1.1

Propositions and Connectives

Logic is concerned with all kinds of reasoning, whether they be legal arguments or mathematical proofs or conclusions in a scientific theory based on a set of hypotheses. An aim of logic is to provide rules by which one can determine whether any particular argument or reasoning is valid (correct). Due to the diversity of their application, these rules, called ***rules of inference***, must be stated in general terms and must be independent of any particular language used in the arguments. The theory of inference is formulated in such a way that we should be able to decide the validity of the argument by following the rules mechanically and independently of our own feelings about the argument.

We require a language to state rules. Natural languages are not suitable for this purpose since they are not precise and are ambiguous. It is therefore necessary to develop a formal language called **object language**. A formal language is one in which the *syntax* is well defined. In order to avoid ambiguity, we use symbols which have been clearly defined in the object language. An advantage of the use of symbols is that they are easy to write and manipulate. Because of the use of symbols, the logic that we shall study is also called **symbolic logic**.

The study of object language requires the use of a natural language (and we choose English in our case). This natural language will then be called ***meta language.***

Propositions:

A ***declarative sentence*** is a sentence that declares a fact. We begin by assuming that the object language contains a set of declarative sentences which cannot be further broken down or analysed into simpler sentences. These are ***primary declarative sentences***. Only those declarative sentences will be admitted in the object languages which have one and only one of two possible values called ***truth values***. The two truth values are ***true*** and ***false*** which are denoted by the symbols T and F respectively. Occasionally T and F are respectively denoted by 1 and 0 .

Since only two possible truth values are admitted, our logic is sometimes called a ***two-valued logic***.

Declarative sentences in the object language are of two types. The first type includes those declarative sentences which are considered to be primitive in the object language. The declarative sentences of the second type are obtained from the primitive declarative sentences by using certain symbols called ***connectives (logical operators)*** and certain punctuation marks, such as parentheses to join primitive declarative sentences. In any case, all the declarative sentences to which it is possible to assign one and only of the two truth values are called ***propositions*** (or ***statements***).

A ***proposition*** (or ***statement***) is a declarative sentence that is either true or false, but not both.

Example 1:

(i) Delhi is the capital of India
(ii) $2 + 3 = 4$
(iii) What is it?
(iv) $x + y = z$
(v) This statement is false

(i) and (ii) are declarative sentences and they are propositions. Proposition (i) is true and the proposition (ii) is false. Sentence (iii) is not a proposition since it is not a declarative sentence. Sentence (iv) is not a proposition, since it is neither true nor false. Note that the sentence (iv) can be turned into a proposition if we assign values to the variables. Sentence (v) is not a proposition since we cannot properly assign a definite truth value. If we assign the value true then (v) implies that the sentence (v) is false. On the other hand, if we assign it the value false then (v) implies that the sentence (v) is true. This example illustrates a ***semantic paradox***.

A ***proposition variable*** (or ***statement variable***) is a variable that represents a proposition. The propositional variables are denoted by lowercase letters. The conventional letters used for propositional variables are $p, q, r, s, \ldots$

The area of logic that deals with propositions is called ***propositional calculus*** or ***propositional logic***. It was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago.

A proposition which does not contain any connectives is called a **atomic** (or **primary** or **simple**) ***proposition***. A proposition which contains one or more atomic propositions as well as some connectives is called a ***molecular*** (or ***compound*** or ***composite***) ***proposition***.

Negation:

Let p be a proposition. The negation of p, denoted by $\sim p$ (also denoted by $\neg p$ or $\bar{p}$), is the proposition " it is not the case that p". The proposition $\sim p$ is read as "not p". The truth value of $\sim p$ is the opposite of the truth value of p.

The truth table of $\sim p$ is a tabular form in which the truth values of $\sim p$ are given for arbitrary truth values of p.

Truth table for $\sim p$	
p	$\sim p$
T	F
F	T

Consider the proposition p : Delhi is a city

Then $\sim p$ is the proposition $\sim p$: It is not the case that Delhi is a city

The above proposition can also be written as $\sim p$: Delhi is not a city

Note that the two propositions "It is not the case that Delhi is a city" and "Delhi is not a city" are not identical but we have denoted both of them by $\sim p$. The reason to denote them by the same $\sim p$ is that they mean the same in English (meta language).

Note: A given proposition in object language is denoted by a symbol and it may correspond to several statements in meta language. This multiplicity happens because in a meta language one can express oneself in a variety of ways.

Note that the negation constructs a new proposition from a single existing proposition. We will now introduce connectives (***and***, ***or***, ***if…then*** … and ***if and only if)*** that are used to form new propositions from two or more existing propositions.

Conjunction:

If p and q are propositions, then the conjunction of p and q is the compound proposition "p and q" denoted by $p \wedge q$. The connective **and** is denoted by the symbol $\wedge$.

The compound proposition $p \wedge q$ is true when both p and q are true and is false otherwise.

The truth table of a compound proposition is a tabular form in which the truth values of the compound proposition are given interms of its component parts.

Truth table for $p \wedge q$		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note that in logic the word "but" sometimes is used instead of "and" in a conjunction. (In meta language, there is a difference in meaning between "and" and "but"; we will not be concerned with this nuance here.)

Disjunction:

If p and q are propositions, the disjunction of p and q is the compound statement "p **or** q" denoted by $p \vee q$. The connective **or** is denoted by the symbol $\vee$.

The compound proposition $p \vee q$ is true if at least one of p or q is true and is false otherwise.

Truth table for $p \vee q$		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note: In English the word "or" is commonly used in two distinct ways. It is used (i) in the *inclusive* sense, i.e., "p or q or both", and (ii) in the *exclusive* sense, i.e., "p or q but not both". For example, "He will go to Harvard University or Yale University" uses "or" in the exclusive sense. In logic $p \vee q$ always means "p and/or q".

Example 2:

Let p and q be propositions given by

p : Sam is rich $\qquad\qquad$ q : Sam is happy

Write each of the following in symbolic form:

a. Sam is poor but happy.
b. Sam is either rich or unhappy.
c. Sam is neither rich nor happy.

Solution:

a. $\sim p \wedge q$

b. $p \vee \sim q$
c. $\sim p \wedge \sim q$

Conditional propositions:

If p and q are propositions, then the compound statement "if p, then q" ("if p, q") denoted by $p \rightarrow q$, is called a ***conditional proposition*** or ***implication***.

In $p \rightarrow q$, the proposition p is called ***antecedent*** (or ***hypothesis***) and the proposition q is called ***consequent*** (or ***conclusion***). The connective ***if … then …*** is denoted by $\rightarrow$.

The proposition $p \rightarrow q$ has a truth value F when q has truth value F and p has truth value T and has truth value T otherwise.

Truth table for $p \rightarrow q$		
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note:

(i) According to the definition, it is not necessary that there be any kind of relation between p and q in order to form $p \rightarrow q$.

(ii) It is customary to represent any one of the following expressions by $p \rightarrow q$:

q is necessary for p	q when p
p is sufficient for q	a necessary condition for p is q
q if p	q unless $\sim p$
p only if q	a sufficient condition for q is p
q whenever p	q follows from p

(iii) In mathematics "if p then q" and "p implies q" are used interchangeably, but in this text we use the word implies in different way.

The **converse** of the implication $p \rightarrow q$ is the implication $q \rightarrow p$.

The **inverse** of the implication $p \rightarrow q$ is the implication $\sim p \rightarrow \sim q$ (read as if not p then not q).

The **contra positive** of the implication $p \rightarrow q$ is the implication $\sim q \rightarrow \sim p$.

Truth tables for converse, inverse and contrapositive of $p \longrightarrow q$

p	q	$Conditional$ $p \longrightarrow q$	$Converse$ $q \longrightarrow p$	$\sim p$	$\sim q$	$Inverse$ $\sim p \longrightarrow \sim q$	$Contrapositive$ $\sim q \longrightarrow \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Example 3:

Write the converse, inverse and contrapositive of the implication "If today is Sunday, then I will go for a walk".

Solution: Let p and q be propositions

p : Today is Sunday $\qquad$ q : I will go for a walk

Then the given proposition is written as $p \longrightarrow q$.

The converse of this implication is

$q \longrightarrow p$: If I will go for a walk, then Today is Sunday

The inverse of the above implication is

$\sim p \longrightarrow \sim q$: If Today is not Sunday then I will not go for a walk

The contrapositive of $p \longrightarrow q$ is

$\sim q \longrightarrow \sim p$: If I will not go for a walk, then Today is not Sunday

Biconditional Propositions:

If p and q are propositions, then the compound statement "p if and only if q", denoted by $\boldsymbol{p \leftrightarrow q}$ **(**or $\boldsymbol{p \rightleftarrows q}$), is called a **biconditional proposition**. The connective ***if and only if*** is denoted by the symbol $\leftrightarrow$ (or $\rightleftarrows$).

The proposition $p \leftrightarrow q$ can also be stated as "p is necessary and sufficient condition for q" (or "if p then q, and conversely" or "p iff q")

The proposition $p \leftrightarrow q$ has truth value T whenever both p and q have identical truth values.

Truth table for $p \leftrightarrow q$		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 4:

Construct the truth tables for $\boldsymbol{p \wedge \sim p}$ and $\boldsymbol{p \vee \sim p}$

Truth table for $p \wedge \sim p$		
p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Truth table for $p \vee \sim p$		
p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

P1:

Which of the following are propositions?

a. **2 is an even integer.**
b. **Why should we study discrete structures?**
c. **There is an integer x such that $x^2 = 3$.**
d. **Please be quiet.**
e. **Dogs can fly.**
f. **There will be snow in December.**
g. **What a beautiful evening!**
h. **Get up and do your exercises.**

Solution:

a. This is a declarative sentence and its truth value is T. Therefore, it is a proposition.
b. This is not a declarative sentence. Therefore, it is not a proposition.
c. This is a declarative sentence, since there is no integer x such that $x^2 = 3$. Its truth value is F. Therefore, it is a proposition.
d. This is not a declarative sentence. Therefore, it is not a proposition.
e. This is a declarative sentence and its truth value is F.
f. This is a declarative sentence and it is either true or false but not both. Therefore, it is a proposition.
g. This is not a declarative sentence. Therefore, it is not a proposition.
h. This is not a declarative sentence. Therefore, it is not a proposition.

P2:

If p and q are the following propositions:

p : 2 is an even integer **q : -3 is a negative integer**

then write the following propositions interms of p, q and logical connectives and find their truth values:

a. **2 is an even integer and -3 is a negative integer.**
b. **2 is not an even integer and -3 is a negative integer.**
c. **2 is not an even integer and -3 is not a negative integer.**
d. **If 2 is not an even integer, then -3 is not a negative integer.**
e. **If 2 is an even integer, then -3 is not a negative integer.**
f. **2 is an even integer if and only if -3 is a negative integer.**
g. **2 is not an even integer if and only if -3 is a negative integer.**

Solution:

First, note that the truth value of p is T and the truth value of q is T.

a. $p \land q$, T
b. $\sim p \land q$, F
c. $\sim p \land \sim q$, F
d. $\sim p \longrightarrow \sim q$, T
e. $p \longrightarrow \sim q$, F
f. $p \leftrightarrow q$, T
g. $\sim p \leftrightarrow q$, F

P3:

Write the converse, the inverse and the contrapositive of the following conditional proposition

“The home team wins whenever it is raining”

Solution:

The given proposition can be rewritten as “If it is raining, then the home team wins”. It is written in symbolic form as $p \longrightarrow q$ where

p : It is raining $\qquad$ q : The home team wins

The converse of $p \longrightarrow q$ is $q \longrightarrow p$, i.e., “If the home team wins, then it is raining”.

The inverse of $p \longrightarrow q$ is $\sim p \longrightarrow \sim q$, i.e., “If it is not raining, then the home team does not win”.

The contrapositive of $p \longrightarrow q$ is $\sim q \longrightarrow \sim p$, i.e., “If the home team does not win, then it is not raining”.

P4:

Construct the truth tables for $\sim p \vee q$ and $\sim p \vee \sim q$.

Solution:

Truth table for $\sim p \vee q$			
p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Truth table for $\sim p \vee \sim q$				
p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

P5:

Construct the truth tables for $\sim p \wedge q$ and $p \wedge \sim q$.

Solution:

Truth table for $\sim p \wedge q$			
p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Truth table for $p \wedge \sim q$			
p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

P6:

Determine the truth values of the following propositions:

a. If Paris is in France then $2 + 2 = 4$
b. If Paris is in France then $2 + 2 = 5$
c. If Paris is in England then $2 + 2 = 4$
d. If Paris is in England then $2 + 2 = 5$

Solution:

Note that the proposition "If p, then q" is false only when p is true and q is false. Therefore, the truth values of the above are

a. T b. F c. T d. T

P7:

Determine the truth values of the following propositions:

a. Paris is in France if and only if $2 + 2 = 4$
b. Paris is in France if and only if $2 + 2 = 5$
c. Paris is in England if and only if $2 + 2 = 4$
d. Paris is in England if and only if $2 + 2 = 5$

Solution:

The propositions (a) and (d) are true since the primary (atomic) propositions are both true in (a) and both false in (d). On the other hand (b) and (d) are false since their atomic propositions have opposite truth values.

