系級............... 學號................... 姓名.........................

微積分 (II) Quiz #1 (40 minutes) 2020/3/9

1. (2 points) Let C be the curve defined by $x(t) = 3t - t^3$, $y(t) = 3t^2$, $0 \le t \le 1$. Find the area of the surface obtained by rotating the curve C about the x-axis.

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(3-3t^2)^2+(6t)^2=9(1+2t^2+t^4)=\left[3(1+t^2)\right]^2, \Rightarrow ds=3(1+t^2)dt$$

$$\text{Surface area} = \int_0^1 2\pi y\, ds = \int_0^1 2\pi(3t^2)\, 3(1+t^2)dt$$

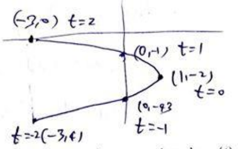
$$= 18\pi\int_0^1 (t^2+t^4)dt = 18\pi\left(\frac{1}{3}+\frac{1}{5}\right)$$

$$= \left(6+\frac{18}{5}\right)\pi = \frac{48}{5}\pi$$

2. (2 points) Consider parametric curve defined by $x = 1 - t^2$ and $y = (t - 2)$, $-2 \le t \le 2$. Eliminate the paremeter to find a Cartesian equation of the curve and sketch the graph. (求在$x-y$座標系之方程式並且繪圖)

$$\begin{cases} x = 1-t^2 \quad \text{①} \\ y = t-2 \Rightarrow t=(y+2) \text{代入①} \end{cases} \Rightarrow x = 1-t^2 = 1-(y+2)^2 = -y^2-4y-3,$$

$$-4 \le y \le 0$$



3. Find an equation of the tangent line to the curve given by $x(t) = e^t \sin(\pi t)$, $y(t) = e^{2t}$ at the point $t = 0$.

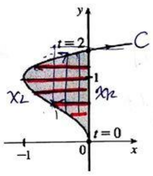
$$\begin{cases} \frac{dx}{dt} = e^t \sin(\pi t) + e^t \cos(\pi t)\cdot\pi = e^t(\pi\cos(\pi t)+\sin(\pi t)) \\ \frac{dy}{dt} = e^{2t}\cdot 2 \end{cases}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2e^{2t}}{e^t(\pi\cos(\pi t)+\sin(\pi t))} = \frac{2e^t}{\pi\cos(\pi t)+\sin(\pi t)}$$

$t=0$ 時 $(x,y)=(0,1) \Rightarrow$ 斜率 $= \frac{2}{\pi}$

∴ 切線方程式: $y-1 = \frac{2}{\pi}(x-0)$ or $y = \frac{2}{\pi}x+1$

4. (2 points) Find the area inclosed by the curve $x(t) = t^2 - 2t$, $y(t) = \sqrt{t}$ and the y-axis ($0 \le t \le 2$).



圍出灰色區域的曲線C: x是y的函數, 因此面積計算公式為 $\int x\,dy$. 曲線C與y軸之交點:

$0 = x(t) = t^2-2t = t(t-2)=0$, ∴ $t=0$, $t=2$

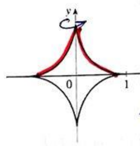
$\begin{cases} t=0 \text{ 時}, x(0)=0, y(0)=0 \\ t=2 \text{ 時}, x(2)=0, y(2)=\sqrt{2} \end{cases}$, $\quad dy = \frac{1}{2}\frac{1}{\sqrt{t}}dt$

$$\therefore \text{所圍面積} = \int_{t=0}^{t=2}(\underset{0}{x_R}-x_L)dy = \int_0^2\left[0-(t^2-2t)\right]\frac{1}{2}\frac{1}{\sqrt{t}}dt$$

$$= -\int_0^2\left(\frac{t^{3/2}}{2}-t^{1/2}\right)dt = -\left(\frac{1}{5}t^{5/2}-\frac{2}{3}t^{\frac{3}{2}}\right)\Big|_0^2$$

$$= -\left(\frac{1}{5}2^{\frac{5}{2}}-\frac{2}{3}2^{\frac{3}{2}}\right) = -2^{1/2}\left(\frac{4}{5}-\frac{4}{3}\right) = \frac{8}{15}\sqrt{2}$$

5. (2 points) Find the area of the surface generated by rotating the curve C: $x^{2/3} + y^{2/3} = 1$, $0 \le y \le 1$ about the y-axis. → 曲面面積 A



$$\boxed{\begin{aligned} y &= (1-x^{2/3})^{3/2} \\ y' &= -\frac{\sqrt{1-x^{2/3}}}{x^{1/3}} \end{aligned}}$$

$= \int 2\pi\cdot x\, ds$, 為 $0 \le x \le 1$, 曲線 $y=(1-x^{2/3})^{3/2}$ 繞y軸旋轉產生.

利用 $ds = \sqrt{1+(y')^2}\,dx = \sqrt{1+\frac{1-x^{2/3}}{x^{2/3}}}\,dx = \sqrt{\frac{1}{x^{2/3}}}\,dx = x^{-1/3}dx$

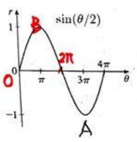
$$\Rightarrow A = \int_0^1 2\pi x\cdot x^{-1/3}dx = 2\pi\int_0^1 x^{2/3}dx = 2\pi\cdot\frac{3}{5}\cdot x^{\frac{5}{3}}\Big|_0^1$$

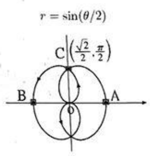
$$= \frac{6}{5}\pi \quad \#$$

系級............... 學號................... 姓名.........................

微積分 (II) Quiz #1 (40 minutes) 2020/3/9

1. (2+1=3 points) 以下是 $r = \sin(\theta/2)$ 的圖形. (a) 求A, B 點之極座標, (b) 求o點到B點之弧長. (設定積分即可)





$A(-1, 3\pi)$

$B(1, \pi)$

$r = \sin(\theta/2)$

$\frac{dr}{d\theta} = \cos(\theta/2)\cdot\frac{1}{2}$

$$\int_0^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^\pi\sqrt{\sin^2\left(\frac{\theta}{2}\right)+\frac{1}{4}\cos^2\left(\frac{\theta}{2}\right)}\,d\theta$$

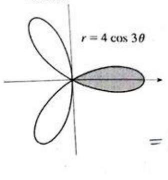
2. (1 points) Use the properties of cross products to evaluate the the vector:

$$(\mathbf{i}+\mathbf{k})\times(\mathbf{i}-\mathbf{k})$$

$$= \hat{i}\times\hat{i} - \hat{i}\times\hat{k} + \hat{k}\times\hat{i} - \hat{k}\times\hat{k}$$

$$= 0-(-\hat{j})+\hat{j}-0 = 2\hat{j} = (0,2,0)$$

3. (2 points) Find the area of the region enclosed by the curve: $r = 4\cos(3\theta)$ (求三片花瓣之總面積)。



灰色部分面積:

$$\int_{-\pi/6}^{\pi/6}\underbrace{\frac{1}{2}(4\cos(3\theta))^2}_{\text{偶函數}}d\theta = 2\int_0^{\pi/6}\frac{1}{2}(16\cos^2(3\theta))d\theta$$

$$= 16\int_0^{\pi/6}\frac{1}{2}(1+\cos 6\theta)d\theta = 8\left[\theta+\frac{1}{6}\sin(6\theta)\right]_0^{\pi/6}$$

$$= 8\times\frac{\pi}{6} = \frac{4}{3}\pi$$

∴ 總面積 $= 3\cdot\left(\frac{4}{3}\pi\right) = 4\pi$

4. (2 points) Find the exact length of the polar curve: $r = \theta^2$, $0 \le \theta \le 2\pi$

$$L = \int_0^{2\pi}\sqrt{(\theta^2)^2+(2\theta)^2}\,d\theta \qquad \frac{dr}{d\theta} = 2\theta$$

$$= \int_0^{2\pi}\sqrt{\theta^4+4\theta^2}\,d\theta = \int_0^{2\pi}\sqrt{\theta^2+4}\,(\theta)d\theta$$

令 $u = \theta^2+4$, $\Rightarrow du = 2\theta d\theta$, ∴ $\theta d\theta = \frac{1}{2}du$

$$= \int_4^{4\pi^2+4}\frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_4^{4\pi^2+4}$$

$$= \frac{1}{3}\left[(4\pi^2+4)^{3/2} - 4^{3/2}\right] = \frac{1}{3}\left[4^{\frac{3}{2}}(\pi^2+1)^{\frac{3}{2}}-4^{\frac{3}{2}}\right] = \frac{8}{3}\left[(\pi^2+1)^{\frac{3}{2}}-1\right]$$

到此為止亦可

5. (2 points) Find the volume of the parallelepiped with adjacent edges PQ, PR and PS.

$$P(3,0,1),\quad Q(-1,2,5),\quad R(5,1,-1),\quad S(0,4,2).$$

令 $\vec{a} = \overrightarrow{PQ} = (-4,2,4)$

$\vec{b} = \overrightarrow{PR} = (2,1,-2)$

$\vec{c} = \overrightarrow{PS} = (-3,4,1)$

$$\text{所求之體積} = \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

$$= -4\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix}$$

$$= (-4)(9) - 2(-4) + 4(11)$$

$$= -36+8+44 = 16 \quad (\text{立方單位})$$

(Solution)

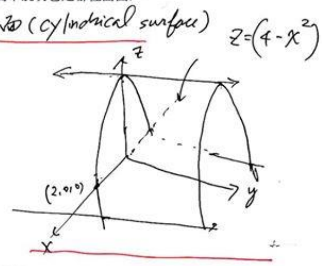
系級............... 學號 .... 姓名.........................

微積分 (II) Quiz #3 openbook (40 minutes) 2020/3/23

1. (2 points) 描繪 $z = 4 - x^2$ 之圖形, 並且簡單說明它是哪種曲面.

變數 y 沒有出現 $\Rightarrow$ 柱面 (cylindrical surface)

(在 x-z 平面上先描繪 $z=4-x^2$ 之圖形, 再往 y 軸方向延伸)

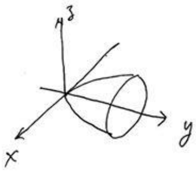


2. (2 points) 將以下方程式化簡成標準式, 判別是哪種曲面並且描繪 其圖形: $4x^2 - y + 2z^2 = 0$.

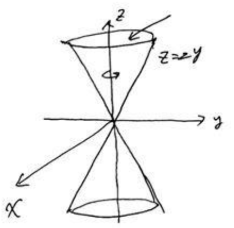
$$y = 4x^2 + 2z^2$$

$$y = \frac{x^2}{(1/2)^2} + \frac{z^2}{(1/\sqrt{2})^2}$$

$\Rightarrow$ elliptic paraboloid



3. (2 points) 令S是將$y-z$平面上之直線 $z = 2y$ 對z 軸旋轉產生之曲面, 描繪 S 之圖形, 並且求其方程式。



$z=C$ 的平面上的 trace 是以 $(0,0,c)$ 為圓心, 半徑為 $y=\frac{z}{2}=\frac{c}{2}$ 的圓, 其方程式為 $x^2+y^2=\left(\frac{c}{2}\right)^2,\ z=c$

$\Rightarrow$ 此曲面為

$x^2+y^2=\left(\frac{z}{2}\right)^2$ 或

$z^2 = 4x^2+4y^2$

4. (2 points) Find the limit: $\lim_{t\to\infty}\left(te^{-t}, \frac{t^3+5t}{2t^3-1}, t\sin\frac{1}{t}\right)$ $= \left(0, \frac{1}{2}, 1\right)$

$$\lim_{t\to\infty}\frac{t}{e^t}\left(\frac{\infty}{\infty}\right) = \lim_{t\to\infty}\frac{1}{e^t} = 0$$

$$\lim_{t\to\infty}\frac{t^3+5t}{2t^3-1} = \lim_{t\to\infty}\frac{1+5/t^2}{2-1/t^3} = \frac{1}{2}$$

$$\lim_{t\to\infty} t\sin\left(\frac{1}{t}\right) = \lim_{t\to\infty}\frac{\sin(1/t)}{1/t} = \lim_{u\to 0^+}\frac{\sin(u)}{u} = 1$$

(令 $u=\frac{1}{t}$)

5. (1+1=2 points) Use properties of cross product to show the identity:

(a) $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2(\mathbf{a}\times\mathbf{b})$

(b) $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$

ⓐ
$$(a-b)\times(a+b) = (a-b)\times a + (a-b)\times b$$
$$= a\times a - b\times a + a\times b - \underbrace{b\times b}_{0}$$
$$= \underbrace{\;}_{0}\ a\times b + a\times b$$
$$= 2a\times b$$

(Note $b\times a = -a\times b$)

ⓑ $|a\times b| = |a||b|\sin\theta$, θ 為 a,b 之夾角

$$\therefore |a\times b|^2 = |a|^2|b|^2\sin^2\theta$$
$$= |a|^2|b|^2(1-\cos^2\theta)$$
$$= |a|^2|b|^2 - |a|^2|b|^2\cos^2\theta$$
$$= |a|^2|b|^2 - (a\cdot b)^2$$

(Note $a\cdot b = |a||b|\cos\theta$)

系級…………… 學號 ……………… 姓名……………………

微積分 (II) Quiz #4–openbook (40 minutes) 2020/3/30

1. (1+1=2 points) (a) Assume that $f(t) = \Big(\mathbf{u}(t)\cdot\mathbf{v}(t)\Big)$, $\mathbf{u}(2) = (1, 2, -1)$, $\mathbf{u}'(2) = (3, 0, 4)$ and $\mathbf{v}(t) = (t, t^2, t^3)$. Find $f'(2)$.

$f'(t) = u'\cdot v + u\cdot v' \Rightarrow f'(2) = u'(2)\cdot v(2) + u(2)\cdot v'(2)$

$v' = (1, 2t, 3t^2)$, $v'(2) = (1, 4, 12)$, 又 $v(2) = (2, 4, 8)$

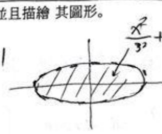
$\therefore f'(2) = \begin{pmatrix}3\\0\\4\end{pmatrix}\cdot\begin{pmatrix}2\\4\\8\end{pmatrix} + \begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\4\\12\end{pmatrix} = 6+32+1+8-12 = 35$

(b) Let $\mathbf{u}(t)$, $\mathbf{v}(t)$ be differentiable vector functions and $\mathbf{a}$ is a given constant vector. Find the derivative of the vector function $\mathbf{r}(t) = \mathbf{u}(t)\times(\mathbf{v}(t) + t\mathbf{a})$. $= \vec{u}(t)\times\vec{v}(t) + t\,\vec{u}(t)\times\vec{a}$

$$r' = \frac{d}{dt}\big(\vec{u}(t)\times\vec{v}(t)\big) + \frac{d}{dt}\big(t\,\vec{u}(t)\times\vec{a}\big)$$
$$= \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t) + \underbrace{\frac{d}{dt}\big(t\,\vec{u}(t)\big)\times\vec{a}}$$
$$= \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t) + \vec{u}(t)\times\vec{a} + t\,\vec{u}'(t)\times\vec{a}$$

2. (1+1=2 points) (a) 求函數 $f(x, y) = \ln(9 - x^2 - 9y^2)$ 之定義域並且描繪 其圖形。

$9 - x^2 - 9y^2 > 0 \Rightarrow x^2 + 9y^2 < 9 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{1} < 1$



(b) 描述以下函數 $f(x, y, z) = y^2 + z^2$ 之level surfaces.

$y^2 + z^2 = k \geq 0$, $k \in \mathbb{R}$

半徑為 $\sqrt{k}$ 之柱面 (Cylindrical surfaces)

3. (2 points) Find parametric equations for the tangent line to the curve: $x(t) = 1 + 2\sqrt{t}$, $y(t) = t^3 - t$, $z(t) = t^3 + t$, at the point $(3, 0, 2)$.

$r'(t) = \left(2\frac{1}{2\sqrt{t}}, 3t^2 - 1, 3t^2 + 1\right)$, $\quad (1 + 2\sqrt{t}, t^3 - t, t^3 + t) = (3, 0, 2)$

$\vec{r}'(1) = (1, 2, 4)$ $\quad t^3 - t = 0 \Rightarrow t(t+1)(t-1) = 0 \Rightarrow t = 0, -1, 1$

$\boxed{\Rightarrow t = 1 \text{ 才會同時滿足三方程式}}$

切線方程式：

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\0\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\4\end{pmatrix}$

$\therefore x = 3 + t$, $y = 2t$, $z = 2 + 4t$

4. (2 points) Find the unit tangent vector $\mathrm{T}(t)$, normal vector $\mathrm{N}(t)$ and curvature for the curve defined by $\mathbf{r}(t) = (t, \frac{t^2}{2}, t^2)$. $\Rightarrow \vec{r}'(t) = (1, t, 2t)$, $|\vec{r}'| = \sqrt{1+5t^2}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'|} = \frac{1}{\sqrt{1+5t^2}}(1, t, 2t)$$
$$\vec{T}' = \frac{-1}{2}(1+5t^2)^{-3/2}(10t)(1, t, 2t) + \frac{1}{\sqrt{1+5t^2}}(0, 1, 2)$$
$$= \frac{-5t}{(1+5t^2)^{3/2}}(1, t, 2t) + \frac{1}{(1+5t^2)^{3/2}}(0, 1+5t^2, 2+10t^2)$$
$$= \frac{1}{(1+5t^2)^{3/2}}(-5t, 1, 2) \Rightarrow |\vec{T}'| = \frac{1}{(\sqrt{1+5t^2})^3}\sqrt{(25t^2+5)} = \frac{\sqrt{5}}{1+5t^2}$$
$$\therefore \vec{N}(t) = \frac{\vec{T}'}{|\vec{T}'|} = \frac{1+5t^2}{\sqrt{5}}\cdot\frac{1}{(1+5t^2)^{3/2}}(-5t, 1, 2)$$
$$= \frac{1}{\sqrt{5}\sqrt{1+5t^2}}(-5t, 1, 2) = \frac{1}{\sqrt{5+25t^2}}(-5t, 1, 2)$$
$$\text{curvature} = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\frac{\sqrt{5}}{1+5t^2}}{\sqrt{1+5t^2}} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}$$

5. (1+1=2 points) (a) Find the length of the curve: $\mathbf{r}(t) = (2\cos t, \sqrt{5}t, 2\sin t)$, $-2 \leq t \leq 2$.

$(\vec{r}'(t)) = (-2\sin t, \sqrt{5}, 2\cos t) \Rightarrow |\vec{r}'| = \sqrt{4\sin^2 t + 5 + 4\cos^2 t} = \sqrt{9} = 3$

$\Rightarrow \text{arc-length} = \int_{-2}^{2}|\vec{r}'(t)|\,dt = \int_{-2}^{2} 3\,dt = 3t\Big|_{-2}^{2} = 12$

(b) Find the curvature for the curve defined by $y = x^3$. $\Rightarrow f(x) = x^3$

$\Rightarrow f'(x) = 3x^2$, $f''(x) = 6x$

$$k(x) = \frac{|6x|}{(1 + (3x^2)^2)^{3/2}} = \frac{6|x|}{(1+9x^4)^{3/2}}$$

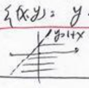
系級…………… 學號 ……………… 姓名……………………

微積分 (II) Quiz #5–openbook (40 minutes) 2020/4/13

1. (1+1=2 points) (a) Determine the set of points at which $f(x, y) = \ln(1 + x - y)$ is continuous.

$\{(x,y) : 1 + x - y > 0\}$ 或 $\{(x,y) : y < 1 + x\}$

或繪圖：



(b) Find the limit, if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$$

$0 \leq \frac{x^2}{x^2+2y^2} \leq 1 \Rightarrow 0 \leq \sin^2(y)\cdot\frac{x^2}{x^2+2y^2} \leq \sin^2(y)$, $\lim_{(x,y)\to(0,0)} \sin^2(y) = 0$,

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{x^2+y^2} = 0$

2. (1+1=2 points) (a) Find an equation of tangent plane for the surface $z = x/y^2$ at the point (-4,2,-1).

$= f(x,y)$, $\begin{cases} f_x = \frac{1}{y^2} \\ f_y = \frac{-2x}{y^3} \end{cases}$ $x = -4$, $y = 2$ $\Rightarrow f_x(-4,2) = \frac{1}{4}$, $f_y(-4,2) = \frac{-2\cdot(-4)}{2^3} = 1$

$\therefore$ tangent plane is: $z - (-1) = \frac{1}{4}(x - (-4)) + 1(y - 2)$

或 $z + 1 = \frac{1}{4}(x+4) + (y-2)$

或 $z = \frac{x}{4} + y - 2$

Find the linearization $L(x, y)$ for the function $f(x, y) = \frac{1+y}{1+x}$ at (1,3).

$\begin{cases} f_x = (1+y)(-1)(1+x)^{-2} = \frac{-(1+y)}{(1+x)^2} \Rightarrow f_x(1,3) = \frac{-4}{2^2} = -1 \\ f_y = \frac{1}{1+x} \Rightarrow f_y(1,3) = \frac{1}{1+1} = \frac{1}{2} \end{cases}$

$f(1,3) = \frac{1+3}{1+1} = 2$

$\therefore L(x,y) = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)$

$= 2 + (-1)(x-1) + \frac{1}{2}(y-3)$

或 $= -x + y/2 + \frac{3}{2}$

3. (2 points) Use differential to estimate the number: $\left(\sqrt{99} + \sqrt[3]{124}\right)^4$.

令 $f(x,y) = \left(\sqrt{x} + \sqrt[3]{y}\right)^4$

$\left(\sqrt{99} + \sqrt[3]{124}\right)^4 = \left(\sqrt{100 + (-1)} + \sqrt[3]{125 + (-1)}\right)^4$

$= f(100, 125) + f_x(100,125)\,dx + f_y(100,125)\,dy$ $\quad (dx = -1, dy = -1)$

$= 15^4 + 675\cdot(-1) + 180\cdot(-1)$

$= 50625 - 675 - 180 = 49700$

$f_x = 4(\sqrt{x} + \sqrt[3]{y})^3\cdot\frac{1}{2}\frac{1}{\sqrt{x}}$

$\Rightarrow f_x(100,125) = 4(10+5)^3\cdot\frac{1}{2\cdot 10} = 675$

$f_y = 4(\sqrt{x} + \sqrt[3]{y})^3\cdot\frac{1}{3}\frac{1}{\sqrt[3]{y^2}}$

$f_y(100,125) = 4(10+5)^3\cdot\frac{1}{3}\frac{1}{5^2} = 180$

4. (2 points) Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$: $yz + x\ln y = z^2$.

$\frac{\partial}{\partial x}(yz + x\ln y) = \frac{\partial}{\partial x}(z^2)$

$y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}$

$\Rightarrow (y - 2z)\frac{\partial z}{\partial x} = -\ln y$

$\therefore \frac{\partial z}{\partial x} = \frac{\ln y}{(2z - y)}$

$\frac{\partial}{\partial y}(yz + x\ln y) = \frac{\partial}{\partial y}(z^2)$

$z + y\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$

$\Rightarrow z + \frac{x}{y} = (2z - y)\frac{\partial z}{\partial y}$

$\frac{yz + x}{y} = (2z - y)\frac{\partial z}{\partial y}$

$\therefore \frac{\partial z}{\partial y} = \frac{x + yz}{y(2z - y)}$

5. (1+1=2 points) (a) Let $f(x, y) = \int_{\sqrt{y}}^{x^2} \cos(e^t)dt$, Compute f_y

$f_y = \frac{\partial}{\partial y}\int_{\sqrt{y}}^{x^2} \cos(t)\,dt = -\frac{\partial}{\partial y}\int_{x^2}^{\sqrt{y}} \cos(t)\,dt$

$= -\cos(\sqrt{y})\cdot\frac{d\sqrt{y}}{dy} = -\cos(\sqrt{y})\cdot\frac{1}{2}\frac{1}{\sqrt{y}}$. $\left(\text{or } \frac{-\cos(\sqrt{y})}{2\sqrt{y}}\right)$

(b) Let $f(x, y) = \frac{y}{2x + 3y}$. Find $\frac{\partial^2 f}{\partial y \partial x}$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{2x+3y}\right) = y\cdot(-1)(2x+3y)^{-2}\cdot 2 = \frac{-2y}{(2x+3y)^2}$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-2y}{(2x+3y)^2}\right)$

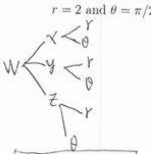
$= -\frac{2(2x+3y)^2 - 2y(2)(2x+3y)\cdot 3}{(2x+3y)^4} = \frac{6y - 4x}{(2x+3y)^3}$

(Solution)

系級.............. 學號.................. 姓名........................

微積分 (II) Quiz #6 (40 minutes) 2020/4/20

1. (1+1=2 points) Let $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ for $r = 2$ and $\theta = \pi/2$.



$$\frac{\partial W}{\partial r} = W_x \cdot x_r + W_y y_r + W_z z_r = (y+z)\cos\theta + (x+z)\sin\theta + (x+y)\theta$$

$$\frac{\partial W}{\partial r}(2, \pi/2) = (2+\pi)\underbrace{\cos(\tfrac{\pi}{2})}_{=0} + \pi \underbrace{\cos(\tfrac{\pi}{2})}_{=1} + 2 \cdot \frac{\pi}{2}$$

$$= \pi + \pi = 2\pi$$

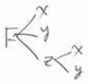
$$\frac{\partial W}{\partial \theta} = W_x x_\theta + W_y y_\theta + W_z z_\theta = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (x+y)r$$

$\boxed{r=2, \theta=\pi/2 \text{ 時}}$

$\boxed{x = 2\cos(\tfrac{\pi}{2}) = 0,\ y = 2\sin(\tfrac{\pi}{2}) = 2,\ z = 2 \cdot \tfrac{\pi}{2} = \pi}$

$$\Rightarrow \frac{\partial W}{\partial \theta}(2, \pi/2) = -2\pi$$

2. (1+1=2 points) Suppose that z is given implicit as a function $z(x,y)$ by an equation of the form $F(x,y,z(x,y)) = 0$. (a) Derive the formula: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$



$$\frac{\partial F}{\partial x} = \frac{\partial 0}{\partial x} = 0 \Rightarrow F_x + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$\therefore \frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \text{ 故得証}$$

(b) Find $\frac{\partial z}{\partial y}$ if x, y, $z(x,y)$ satisfy the equation: $x^2 - y^2 + z^2 - 2z = 4$.

$$\frac{\partial}{\partial y}(x^2 - y^2 + z^2 - 2z) = \frac{\partial 4}{\partial y} = 0$$

$$-2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow (2z-2)\frac{\partial z}{\partial y} = 2y \qquad \therefore \frac{\partial z}{\partial y} = \frac{y}{(z-1)}$$

另解: $F(x,y,z) = x^2 - y^2 + z^2 - 2z - 4 = 0$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-(-2y)}{2z-2} = \frac{y}{z-1}$$

3. (1+1=2 points) (a) Find the direction derivative $g(u,v) = u^2 e^{-v}$ at (3,0) in the direction of $\mathbf{v} = (3,4)$.

$$\nabla g = \begin{pmatrix} g_u \\ g_v \end{pmatrix} = \begin{pmatrix} 2ue^{-v} \\ -u^2 e^{-v} \end{pmatrix}, \quad |\vec{v}| = 5, \quad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

方向導數 $= \nabla g(3,0) \cdot \vec{u}$

$$= \begin{pmatrix} 6 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \frac{1}{5}(18 - 36) = -\frac{18}{5}$$

(b) Find equation of the normal line to the surface $y = x^2 - z^2$ at the point (4,7,3).

令 $F = y - x^2 + z^2$

$$\nabla F = \begin{pmatrix} -2x \\ 1 \\ 2z \end{pmatrix}, \quad \nabla F(4,7,3) = \begin{pmatrix} -8 \\ 1 \\ 6 \end{pmatrix}$$

normal line:

$$\frac{x-4}{-8} = \frac{y-7}{1} = \frac{z-3}{6}$$

或

$$\frac{x-4}{8} = \frac{y-7}{-1} = \frac{z-3}{-6}$$

4. (2 points) Fin the local maximum and minimum and saddle point(s) of the function $f(x,y) = y(e^x - 1)$.

$f_x = ye^x = 0 \Rightarrow y = 0$

$f_y = e^x - 1 = 0 \Rightarrow e^x = 1 \Rightarrow x = 0$

$\therefore$ 唯一的 critical point 為 (0,0)

$f_{xx} = ye^x$, $f_{xy} = e^x$, $f_{yy} = e^x$

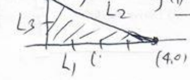
$\therefore D(x,y) = y(e^x)^2 - (e^x)^2$

$\Rightarrow D(0,0) = 0 - 1 = -1 < 0$

$\therefore$ 在(0,0)處為 saddle point.

5. (2 points) Find the absolute maximum and minimum of $f(x,y) = x + y - xy$ on the closed triangular region D with vertices (0,0), (0,2) and (4,0).

$$\begin{cases} f_x = 1 - y = 0 \\ f_y = 1 - x = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$



$f(1,1) = 1 + 1 - 1 = 1$

L_1: $y = 0$, $0 \le x \le 4$.

$g_1(x) = f(x,0) = x$. 直線, 端點值為絕對極大.極小值

$f(4,0) = 4$, $\boxed{f(0,0) = 0}$

L_2: 直線方程式為 $\frac{x}{4} + \frac{y}{2} = 1 \Rightarrow y = 2 - x/2$, $0 \le x \le 4$

$$g_2(x) = f(x, 2 - \tfrac{x}{2}) = x + (2 - \tfrac{x}{2}) - x(2 - \tfrac{x}{2})$$

$$= \frac{+x^2}{2} - \frac{3}{2}x + 2 = \frac{1}{2}(x^2 - 3x) + 2 = \frac{1}{2}(x - \tfrac{3}{2})^2 - (\tfrac{1}{2})(\tfrac{9}{4}) + 2$$

$$= \frac{1}{2}(x - \tfrac{3}{2})^2 + \frac{7}{8}, \quad \boxed{\min = \tfrac{7}{8},\ \max \text{ at } x = 4,\ f(4,0) = 4}$$

L_3: $x = 0$, $0 \le y \le 2$

$g_3(y) = f(0,y) = y$, $0 \le y \le 2$, 直線. $\boxed{\begin{array}{l} \min = f(0,0) = 0 \\ \max = f(0,2) = 2 \end{array}}$

$\boxed{\begin{array}{l} \min = 0 \\ \max = 2 \end{array}}$

$\therefore$ 在D上之 absolute max $= 4 = f(4,0)$

absolute min $= 0 = f(0,0)$

(Solution)

系級.............. 學號.................. 姓名........................

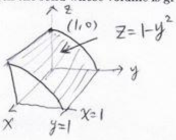
微積分 (II) Quiz #7–openbook (40 minutes) 2020/4/27

1. (2 points) If $f(x,y)$ is continuous on $[a,b] \times [c,d]$ and $g(x,y) = \int_a^x \int_b^y f(u,v)\,dv\,du$ $a < x < b$, $c < y < d$. Compute g_{xy}.

先算 $g_x = \frac{\partial}{\partial x}\left(\int_a^x \int_b^y f(u,v)\,dv\,du\right) = \left(\int_b^y f(x,v)\,dv\right)$

$$\Rightarrow g_{xy} = (g_x)_y = \frac{\partial}{\partial y}\left(\int_b^y f(x,v)\,dv\right) = f(x,y)$$

2. (2 points) Sketch the solid whose volume is given by the iterated integral: $\int_0^1 \int_0^1 (1-y^2)\,dy\,dx$.



$z = 1 - y^2$ (柱面) 往x方向延伸

3. (2 points) Use Lagrange multiplier method to find the extreme values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x + y + z = 12$. 令 $g(x,y,z) = x + y + z$

$\nabla f = (2x, 2y, 2z)$, $\nabla g = (1,1,1)$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 12 \end{cases} \Rightarrow \begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x + y + z = 12 \end{cases} \Rightarrow x = y = z = \frac{\lambda}{2}$$

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = \frac{3}{2}\lambda = 12, \Rightarrow \lambda = 8;$$

$$\Rightarrow (x,y,z) = (4,4,4)$$

$$\Rightarrow \text{Min} = 4^2 + 4^2 + 4^2 = 48$$

4. (3 points) Find the extreme values (using Lagrange multiplier method if necessary) of $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ on the region

$$D = \{(x,y) : x^2 + y^2 \le 16\}.$$

Ⓐ 先求 $\nabla f = 0$ 之解:

(找 $f(x,y)$ 之 critical point) $\begin{cases} f_x = 4x - 4 = 0 \\ f_y = 6y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$

(1,0) 在D中, $f(1,0) = 2 - 4 - 5 = -7$

Ⓑ 求 $f(x,y)$ 在 $\underbrace{x^2 + y^2}_{g(x,y)} = 16$ 之上的極值:

$\nabla f = (4x - 4, 6y)$, $\nabla g = (2x, 2y)$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 16 \end{cases} \Rightarrow \begin{cases} 4x - 4 = \lambda(2x) \\ 6y = \lambda(2y) \\ x^2 + y^2 = 16 \end{cases} \Rightarrow \begin{cases} 2x - 2 = \lambda x \\ 3y = \lambda y \\ x^2 + y^2 = 16 \end{cases}$$

$$\Rightarrow \begin{cases} (2-\lambda)x = 2 & ① \\ (3-\lambda)y = 0 & ② \\ x^2 + y^2 = 16 & ③ \end{cases}$$

ⓘ $\lambda = 2$, ①式變成 $0 = 2$ ✗, 無解

ⓘⓘ $\lambda = 3$, 代入①式, $\Rightarrow x = -2$, 代入③式

$(-2)^2 + y^2 = 16 \Rightarrow y^2 = 12$,

$y = \pm 2\sqrt{3}$

$$\begin{cases} f(-2, +2\sqrt{3}) = 8 + 36 + 8 - 5 = 47 \\ f(-2, -2\sqrt{3}) = 8 + 36 + 8 - 5 = 47 \end{cases}$$

ⓘⓘⓘ $\lambda \ne 3 \Rightarrow \begin{cases} x = \frac{2}{2-\lambda} \\ y = 0 \end{cases}$ 代入③ $x^2 = 16$, $\Rightarrow x = \pm 4$

$f(4,0) = 32 - 16 - 5 = 11$

$f(-4,0) = 32 + 16 - 5 = 43$

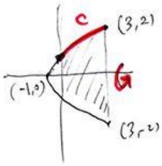
絕對極大值: 47

絕對極小值: −7

系級............... 學號.................. 姓名.........................

微積分 (II) 課堂練習 #1 Solution 2020/3/4

1. (4 points) Find the area of the surface obtained by rotating the curve $y^2 = x + 1$, $0 \le x \le 3$ about the x-axis.



area $= \int 2\pi y\, ds$, 其中, $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

$\because x = y^2 - 1 \Rightarrow \frac{dx}{dy} = (2y) \Rightarrow \boxed{ds = \sqrt{1+(2y)^2}\, dy = \sqrt{1+4y^2}\, dy}$

$x = 0$ 時, $y = \pm 1$ (取 $+1$)

$y = 3$ 時, $y = \pm 2$ (取 $+2$)

$\Rightarrow$ area $= \int_1^2 2\pi y \sqrt{1+4y^2}\, dy$, 令 $u = 1+4y^2 \Rightarrow \boxed{du = 8y\,dy \Rightarrow y\,dy = \frac{1}{8} du}$

$= \int_5^{17} (2\pi) \cdot \sqrt{u} \cdot \frac{1}{8} du = \frac{\pi}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_5^{17} = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$

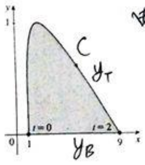
2. (3 points) Consider the parametric curve defined by $x(t) = t^3 - 12t$, $y(t) = t^2 - 1$. Find $\frac{d^2y}{dx^2}$. For which values of t the curve concave upward?

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{(3t^2-12)}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{2t}{3t^2-12}\right) = \frac{\frac{d}{dt}\left(\frac{2t}{3t^2-12}\right)}{\frac{dx}{dt}}$

$= \frac{\frac{2(3t^2-12) - 2t(6t)}{(3t^2-12)^2}}{(3t^2-12)} = \frac{-6t^2-24}{(3t^2-12)^3}$ or $= \frac{-2(t^2+4)}{9(t^2-4)^3}$

3. (3 points) Find the area enclosed by the x-axis and the curve: $x(t) = t^3 + 1$, $y(t) = 2t - t^2$.



曲線 C: y 是 x 的函數 $\Rightarrow$ 面積 $= \int y\, dx$

$= \int_{t=0}^{t=2} (y_T(t) - 0)\, dx = \int_0^2 (2t - t^2)\, \underbrace{x'(t)}_{3t^2} dt$

$= 3\int_0^2 (2t^3 - t^4)\, dt = 3\left[\frac{2t^4}{4} - \frac{t^5}{5}\right]_0^2$

$= 3\left[\frac{1}{2} \cdot 16 - \frac{32}{5}\right] = 3\left(\frac{8}{5}\right) = \frac{24}{5}$

系級............... 學號.................. 姓名.........................

微積分 (II) 課堂練習 #2 Solution 2020/3/11

1. (3 points) Find the points on the curve $r = 3\cos\theta$ where the tangent line is horizontal or vertical.

$x(\theta) = r \cdot \cos\theta = 3\cos^2\theta$, $y(\theta) = r\sin\theta = 3\cos\theta\sin\theta$

$\Rightarrow x'(\theta) = 6\cos\theta(-\sin\theta)$, $y'(\theta) = 3(-\sin^2\theta) + 3\cos^2\theta$

$= -3\sin 2\theta = 0$ $\qquad = 3\cos(2\theta) = 0$

if $2\theta = 0, \pi$ $\qquad$ if $2\theta = \frac{\pi}{2}, \frac{3}{2}\pi \Rightarrow \frac{\pi}{4}, \frac{3\pi}{4}$

$\Rightarrow \theta = 0, \frac{\pi}{2}$, $3\cos\theta = 3, 0$

$\therefore$ horizontal tangent: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$, 和 $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{4}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\pi\right)$

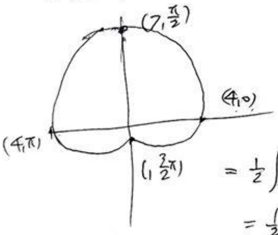
$3\cos\theta = \frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}$

vertical tangent: $(3, 0)$, $(0, \frac{\pi}{2})$

$\theta = 0$, $r = 3\cos\theta = 3$,

$\theta = \frac{\pi}{2}$, $r = 3\cos(\frac{\pi}{2}) = 0$,

2. (4points) Sketch the curve $r = 4 + 3\sin\theta$ and find the area it encloses.



Area $= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (4 + 3\sin\theta)^2 d\theta$

$= \frac{1}{2}\int_0^{2\pi} (16 + 24\sin\theta + 9\sin^2\theta)\, d\theta$, $\left(\frac{1-\cos 2\theta}{2}\right)$ 代入

$= \frac{1}{2}\int_0^{2\pi} \left(\frac{41}{2} + 24\sin\theta - \frac{9}{2}\cos 2\theta\right) d\theta$

$= \frac{1}{2}\left(\frac{41}{2}\right)\theta \Big|_0^{2\pi} - 12\cos\theta\Big|_0^{2\pi} - \frac{9}{4} \cdot \frac{1}{2}\sin 2\theta \Big|_0^{2\pi} = \frac{41}{2}\pi$

3. (3 points) Find the arc length of the polar curve: $r = 5^\theta$, $0 \le \theta \le 2\pi$. $\quad \frac{dr}{d\theta} = 5^\theta \cdot \ln 5$

$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(5^\theta)^2 + (5^\theta \ln 5)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{1 + (\ln 5)^2} \cdot 5^\theta\, d\theta = \sqrt{1 + (\ln 5)^2} \cdot \frac{5^\theta}{\ln 5}\Big|_0^{2\pi}$

$= \sqrt{1 + (\ln 5)^2} \cdot \frac{1}{\ln 5}\left(5^{2\pi} - 1\right)$

系級............... 學號.................. 姓名.........................

微積分 (II) 課堂練習 #3 Solution 2020/3/18

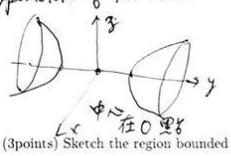
1. (4 points) Reduce the equation to one of the standard forms, classify the surfaces:

(a) $y^2 = x^2 + 4z^2 + 4$, (b) $x^2 - y^2 + z^2 - 4x - 2z = 0$

(a) $-x^2 + y^2 - 4z^2 = 4$

$-\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1$

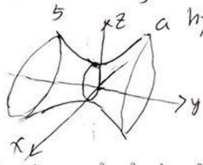
hyperboloid of two sheets



(b) $\underbrace{(x^2 - 4x + 4)}_{(x-2)^2} - y^2 + \underbrace{(z^2 - 2z + 1)}_{(z-1)^2} = 4 + 1 = 5$

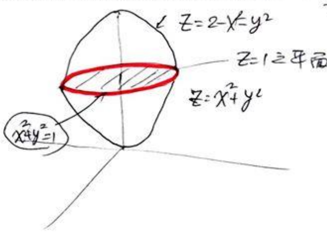
$\frac{(x-2)^2}{5} - \frac{y^2}{5} + \frac{(z-1)^2}{5} = 1$

a hyperboloid of one sheet



中心在 $(2, 0, 1)$ 點

2. (3points) Sketch the region bounded by the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.



交集之曲線

$x^2 + y^2 = 2 - x^2 - y^2$

$\Rightarrow x^2 + y^2 = 1$

$(z = 1)$

3. (3 points) Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

令相交之曲線為C, 將C投影在 xy 平面上之方程式為 $x^2 + y^2 = 4$, $z = 0$, $\Rightarrow x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

因為C也在曲面 $z = xy$ 上, $\Rightarrow z = (2\cos\theta)(2\sin\theta)$

$= 4\cos\theta\sin\theta$ 或 $= 2\sin(2\theta)$

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2\sin(2\theta))$

$0 \le \theta \le 2\pi$.

系級............... 學號.................. 姓名.........................

微積分 (II) 課堂練習 #4 Solution 2020/3/25

1. (2 points) Find parametric equations for the tangent line to the curve C: $x(t) = \ln(t+1)$, $y(t) = t\cos(2t)$, $z(t) = 2^t$ at the point $(0,0,1)$.

$\vec{r}(t) = (\ln(t+1), t\cos(2t), 2^t) \Rightarrow \vec{r}'(t) = \left(\frac{1}{t+1}, \cos(2t) + (-2)t\sin(2t), 2^t \ln 2\right)$

$(0,0,1)$ 點對應 $t = 0 \Rightarrow$ 切向量 $\vec{r}'(0) = (1, 1, \ln 2)$

切線方程式:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + u\begin{pmatrix} 1 \\ 1 \\ \ln 2 \end{pmatrix} = \begin{pmatrix} u \\ u \\ 1 + u(\ln 2) \end{pmatrix}$, $u \in \mathbb{R}$

$(u \in \mathbb{R})$ (u 為參數)

2. (6 points) Let $\mathbf{r}(t) = (t, 3\cos t, 3\sin t)$.

(a) Find the unit tangent vector $\mathbf{T}(t)$ and normal vector $\mathbf{N}(t)$.

(b) Find the curvature.

(a) $\vec{r}'(t) = (1, -3\sin t, 3\cos t)$,

$|\vec{r}'| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$

$\therefore \vec{T} = \frac{1}{\sqrt{10}}(1, -3\sin t, 3\cos t)$. $\Rightarrow \vec{T}' = \frac{1}{\sqrt{10}}(0, -3\cos t, -3\sin t)$

$|\vec{T}'| = \frac{1}{\sqrt{10}}(9\cos^2 t + 9\sin^2 t)^{\frac{1}{2}} = \frac{3}{\sqrt{10}}$

$\therefore \vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \frac{\frac{1}{\sqrt{10}}(0, -3\cos t, -3\sin t)}{3/\sqrt{10}} = (0, -\cos t, -\sin t)$

(b) $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$

3. (2 points) Find the curvature for the curve: $\mathbf{r}(t) = (t^2, t^3)$.

$x = t^2$, $\dot{x} = 2t$, $\ddot{x} = 2$

$y = t^3$, $\dot{y} = 3t^2$, $\ddot{y} = 6t$

$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(2t)(6t) - (2)(3t^2)|}{((2t)^2 + (3t^2)^2)^{3/2}}$

$= \frac{|12t^2 - 6t^2|}{(4t^2 + 9t^4)^{3/2}} = \frac{6t^2}{(4t^2 + 9t^4)^{3/2}}$

系級…………… 學號………………… 姓名……………………

微積分 (II) 課堂練習 #5 2020/4/8

繳交期限: 4/13 (週一) 中午 12:10

1. (2 points) (a) Let $f(x,y)=x^y$, Compute $\frac{\partial^2 f}{\partial x \partial y}$

(b) Let $f(x,y)=\int_{\sqrt{y}}^{x} \cos(e^t)dt$, Compute f_x

(a) $f_y = \frac{\partial}{\partial y}(x^y) = x^y \cdot \ln x$

(y視為常數)

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(x^y \cdot \ln x)$

(此時 x 視為常數

$= \frac{\partial}{\partial x}(x^y) \cdot \ln y + x^y \cdot \frac{\partial}{\partial x}(\ln x)$

$= y \cdot x^{y-1} \cdot \ln y + x^y \cdot \frac{1}{x}$

$= x^{y-1} \cdot y \cdot \ln y + x^{y-1}$

(b) $\frac{\partial}{\partial x} \int_{\sqrt{y}}^{x} \cos(e^t)dt$

$= \cos(e^x)$

(由微積分基本定理)

2. Find an equation of tangent plane to the surface $z=\sqrt{x+e^{4y}}$ at the point (3,0).

$\frac{\partial z}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x+e^{4y}}} \cdot \frac{\partial}{\partial x}(x+e^{4y}) = \frac{1}{2} \frac{1}{\sqrt{x+e^{4y}}} \cdot 1$

$\frac{\partial z}{\partial y} = \frac{1}{2} \frac{1}{\sqrt{x+e^{4y}}} \frac{\partial}{\partial y}(x+e^{4y}) = \frac{1}{2} \frac{1}{\sqrt{x+e^{4y}}} \cdot e^{4y} \cdot 4 = \frac{2e^{4y}}{\sqrt{x+e^{4y}}}$

在 (3,0) 處: $z = \sqrt{3+e^0} = \sqrt{4} = 2$

$\frac{\partial z}{\partial x}(3,0) = \frac{1}{2\sqrt{3+e^0}} = \frac{1}{4}$

$\frac{\partial z}{\partial y}(3,0) = \frac{2 \cdot e^0}{\sqrt{3+e^0}} = 1$

⇒ 切平面方程式:

$(z-2) = \frac{1}{4}(x-3) + (y-0)$

$\Rightarrow z = \frac{1}{4}(x-3) + y + 2$

3. (2 points) If $f(x,y)=\sqrt[3]{x^3+y^3}$, find $f_x(0,0)$.

hint: use definition of the partial derivative (p.913, formula 2)

$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\sqrt[3]{(\Delta x)^3 + 0^3} - 0}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

4. (2 points) Let $f(x,y) = \begin{cases} \frac{\sin(xy)}{xy}, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$ Is $f(x,y)$ continuous on $\mathbb{R}^2$? Why or why not?

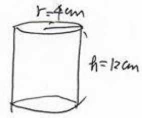
$f(x,y) = \frac{\sin(xy)}{xy}$ 只要有定義即為連續. 其不連續點只可能出現在 $xy=0$ 處., 令 $D = \{(a,b) \in \mathbb{R}^2, ab=0\}$

假設 $(a,b) \in D$ (亦即 $ab=0$), 且 $(x,y) \to (a,b)$. 因為 xy 是連續函數 ⇒ $xy \to ab$. 令 $t = xy$, 則 當 $(x,y) \to (a,b)$ 時 $t \to ab = 0$ ⇒ $\lim_{(x,y)\to(a,b)} \frac{\sin(xy)}{xy} = \lim_{t \to 0} \frac{\sin(t)}{t} = 1 = f(a,b)$

that is, $f(x,y)$ is continuous on $\mathbb{R}^2$.

5. Use differential to estimate the amount of tin in a closed tin can with diameter 8cm and height 12cm if the tin is 0.04cm thick.

直徑8cm



V: volume of the can

$V = \pi r^2 h$

$\Delta V \approx dV$ is an estimate amount of tin.

罐頭材料厚度為 0.04 cm

$dr = 0.04$cm

$dh = 0.08$cm (top + bottom)

$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$,

$= (2\pi r h)dr + (\pi r^2)dh$ $\begin{cases} h = 12\text{cm} \\ r = 4\text{cm} \end{cases}$

$= (2\pi \cdot 4 \cdot 12)0.04 + (\pi \cdot 4^2) \cdot 0.08$

$= (3.84 + 1.28)\pi$

$= 5.12\pi$

系級…………… 學號………………… 姓名……………………

微積分 (II) 課堂練習 #6 2020/4/15

1. (3 points) Find the directional derivative of the function $f(x,y,z) = xy^2\tan^{-1}(z)$ at the point $P(2,1,1)$ in the direction of the vector $\mathbf{v} = (1,1,1)$.

$\nabla f(x,y,z) = (f_x, f_y, f_z) = \left(y^2 \tan^{-1}(z),\ 2xy\tan^{-1} z,\ \frac{xy^2}{1+z^2}\right)$

$\nabla f(2,1,1) = \left(\tan^{-1}(1),\ 4\tan^{-1}(1),\ \frac{2}{1+1^2}\right) = \left(\frac{\pi}{4}, \pi, 1\right)$

$u = \frac{v}{|v|} = \frac{(1,1,1)}{\sqrt{3}}$

∴ 方向導數 $= \nabla f(2,1,1) \cdot u = \begin{pmatrix} \frac{\pi}{4} \\ \pi \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} + \pi + 1\right) = \frac{1}{\sqrt{3}}\left(\frac{5\pi}{4} + 1\right)$

2. (4 points) Find the equations of (a) the tangent plane and (b) the normal line to the surface $y = x^2 - z^2$ at the point (4,7,3).

令 $F(x,y,z) = x^2 - z^2 - y$, then, $y = x^2 - z^2 \Leftrightarrow F(x,y,z) = 0$

$F_x = 2x,\ F_y = -1,\ F_z = -2z \Rightarrow \nabla F(4,7,3) = \begin{pmatrix} 8 \\ -1 \\ -6 \end{pmatrix}$

(a) 令 (x,y,z) 為切平面上任意點

$\Rightarrow \begin{pmatrix} x-4 \\ y-7 \\ z-3 \end{pmatrix} \cdot \nabla F(4,7,3) = 0$

$\Rightarrow 8(x-4) - (y-7) - 6(z-3) = 0$ (or $8x - y - 6z = 7$)

normal line 方程式:

$\frac{x-4}{8} = \frac{y-7}{-1} = \frac{z-3}{-6}$ or $\left(\frac{x-4}{-8} = \frac{y-7}{1} = \frac{z-3}{6}\right)$

3. Find the local maximum and minimum values and saddle point(s) of the function

$$f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2.$$

$\begin{cases} f_x = 6xy - 12x = 0 \text{ ①} \\ f_y = 3y^2 + 3x^2 - 12y = 0 \text{ ②} \end{cases}$

① ⇒ $6x(y-2) = 0 \Rightarrow x = 0$ or $y = 2$

(i) $x = 0$ 代入② ⇒ $3y^2 - 12y = 3y(y-4) = 0 \Rightarrow y = 0, 4$

⇒ critical points: (0,0), (0,4)

(ii) $y = 2$ 代入②式 ⇒ $12 + 3x^2 - 24 = 3x^2 - 12 = 0$

$3(x^2 - 4) = 0 \Rightarrow x = \pm 2$ ⇒ critical points: $(\pm 2, 2)$

$f_{xx} = 6y - 12$

$f_{xy} = 6x$

$f_{yy} = 6y - 12$

$D(x,y) = (6y-12)^2 - 6x^2$

$D(0,0) = 144 > 0$, $f_{xx}(0,0) = -12 < 0$, $f(0,0)$ local max

$D(0,4) = 12^2 = 144 > 0$. $f_{xx}(0,4) = 24 - 12 = 12 > 0$ $f(0,4)$ local min

$D(\pm 2, 2) = 0^2 - 24 < 0$. ⇒ $(\pm 2, 2)$ are saddle points

系級…………… 學號………………… 姓名……………………

微積分 (II) 課堂練習 #7 (Solution) 2020/4/22

1. (4 points) Use Lagrange multipliers to find the extreme values of the function $f(x,y) = xy$ subject to the constraint $4x^2 + y^2 = 8$. , $g(x,y) = 4x^2 + y^2$

解聯立方程組: $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 8 \end{cases}$

$f_x = y$

$f_y = x$

$g_x = 8x$

$g_y = 2y$

$\Rightarrow \begin{cases} y = \lambda(8x) & \text{—①} \\ x = \lambda(2y) & \text{—②} \\ 4x^2 + y^2 = 8 & \text{—③} \end{cases}$

(1) $\lambda = 0$, ⇒ $x = y = 0$

無法滿足③式

不合.

(2) $\lambda \neq 0$,

將①代入②式

$x = \lambda(2y) = 2\lambda \cdot y = 16\lambda^2 x$

$\Rightarrow (1 - 16\lambda^2)x = 0$

$\Rightarrow x = 0$

或 $\lambda = \pm\frac{1}{4}$

ⓐ $x = 0$ 代入③

$\Rightarrow y^2 = 8$

$\Rightarrow y = \pm 2\sqrt{2}$

$\Rightarrow (0, \pm 2\sqrt{2})$

$f(0, \pm 2\sqrt{2}) = 0$

ⓑ $\lambda = \frac{1}{4}$

$\Rightarrow y = 2x$

代入③ $4x^2 + 4x^2 = 8$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\boxed{\begin{matrix} (1,2) \\ (-1,-2) \end{matrix}}$, $\Rightarrow f(x,y) = 2$

$\lambda = -\frac{1}{4} \Rightarrow y = -2x$

代入③ $4x^2 + 4x^2 = 8$

$\Rightarrow x^2 = 1, x = \pm 1$

$\boxed{\begin{matrix} (1,-2) \\ (-1,2) \end{matrix}}$;

$f(x,y) = -2$

極大 2

極小 -2

2. (2 points) Calculate the double integral: $\iint_R \frac{xy^2}{1+x^2}\,dA$, where $R = \{(x,y):\ 0 \le x \le 1,\ -3 \le y \le 3\}$.

$= \int_0^1 \int_{-3}^3 \frac{xy^2}{1+x^2}\,dy\,dx = \int_0^1 \frac{x}{1+x^2}\left(\int_{-3}^3 y^2 dy\right)dx$

$= \frac{y^3}{3}\Big|_{-3}^3 \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{3}(27 - (-27)) \cdot \frac{\ln(1+x^2)}{2}\Big|_0^1$

$= 18 \cdot \frac{1}{2} \cdot (\ln 2 - \ln 1) = 9(\ln 2)$

3. (2 points) Sketch the solid whose volume is given by the iterated integral: $\int_0^1 \int_0^1 (2 - x^2 - y^2)\,dx\,dy$.