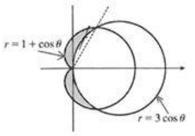
系級............... 學號.................. 姓名.........................

微積分 (II) Quiz #9 (40 minutes) 2020/5/25

1. (1+1=2 points) (a) Set up an iterated integral (設定積分即可) to find the area of the shaded region of the following figure.



$1+\cos\theta = 3\cos\theta$

$\Rightarrow 2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$$\Rightarrow 2\int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta$$

$$+\ 2\int_{\pi/2}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta$$ ↙ 少寫這一項

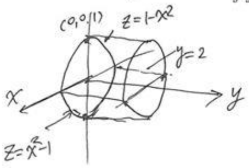
(b) Let S be the solid below the cone $z = \sqrt{x^2+y^2}$ and above the ring $1 \le x^2+y^2 \le 4$. Set up an iterated integral (設定積分即可) to find the volume of S using polar coordinates.

$D = \{(x,y) : 1 \le x^2+y^2 \le 4\} = \{(r,\theta) ; 1 \le r \le 2, 0 \le \theta \le 2\pi\}$

$$\text{volume} = \iint_D z\,dA = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_1^2 \sqrt{r^2}\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_1^2 r^2\,dr\,d\theta$$

或 皆可

2. (2 points) Let E be the solid bounded by the surfaces $z = x^2-1$, $z = 1-x^2$, $y = 0$, and $y = 2$. Sketch E and evaluate $\iiint_E x\,dV$.

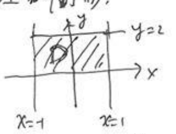


投影到 x-y 平面上的圖形:

$z = 1-x^2 = x^2-1$

$\Rightarrow 2x^2-2=0$

$\Rightarrow x = \pm 1$

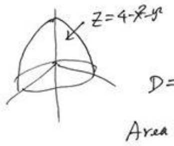


$$\iiint_E x\,dV = \iint_D\left(\int_{x^2-1}^{1-x^2} x\,dz\right)dA = \int_0^2\int_{-1}^1\int_{x^2-1}^{1-x^2} x\,dz\,dx\,dy$$

$$= \int_0^2\int_{-1}^1 x\big((1-x^2)-(x^2-1)\big)\,dx\,dy = \int_0^2\int_{-1}^1 x(2-2x^2) = \int_0^2\int_{-1}^1 \underbrace{(2x-2x^3)}_{\text{odd function}}\,dx\,dy$$

$$= \int_0^2 0\,dy = 0$$

3. (2 points) Find the surface area of the paraboloid $z = 4-x^2-y^2$ that lies above the xy-plane.



$z = 4-x^2-y^2 = 0$, $z_x = -2x$, $z_y = -2y$

$\Rightarrow x^2+y^2 = 4$

$D = \{(x,y) : x^2+y^2 \le 4\}$

$$\text{Area} = \iint_D \sqrt{1+(z_x)^2+(z_y)^2}\,dA$$

$$= \iint_D \sqrt{1+4x^2+4y^2}\,dA$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\,r\,dr\,d\theta$$

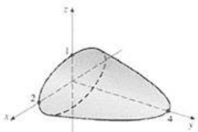
$$= \int_0^{2\pi} d\theta \int_0^2 \sqrt{1+4r^2}\,r\,dr$$

$$= 2\pi \cdot \frac{1}{12}(4r^2+1)^{3/2}\Big|_0^2$$

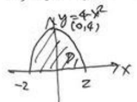
$$= \frac{\pi}{6}(17\sqrt{17}-1)$$

4. (2 points) Let S be the solid bounded by the surfaces $y = 4-x^2-z^2$ and $y = 0$ (See the figure below). Express the integral $\iiint_S f(x,y,z)\,dV$ as the following iterated integrals:

(a) $\iiint \rho(x,y,z)\,dz\,dy\,dx$ (b) $\iiint \rho(x,y,z)\,dy\,dz\,dx$



(a) 投影在 x-y plane $y = 4-x^2$, $y=0$



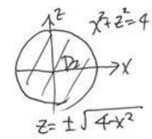
$$\text{原式} = \iint_{D_1}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} f\,dz\,dA$$

$y = 4-x^2-z^2$ solve for z: $z = \pm\sqrt{4-x^2-y^2}$

$$= \int_{-2}^{2}\int_0^{4-x^2}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

(b) 投影到 xz-plane: 令 $y=0$, $4-x^2-z^2=0$ $x^2+z^2=4$

$0 \le y \le 4-x^2-z^2$



$$\text{原式} = \iint_{D_2}\left(\int_0^{4-x^2-z^2} f\,dy\right)dA$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-z^2} f\,dy\,dz\,dx$$

系級............... (Solution) 學號.................. 姓名.........................

微積分 (II) Quiz #10 (45 minutes) 2020/6/1

1. (2 points) Let E be the solid that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=16$, above the xy-plane, and below the plane $z = y+4$. Find the volume of the solid E using cylindrical coordinates.

$z = (r\sin\theta+4)$

$E = \{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 1 \le r \le 4,\ 0 \le z \le (r\sin\theta+4)\}$

$$\text{volume} = \iiint_E dV = \iint_{D(x,y)}\left(\int_0^{y+4} dz\right)dA = \int_0^{2\pi}\int_1^4\left(\int_0^{r\sin\theta+4} dz\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^4 r(r\sin\theta+4)\,dr\,d\theta = \int_0^{2\pi}\int_1^4 (r^2\sin\theta+4r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left((\sin\theta)\frac{r^3}{3}\Big|_1^4 + 2r^2\Big|_1^4\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{63}{3}\sin\theta + 30\right)d\theta = 21\left(\int_0^{2\pi}\sin\theta\,d\theta\right) + 60\pi$$

$$= \underbrace{21(-\cos\theta)\Big|_0^{2\pi}}_{=0} + 60\pi = 60\pi$$

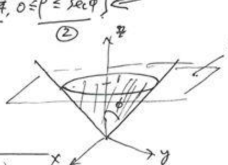
2. (2 points) Sketch the solid whose volume is given by the integral: $\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi}\left(\rho^2\sin\phi\right)d\rho\,d\theta\,d\phi$.

$\{(\rho,\theta,\phi) : 0 \le \theta \le 2\pi,\ \underbrace{0 \le \phi \le \tfrac{\pi}{4}}_{①},\ \underbrace{0 \le \rho \le \sec\phi}_{②}\}$

① $\phi = \frac{\pi}{4} \Leftrightarrow z = \sqrt{x^2+y^2}$ 的 cone.

② $\rho = \sec\phi = \frac{1}{\cos\phi}$

$\Leftrightarrow \rho\cos\phi = 1 \Leftrightarrow \boxed{z=1}$



3. (2 points) Find the Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ for the transformation:

$$\begin{cases} x(u,v,w) &= u+vw \\ y(u,v,w) &= v+wu \\ z(u,v,w) &= w+uv \end{cases}$$

$x_u = 1,\ x_v = w,\ x_w = v$

$y_u = w,\ y_v = 1,\ y_w = u$

$z_u = v,\ z_v = u,\ z_w = 1$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & w & v \\ w & 1 & u \\ v & u & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & u \\ u & 1 \end{vmatrix} - w\begin{vmatrix} w & u \\ v & 1 \end{vmatrix} + v\begin{vmatrix} w & 1 \\ v & u \end{vmatrix}$$

$$= (1-u^2) - w(w-uv) + v(wu-v)$$

$$= 1-u^2-v^2-w^2+2(uvw)$$

4. (2 points) Let E be the solid above the cone $z = \sqrt{x^2+y^2}$ (這個條件是說$\phi \le \pi/4$) and between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$. Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ using spherical coordinates.

$E = \{(\rho,\theta,\phi) : 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4},\ 1 \le \rho \le 2\}$

$\sqrt{x^2+y^2+z^2} = \rho$

$$\iiint_E \sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 (\rho)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho^3\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/4}\sin\phi\right)\left(\int_1^2 \rho^3\,d\rho\right) = 2\pi\cdot\left(-\cos\phi\Big|_0^{\pi/4}\right)\cdot\left(\frac{\rho^4}{4}\right)\Big|_1^2$$

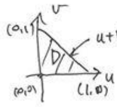
$$= \frac{15}{2}\pi\left(\cos 0 - \cos\left(\tfrac{\pi}{4}\right)\right) = \frac{15}{2}\pi\left(1-\frac{1}{\sqrt{2}}\right) \text{ 或 } \frac{15}{2}\left(1-\frac{\sqrt{2}}{2}\right)\pi$$

5. (2 points) Use the transformation $x(u,v) = 2u+v$, $y(u,v) = u+2v$ to evaluate the integral $\iint_R (x-3y)\,dA$, where R is the triangular region with vertices (0,0), (2,1) and (1,2).

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3, \quad (x-3y) = (2u+v)-3(u+2v) = (-u-5v)$$

$$\begin{matrix} ⓐ\ x = 2u+v \\ ⓑ\ y = u+2v \end{matrix}\Bigg\} \Rightarrow \begin{cases} 2x-y = 3u \\ 2y-x = 3v \end{cases} \Rightarrow \begin{cases} u = \frac{2x-y}{3} \\ v = \frac{2y-x}{3} \end{cases}$$

x	y	u	v
0	0	0	0
2	1	1	0
1	2	0	1



$$\Rightarrow \iint_R (x-3y)\,dA = \iint_{D(u,v)} (-u-5v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dA$$

$$= \int_0^1\int_0^{1-u}(-u-5v)|3|\,dv\,du = -3\int_0^1\left[uv\Big|_0^{1-u} + \frac{5}{2}v^2\Big|_0^{1-u}\right]dy$$

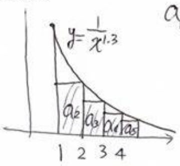
$$= -3\int_0^1\left[(u-u^2)+\frac{5}{2}(1-u)^2\right]du = -3\left[\frac{u^2}{2}-\frac{u^3}{3}-\frac{5}{6}(1-u)^3\right]\Big|_0^1$$

$$= -3\left(\frac{1}{2}-\frac{1}{3}+\frac{5}{6}\right) = -3$$

系級…………… 學號 (Solution) 姓名……………………

微積分 (II) Quiz #11(open-book) (45 minutes) 2020/6/1

1. (2 points) Draw a picture to show that $\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}} dx$. What can you conclude about the series?



$a_n = \frac{1}{n^{1.3}}$ $\quad \sum_{n=2}^{\infty} a_n < \int_1^{\infty} \frac{1}{x^{1.3}} dx$

矩形面積之和 $\quad$ $y=\frac{1}{x^{1.3}}$, $x \geq 1$ 之面積

2. (2 points) Test the series for convergence or divergence: $\sum_{n=1}^{\infty} (-1)^{n+1} n\, e^{-n}$.

$b_n = \frac{n}{e^n}$, $n \geq 1$, 令 $f(x) = \frac{x}{e^x}$.

$f'(x) = \frac{e^x - xe^x}{(e^x)^2} = e^{-x} - xe^{-x} = (1-x)e^{-x} < 0$, if $x > 1$

$\therefore \{b_n\} \downarrow$ (decreasing)

$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{x\to\infty} \frac{x}{e^x} \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{1}{e^x} = 0$

$\therefore \{b_n\} \to 0$ $\quad$ ∴ 交錯級數收斂

3. (2 points) How many terms are needed if we want to find the sum of the following convergent alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n)!}$ correct to 4 decimal places?

$b_n = \frac{1}{(2n)!}$

$\left| \sum_{n=1}^{M} b_n - s \right| < b_{M+1} = \frac{1}{(2(M+1))!} < 10^{-4} = 0.0001$

$8! = 40320$, $2(M+1) = 8 \Rightarrow M+1 = 4$, $\therefore M = 3$.

4. (2 points) Find the radius of convergence of the power series: $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$.

令 $a_n = \frac{10^n x^n}{n^3}$, 利用 ratio test,

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{10^{n+1} x^{n+1}}{(n+1)^3} \right| \left| \frac{n^3}{10^n x^n} \right|$

$= \lim_{n\to\infty} \frac{10\, n^3}{(n+1)^3} \cdot |x| = \lim_{n\to\infty} \frac{10}{\left(1+\frac{1}{n}\right)^3} |x|$

$= 10|x| < 1 \Rightarrow |x| < \frac{1}{10}$

$\therefore$ radius of convergence $R = \frac{1}{10}$

5. (2 points) Find the interval of convergence of the series: $\sum_{n=1}^{\infty} n!(2x-1)^n$.

令 $a_n = n!(2x-1)^n$, 利用 ratio test

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(n+1)!(2x-1)^{n+1}|}{|n!(2x-1)^n|}$

$= \lim_{n\to\infty} (n+1) \cdot |2x-1| = \infty$ if $2x-1 \neq 0$ (or $x = \frac{1}{2}$)

$\therefore$ interval of convergence $= \{\frac{1}{2}\}$ (single point)

系級…………… 學號 (Solution) 姓名……………………

微積分 (II) Quiz #12(closed-book) (45 minutes) 2020/6/15

1. (1+1=2 points) Find Maclaurin series for the functions

(a) $\sin(\pi x/4)$ $\quad$ (a) $\sin(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$

$\Rightarrow \sin\left(\frac{\pi x}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi x}{4}\right)^{2n+1}}{(2n+1)!}$ 或

$= \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} x^{2n+1}$.

(b) $\frac{x}{\sqrt{4+x^2}}$.

$= \frac{x}{\sqrt{4\left(1+\frac{x^2}{4}\right)}}$

$= \frac{x}{2} \frac{1}{\left(1+\frac{x^2}{4}\right)^{1/2}} = \frac{x}{2}\left(1+\frac{x^2}{4}\right)^{-1/2}$

$= \frac{x}{2} \cdot \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\frac{x^2}{4}\right)^n$ or

$= \sum_{n=0}^{\infty} \frac{\binom{-1/2}{n}}{2 \cdot 4^n} x^{2n+1}$

2. (2 points) Find the first three nonzero terms of the series: $f(x) = e^x \cdot \ln(1+x)$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$

$x + x^2 + x^3/2! + x^4/3!$

$-\frac{x^2}{2} - x^3/2 - x^4/4$

$x^3/3 + x^4/3$

$x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

3. (2 points) Approximate $f(x) = 1/x$ by a Taylor polynomial with degree $n = 2$ at the number $a = 1$. Then, use Taylor's inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the interval [0.7,1.3].

$f(x) = \frac{1}{x}$, $\quad f(1) = 1$

$f'(x) = \frac{-1}{x^2}$, $\quad f'(1) = -1$

$f''(x) = \frac{2}{x^3}$, $\quad f''(1) = 2$

$f'''(x) = \frac{-6}{x^4}$

$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$

$= 1 - (x-1) + \frac{2}{2!}(x-1)^2$ or

$= 2 - x + (x-1)^2$

$|f'''(x)|$ 在 $[0.7, 1.3]$ 之最大值 $M = \frac{6}{(0.7)^4}$

$\therefore |R_2(x)| \leq \frac{M}{3!}(0.3)^3 = \frac{\cancel{6}(0.3)^3}{\cancel{3!}(0.7)^4}$ or $= \frac{(0.3)^3}{(0.7)^4}$

4. (~~2~~ 4 points) Use a power series to approximate the definite integral $\int_0^{0.1} \tan^{-1}\left(\frac{x}{2}\right) dx$ to eight decimal places. How many terms do we need?

$\tan^{-1}(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{2n+1}$ +1

$\Rightarrow \tan^{-1}\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{2n+1}$ +1 $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)}$

$\int_0^{0.1} \tan^{-1}\left(\frac{x}{2}\right) dx = \int_0^{0.1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)}$

$= \left(C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n+1}(2n+1)(2n+2)} \right) \Big|_0^{0.1}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{2n+2}}{2^{2n+1}(2n+1)(2n+2)}$ +1

需要前 3 項 ($n=0,1,2$)

$n=1$, $\frac{(0.1)^4}{3 \cdot 4}$

$n=2$, $\frac{(0.1)^6}{5 \cdot 6} > 10^{-8}$ +1

$n=3$ $\frac{(0.1)^8}{7 \cdot 8}$

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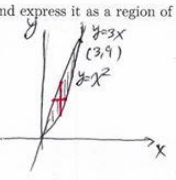
微積分 (II) 課堂練習 #8 (Solution) 2020/4/29

1. (3 points) Let D be the region enclosed by the lines $y=x^2$, $y=3x$. Sketch the region D and express it as a region of type I and also as a region of type II.



$x^2=3x$
$x^2-3x=0$
$x(x-3)=0$
$\Rightarrow x=0, x=3$

type(I): $\{0\le x\le 3,\ x^2\le y\le 3x\}$

type(II): $\{0\le y\le 9,\ \frac{y}{3}\le x\le\sqrt{y}\}$

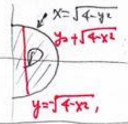
$y=3x \Leftrightarrow x=\frac{y}{3}$

$y=x^2 \Leftrightarrow x=\pm\sqrt{y}$ ($-\sqrt{y}$ 不合)

2. (4 points) Sketch the region of integration: $\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} f(x,y)\,dxdy$, and then change the order of integration.

