{|S_j|} \rceil$ for all $i \in [k]$, so the ceiling of the expected number person i drives is never exceeded, and graph G only has edges betwen person i and day j if i is in S_j, so the driver on day j is always carpooling on that day. Given a driving schedule S we can take the flow $F = \{f((i, j)) = 1 : p_{i_j}\} \cup \{f((s, i)) = \sum_{j: i_j = i} f((i, j))\} \cup \{f((j, t)) = 1\}$. So there is a bijection between driving schedules and integer valued max-flows, and the problem is reducible to max-flow as needed.

Now, we will show that a driving schedule always exsits by proving the max-flow has value at least n, i.e. there is someone driving on every day. Consider the following fractional flow F' on G. From s we send $\sum_{j \in [n]: i \in S_j} \frac{1}{|S_j|}$ flow to each $i \in [k]$, and from each $i \in [k]$, send $\frac{1}{|S_j|}$ flow to each $j \in [n]$ for which they share an edge. Since each vertex j receives $\frac{1}{|S_j|}$ flow from each

i in S_j, $f_{in}(j) = 1$ for all $j \in [n]$. Thus, t receives 1 flow from each j for a total of n flow. Since s will always send at least $\sum_{j \in [n]: i \in S_j} \frac{1}{|S_j|}$ flow to each $i \in [k]$, n is a lower bound for the max-flow in G as needed.

Runtime: The graph can be constructed in linear time. The maximum integer valued flow is of value n, since only one person can drive on each of the n days. There are at most $k + kn + n$ edges in the graph, so running Ford-Fulkerson will give a total runtime of $O(k^2 n)$ for the algorithm.