P8:

Let p, q be primitive propositions for which the implication $p \longrightarrow q$ is false. Determine truth values for

a. $p \land q$
b. $\sim p \lor q$
c. $q \longrightarrow p$
d. $\sim q \longrightarrow \sim p$
e. $p \leftrightarrow q$
f. $\sim p \longrightarrow \sim q$

Answers:

Note that the proposition "If p then q" is false only when p is true and q is false. Therefore, the truth values of the above are

a. F
b. F
c. T
d. F
e. F
f. T

1.1

Propositions and Connectives

Answers:

1.
    a. Prop, F
    b. Not a prop (since it is not a declarative sentence)
    c. Prop, T
    d. Not a prop, since it is neither true nor false

2.
    a. Today is not Monday
    b. There is pollution in Jaipur
    c. $2 + 5 \neq 7$

3.
    a. $p \land q$
    b. $p \land \sim q$
    c. $\sim p \land \sim q$
    d. $p \lor q$
    e. $p \longrightarrow q$
    f. $p \leftrightarrow q$

4.
    a. Sharks have not been spotted near the shore
    b. Swimming at the New Jersey shore is allowed and sharks have been spotted near the shore
    c. Swimming at the New Jersey shore is not allowed or sharks have been spotted near the shore
    d. If swimming at the New Jersey shore is allowed then sharks have not been spotted near the shore

e. If sharks have not been spotted near the shore then swimming at the New Jersey shore is allowed
f. If swimming at the New Jersey shore is not allowed then sharks have not been spotted near the shore
g. Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore.

5.

a. F
b. T
c. T
d. T

6.

a. T
b. F
c. T
d. F

7.

a.

Truth table for $p \vee \sim q$			
p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

b.

Truth table for $\sim p \wedge \sim q$				
p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

1.1

Propositions and Connectives

Exercise:

1. Which of the following sentences are propositions? What are the truth values of those that are propositions?
    a. Kharagpur is in Andhra Pradesh
    b. Answer this question
    c. $5 + 8 = 13$
    d. $x + 2 = 9$

2. What is the negation of each of these propositions?
    a. Today is Monday
    b. There is no pollution in Jaipur
    c. $2 + 5 = 7$

3. Let p and q be propositions, where

p : It is below freezing $\qquad$ q : It is snowing

Write the following propositions using p and q and connectives:

a. It is below freezing and snowing
b. It is below freezing but not snowing
c. It is not below freezing and it is not snowing
d. It is either snowing or below freezing (or both)
e. If it is below freezing, it is also snowing
f. That it is below freezing is necessary and sufficient for it to be snowing

4. Let p and q be propositions, where

p : Swimming at the New Jersey shore is allowed
q : Sharks have been spotted near the shore

Express each of these compound propositions as an English sentence.

a. $\sim q$
b. $p \wedge q$
c. $\sim p \vee q$
d. $p \longrightarrow \sim q$
e. $\sim q \longrightarrow p$
f. $\sim p \longrightarrow \sim q$
g. $p \leftrightarrow \sim q$

5. Determine whether the following conditional propositions are true of false.
   a. If $1 + 1 = 2$ then $2 + 2 = 5$
   b. If $1 + 1 = 3$ then $2 + 2 = 4$
   c. If $1 + 1 = 3$ then $2 + 2 = 5$
   d. If monkey can fly then $1 + 1 = 3$

6. Determine whether the following biconditionals are true or false:
   a. $2 + 2 = 4$ if and only if $1 + 1 = 2$
   b. $1 + 1 = 2$ if and only if $2 + 3 = 4$
   c. $1 + 1 = 3$ if and only if monkeys can fly
   d. $0 > 1$ if and only if $2 > 1$

7. Construct the truth tables for following:
   a. $p \vee \sim q$
   b. $\sim p \wedge \sim q$

1.2. Well – formed Formulas

A ***well-formed formula*** (***wff***) is defined recursively as follows:

(i) *A proposition variable alone is a wff*
(ii) *If a is a wff then $\sim a$ is a wff*
(iii) *If a and b are wffs, then $(a \wedge b)$, $(a \vee b)$, $(a \longrightarrow b)$ and $(a \leftrightarrow b)$ are wffs.*
(iv) *A string of symbols containing the proposition variables, connectives and parentheses is a wff if and only if it can be obtained by finitely many applications of the above rules (i), (ii) and (iii).*

By the above definitions the following are wffs:

$$(p \wedge q)\ , \big((p \wedge q) \longrightarrow q\big), \Big(\big((p \longrightarrow q) \wedge (q \longrightarrow r)\big) \leftrightarrow (p \longrightarrow r)\Big)$$

We use parentheses to avoid ambiguity. For the sake of convenience we shall omit the outer parentheses. Thus we write the above wffs respectively as

$$p \wedge q\ , (p \wedge q) \longrightarrow q, \big((p \longrightarrow q) \wedge (q \longrightarrow r)\big) \leftrightarrow (p \longrightarrow r)$$

Since we deal with only wffs, we refer wffs as *formulas*.

The following are not formulas

$$p \vee q \wedge r\ ,\ \ (p \longrightarrow q) \longrightarrow (\wedge\, r), (p \wedge q) \longrightarrow q)$$

Note:

(i) In the construction of formulas, the parentheses will be used in the same sense in which they are used in elementary algebra or some times in a programming language.
(ii) The following is the hierarchy of operations and parentheses:
1. Connectives within parentheses; among parentheses innermost first
2. Negation $\sim$
3. $\wedge$ and $\vee$
4. $\longrightarrow$
5. $\leftrightarrow$

Truth tables of well-formed Formulas

If there are $\boldsymbol{n}$ distinct variables in a formula, then we need to consider $\mathbf{2}^{\boldsymbol{n}}$ possible combinations of truth values in order to obtain the truth table of the formula.

Construction of truth tables

There are **two** methods of construction of truth tables.

The first columns of the table are for the proposition variables p, q, r, ... and create enough number of rows ($i.e.$, 2^n rows if there are n *variables*) in the table for all possible combinations of T and F for the variables.

Method – 1

Devote a column for each elementary stage of the construction of the formula. The truth value at each step is determined from the previous stages by the definition of connectives. Finally the truth value of the formula is given in the last column.

Method – 2

Write the formula on the top row to the right of its variables. A column is drawn for each variable as well as for the connectives that appear in the formula. The truth values are entered step by step. The step numbers at the bottom of the table show the sequence followed in arriving the final step.

Example 1:

Construct the truth table for the formula

$$(p \vee q) \longrightarrow \big((r \vee p) \wedge (\sim r \vee q)\big)$$

Solution:

Method- 1

Let β be $(r \vee p) \wedge (\sim r \vee q)$. Then the given formula is $(p \vee q) \longrightarrow \beta$.

Truth table for $(p \vee q) \longrightarrow \big((r \vee p) \wedge (\sim r \vee q)\big)$								
p	q	r	$p \vee q$	$r \vee p$	$\sim r$	$\sim r \vee q$	β	$(p \vee q) \longrightarrow \beta$
T	T	T	T	T	F	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	T	F	F	F	F
T	F	F	T	T	T	T	T	T
F	T	T	T	T	F	T	T	T
F	T	F	T	F	T	T	F	F
F	F	T	F	T	F	F	F	T
F	F	F	F	F	T	T	F	T

Method- 2

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$(\boldsymbol{p}$	$\vee$	$\boldsymbol{q})$	$\longrightarrow$	$((\boldsymbol{r}$	$\vee$	$\boldsymbol{p})$	$\wedge$	$(\sim$	$\boldsymbol{r}$	$\vee$	$\boldsymbol{q}))$
T	T	T	T	T	T	T	T	T	T	T	F	T	T	T
T	T	F	T	T	T	T	F	T	T	T	T	F	T	T
T	F	T	T	T	F	F	T	T	T	F	F	T	F	F
T	F	F	T	T	F	T	F	T	T	T	T	F	T	F
F	T	T	F	T	T	T	T	T	F	T	F	T	T	T
F	T	F	F	T	T	F	F	F	F	F	T	F	T	T
F	F	T	F	F	F	T	T	T	F	F	F	T	F	F
F	F	F	F	F	F	T	F	F	F	F	T	F	T	F
1	**1**	**1**	**1**	**2**	**1**	**7**	**1**	**3**	**1**	**6**	**4**	**1**	**5**	**1**

Step 7 is the final step and it gives the truth values of the given formula

Tautology

*A wff is said to be a **tautology** (a **universally valid formula** or **logical truth**) if its truth value is T for all possible assignments of truth values to the proposition variables of the formula.*

Note:

(i) If p is any proposition then the formula $p \vee \sim p$ is a tautology
(ii) The conjunction of two tautologies is also a tautology

The proof of the above result (ii) is given below:

Let a and b be formulas which are tautologies. For all possible assignments of truth values to the variables of a and b, the truth values of both a and b will be T and hence the truth value of $a \wedge b$ will be T. Thus $a \wedge b$ is a tautology.

Contradiction

A wff is said to be a ***contradiction*** *(or* ***absurdity****) if its truth value is F for all possible assignments of truth values of the proposition variables of the formula.*

Note:

(i) If p is any proposition then the formula $p \wedge \sim p$ is a contradiction
(ii) The negation of a contradiction is a tautology

Contingency

A wff is said to be ***satisfiable*** *or* ***contingency*** *if it is neither a tautology nor a contradiction.*

Ex: If p and q are propositions then $p \wedge q$ and $p \vee q$ are satisfiable (contingency).

Example 2:

Show that the formula $\big(p \longrightarrow (q \longrightarrow r)\big) \longrightarrow \big((p \longrightarrow q) \longrightarrow (p \longrightarrow r)\big)$
is a tautology.

Solution:

Method-1

Let α be the formula $\big(p \longrightarrow (q \longrightarrow r)\big)$ and β be the formula $(p \longrightarrow q) \longrightarrow (p \longrightarrow r)$. Then the given formula is $\alpha \longrightarrow \beta$.

Truth table for $\alpha \longrightarrow \beta$								
p	q	r	$q \longrightarrow r$	α	$p \longrightarrow q$	$p \longrightarrow r$	β	$\alpha \longrightarrow \beta$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

It is a tautology since its truth value is T for all possible assignments of truth values to the proposition variables of the formula.

Method- 2

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$(\boldsymbol{p}$	$\longrightarrow$	$(\boldsymbol{q}$	$\longrightarrow$	$\boldsymbol{r}))$	$\longrightarrow$	$((\boldsymbol{p}$	$\longrightarrow$	$\boldsymbol{q})$	$\longrightarrow$	$(\boldsymbol{p}$	$\longrightarrow$	$\boldsymbol{r}))$
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	T	F	T	F	F
T	F	T	T	T	F	T	T	T	T	F	F	T	T	T	T
T	F	F	T	T	F	T	T	T	T	F	F	T	T	F	F
F	T	T	F	T	T	T	T	T	F	T	T	T	F	T	T
F	T	F	F	T	T	F	T	T	F	T	T	T	F	T	F
F	F	T	F	T	F	T	T	T	F	T	F	T	F	T	T
F	F	F	F	T	F	T	T	T	F	T	F	T	F	T	F
1	**1**	**1**	**1**	**3**	**1**	**2**	**1**	**7**	**1**	**4**	**1**	**6**	**1**	**5**	**1**

Step 7 is the final step and it has truth values of T's only. Hence the given formula is a ***tautology***.

Example 3:

Show that the formula $\sim(p \vee q \vee \sim r) \wedge \big((r \longrightarrow p) \vee (r \longrightarrow q)\big)$ is a contradiction.

Solution:

We show this by constructing truth table.

p	q	r	$\sim$	$(p$	$\vee$	q	$\vee$	$\sim$	$r)$	$\wedge$	$((r$	$\longrightarrow$	$p)$	$\vee$	$(r$	$\longrightarrow$	$q))$
T	T	T	F	T	T	T	T	F	T	F	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	F	F	F	T	T	T	F	T	T
T	F	T	F	T	T	F	T	F	T	F	T	T	T	T	T	F	F
T	F	F	F	T	T	F	T	T	F	F	F	T	T	T	F	T	F
F	T	T	F	F	T	T	T	F	T	F	T	F	F	T	T	T	T
F	T	F	F	F	T	T	T	T	F	F	F	T	F	T	F	T	T
F	F	T	T	F	F	F	F	F	T	F	T	F	F	F	T	F	F
F	F	F	F	F	F	F	T	T	F	F	F	T	F	T	F	T	F
1	**1**	**1**	**5**	**1**	**3**	**1**	**4**	**2**	**1**	**9**	**1**	**6**	**1**	**8**	**1**	**7**	**1**

Step 9 is the final step and it has truth values of F only. Hence the given formula is a contradiction.

Example 4:

Is the formula $(p \wedge q) \vee r \longrightarrow p \wedge (q \vee r)$ a tautology, contradiction or contingency?

Solution:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$((\boldsymbol{p}$	$\wedge$	$\boldsymbol{q})$	$\vee$	$\boldsymbol{r})$	$\longrightarrow$	$(\boldsymbol{p}$	$\wedge$	$(\boldsymbol{q}$	$\vee$	$\boldsymbol{r}))$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	T	T	T	T	T	F
T	F	T	T	F	F	T	T	T	T	T	F	T	T
T	F	F	T	F	F	F	F	T	T	F	F	F	F
F	T	T	F	F	T	T	T	F	F	F	T	T	T
F	T	F	F	F	T	F	F	T	F	F	T	T	F
F	F	T	F	F	F	T	T	F	F	F	F	T	T
F	F	F	F	F	F	F	F	T	F	F	F	F	F
1	**1**	**1**	**1**	**2**	**1**	**3**	**1**	**6**	**1**	**5**	**1**	**4**	**1**

Step 6 is the final step and it has truth values of T and F. Hence the given formula is neither a ***tautology*** nor a ***contradiction***. Therefore, it is a ***contingency***.

Substitution instance

*Let a and b be formulas. We say that a is a **substitution instance** of b if a can be obtained from b by substituting formulas for the variables of b, with the condition that the same formula is substituted for the same variable each time it occurs.*

Example: Let b: $p \longrightarrow (q \ \wedge \ p)$. Then

a: $(r \leftrightarrow s) \longrightarrow \big(q \wedge \ (r \leftrightarrow s)\big)$is a substitution instance of a , since $r \leftrightarrow s$ is substituted for p in b.Note that $(r \leftrightarrow s) \longrightarrow (q \ \wedge p)$ is not a substitution instance of b.

Note:

1. In constructing substitution instances of a formula, substitutions are made for the atomic formula and never for the molecular formula:

Ex: $p \longrightarrow q$ is not a substitution instance of $p \longrightarrow {\sim}r$, since it is r which must be substituted and not ${\sim}r$.

2. **Any substitution instance of a tautology is a tautology**

It is known that $p \vee \sim p$ is a tautology. If we substitute any formula for p, the resulting formula will be a tautology.

Example 5:

Determine the formulas which are substitution instances of other formulas in the following list and give the substitutions.

(a) $\big(p \longrightarrow (q \longrightarrow p)\big)$
(b) $\Big(\Big(\big((p \longrightarrow q) \wedge (r \longrightarrow s)\big) \wedge (p \vee r)\Big) \longrightarrow (q \vee s)\Big)$
(c) $\Big(q \longrightarrow \big((p \longrightarrow p) \longrightarrow q\big)\Big)$
(d) $\big(p \longrightarrow \big(p \longrightarrow (q \longrightarrow p)\big) \longrightarrow p\big)$

(e) $\Big(\big(((r \longrightarrow s) \wedge (q \longrightarrow p)\big) \wedge (r \vee q)\Big) \longrightarrow (s \vee p)\Big)$

Solution:

(i) Substitute q for p and $(p \longrightarrow p)$ for q in (a), we get (c).Therefore ,(c) is the substitution instance of (a)

(ii) Substitute $\big(p \longrightarrow (q \longrightarrow p)\big)$for q in (a), we get (d). Therefore, (d) is the substitution instance of (a)

(iii) Substitute r, s, q and p for p, q, r and s respectively in (b) we get(e). Therefore, (e) is the substitution instance of (b).

P1:

Construct the truth table for $(\boldsymbol{p} \lor \boldsymbol{q}) \lor \sim\boldsymbol{p}$.

Solution:

Truth table for $(p \lor q) \lor \sim p$				
p	q	$p \lor q$	$\sim p$	$(p \lor q) \lor \sim p$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

P2:

Construct the truth table for $(\boldsymbol{p} \longrightarrow \boldsymbol{q}) \wedge (\boldsymbol{q} \longrightarrow \boldsymbol{p})$.

Solution:

Truth table for $(p \longrightarrow q) \wedge (q \longrightarrow p)$				
p	q	$p \longrightarrow q$	$q \longrightarrow p$	$(p \longrightarrow q) \wedge (q \longrightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

P3:

Show that the formula $\sim(\boldsymbol{p} \wedge \boldsymbol{q}) \leftrightarrow (\sim\boldsymbol{p} \vee \sim\boldsymbol{q})$ is a tautology.

Solution:

We show this by constructing its truth table

Method- 1

Truth table for $\alpha : \sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$							
p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$(\sim p \vee \sim q)$	α
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

It is a tautology since its truth value is T for all possible assignments of truth values to the proposition variables of the formula.

Method- 2

$\boldsymbol{p}$	$\boldsymbol{q}$	$\sim$	$(\boldsymbol{p}$	$\wedge$	$\boldsymbol{q})$	$\leftrightarrow$	$(\sim$	$\boldsymbol{p}$	$\vee$	$\sim$	$\boldsymbol{q})$
T	T	F	T	T	T	T	F	T	F	F	T
T	F	T	T	F	F	T	F	T	T	T	F
F	T	T	F	F	T	T	T	F	T	F	T
F	F	T	F	F	F	T	T	F	T	T	F
1	**1**	**3**	**1**	**2**	**1**	**7**	**4**	**1**	**6**	**5**	**1**

Step 7 is the final step and it has truth values of T only. Hence the given formula is a ***tautology*.**

$\therefore \sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$ is a tautology

P4:

If p, q and r are proposition variables then show that $(p \longrightarrow q) \vee (\sim p \longrightarrow r)$ is a tautology.

Solution:

We show this by constructing truth table.

Method-1

Truth Table for $(p \longrightarrow q) \vee (\sim p \longrightarrow r)$							
p	q	r	$a: p \longrightarrow q$	$\sim p$	r	$b: \sim p \longrightarrow r$	$a \vee b$
T	T	T	T	F	T	T	T
T	T	F	T	F	F	T	T
T	F	T	F	F	T	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	F	F	T

It is a tautology since its truth value is T for all possible assignments of truth values to the proposition variables of the formula.

Method-2

p	q	r	$(p$	$\longrightarrow$	$q)$	$\vee$	$(\sim$	p	$\longrightarrow$	$r)$
T	T	T	T	T	T	T	F	T	T	T
T	T	F	T	T	T	T	F	T	T	F
T	F	T	T	F	F	T	F	T	T	T
T	F	F	T	F	F	T	F	T	T	F
F	T	T	F	T	T	T	T	F	T	T
F	T	F	F	T	T	T	T	F	F	F
F	F	T	F	T	F	T	T	F	T	T
F	F	F	F	T	F	T	T	F	F	F
1	**1**	**1**	**1**	**2**	**1**	**5**	**3**	**1**	**4**	**1**

Step 5 is the final step and it has truth values of T only. Hence the given formula is a ***tautology***.

P5:

Show that the formula $(\sim q \wedge p) \wedge (p \longrightarrow q)$ is a contradiction.

Solution:

p	q	$(\sim$	q	$\wedge$	$p)$	$\wedge$	$(p$	$\longrightarrow$	$q)$
T	T	F	T	F	T	F	T	T	T
T	F	T	F	T	T	F	T	F	F
F	T	F	T	F	F	F	F	T	T
F	F	T	F	F	F	F	F	T	F
1	**1**	**2**	**1**	**3**	**1**	**5**	**1**	**4**	**1**

Step 5 is the final step and it has truth values F only. Hence the given formula is a Contradiction

P6:

Show that the formula for $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction.

Solution:

p	q	$(p$	$\wedge$	$q)$	$\wedge$	$\sim$	$(p$	$\vee$	$q)$
T	T	T	T	T	F	F	T	T	T
T	F	T	F	F	F	F	T	T	F
F	T	F	F	T	F	F	F	T	T
F	F	F	F	F	F	F	F	T	F
1	**1**	**1**	**2**	**1**	**5**	**4**	**1**	**3**	**1**

Step 5 is the final step and it has truth values F only. Hence the given formula is a Contradiction

P7:

Show that the formula $\left(p \longrightarrow (q \longrightarrow r)\right) \longrightarrow \left((p \longrightarrow q) \longrightarrow r\right)$ is a contingency.

Solution:

p	q	r	$(p$	$\longrightarrow$	$(q$	$\longrightarrow$	$r))$	$\longrightarrow$	$((p$	$\longrightarrow$	$q)$	$\longrightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	T	F	F
T	F	T	T	T	F	T	T	T	T	F	F	T	T
T	F	F	T	T	F	T	F	T	T	F	F	T	F
F	T	T	F	T	T	T	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F	F	F	T	T	F	F
F	F	T	F	T	F	T	T	T	F	T	F	T	T
F	F	F	F	T	F	T	F	F	F	T	F	F	F
1	**1**	**1**	**1**	**3**	**1**	**2**	**1**	**6**	**1**	**4**	**1**	**5**	**1**

Step 6 is the final step and it has truth values T and F. Thus it is neither a Tautology nor a contradiction. Therefore it is a contingency.

P8:

Show that the formula $(\boldsymbol{p} \longrightarrow (\boldsymbol{q} \vee \boldsymbol{r})) \longrightarrow ((\boldsymbol{p} \wedge \boldsymbol{q}) \longrightarrow \boldsymbol{r})$ is a contingency.

Solution:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$(\boldsymbol{p}$	$\longrightarrow$	$(\boldsymbol{q}$	$\vee$	$\boldsymbol{r}))$	$\longrightarrow$	$((\boldsymbol{p}$	$\wedge$	$\boldsymbol{q})$	$\longrightarrow$	$\boldsymbol{r})$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	F	T	T	T	F	F
T	F	T	T	T	F	T	T	T	T	F	F	T	T
T	F	F	T	F	F	F	F	T	T	F	F	T	F
F	T	T	F	T	T	T	T	T	F	F	T	T	T
F	T	F	F	T	T	T	F	T	F	F	T	T	F
F	F	T	F	T	F	T	T	T	F	F	F	T	T
F	F	F	F	T	F	F	F	T	F	F	F	T	F
1	**1**	**1**	**1**	**3**	**1**	**2**	**1**	**6**	**1**	**4**	**1**	**5**	**1**

Step 6 is the final step and it has truth values T and F. Thus it is neither a Tautology nor a contradiction. Therefore it is a contingency.

1.2. Well-Formed Formulas

Exercises:

Which of the following formulas is a tautology, contradiction, contingency?

a. $\sim\big((p \longrightarrow q) \longrightarrow ((r \vee p) \longrightarrow (r \vee q))\big)$

b. $(q \wedge (p \longrightarrow q)) \longrightarrow p$

c. $(p \vee q) \longrightarrow ((p \vee r) \vee (r \vee q))$

d. $\sim(p \vee (q \wedge r)) \longleftrightarrow ((p \vee q) \wedge (p \vee r))$

e. $(p \vee (q \longrightarrow r)) \longleftrightarrow ((p \vee \sim r) \longrightarrow q)$

f. $p \longrightarrow \Big(q \longrightarrow \big(r \longrightarrow (\sim p \longrightarrow (\sim q \longrightarrow \sim r))\big)\Big)$

g. $(p \longrightarrow \sim p) \longrightarrow \sim p$

h. $\big((p \wedge (p \longrightarrow \sim q)) \vee (q \longrightarrow \sim q)\big) \longrightarrow \sim q$

i. $(p \longrightarrow q) \longrightarrow ((p \vee r) \longrightarrow (q \vee r))$

j. $((p \vee q) \wedge \sim r) \longrightarrow \sim p \vee r$

k. $(\sim(p \vee q \vee \sim r)) \wedge ((r \longrightarrow p) \vee (r \longrightarrow q))$

1.3

Logical Equivalence and the Laws of Logic

In this module we define the logical equivalence of wffs, state the principle of duality and prove the laws of logic.

Logical Equivalence

Let a and b be two wffs and let $p_1, p_2, \dots, p_n$ be all the propositional variables occurring in a and b.

We say that a and b are ***equivalent*** (or ***logically equivalent***) if the truth values of a and b are equal for each of the 2^n possible combinations of truth values assigned to $p_1, p_2, \dots, p_n$.

Theorem 1:

a and b are equivalent if and only if a ⟷ b is a tautology.

Proof: Suppose a and b are equivalent. That is, a and b have the same truth value for each of the 2^n possible combinations of truth values assigned to proposition variables $p_1, p_2, \dots, p_n$ and therefore, $a \longleftrightarrow b$ is T for each of the 2^n combinations of truth values. Thus, $a \longleftrightarrow b$ is a tautology. Conversely, if $a \longleftrightarrow b$ is a tautology, then a and b have the same truth value for each of the 2^n combinations of truth values assigned to its n propositional variables. Thus a and b are equivalent.

Representation: We represent the equivalence of a and b by $a \Leftrightarrow b$ or $a \equiv b$,read as a is *equivalent* to b.

Note:

1. The symbol ⟺ (or ≡) is not a connective in the object language but a symbol in the meta language and it is a relation.
2. Note that $a \equiv b$ is same as $b \equiv a$ ($i.e.$, a and b have the same truth values for each assignment of the truth values for the proposition variables). Thus, equivalence of wffs is a *Symmetric relation*. Further, if $a \equiv b$ and $b \equiv c$ then $a \equiv c$. That is, the equivalence of wffs is a *Transitive relation.*

Duality Law (Principle of Duality)

We shall consider wffs which contain the connectives ∧,∨ and ∼ . (There is no loss of generality in restricting our consideration to these three connectives since any formula containing any other connectives can be replaced by an equivalent formula containing only these three connectives). We introduce two special variables $\boldsymbol{T_0}$ and $\boldsymbol{F_0}$ denoting a tautology and contradiction respectively.

Dual of a formula

Let a be a formula containing logical connectives ∧, ∨, ∼ and special variables T_0 and F_0. *The **dual** of a, denoted by a^*, is the formula obtained from a by replacing each occurrence of ∧ and ∨ by ∨ and ∧ respectively and each occurrence of T_0 and F_0 by F_0 and T_0 respectively*.

The connectives ∧ and ∨ are also called *duals* of each other.

Example: The duals of $p \vee \sim p$, $p \vee T_0$ and $(p \vee \sim q) \wedge (r \vee F_0)$ are $p \wedge \sim p$, $p \wedge F_0$ and $(p \wedge \sim q) \vee (r \wedge T_0)$ respectively.

Note: $(a^*)^* = a$

The following is an interesting theorem which states that if any two formulas (containing ∧, ∨, ∼ , T_0 and F_0) are equivalent then their duals are also equivalent to each other.

Theorem 2: Principle of Duality

Let a and b be formulas (containing ∧,∨,∼,T_0 and F_0). If $a \equiv b$ then $a^* \equiv b^*$.

Basic equivalent formulas:

Theorem 3:

If p and q are proposition variables then the following properties hold:

- *Idempotent properties*

  $p \vee p \equiv p$; $p \wedge p \equiv p$
- *Commutative properties*

  $p \vee q \equiv q \vee p$; $p \wedge q \equiv q \wedge p$

- *Absorption properties*

$$p \wedge (p \vee q) \equiv p \quad ; \qquad p \vee (p \wedge q) \equiv p$$

Proof: We prove $p \vee p \equiv p$, $p \vee q \equiv q \vee p$ and $p \vee (p \wedge q) \equiv p$ through truth tables.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p}$	$\boldsymbol{p \vee p}$	$\boldsymbol{p \vee q}$	$\boldsymbol{q \vee p}$	$\boldsymbol{p \wedge (p \vee q)}$
T	T	T	T	T	T	T
T	F	T	T	T	T	T
F	T	F	F	T	T	F
F	F	F	F	F	F	F

Note that the truth values of $p \vee p$ and p are identical. Therefore $p \vee p \equiv p$.By the same reason $p \vee q \equiv q \vee p$ and $p \wedge (p \vee q) \equiv p$. The other equivalences follow by the principle of duality.

Theorem 4: Negation properties:

If p and q are proposition variables, then the following properties hold:

- *Double negation property*

  $$\sim(\sim p) \equiv p$$

- *De Morgan's properties*

  $$\sim(p \vee q) \equiv \sim p \wedge \sim q \quad ; \qquad \sim(p \wedge q) \equiv \sim p \vee \sim q$$

proof: We prove $\sim(\sim p) \equiv p$ and $\sim(p \vee q) \equiv \sim p \wedge \sim q$ through truth tables.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \vee q}$	$\boldsymbol{\sim(p \vee q)}$	$\boldsymbol{\sim p}$	$\boldsymbol{\sim q}$	$\boldsymbol{\sim p \wedge \sim q}$	$\boldsymbol{\sim(\sim p)}$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	F
F	F	F	T	T	T	T	F

Note that the truth values of $\sim(\sim p)$ and p are identical. Therefore $\sim(\sim p) \equiv p$. By the same reason$\sim(p \vee q) \equiv \sim p \wedge \sim q$.The other De Morgan's law follows by the principle of duality.

Theorem 5:

If p, q and r are proposition variables, then the following properties hold:

- *Associative properties*

  $p \vee (q \vee r) \equiv (p \vee q) \vee r \quad ; \quad p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
- *Distributive properties*

  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \quad ; \quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Proof: We prove $p \vee (q \vee r) \equiv (p \vee q) \vee r$ and $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ through truth tables.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$(\boldsymbol{p}$	$\vee$	$(\boldsymbol{q}$	$\vee$	$\boldsymbol{r}))$	$((\boldsymbol{p}$	$\vee$	$\boldsymbol{q})$	$\vee$	$\boldsymbol{r})$
T	T	T		T		T			T		T	
T	T	F		T		T			T		T	
T	F	T		T		T			T		T	
T	F	F		T		F			T		T	
F	T	T		T		T			T		T	
F	T	F		T		T			T		T	
F	F	T		T		T			F		T	
F	F	F		F		F			F		F	
1	**1**	**1**	**1**	**3**	**1**	**2**	**1**	**1**	**4**	**1**	**5**	**1**

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$(\boldsymbol{p}$	$\wedge$	$(\boldsymbol{q}$	$\vee$	$\boldsymbol{r}))$	$((\boldsymbol{p}$	$\wedge$	$\boldsymbol{q})$	$\vee$	$(\boldsymbol{p}$	$\wedge$	$\boldsymbol{r}))$
T	T	T		T		T			T		T		T	
T	T	F		T		T			T		T		F	
T	F	T		T		T			F		T		T	
T	F	F		F		F			F		F		F	
F	T	T		F		T			F		F		F	
F	T	F		F		T			F		F		F	
F	F	T		F		T			F		F		F	
F	F	F		F		F			F		F		F	
1	**1**	**1**		**3**		**2**			**4**		**6**		**5**	

Note that the truth values of $p \vee (q \vee r)$ and $(p \vee q) \vee r$ are identical. Therefore $p \vee (q \vee r) \equiv (p \vee q) \vee r$. By the same reason $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$. The other equivalences follow by the principle of duality.

Theorem 6:

If p is a proposition variable, T_0 and F_0 are special variables denoting a tautology and contradiction respectively, then the following properties hold:

- *Identity properties*

  $p \wedge T_0 \equiv p$; $p \vee F_0 \equiv p$
- *Domination properties*

  $p \vee T_0 \equiv T_0$; $p \wedge F_0 \equiv F_0$
- *Inverse properties*

  $p \vee \sim p \equiv T_0$; $p \wedge \sim p \equiv F_0$

Proof: We prove $p \wedge T_0 \equiv p$, $p \vee T_0 \equiv T_0$ and $p \vee \sim p \equiv T_0$ through the truth tables.

$\boldsymbol{p}$	$\boldsymbol{T_0}$	$\boldsymbol{F_0}$		$(\boldsymbol{p}$	$\wedge$	$\boldsymbol{T_0})$		$(\boldsymbol{p}$	$\vee$	$\boldsymbol{T_0})$		$(\boldsymbol{p}$	$\vee$	$\sim\boldsymbol{p})$
T	T	F			T				T				T	F
F	T	F			F				T				T	T
1					**2**				**3**				**4**	

Note that the truth values of $p \wedge T_0$ and p are identical. Therefore $p \wedge T_0 \equiv p$. By the same reason $p \vee T_0 \equiv T_0$ and $p \vee \sim p \equiv T_0$. The other equivalences follow by the principle of duality.

The equivalences so far given are the properties of the operators $\wedge$,$\vee$ and $\sim$ on the set of propositions in symbolic logic. The set of all propositions under the operations $\wedge$,$\vee$ and $\sim$ is an algebra called *algebra of propositions* which is a particular example of a *Boolean algebra*. The following is the list of laws for the algebra of propositions:

The laws of Logic (Logical Equivalences):

If p, q, r are any proposition variables, T_0 is any tautology and F_0 is any contradiction, then the following laws of logic are valid.

Equivalence	Name
$p \vee p \equiv p$; $p \wedge p \equiv p$	Idempotent laws
$p \vee q \equiv q \vee p$; $p \wedge q \equiv q \wedge p$	Commutative laws
$p \wedge (p \vee q) \equiv p$; $p \vee (p \wedge q) \equiv p$	Absorption laws
$\sim(\sim p) \equiv p$	Double negation law
$\sim(p \vee q) \equiv \sim p \wedge \sim q$; $\sim(p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's laws
$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associative laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
$p \wedge T_0 \equiv p$; $p \vee F_0 \equiv p$	Identity laws
$p \vee T_0 \equiv T_0$; $p \wedge F_0 \equiv F_0$	Domination laws
$p \vee \sim p \equiv T_0$; $p \wedge \sim p \equiv F_0$	Inverse laws

Note the logical equivalences in the above table, except the double negation law, are in pairs where each pair of formulas are duals of each other.

Replacement Process:

It is a process in which we replace any part of the *wff* which is itself a formula (be it *atomic* or *molecular*) by any other formula. In general, a replacement yields a new formula.

Replacement rule: Let a be a formula, b be a formula that appears in a, and let c be a formula such that $c \equiv b$. Let a_1 be the formula obtained by replacing one or more occurrences of b by c in a. Then $a_1 \equiv a$.
Further if a is a tautology then a_1 is also a tautology. That is, if we replace any part or parts of a tautology by formulas that are equivalent to these parts we again get a tautology.