積分區域 $\{-2\le y\le 2,\ 0\le x\le\sqrt{4-y^2}\}$

$x=\sqrt{4-y^2}\ge 0$ ($x^2+y^2=4$, $x\ge0$)



它是將區域D描述成type(II).

改變積分次序 ⇒ 用type(I)方式描述D:

$0\le x\le 2$,

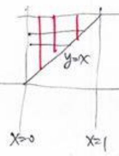
$-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}$

$\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dydx$

3. (3 points) Evaluate the integral $\int_0^1\int_x^1 e^{x/y}\,dydx$ by reversing the order of integration.

積分區域 $R=\{0\le x\le 1,\ x\le y\le 1\}$: type (I) 方式描述

$x\le y$



reverse 積分次序 ⇒ 改用type (II) 方式描述

$R=\{0\le y\le 1,\ 0\le x\le y\}$

$\int_0^1\int_0^y e^{x/y}\,dxdy = \int_0^1 ye^{x/y}\Big|_0^y dy$

$=\int_0^1 y(e-1)\,dy = (e-1)\frac{y^2}{2}\Big|_0^1$

$=\frac{1}{2}(e-1)$

$\boxed{\int e^{x/y}dx \text{ (y當做常數)} = ye^{x/y}+C}$

系級............ 學號.................. 姓名........................

微積分 (II) 課堂練習 #9 (5/18繳交) 2020/5/13

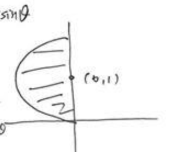
1. (2points) Sketch the region whose area is given by the integral $\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r\,drd\theta$ and evaluate the integral.

$\{(r,\theta): 0\le r\le 2\sin\theta,\ \frac{\pi}{2}\le\theta\le\pi\}$

$r=2\sin\theta \Leftrightarrow r^2=2r\sin\theta$

$x^2+y^2=2y$

$x^2+(y-1)^2=1$

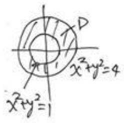


$=\int_{\pi/2}^{\pi}\frac{r^2}{2}\Big|_0^{2\sin\theta}d\theta=\int_{\pi/2}^{\pi}2\sin^2\theta\,d\theta=\int_{\pi/2}^{\pi}(1-\cos2\theta)\,d\theta$

$=\frac{\pi}{2}-\frac{1}{2}\cos2\theta\Big|_{\pi/2}^{\pi}=\frac{\pi}{2}$ #

2. (2 points) Let E be the solid below the cone $z=\sqrt{x^2+y^2}$ and above the ring $1\le x^2+y^2\le4$. Sketch the solid E and use polar coordinates to find the volume of E.

D



$\iint_D\sqrt{x^2+y^2}\,dA=\int_0^{2\pi}\int_1^2\sqrt{r^2}\,r\,drd\theta=\int_0^{2\pi}d\theta\int_1^2 r^2\,dr$

$=2\pi\cdot\left(\frac{r^3}{3}\Big|_1^2\right)=2\pi\cdot\frac{7}{3}=\frac{14}{3}\pi$

3. (2 points) Let D_a be the disk with radius a and center the origin. Show that

$$\iint_{\mathbb{R}^2}e^{-(x^2+y^2)}\,dA=\lim_{a\to\infty}\iint_{D_a}e^{-(x^2+y^2)}\,dA=\pi$$

$\iint_{D_a}e^{-(x^2+y^2)}dA=\int_0^{2\pi}\int_0^a e^{-r^2}r\,drd\theta=\int_0^{2\pi}d\theta\int_0^a e^{-r^2}\cdot r\,dr$

$\boxed{u=-r^2,\ du=-2r\,dr \quad \therefore r\,dr=-\frac{1}{2}du}$

$=2\pi\cdot\left(-\frac{1}{2}\right)e^{-r^2}\Big|_0^a$

$=-\pi(e^{-a^2}-1)=\pi(1-e^{-a^2})$

$\therefore\iint_{\mathbb{R}^2}e^{-(x^2+y^2)}dA=\lim_{a\to\infty}\iint_{D_a}e^{-(x^2+y^2)}dA=\lim_{a\to\infty}\pi(1-e^{-a^2})$

$=\pi$ #

4. (1+1=2 points) (a) Change from rectangular to cylindrical coordinates: $(1,\sqrt{3},-1)$

$r^2=1^2+(\sqrt{3})^2\Rightarrow r=\sqrt{4}=2$

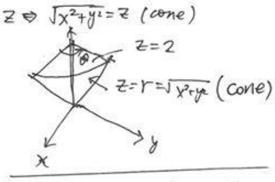
$\tan\theta=\frac{y}{x}=\frac{\sqrt{3}}{1}$, $\theta=\frac{\pi}{3}$

$\Rightarrow(2,\frac{\pi}{3},-1)$

(b) Sketch the solid described by the inequalities:

$0\le\theta\le\pi/2,\ r\le z\le 2$.

$r=z\Leftrightarrow\sqrt{x^2+y^2}=z$ (cone)



5. (2 points) (1+1=2 points)

(a) Change from rectangular to spherical coordinates: $(\sqrt{3},-1,2\sqrt{3})$

$\rho=\sqrt{(\sqrt{3})^2+(-1)^2+(2\sqrt{3})^2}=\sqrt{3+1+12}=\sqrt{16}=4$

由 $z=\rho\cos\phi\Rightarrow\cos\phi=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$, $\therefore\phi=\pi/6$ $\Rightarrow(4,\frac{11\pi}{6},\frac{\pi}{6})$

由 $x=\rho\sin\phi\cos\theta\Rightarrow\sqrt{3}=4\cdot\frac{1}{2}\cdot\cos\theta=2\cos\theta$

$\Rightarrow\cos\theta=\frac{\sqrt{3}}{2}$, $(\sqrt{3},-1)$在第四象限 $\Rightarrow\theta=\frac{11}{6}\pi$

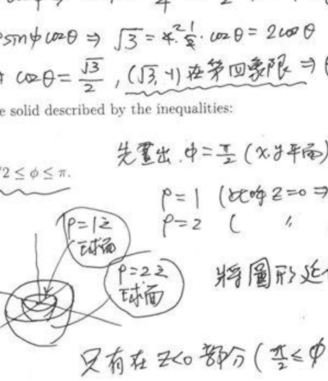
(b) Sketch the solid described by the inequalities:

$1\le\rho\le2,\ \pi/2\le\phi\le\pi$.

先畫出 $\phi=\frac{\pi}{2}$ (xy平面) 的圖形.

$\rho=1$ (此時 $z=0\Rightarrow\sqrt{x^2+y^2}=1$)

$\rho=2$ (〃 $\sqrt{x^2+y^2}=2$)



將圖形延伸到 $\frac{\pi}{2}\le\phi\le\pi$

只有在 z<0 部分 ($\frac{\pi}{2}\le\phi\le\pi$)

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微積分 (II) 課堂練習 #10 (5/25繳交) 2020/5/20

1. (2 points) Let S be the solid inside the cylinder $x^2+y^2=4$ and the ellipsoid $4x^2+4y^2+z^2=64$. Find the volume of S using polar coordinates.

$4x^2+4y^2+z^2=64$

$\Rightarrow z=\pm\sqrt{64-4x^2-4y^2}$

$z_{top}=\sqrt{64-4x^2-4y^2}$

$z_{bottom}=-\sqrt{64-4x^2-4y^2}$

volume $=\iint_{\{x^2+y^2\le4\}}\left(\sqrt{64-4x^2-4y^2}-\left(-\sqrt{64-4x^2-4y^2}\right)\right)dA$

$=\iint_{x^2+y^2\le4}4\sqrt{16-x^2-y^2}\,dA=4\int_0^{2\pi}\int_0^2\sqrt{16-r^2}\,r\,drd\theta$

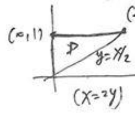
$=4\int_0^{2\pi}d\theta\int_0^2\sqrt{16-r^2}\,r\,drd\theta$

$=8\pi\cdot\left(\frac{-1}{3}\right)(16-r^2)^{\frac{3}{2}}\Big|_0^2$

$=\frac{-8\pi}{3}\left(12^{\frac{3}{2}}-16^{\frac{3}{2}}\right)$ 或

$=\frac{8\pi}{3}\left(64-12^{\frac{3}{2}}\right)$ 皆可

2. (2 points) Find the area of the surface $z=1+3x+2y^2$ that lies above the triangle with vertices (0,0), (0,1) and (2,1). $z_x=3$, $z_y=4y$



Area $=\iint_D\sqrt{1+3^2+(4y)^2}=\iint_D\sqrt{10+16y^2}\,dA$ (先對x積分較容易)

$=\int_0^1\int_0^{2y}\sqrt{10+16y^2}\,dx\,dy=\int_0^1 2y\sqrt{10+16y^2}\,dy$

$u=16y^2$, $du=32y\,dy$

$=\frac{1}{16}\int_0^1 32y\sqrt{10+16y^2}\,dy=\frac{1}{16}\cdot\frac{2}{3}(10+16y^2)^{3/2}\Big|_0^1$

$=\frac{1}{24}\cdot\left(26^{\frac{3}{2}}-10^{3/2}\right)$

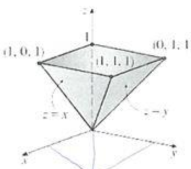
3. (1 points) Find the Jacobian of the transformation: $x=u^2+uv$, $y=uv^2$.

$x=u^2+uv$, $y=uv^2$, $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}x_u&x_v\\y_u&y_v\end{vmatrix}$

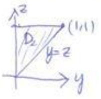
$=\begin{vmatrix}2u+v&u\\v^2&2uv\end{vmatrix}=4u^2v+2uv^2-uv^2$

$=4u^2v+uv^2$

4. (5 points) Write five other iterated integrals that are equal to the integral: $\int_0^1\int_y^1\int_0^z f(x,y,z)\,dx\,dz\,dy$ (the region of integration is given below).



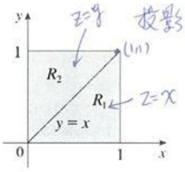
投影到 yz plane



$\{(y,z)\in D_2:\ 0\le x\le z\}$

$$=\iint_{D_2}\left(\int_0^z f\,dx\right)dA$$
$$=\int_0^1\int_y^1\int_0^z f\,dx\,dz\,dy \quad (原式)$$
$$=\int_0^1\int_0^z\int_0^z f\,dx\,dy\,dz \quad ①$$

hint: the region obtained by projection onto xy-plane:



投影到 xy-plane

$$\iint_{R_1}\left(\int_x^1 f\,dz\right)dA+\iint_{R_2}\left(\int_y^1 f\,dz\right)dA$$

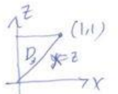
R_1: $\{0\le x\le 1,\ 0\le y\le x\}$, R_2: $\{0\le x\le 1,\ x\le y\le 1\}$

$$=\int_0^1\int_0^x\left(\int_x^1 f\,dz\right)dy\,dx+\int_0^1\int_x^1\left(\int_y^1 f\,dz\right)dy\,dx \quad ②$$

R_1: $\{0\le y\le 1,\ y\le x\le 1\}$, $R_2 = \{0\le y\le 1,\ 0\le x\le y\}$

$$=\int_0^1\int_y^1\left(\int_x^1 f\,dz\right)dx\,dy+\int_0^1\int_0^y\left(\int_x^1 f\,dz\right)dx\,dy \quad ③$$

投影到 xz-plane $\{(x,z)\in D_3,\ 0\le y\le z\}$



$$=\iint_{D_3}\left(\int_0^z f\,dy\right)dA$$
$$=\int_0^1\int_x^1\left(\int_0^z f\,dy\right)dz\,dx \quad ④$$
$$=\int_0^1\int_0^z\left(\int_0^z f\,dy\right)dx\,dz \quad ⑤$$

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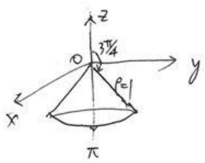
微積分 (II) 課堂練習 #11 (6/1繳交) Solution 2020/5/27

1. (2 points) (15.8.#13) Sketch the solid E described by $\{(\rho,\theta,\phi):\ \rho\le 1,\ 3\pi/4\le\phi\le\pi,\ 0\le\theta\le 2\pi\ \}$, then find its volume using spherical coordinates.



$$\int_0^{2\pi}\int_{3\pi/4}^{\pi}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$=\int_0^{2\pi}d\theta\cdot\int_{3\pi/4}^{\pi}\sin\phi\,d\phi\int_0^1\rho^2 d\rho$$
$$=2\pi\cdot\left(-\cos\phi\Big|_{3\pi/4}^{\pi}\right)\cdot\frac{1}{3}=\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

2. (3 points) (15.7 #19) Use cylindrical coordinates to evaluate the integral $\iiint_E (x+y+z)dV$, where E is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$.

$z=4-x^2-y^2=(4-r^2)$

$$=\iint_D\int_{z=0}^{z=4-r^2}(r\cos\theta+r\sin\theta+z)\,dz\,dA$$
$$=\int_0^{\pi/2}\int_0^2\int_0^{4-r^2}(r\cos\theta+r\sin\theta+z)\,r\,dz\,dr\,d\theta=\int_0^{\pi/2}\int_0^2\left[r^2(\cos\theta+\sin\theta)z\Big|_0^{4-r^2}+\frac{1}{2}rz^2\Big|_0^{4-r^2}\right]dr\,d\theta$$
$$=\int_0^{\pi/2}\int_0^2\left[(4r^2-r^4)(\cos\theta+\sin\theta)+\frac{r}{2}(4-r^2)^2\right]dr\,d\theta$$
$$=\int_0^{\pi/2}\left(\left[\frac{4}{3}r^3-\frac{r^5}{5}\right]\Big|_0^2(\cos\theta+\sin\theta)+\frac{1}{12}(4-r^2)^3\Big|_0^2\right)d\theta=\int_0^{\pi/2}\left(\frac{64}{15}(\cos\theta+\sin\theta)+\frac{16}{3}\right)d\theta$$
$$=\frac{64}{15}(\sin\theta-\cos\theta)+\frac{16}{3}\theta\Big|_0^{\pi/2}=\frac{8}{3}\pi+\frac{128}{15}$$

3. (4 points) (15.9 #23) Evaluate the integral $\iint_R \frac{x-2y}{3x-y}\,dA$ by making an appropriate change of variables, where R is the region enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, and $3x-y=8$.

令 $\begin{cases}u=x-2y\\ v=3x-y\end{cases}\Rightarrow\begin{cases}x=\frac{1}{5}(2v-u),\\ y=\frac{1}{5}(v-3u)\end{cases}$ $\quad \frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}-1/5 & 2/5\\ -3/5 & 1/5\end{vmatrix}=\frac{1}{5}$

$x-2y=0 \Leftrightarrow u=0$
$x-2y=4 \Leftrightarrow u=4$
$3x-y=1 \Leftrightarrow v=1$
$3x-y=8 \Leftrightarrow v=8$
D

$$=\iint_D\frac{u}{v}\left|\frac{1}{5}\right|dA=\frac{1}{5}\int_0^4\int_1^8\frac{u}{v}\,dv\,du$$
$$=\frac{1}{5}\left(\int_0^4 u\,du\right)\left(\int_1^8\frac{1}{v}\,dv\right)=\frac{1}{5}\frac{u^2}{2}\Big|_0^4\cdot\ln|v|\Big|_1^8$$
$$=\frac{8}{5}\ln 8$$

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微積分 (II) Quiz #8 (40 minutes) 2020/5/18

1. (1+1=2 points) (a) Change from rectangular to spherical coordinates: $(\sqrt{3},\sqrt{3},\sqrt{2})$

$\rho=\left[(\sqrt{3})^2+(\sqrt{3})^2+(\sqrt{2})^2\right]^{1/2}=\sqrt{8}=2\sqrt{2}$, 由 $z=\rho\cos\phi\Rightarrow\cos\phi=\frac{z}{\rho}=\frac{\sqrt{2}}{2\sqrt{2}}=\frac{1}{2}$

由 $x=\rho\sin\phi\cos\theta\Rightarrow\sqrt{3}=2\sqrt{2}\cdot\frac{\sqrt{3}}{2}\cdot\cos\theta$,

$\therefore\cos\theta=\frac{1}{\sqrt{2}}$, 且 $(\sqrt{3},\sqrt{3})$ 在第一象限 $\Rightarrow\theta=\frac{\pi}{4}$ $\quad\boxed{\therefore\phi=\frac{\pi}{3}}$

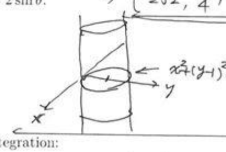
$\Rightarrow\left(2\sqrt{2},\frac{\pi}{4},\frac{\pi}{3}\right)$

(b) sketch the surface whose equation is (in cylindrical coordinates) $r=2\sin\theta$.

