One-dimensional forest fire model in the critical regime

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1 Introduction

In these notes, we present a method to find the cluster size distribution $n(s)$ of a one-dimensional forest fire model analytically when the model behaves critically. The cluster size distribution depends on the size in a power-law fashion. We find the exponent as well as the cutoff at which this behaviour breaks down.

2 Model

Consider lattice points in one dimension with possible states : "e", "t" and "b", representing an empty site, a site with a tree or a burning tree. At each time step, the sites transition probabilistically to a new state. An empty site grows a tree $e \to t$ with probability p. A tree catches fire $t \to b$ with probability f, and the fire spreads from one site to a neighboring tree with probability $1 - g$. Conversely, the time scale for an empty site to grow a tree is p^{-1} and for a tree to catch fire is f^{-1}. We set $g = 0$. We will be looking at equilibrium dynamics of the tree with densities ρ_t, ρ_e and ρ_b.

3 Limits for critical behaviour

Consider a situation where a lot of trees grow between two consecutive fires. This happens when $p >> f$, or $f/p \to 0$. Such a system has large clusters.
If we look at the largest cluster with trees s_{max} and call the time that it takes for this cluster to burn $T(s_{max})$, then assume very fast burn dynamics compared with tree growth so that $T(s_{max}) << p^{-1}$. Together, we have the limits,

$$T(s_{max}) << p^{-1} << f^{-1} \tag{3.1}$$

So we can effectively burn entire clusters whenever there is a lightning strike at every time step so that ρ_b is negligible. We also know from the limit that the model has very large clusters.

4 Size Distribution

At equilibrium, the number of trees destroyed on average is comparable with the number of trees that grow so that ρ_t is a constant.

$$s_{avg} f \rho_t = p(1 - \rho_t) \tag{4.1}$$

This gives us the average number of trees burnt in a single lightning strike

$$s_{avg} = \frac{p}{f}\Big(\frac{1-\rho_t}{\rho_t}\Big) \tag{4.2}$$

At equilibrium, we find that the average number of trees lost in a fire diverges with exponent -1, $s_{avg} \sim (\frac{f}{p})^{-1}$.
To calculate size distributions, we use the fact that our system has very large clusters. We look at some size - k segment of the cluster which is much smaller than the size of the cluster. We denote by $P_k(m)$ the probability that the k-cluster has m trees. This is independent of where the trees are located in the cluster since the tree growth probability is the same at every site. The size distribution $n(s)$ is one particular scenario with $k = s+2$ and $m = s$, namely with empty endpoints in the cluster filled with trees.

$$n(s) = \frac{P_{s+2}(s)}{\binom{s+2}{s}} \tag{4.3}$$

Since fires are rare, a cluster with 0 trees slowly fills up with trees and a lightning strike only occurs after the cluster is full. Since the rate at which a configuration with m trees is created is the same as the rate at which it is destroyed, we have

$$(k-m+1)pP_k(m-1) = (k-m)pP_k(m) \tag{4.4}$$

with $m = 1...k$. The $m = 0$ configuration satisfies

$$kpP_k(0) = p(1-\rho_t) \tag{4.5}$$

These relations give us

$$P_k(m) = \frac{1}{k-m}(1-\rho_t) \tag{4.6}$$
$$P_k(k) \approx 1-(1-\rho_t)\log k \tag{4.7}$$

Plugging in the expression for $n(s)$, we have

$$n(s) = \frac{(1-\rho_t)}{(s+1)(s+2)} \tag{4.8}$$
$$\approx \frac{(1-\rho_t)}{s^2} \tag{4.9}$$

Note the exponent -2. This behaviour breaks down when the probability of a lightning strike destroying the cluster is $\mathcal{O}(1)$ before the cluster is fully dense. A k-cluster takes time $t(m)$ to go from 0 trees to a m-tree configuration.

$$pt(m) = \frac{1}{k} + \frac{1}{k-1} + \dots \frac{1}{k-m} \tag{4.10}$$

The number of trees in the cluster as a function of the time is given by

$$m(t) = k(1-e^{-pt}) \tag{4.11}$$

and the probability that fire strikes before $t(k)$ being $\mathcal{O}(1)$ is the condition

$$s_{cut}\log s_{cut} \sim p/f \tag{4.12}$$