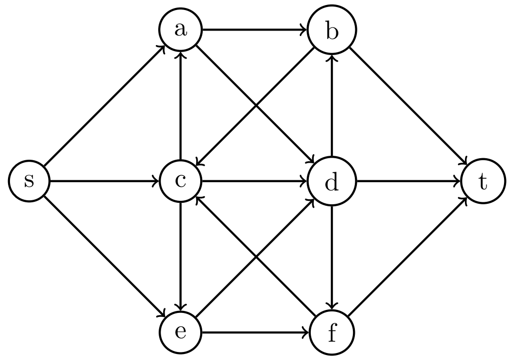
Problem Set 6 Solutions

Calvin Walker

Problem 1:



Problem 2:

Algorithm: If e was not at full capacity, such that $f(e) < c_e$, then reducing c_e by 1 will have no effect, so F is unchanged. If e is at full capacityy, there must be a path from u to s and a path from t to v in the residual graph. Traverse a path in the residual graph $t \to \cdots \to v$ and a path $u \to \cdots \to s$, and for each reverse edge, undo one unit of flow by decrementing the reverse edges and incrementing the forward edges. Do the same for e unless the new capacity is zero, in which case both the forward and reverse edge $e^{\leftarrow}$ have capacity zero. Then, use one iteration of Ford-Fulkerson to find the new max-flow in the updated graph.

Correctness: Consider the algorithm in each of the possible scenarios

1. e was not at full capacity, so $f(e) < c_e$ and $f(e) \leq c_e - 1$, therefore the max-flow is unchanged.

2. e was at full capacity, and there is a path from u to v in the residual graph, so if capacity c_e is decremented by one, we can route this unit of flow along the alternate path from u to v using Ford-Fulkerson, and the max-flow will be unchanged.

3. e was at full capacity, and there is not a path from u to v in the residual graph.

   Since the flow into u must equal the flow out of u, and e has capacity at least 1, there must be at least one path from s to u having positive flow. Thus, there is a path in the residual graph using reverse edges from u to s. Similarly, the flow into v must equal the flow out of v, and the same can be said for all verticies on a path from v to t, so there must be a path from v to t having positive flow. Thus, there is a path in the residual graph using reverse edges from t to v.

   If we undo one unit of flow along each of these paths and undo one unit of flow through e, then we have a valid flow for the updated graph, since we have reduced the flow into and out of e by one to satisfy its new capacity, and the Ford-Fulkerson algorithm can now find the updated max-flow.

Runtime: Since each edge in the updated graph had at most one unit of flow undone, the Ford-Fulkerson algorithm can find the new max-flow in one iteration, or $O(|E|)$, as it is at most one unit less than the original graph. We can traverse the residual graph to find and update the paths from t to v and from u to s in $O(|V| + |E|)$ time. So the total runtime is $O(|V| + |E|)$. *change to 2 iterations?*

Problem 3:

Algorithm: Initialize a directed graph $G = (V, E)$ such that there is a vertex s with edges to all $i \in [m]$ of capacity s_i, each vertex $i \in [m]$ has edges to all $j \in [n]$ with capacity c_{ij}, and all $j \in [n]$ have edges to a vertex t with capacity b_j. Then, compute the max-flow from s to t using Ford-Fulkerson, and return the integer value of the flow as T.

Proof: Since the correctness of Ford-Fulkerson is accepted, we will prove the correctess of the algorithm by proving that this problem is reducible to max flow. Consider a sales plan S with total sales $T = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}$. Let (u, v) denote the edge between to verticies in G, where $c_{(u,v)}$ is the capactiy of (u, v). If S is a valid sales plan, then:

- For all $i \in [m]$, $\sum_{j}^{n} x_{ij} \leq s_i = c_{(s,i)}$, so the flow $f((s,i)) \leq c_{(s,i)}$
- For all $i \in [m]$ and $j \in [n]$, $x_{ij} \leq c_{ij} = c_{(i,j)}$, so the flow $f((i,j)) \leq c_{(i,j)}$
- For all $j \in [n]$, $\sum_{i}^{m} x_{ij} \leq b_j = c_{(j,t)}$, so the flow $f((j,t)) \leq c_{(j,t)}$

So for every edge in G, no capacity is exceeded for sales plan S, so the flow where $f((i,j)) = x_{ij}$ has value T in G. Now, assume there is an integer-valued flow with value T in G. Then we take the sales plan $S = \{x_{ij} = f((i,j))\}$, which has total sales T. So there is a bijection between integer valued flows on G and sales plans. Thus, the problem is reducible to max flow. Runtime: We can initialize the graph G in linear time. The maximum flow on G is bound by the supply $S = \sum_i^m s_i$ and there are $mn + m + n$ edges in G. So Ford-Fulkerson will run in $O(Smn)$ time, giving the algorithim a runtime of $O(Smn)$.

Problem 4: