Enumerating Error Bounded Polytime Algorithms Through Arithmetical Theories

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informatiques mathématiques

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Part I

Context and Motivation

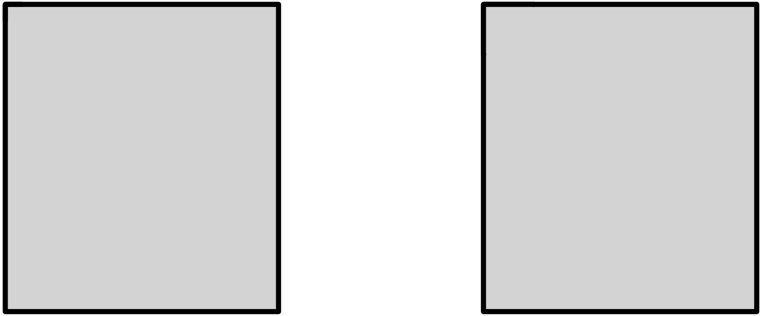
Implicit Computational Complexity

Programs

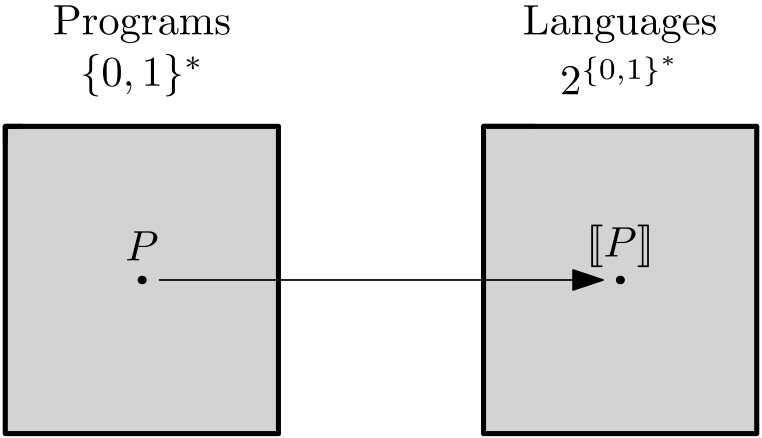
$\{0,1\}^*$

Languages

$2^{\{0,1\}^*}$



Implicit Computational Complexity



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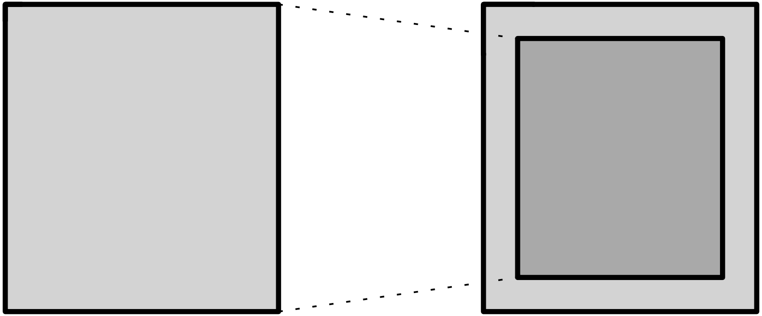

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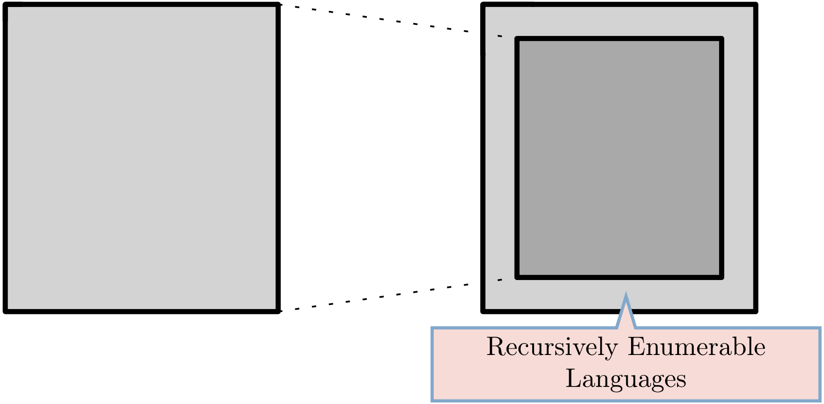

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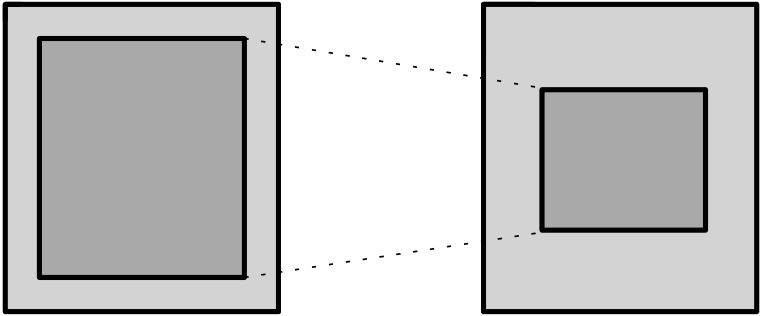