$\Rightarrow r^2=2r\sin\theta\Rightarrow x^2+y^2=2y$

$\Rightarrow x^2+(y-1)^2=1$

圓心: $(0,1)$, 半徑 $=1$

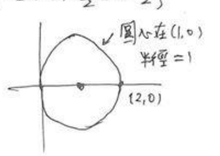


2. (2 points) Sketch the region of integration:

(a) $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta$

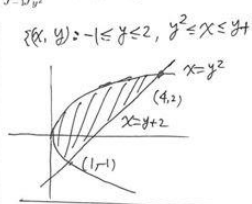
$\left\{(r,\theta):\ 0\le r\le 2\cos\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right\}$

$r=2\cos\theta$
$\Rightarrow r^2=2r\cos\theta$
$x^2+y^2=2x$
$(x-1)^2+y^2=1$



(b) $\int_{-1}^2\int_{y^2}^{y+2} f(x,y)\,dx\,dy$

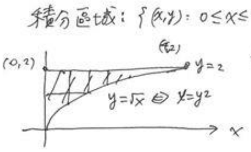
$\{(x,y):-1\le y\le 2,\ y^2\le x\le y+2\}$



3. (2 points) Evaluate the integral by reversing the order of integration: $\int_0^4\int_{\sqrt{x}}^2\frac{1}{y^3+1}\,dy\,dx$

將積分次序調換成 $dx\,dy$

積分區域: $\{(x,y):\ 0\le x\le 4,\ \sqrt{x}\le y\le 2\}$ 這是 type (II): horizontal strip.

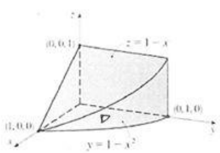


$\Rightarrow$ 使用 type (II) 描述: $\{0\le y\le 2,\ 0\le x\le y^2\}$

$$原式=\int_0^2\int_0^{y^2}\frac{1}{y^3+1}\,dx\,dy$$
$$=\int_0^2\frac{1}{y^3+1}\left(x\Big|_0^{y^2}\right)dy=\int_0^2\frac{y^2}{y^3+1}\,dy,$$ 利用代換 $u=y^3+1$, $du=3y^2dy$
$$=\frac{1}{3}\ln(y^3+1)\Big|_0^2=\frac{1}{3}(\ln 9-\ln 1)$$
$$=\frac{1}{3}\ln 9\quad(或\ \frac{2}{3}\ln 3)$$

(本題打字錯誤,因此送分)

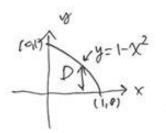
4. (2 points) Let S be the solid bounded by the surface $x^2+y=1$ and the planes $x=0$, $y=0$, $z=0$ and $z=1-x$ (see the figure below). Use double integral to find the volume of S.



此三維物體在平面 $z=1-x$ 之下, xy 平面的區域 D 之上

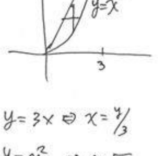
$D=\{(x,y):\ 0\le x\le 1,\ 0\le y\le 1-x^2\}$

$$\therefore \text{volume of } S=\int_0^1\int_0^{1-x^2}(1-x)\,dy\,dx$$



$$=\int_0^1 (1-x)y\Big|_0^{1-x^2}dx=\int_0^1(1-x)(1-x^2)\,dx=\int_0^1(1-x-x^2+x^3)\,dx$$
$$=\left(x-\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}\right)\Big|_0^1$$
$$=1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}=\frac{12-6-4+3}{12}=\frac{5}{12}$$

5. (2 points) Let D be the region bounded by $y=x^2$, $y=3x$. Set up iterated integrals for both orders of integration to evaluate the double integral $\iint_D f(x,y)\,dA$.



type (I): $\{0\le x\le 3,\ x^2\le y\le 3x\}$

$$\Rightarrow\int_0^3\int_{x^2}^{3x} f(x,y)\,dy\,dx$$

$y=3x\Leftrightarrow x=y/3$
$y=x^2,\ x\ge 0,\ \Rightarrow x=\sqrt{y}$

type (II): $\{0\le y\le 9,\ \frac{y}{3}\le x\le\sqrt{y}\}$

$$\Rightarrow\int_0^9\int_{y/3}^{\sqrt{y}} f(x,y)\,dx\,dy$$

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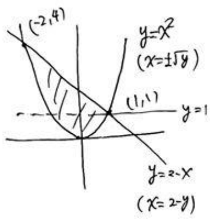
微積分 (II) 期末考加分測驗 #1 (20 minutes) 2020/6/3

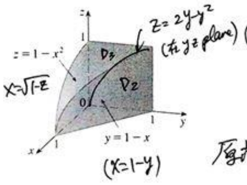
1. (4 points) Sketch the region Ω that gives the repeated integrals $\int_{-2}^{1}\int_{x^2}^{2-x} f(x,y)\,dydx$ and change the order of integration.



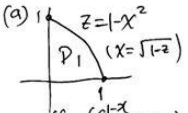
$$\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f\,dxdy + \int_1^4\int_{-\sqrt{y}}^{2-y} f\,dxdy$$

2. (3+3=6 points) Let S be the solid in the first octant bounded by the surfaces $z = 1 - x^2$, $y = 1 - x$ (See the figure below). Express the integral $\iiint_S f(x,y,z)dV$ as the following iterated integrals:

(a) $\iiint f(x,y,z)\ dydxdz$ (b) $\iiint f(x,y,z)\ dxd\cancel{z}dy$

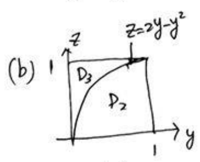


(a)



$$原式 = \iint_{D_1}\left(\int_0^{1-x} f\,dy\right)dA = \int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f\,dydxdz$$

(b)



$$原式 = \iint_{D_2}\left(\int_0^{1-y} f\,dx\right)dA + \iint_{D_3}\left(\int_0^{\sqrt{1-z}} f\,dx\right)dA$$

$$= \int_0^1\int_0^{2y-y^2}\int_0^{1-y} f\,dxdzdy + \int_0^1\int_{2y-y^2}^{1}\int_0^{\sqrt{1-z}} f\,dzdy$$

$\begin{cases} y = 1-x \\ z = 1-x^2 \end{cases}$ 之交集

$\Rightarrow z = 1-(1-y)^2$

$= 1-(1-2y+y^2) = (2y-y^2)$

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微積分 (II) 期末考加分測驗 #2 (20 minutes) 2020/6/10

1. (5 points) Use polar coordinates to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dA = \pi$.

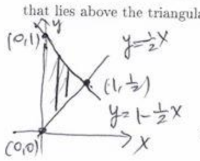
$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}dA = \lim_{a\to\infty}\iint_{B(0,a)} e^{-(x^2+y^2)}dA = \lim_{a\to\infty}\int_0^{2\pi}\int_0^a e^{-r^2}\,r\,drd\theta$$

$$= \lim_{a\to\infty}\left(\int_0^{2\pi}d\theta\right)\left(\int_0^a e^{-r^2}r\,dr\right) = \cancel{2}\pi \lim_{a\to\infty} \frac{-e^{-r^2}}{\cancel{2}}\Big|_0^a$$

$$= \pi \lim_{a\to\infty}\left(1-e^{-a^2}\right) = \pi \lim_{a\to\infty}\left(1-\frac{1}{e^{a^2}}\right)$$

$$= \pi.$$

2. (2 points) Set up an iterated integrals to find the surface area of the surface $z = 1 + 3x + 2y^2$ that lies above the triangular region with vertices (0,0), (1,1/2) and (0,1).

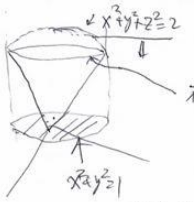


$z_x = 3,\ z_y = 4y$

$$\iint_D \sqrt{1+(z_x)^2+(z_y)^2}\,dA = \iint_D \sqrt{1+9+(4y)^2}\,dA$$

$$= \int_0^1\int_{\frac{1}{2}x}^{1-\frac{1}{2}x} \sqrt{10+16y^2}\,dydx$$

3. (3 points) Set up an iterated integral to find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$ using cylindrical coordinates.



$\begin{cases} z = \sqrt{x^2+y^2} \text{ — ①} \\ x^2+y^2+z^2 = 2 \text{ — ②} \end{cases}$ 之交集: $z^2 = x^2+y^2$ 代入②式: $2z^2 = 2$

$(z \geq 0)$ $\Rightarrow 2(z+1)(z-1) = 0$

在 $z=1$ 平面上, ② ⇒ $x^2+y^2=1$ $z = 1, -1$ (不合)

$$\iint_D\left(\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz\right)dA$$

$z^2 = 2-x^2-y^2$

$z = \pm\sqrt{2-x^2-y^2}$

$D: \{(x,y) : x^2+y^2 \leq 1\}$

$$= \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\cdot r\,drd\theta$$

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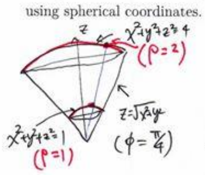
微積分 (II) 期末考加分測驗 #3 (20 minutes) 2020/6/17

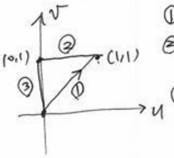
1. (3 points) Let E be the solid lies above the cone $z = \sqrt{x^2+y^2}$ and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Set up an iterated integrals to find the volume of E using spherical coordinates.



$E = \{(\rho,\theta,\phi) : 1 \leq \rho \leq 2,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{4}\}$

$$\iiint_E 1\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

2. (3 points) Let S be the triangular region with vertices (0,0), (1,1), (0,1) on the uv-plane. Find and sketch the image of S under the transformation: $x = u^2$, $y = v$.



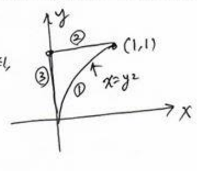
① $u = v,\ 0 \leq u \leq v$, $x = u^2 = y^2$

② $v = 1,\ 0 \leq u \leq 1$, $\Rightarrow \begin{cases} x = u^2,\ 0 \leq x \leq 1, \\ y = 1 \end{cases}$

③ $u = 0,\ 0 \leq v \leq 1$

$\Rightarrow x = 0$

$y = v \in [0,1]$



3. (2+2=4 points) (a) Find the sum of the series: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^n}{n5^n}$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{3}{5}\right)^n}{n}$$

$$\boxed{\text{Note: } \ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}}$$

$$= \ln\left(1+\frac{3}{5}\right)$$

(b) Find the first two nonzero terms in the Maclaurin series for the function $f(x) = \sec(x)$.

$$\sec(x) = \frac{1}{\cos(x)} = \frac{1}{1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots}$$

$$\begin{array}{r|l} & 1+\frac{x^2}{2}+\cdots \\ \hline 1-\frac{x^2}{2}+\frac{x^4}{24} & 1 \\ & 1-\frac{x^2}{2}+\frac{x^4}{24}+\cdots \\ \hline & \frac{x^2}{2}-\frac{x^4}{24}+\cdots \\ & \frac{x^2}{2}-\frac{x^4}{4}+\cdots \\ \hline & \cdots \end{array}$$

$$= 1+\frac{x^2}{2}$$