The following is a brief analysis of the mechanics of an AMM based prediction market with 2 outcomes – YES and NO. Participants express their belief in either of the outcomes by buying the corresponding tokens.

System state variables

L = Initial $ liquidity in the system which is independent of the no. of outcome tokens and which simply determines the spot price and cost of adding/removing liquidity from the AMM.
Y = Initial number of YES tokens
N = Initial number of NO tokens
Y*N = k is a constant

These state variables together determine the probabilities, alongwith various prices, of the 2 outcomes that this AMM supports.

The product of no. of YES and NO tokens changes only when new tokens are minted during liquidity being added to the system by liquidity providers or being removed by the liquidity providers. More on how this works in the liquidity provisioning section.

Probability of YES outcome = $\frac{N}{Y+N}$ (1) – this follows from the fact that the probability of an outcome is inversely proportional to the number of opposite outcome tokens in the pool.
Probability of NO outcome = $\frac{Y}{Y+N}$ (2)

Probability of YES + NO tokens always equals 1 from equations (1) and (2).

Based on the above, we can state a few more important equations

Spot price of YES tokens in terms of NO tokens = $\frac{N}{Y}$ (3)

Spot price of NO tokens in terms of YES tokens = $\frac{Y}{N}$ (4)

$L buys Y+N tokens, hence spot price of a theoretical composite token = $\frac{L}{Y+N}$ (5)

Price of 1 YES token in terms of NO tokens = $\frac{N}{Y}$ from equation (3) above (6)
Price of Y YES tokens in terms of NO tokens = $\left(\frac{N}{Y}\right) * Y = \mathbf{N}$ (7)
Price of N NO tokens in terms of YES tokens = $\left(\frac{Y}{N}\right) * N = \mathbf{Y}$ (8)
Spot price of YES tokens in $ terms = $\frac{L}{Y+N} = \frac{L}{Y+Y} = \frac{L}{2Y}$ (9)
this expresses how many tokens a $ can buy

In the above equation, we are converting N NO tokens to their equivalent value in YES tokens using equation (8) from above.

Similarly, Spot price of NO tokens in $ terms = $\frac{L}{Y+N} = \frac{L}{N+N} = \frac{L}{2N}$ (10)

How is Liquidity Added/Removed to/from the System

\$L buys Y+N tokens, hence \$X buys $\left(\frac{\boldsymbol{Y+N}}{\boldsymbol{L}}\right) * \boldsymbol{X}$ number of tokens with a mixture of YES and NO tokens.

Proportion of YES tokens that \$X buys Y'' = $\left(\frac{Y+N}{L}\right) X \left(\frac{Y}{Y+N}\right)$ = $\boldsymbol{XY/L}$ (11)

Proportion of NO tokens that \$X buys N'' = $\left(\frac{Y+N}{L}\right) X \left(\frac{N}{Y+N}\right)$ = $\boldsymbol{XN/L}$ (12)

When a liquidity provider adds liquidity worth \$X to the pool, we mint YES and NO tokens according to equations (11) and (12) above. Similarly, when liquidity provider withdraws \$X worth of liquidity from the pool, we remove number of tokens given by the equation above.
Spot prices of YES and NO tokens remain unchanged when adding/removing liquidity.
Probabilities of YES and NO outcomes remain unchanged when adding/removing liquidity.

System Dynamics when buying an outcome token

When a user buys \$X worth of tokens, we first mint NO tokens according to equation (12), but this invalidates the invariant Y*N = k, hence number of Y tokens in the pool will depend on this invariant.

New no. of YES tokens allowed Y' = $\frac{\boldsymbol{YN}}{\boldsymbol{N+N''}}$ (13)

The number of YES tokens that would have been there in the pool based on \$X liquidity being added (if invariant was not violated) follows from equation (11) = **Y + Y''** (14)

The difference between equation (14) and (13), i.e. difference between no. of YES tokens that would have been there in the pool with a simple liquidity addition v/s the number of tokens actually there in the pool, represents the number of YES tokens bought by the user spending \$X. Denote this by M. Hence, effective price = **X/M** (15)

Also, **M = Y+Y'' – Y'** (16)

Several proofs now follow which prove that the AMM works as expected based on intuitions of supply and demand for the various tokens and that the probabilities reflect actual belief on the outcomes as shown by trading prices.

Effective price when buying a token is always greater than the spot price

$\boldsymbol{X/M} >= \boldsymbol{L/2Y}$ - this is true if (17)

if $\frac{X}{Y+Y''-Y'} \geq \frac{L}{2Y}$

if $\frac{X}{\mathrm{Y} + \left(\frac{\mathrm{XY}}{\mathrm{L}}\right) - \left(\frac{\mathrm{YN}}{\mathrm{N}+\frac{\mathrm{XN}}{\mathrm{L}}}\right)} >= \frac{\mathrm{L}}{2\mathrm{Y}}$ using equations (11) & (13)

if $\frac{X}{\mathrm{Y}+\left(\frac{\mathrm{XY}}{\mathrm{L}}\right) - \left(\frac{\mathrm{YL}}{\mathrm{L+X}}\right)} >= \frac{\mathrm{L}}{2\mathrm{Y}}$

if $\frac{\mathrm{XL(L+X)}}{2\mathrm{XLY}+\mathrm{X}^2\mathrm{Y}} >= \frac{\mathrm{L}}{2\mathrm{Y}}$

if $\frac{L+X}{2L+X} >= \frac{1}{2}$

if $2L + 2X >= 2L + X$ which is true for X>= 0

This makes intuitive sense based on basic AMM mechanics.
New liquidity in the pool when $X worth of tokens are bought **L' = L+X** (18)