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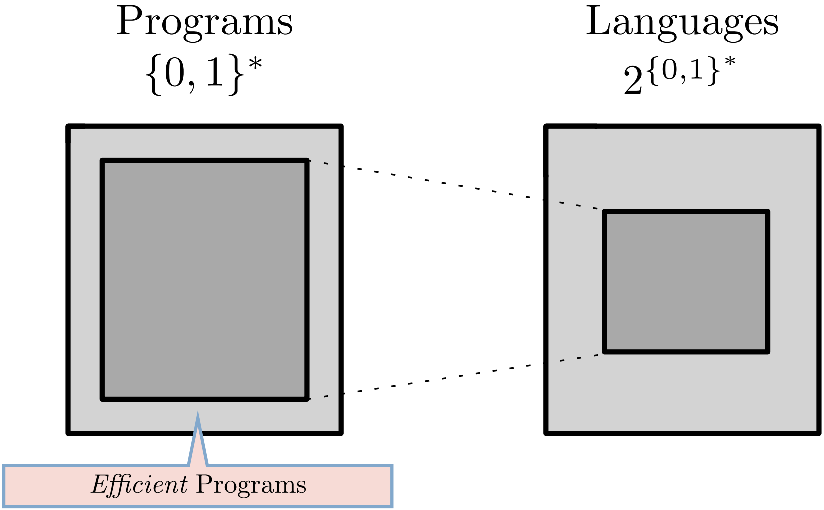

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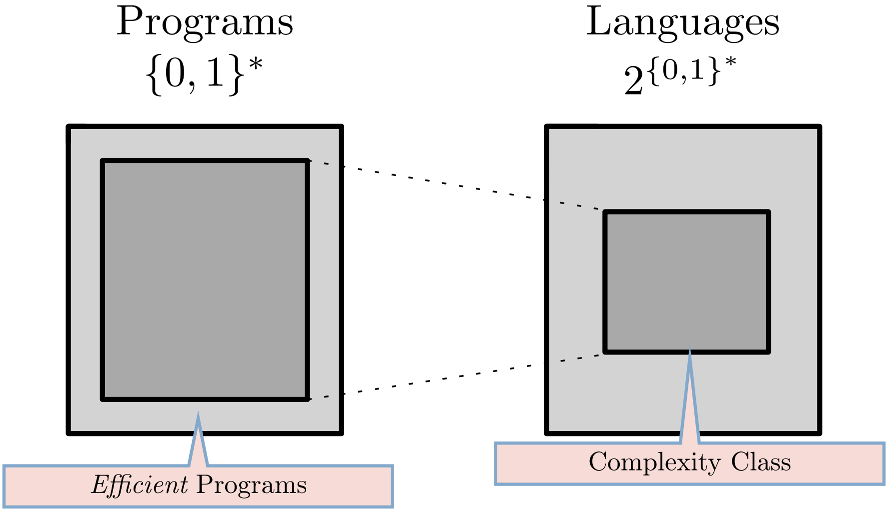

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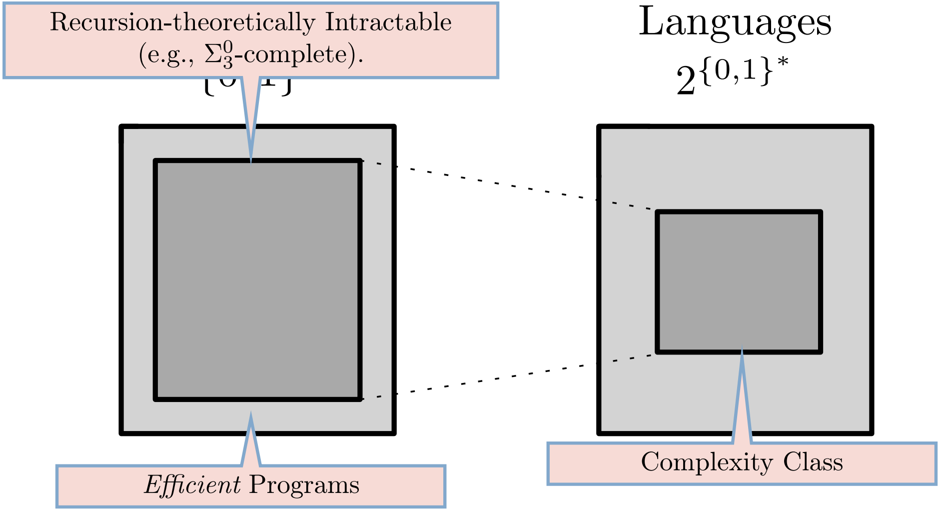

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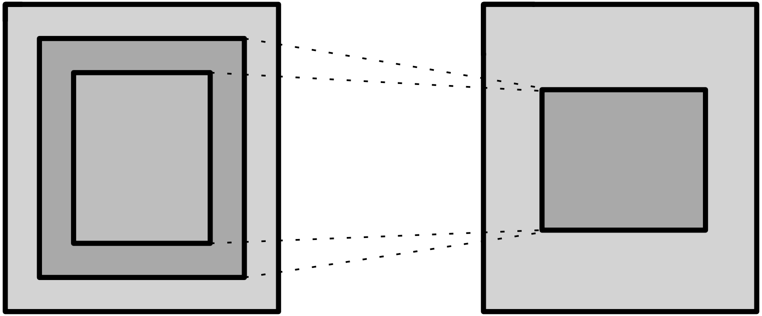

Implicit Computational Complexity

Recursion-theoretically Tractable

Examples: bounded arithmetic, safe recursion, light logics, path orders, etc.

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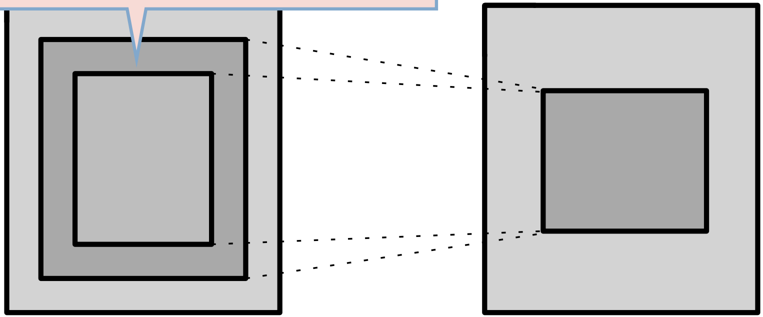
$2^{\{0,1\}^*}$



So What?

- **Pave the Way Towards Formal Methods**
  - Once a class is characterized by a collection of programs which is itself easy to decide, a sound (but not complete!) formal method for complexity analysis is available.
  - Since the class is *characterized*, every feasible language is captured by *a* program.
  - Of course, nothing prevents the caught algorithms to be highly *unnatural*.
  - This path has been followed many times in the past, e.g. in amortized analysis and polynomial interpretations.

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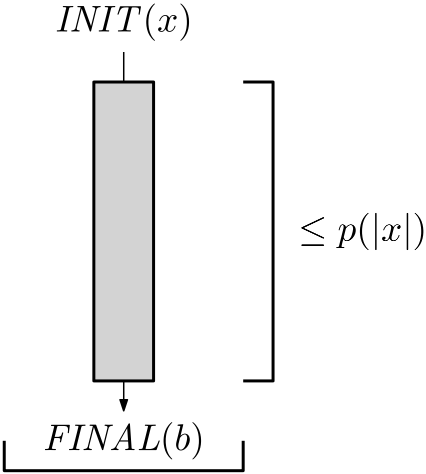
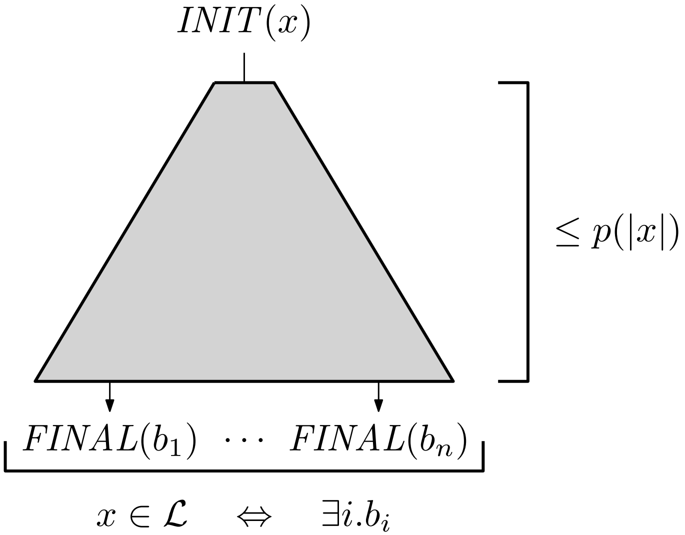
  - This path has been followed many times in the past, e.g. in amortized analysis and polynomial interpretations.
- **Complexity Classes from an Unusual Viewpoint**
  - The obtained definition is *intrinsically different* from the usual one.
  - As such, it could shed light on the nature of the characterized classes, most of them still being mysterious objects.

P

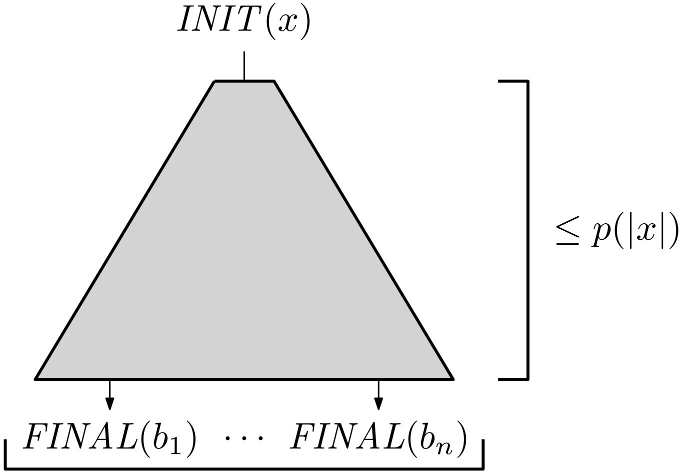


$$x \in \mathcal{L} \quad \Leftrightarrow \quad b$$

NP


$INIT(x)$
$\leq p(|x|)$
$FINAL(b_1) \cdots FINAL(b_n)$
$x \in \mathcal{L} \quad \Leftrightarrow \quad \exists i . b_i$

BPP



$$x \in \mathcal{L} \quad \Rightarrow \quad \Pr[b_i] \geq \tfrac{2}{3}$$

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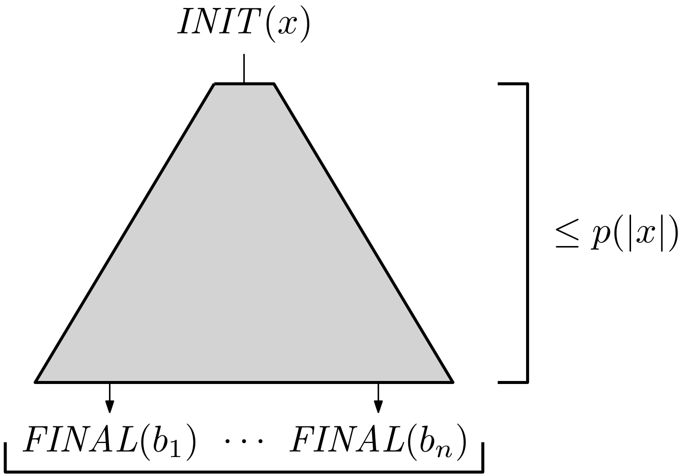

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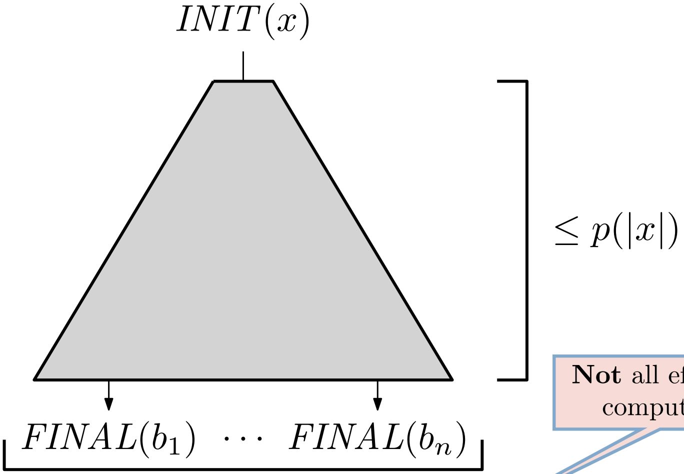

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Not all efficient machines compute *a* language!

Not the same as

$$x \in \mathcal{L} \quad \Leftrightarrow \quad \Pr[b_i] \geq \frac{2}{3}$$

Two Kinds of Classes

Syntactic Classes

P **PSPACE** **L** . . .

coNP **NP** **EXP**

Semantic Classes

BPP . . .

ZPP **BQP**

Two Kinds of Classes

Syntactic Classes

P **PSPACE** **L** **coNP** **NP** **EXP** . . .

- Complete problems exist;
- Hierarchy theorems hold.

Semantic Classes

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Two Kinds of Classes

Syntactic Classes

P **PSPACE** **L** **coNP** **NP** **EXP** ...

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Semantic Classes

BPP **ZPP** **BQP** ...

Nothing is known about complete problems nor about hierarchy theorems.

The Strong Church-Turing Thesis

Just as the theory of computability has its foundations in the Church-Turing thesis, computational complexity theory rests upon a modern strengthening of this thesis, which asserts that any "reasonable" model of computation can be efficiently simulated on a probabilistic Turing machine (an efficient simulation is one whose running time is bounded by some polynomial in the running time of the simulated machine). Here, we take reasonable to mean in principle physically realizable.

Ethan Bernstein & Umesh Varirani

Back to ICC

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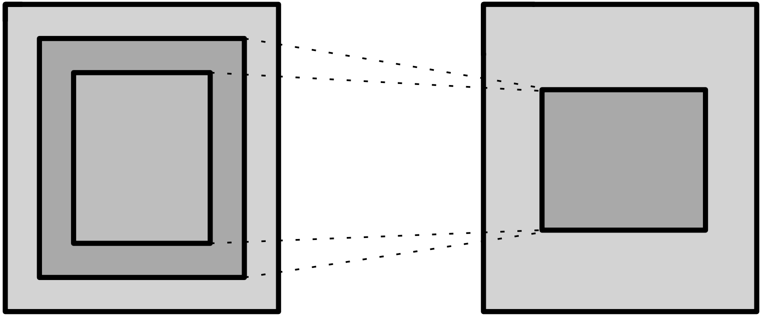

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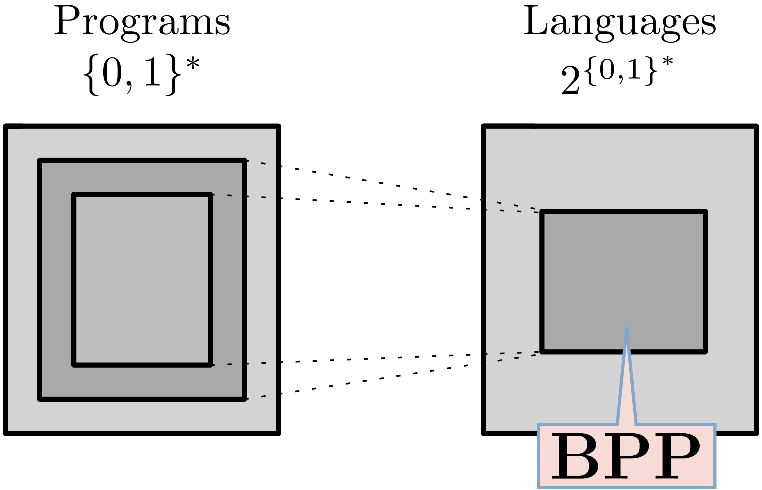

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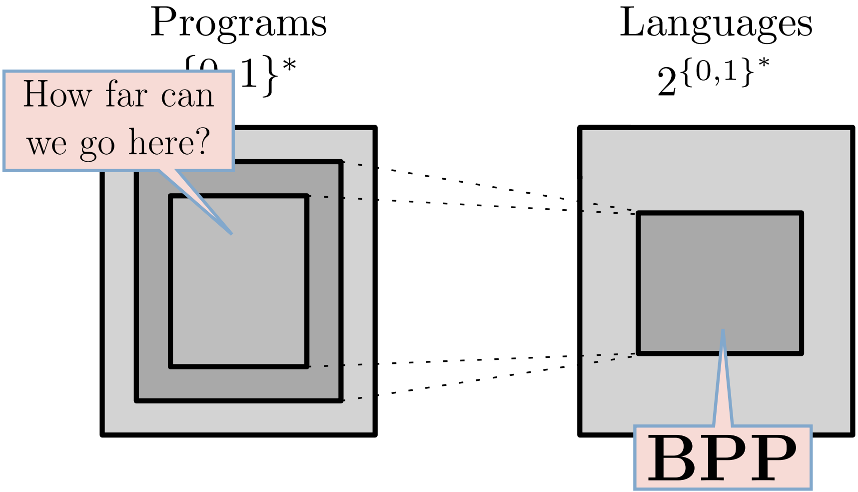

$$\mathbf{P} \stackrel{?}{=} \mathbf{BPP}$$

Complexity theorists believe this is true...

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. . . but the classes' definitions look fundamentally different!

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$$\mathbf{P} \stackrel{?}{=} \mathbf{BPP}$$

...but the classes' definitions look fundamentally different!

Can we somehow reconcile them?

Part II

Bounded Arithmetic

PA as a Way to Represent Functions

$$\mathsf{PA} \vdash \forall x. \exists ! y. A(x, y)$$

PA as a Way to Represent Functions

- Peano Axioms, reformulated on strings.
- Unrestricted induction.

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$$\mathsf{PA} \vdash \forall x.\exists! y.A(x,y) \quad \Longrightarrow \quad \begin{array}{c} f : \mathbb{S} \to \mathbb{S} \\ \models A(s, f(s)) \text{ for every } s \in \mathbb{S} \end{array}$$

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Too large for our purposes!

Restricting PA

$$A(0) \to \forall x.(A(x) \to A(x+1)) \to \forall x.A(x)$$

Restricting PA

- The number of times the induction hypothesis is applied when proving $A(n)$ is $\Theta(n)$
- This, in turn, is exponential in $|n|$.
- Better switch to recursion *on notation*

$$A(\varepsilon) \to \forall x.(A(x) \to A(x0) \wedge A(x1)) \to \forall x.A(x)$$

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- If quantifiers occurs in A, the logic becomes simply too powerful.
- Even a single layer of existential quantifiers is too much.
- Better to consider *bounded* quantification:

$$\forall x \leq p.A \qquad \exists x \leq q.A$$

Characterizing **FP**

$$\llbracket \mathsf{S}^1_2 \rrbracket$$

Characterizing **FP**

$$[\![\mathsf{S}^1_2]\!]$$

- Induction *on notation*.
- Induction formulas are Σ^b_1, namely bounded existential quantifications of sharply bounded formulas.

Characterizing **FP**

- Due to Buss [Buss86].
- Many variations exists.

$$[\![\mathsf{S}^1_2]\!] = \mathbf{FP}$$

Characterizing $\mathbf{FP}$

$$[\![\mathsf{S}^1_2]\!] = \mathbf{FP}$$

$$\begin{tikzcd}
\mathsf{S}^1_2 \arrow[r, bend left] & \mathcal{PR} \arrow[l, bend left] \arrow[r, bend left] & \mathbf{PDTM} \arrow[l, bend left]
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Cobham's *Bounded Recursion on Notation*

Polytime Deterministic TMs.

Characterizing **FP**

- Arguably the most difficult step.
- Can be done in various way, e.g. through cut-elimination process, or by realizability.

$$[\![\mathsf{S}^1_2]\!] = \mathbf{FP}$$

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Part III

Incepting Randomness into Bounded Arithmetic

The Main Idea

$$\begin{array}{c} \mathsf{PA} \\ \big\downarrow \\ \mathsf{S}^1_2 \end{array}$$

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$$\begin{tikzcd} \mathsf{PA} \arrow[r] \arrow[d] & \mathsf{MQPA} \\ \mathsf{S}^1_2 & \end{tikzcd}$$

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- A conservative extension of PA capturing randomized computation.
- All computable random functions from $\mathbb{S}$ to distributions over $\mathbb{S}$ can be represented in MQPA.

The Main Idea

$$\begin{tikzcd} \mathsf{PA} \arrow[r] \arrow[d] & \mathsf{MQPA} \arrow[d] \\ \mathsf{S}^1_2 \arrow[r] & ? \end{tikzcd}$$

What Changes?

The Formulas

- We follow the ideas behind MQPA.
- We endow the grammar of formulas with a *unary predicate* Flip, which models the access to an oracle providing fair random bits, and is interpreted as a measurable subset of $2^{\mathbb{S}}$
- The semantics of a closed formula A is not a truth value but the (measurable!) set $[\![A]\!]$ of all the possible interpretations of `Flip` rendering A true. For example:

$$
\begin{aligned}
[\![\mathsf{Flip}(0) \vee \mathsf{Flip}(1)]\!] &= \{\omega \mid \omega(0) = 1 \text{ or } \omega(1) = 1\} \\
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- MQPA also has counting quantifiers, which we do not need here.

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  - It stays essentially the same as S^1_2, and we call it RS^1_2.
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- **The Underlying Theory**
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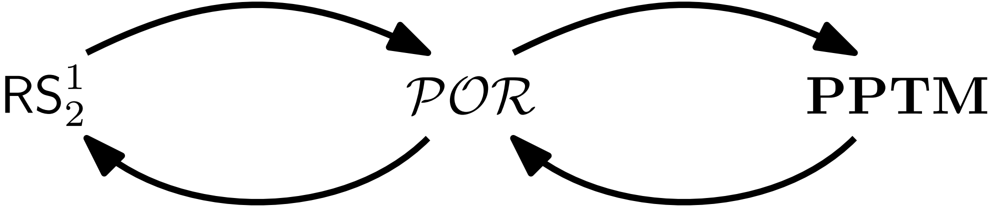
  - The only difference is the fact that `Flip` can occur in any formula in axiom schemas.
- **The Target Computational Model**
  - It is natural to just switch to *probabilistic* TMs.
  - We see them as computing functions from $\mathbb{S}$ to *distributions* over $\mathbb{S}$.
  - We ask that *indipendently* on randomness, the runtime of the machine at hand is polynomially bounded.

The Result

$$[\![\mathsf{RS}^1_2]\!] \;=\; \{f : \mathbb{S} \to \mathbb{D}(\mathbb{S}) \mid f \text{ can be computed by a } \mathbf{PPTM}\}$$

The Proof



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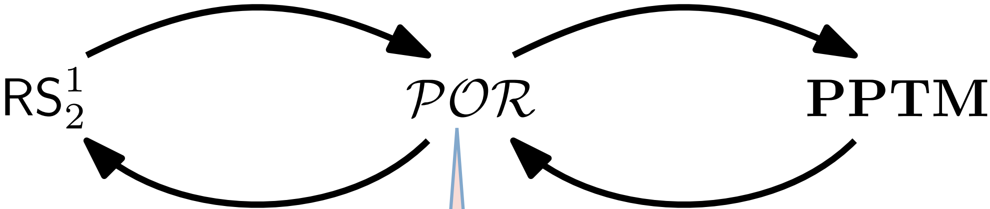

- Obtained by extending $\mathcal{PR}$ with a basic function accessing the random oracle.
- Generates functions from $\mathbb{S} \times 2^{\mathbb{S}}$ to $\mathbb{S}$.

The Proof

- Based on “randomized” realizability.
- Closely follows [CookUrquhart1990].

$$\begin{tikzcd} \mathsf{RS}^1_2 \arrow[rr, bend left] & & \mathcal{POR} \arrow[ll, bend left] \arrow[rr, bend left] & & \mathbf{PPTM} \arrow[ll, bend left] \end{tikzcd}$$

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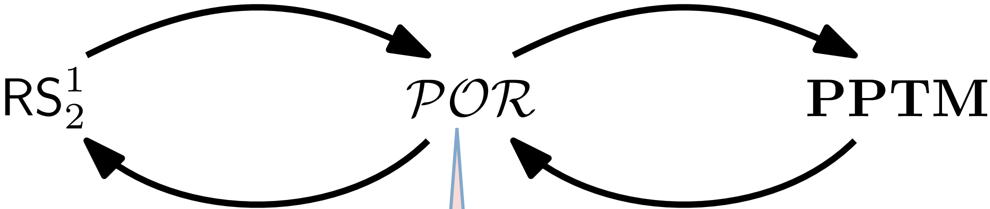
- $\mathcal{POR}$ captures functions in $\mathbb{S}^{\mathbb{S}\times 2^{\mathbb{S}}}$;
- **PPTM** rather captures functions in $\mathbb{S}^{\mathbb{S}\times 2^{\mathbb{N}}}$.



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$$\forall x, y \in \mathbb{S}.\ \mu(\{\omega \mid f(x, \omega) = y\}) = \mu(\{\eta \mid \langle f \rangle(x, \eta) = y\})$$

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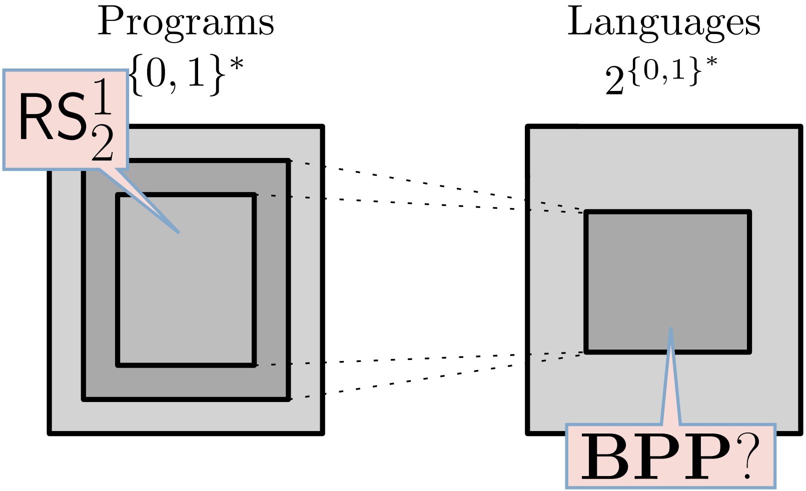
- The way f accesses ω is...random, while $\langle f \rangle$ makes use of η linearly, in an on demand fashion.
- It is convenient to see the encoding as going through *several intermediate steps*, each of them being conceptually very simple.

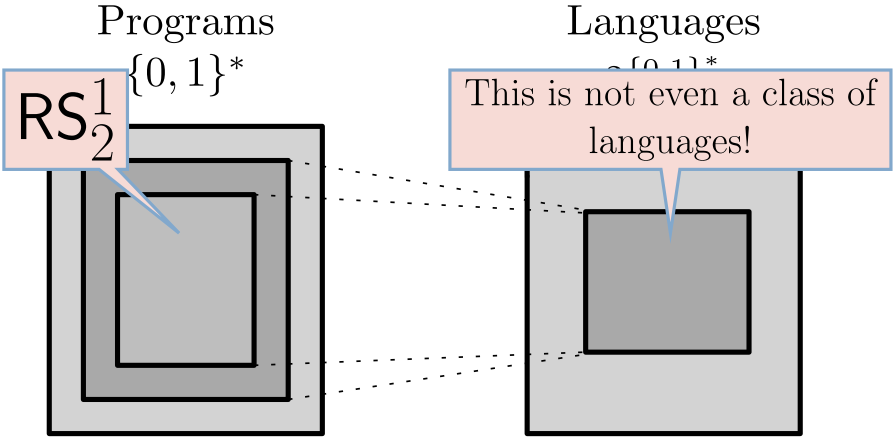
Part IV

Towards **BPP**

Are We There, Yet?