<u>System dynamics when selling an outcome token</u>

When user wants to sell A number of YES tokens, the total no. of YES tokens in the pool becomes Y+A. This invalidates the invariant Y*N = k, so new number of NO tokens in the pool

$\boldsymbol{N' = \frac{YN}{Y+A}}$ (19)

This is necessarily less than N, i.e. N' < N which means that some number of NO tokens are burnt. The $ equivalent of this is returned to the user from the pool.
So amount given to user L'' = (N-N')*Spot price of N in dollar terms

$\boldsymbol{L'' = (N - N') * (\frac{L}{2N})}$ (20) – this uses spot price of NO tokens from equation (10)
New liquidity in the pool L' = L-L'' (21)

Effective selling price = L''/A (22)

Effective selling price is always less than spot price

$\boldsymbol{L''/A <= \frac{L}{2Y}}$, this is true if

if $\frac{(N-N')*\left(\frac{L}{2N}\right)}{A} <= \frac{L}{2Y}$

if $\frac{\left(\frac{AN}{Y+A}\right)*\left(\frac{L}{2N}\right)}{A} <= \frac{L}{2Y}$

if $\left(\frac{NL}{(Y+A)*2N}\right) <= \frac{L}{2Y}$

if $Y <= Y + A$ which is true for A>=0 – and this too makes intuitive sense from an AMM mechanics point of view.

Probability of YES outcome always increases when it is bought and similarly for NO tokens.

Probability of YES outcome before buying YES tokens P = N/Y+N from equation (1)
New probability after YES tokens are bought P' = N'/Y' + N' where Y' and N' are new amounts of YES and NO tokens in the pool
P' > P

if $\frac{N'}{Y'+N'} > \frac{N}{Y+N}$

if $\frac{\left(N+\left(\frac{XN}{L}\right)\right)}{\left(\left(\frac{YL}{L+X}\right)+\left(N+\left(\frac{XN}{L}\right)\right)\right)} > \frac{N}{Y+N}$

if $\frac{N(L+X)^2}{YL^2+N(L+X)^2} > \frac{N}{Y+N}$

if $NY(L+X)^2 + N^2(L+X)^2 > NYL^2 + N^2(L+X)^2$
which is true for X>0
The proof is similar for the situation when NO tokens are bought.

Probability of an outcome always decreases when opposite outcome token is bought
Since the probabilities sum to 1, this directly follows from the previous proof. If probability of YES outcome increases then probability of NO outcome has to decrease.

Probability of an outcome always decreases when selling that token

Let user sell A number of YES tokens, new YES tokens Y' = Y+A
New number of NO tokens N' = $\frac{YN}{Y+A}$
P' < P
if $\frac{N'}{Y'+N'} < \frac{N}{Y+N}$
if $\frac{\frac{YN}{Y+A}}{Y+A+\frac{YN}{Y+A}} < \frac{N}{Y+N}$
if $\frac{YN}{(Y+A)^2+YN} < \frac{N}{Y+N}$
if $YN(Y+N) < YN^2 + N(Y+A)^2$
if $NY^2 < NY^2 + 2YAN + NA^2$, which is true for A>0

Probability of an outcome always increases when selling opposite outcome token
This directly follows from the proof above since the probabilities are mutually exclusive and exhaustive. Hence, probability of YES outcome decreasing means that probability of NO outcome increases.

Spot price of a token always increases on buying that token
Spot price of YES tokens = L/2Y
New spot price after buying $X worth of YES tokens = (L+X)/2Y'
L+X > L for X>0
Y' < Y since number of YES tokens in the pool decreases when buying YES tokens. This means new spot price > old spot price.

Spot price of a token always decreases when selling that token

Spot price of YES tokens before selling = L/2Y
New spot price after selling A number of YES tokens = (L-X)/2Y'
L-X < L for X >0
Y' = Y+A > Y for A > 0
Hence, new spot price is always less than old spot price before selling an outcome token.

Fees are collected separately from the liquidity pool – when when wants to buy $X worth of tokens, user pays corresponding fee amount **X*fee_rate** which is collected. Hence, introduction of fees does not change any of the pricing dynamics.

How are rewards distributed to liquidity providers

The liquidity providers get a share of the fee collected based on their share in the $ liquidity in the pool. To account for inflation, we increase the reward pool by $R * \left(\frac{s(m+1)}{Sum(s(1)....s(m))}\right)$ and we also mark this as withdrawn by the user giving (m+1)th liquidity share. This ensures that a user adding liquidity to the pool only collects share of fees from that point onwards and no part of fees that has already been collected.

References – The mechanics draw inspiration from work on conditional tokens done by Gnosis and the general concept of constant function market makers.