Are We There, Yet? Actually, No!



BPP Through Counting Quantifiers

From...

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. . . To

$$(L \subseteq \mathbb{S}) \in [\![\mathbf{C}\mathsf{RS}^1_2]\!] \quad \Leftrightarrow \quad \begin{array}{c} \mathsf{RS}^1_2 \vdash \forall x.\exists! y.A(x,y) \\ \models \forall x.\exists y.\mathbf{C}^{\frac{2}{3}} A(x,y) \\ L = Language(A) \end{array}$$

BPP Through Counting Quantifiers

From...

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Counting Quantifier

$$[\![\mathbf{C}^{\frac{t}{s}} B]\!] = \begin{cases} 2^{\mathbb{S}} & \text{if } \mu[\![B]\!] \geq \frac{[\![t]\!]}{[\![s]\!]} \\ \emptyset & \text{otherwise} \end{cases}$$

$$(L \subseteq \mathbb{S}) \in [\![\mathsf{CRS}^1_2]\!] \quad \Leftrightarrow \quad \begin{array}{c} \mathsf{RS}_2 \vdash \ldots \exists! y.A(x,y) \\ \models \forall x.\exists y.\mathbf{C}^{\frac{2}{3}} A(x,y) \\ L = \mathit{Language}(A) \end{array}$$

$\mathbf{BPP}$ Through Counting Quantifiers

From. . .

$$f : \mathbb{S} \to \mathbb{D}(\mathbb{S}) \in [\![\mathsf{RS}_2^1]\!] \quad \Leftrightarrow \quad \begin{array}{c} \mathsf{RS}_2^1 \vdash \forall x.\exists! y.A(x,y) \\ f = \mathit{RandomFunction}(A) \end{array}$$

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Theorem

$$[\![\mathbf{C}\mathsf{RS}_2^1]\!] = \mathbf{BPP}$$

Getting Rid of Counting Quantification

From. . .

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... To

$$(L \subseteq \mathbb{S}) \in [\![\mathsf{T} \oplus \mathsf{RS}^1_2]\!] \quad \Leftrightarrow \quad \begin{array}{c} \mathsf{RS}^1_2 \vdash \forall x.\exists! y.A(x,y) \\ \mathsf{T} \vdash \forall x.\exists y.\mathsf{TwoThirds}[A](x,y) \\ L = \mathit{Language}(A) \end{array}$$

Getting Rid of Counting Quantification

From...

$$\mathsf{RS}^1_2 \vdash \forall x.\exists! y.A(x,y)$$
$$(L \subseteq \mathbb{S}) \in [\![\mathbf{C}\mathsf{RS}^1_2]\!] \quad \Leftrightarrow \quad \models \forall x.\exists y.\mathbf{C}^{\frac{2}{3}} A(x,y)$$

- The formula A, once the value of its parameters are fixed, only depends on finitely many coordinates of `Flip`
- We can then internalize the Error Bound inside the language of plain arithmetic, making `Flip` to disappear.
- This go via threshold quantifiers.

$$(L \subseteq \mathbb{S}) \in [\![\mathsf{T} \oplus \mathsf{RS}^1_2]\!] \quad \Leftrightarrow \quad \mathsf{T} \vdash \forall x.\exists y.\mathsf{TwoThirds}[A](x,y)$$
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Theorem

$$[\![\mathsf{T} \oplus \mathsf{RS}^1_2]\!] \subseteq \mathbf{BPP}$$

$\texttt{PIT} \in [\![\mathsf{PA} \oplus \mathsf{RS}^1_2]\!]$

- **Polynomial Identity Testing**, `PIT`.
  - Given two arithmetical circuits, test whether they compute the same polynomial.
  - It admits a naïve decision procedure working in *deterministic exponential time*.
  - Randomization allows the time complexity to be trimmed down to *polynomial*, i.e., `PIT` is well known to be in **BPP**.
  - It is not known whether `PIT` is in **P**.

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$$\text{Randomized Algorithm for } \mathtt{PIT} \quad \Longrightarrow \quad PIT(x, y)$$

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Wrapping Up

- ICC and bounded arithmetic can be seen as ways to *enumerate* complexity classes.
- Semantic classes like **BPP** are not known to be enumerable, due to the *error bound* intrinsic in their definitions.
- We can however enumerate *subclasses* of **BPP** by *internalizing* the error bound check into arithmetic itself!
- Is it that $[\![\mathsf{PA} \oplus \mathsf{RS}^1_2]\!]$ equals **BPP**?

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Thank you! Questions?