Example:

I. Let a be the formula $(p \longrightarrow q) \longrightarrow r$. We have $p \longrightarrow q \equiv \sim p \vee q$. Let a_1 be the formula obtained by replacing $p \longrightarrow q$ by its equivalent formula $\sim p \vee q$
$i.e.,$ a_1: $(\sim p \vee q) \longrightarrow r$. Then $a \equiv a_1$,

$$i.e., (p \longrightarrow q) \longrightarrow r \equiv (\sim p \vee q) \longrightarrow r$$

II. Let b be the formula $p \longrightarrow (p \vee q)$.Since $p \equiv \sim(\sim p)$, the formula $b_1 : p \longrightarrow (\sim(\sim p) \vee q)$ is derived from b by replacing the second occurrence (but not the first occurrence) of p by $\sim(\sim p)$. By replacement rule $b \equiv b_1$

Equivalent formulas for conditional and Biconditional propositions :

Theorem 7: Prove that

(a) $\boldsymbol{p \longrightarrow q \equiv \sim p \vee q}$
(b) $\boldsymbol{p \longleftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p)}$
(c) $\boldsymbol{p \longrightarrow q \equiv \sim q \longrightarrow \sim p}$

Solution: We prove (a) and (b) by constructing truth tables

p	q		p	$\longrightarrow$	q		$(\sim p$	$\vee$	$q)$		p	$\longleftrightarrow$	q		$(p$	$\longrightarrow$	$q)$	$\wedge$	$(q$	$\longrightarrow$	$p)$
T	T			T			F	T				T				T		T		T	
T	F			F			F	F				F				F		F		T	
F	T			T			T	T				F				T		F		F	
F	F			T			T	T				T				T		T		T	
1	**1**			**2**			**3**	**4**				**5**				**6**		**8**		**7**	

(c) We have $p \longrightarrow q \equiv \sim p \vee q$

Now, $\sim q \longrightarrow \sim p \equiv \sim(\sim q) \vee \sim p$

$\equiv q \vee \sim p$ (Double negation property, Replacement rule)

$\equiv \sim p \vee q$ (Commutative law for disjunction)

$\equiv p \longrightarrow q$

Therefore $p \longrightarrow q \equiv \sim q \longrightarrow \sim p$ (Symmetry of equivalence)

Note: A conditional proposition $p \longrightarrow q$ and its contrapositive $\sim q \longrightarrow \sim p$ are logically equivalent.

Example 1: Show that $(\boldsymbol{p} \longrightarrow \boldsymbol{q}) \wedge \left(\sim\boldsymbol{q} \wedge (\boldsymbol{r} \vee \sim\boldsymbol{q})\right) \equiv \sim(\boldsymbol{p} \vee \boldsymbol{q})$.

Solution: $(p \longrightarrow q) \wedge \left(\sim q \wedge (r \vee \sim q)\right)$

$\equiv (p \longrightarrow q) \wedge \left(\sim q \wedge (\sim q \vee r)\right)$ (Commutative law for $\vee$)

$\equiv (p \longrightarrow q) \wedge \sim q$ (Absorption law)

$\equiv (\sim p \vee q) \wedge \sim q$ $(\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$

$\equiv (\sim p \wedge \sim q) \vee (q \wedge \sim q)$ (Distributive law)

$\equiv \left(\sim(p \vee q)\right) \vee F_0$ (De Morgan's law and $q \wedge \sim q$ is a contradiction)

$\equiv \sim(p \vee q)$ (Identity law)

Example 2: Show that

$\left((\boldsymbol{p} \vee \boldsymbol{q}) \wedge \sim\left(\sim\boldsymbol{p} \wedge (\sim\boldsymbol{q} \vee \sim\boldsymbol{r})\right)\right) \vee (\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}) \vee (\sim\boldsymbol{p} \wedge \sim\boldsymbol{r})$ is a tautology.

Solution:

$(\sim p \wedge \sim q) \vee (\sim p \wedge \sim r) \equiv \sim(p \vee q) \vee \sim(p \vee r)$ (De Morgan's law)

$\equiv \sim\left((p \vee q) \wedge (p \vee r)\right)$ (De Morgan's law)

$\equiv \sim\left(p \vee (q \wedge r)\right)$ (Distributive law)

and

$(p \vee q) \wedge \sim\left(\sim p \wedge (\sim q \vee \sim r)\right)$

$\equiv (p \vee q) \wedge \sim\left(\sim p \wedge \sim(q \wedge r)\right)$ (De Morgan's law)

$\equiv (p \vee q) \wedge \sim\left(\sim\left(p \vee (q \wedge r)\right)\right)$ (De Morgan's law)

$\equiv (p \vee q) \wedge \left(\left(p \vee (q \wedge r)\right)\right)$ (Double negation law)

$\equiv (p \vee q) \wedge \left((p \vee q) \wedge (p \vee r)\right)$ (Distributive law)

$\equiv \left((p \vee q) \wedge (p \vee q)\right) \wedge (p \vee r)$ (Associative law)

$\equiv (p \vee q) \wedge (p \vee r) \equiv p \vee (q \wedge r)$ (Idempotent law)

Now, the given formula is equivalent to $p \vee (q \wedge r) \vee \sim\big(p \vee (q \wedge r)\big) \equiv T_0$
Since it is a substitution instance of a tautology $p \vee \sim p$ (by substituting p by $p \vee (q \wedge r)$).

Example 3: Show that

(i) $\boldsymbol{\sim(p \wedge q) \longrightarrow \big(\sim p \vee (\sim p \vee q)\big) \equiv \sim p \vee q}$

(ii) $\boldsymbol{(p \vee q) \wedge \big(\sim p \wedge (\sim p \wedge q)\big) \equiv \sim p \wedge q}$

Solution: $\sim(p \wedge q) \longrightarrow \big(\sim p \vee (\sim p \vee q)\big)$

$\equiv \sim\big(\sim(p \wedge q)\big) \vee \big(\sim p \vee (\sim p \vee q)\big)$ $\quad (\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$

$\equiv (p \wedge q) \vee \big((\sim p \vee \sim p) \vee q\big)$ (Double negation law and Associative law for $\vee$)

$\equiv (p \wedge q) \vee (\sim p \vee q)$ (Idempotent law)

$\equiv \big((p \wedge q) \vee \sim p\big) \vee q$ (Associative law for $\vee$)

$\equiv \big((p \vee \sim p) \wedge (q \vee \sim p)\big) \vee q$ (Distributive law)

$\equiv \big(T_0 \wedge (q \vee \sim p)\big) \vee q$ (Inverse law)

$\equiv (q \vee \sim p) \vee q$ (Identity law)

$\equiv (\sim p \vee q) \vee q$ (Commutative law for $\vee$)

$\equiv \sim p \vee (q \vee q)$ (Associative law for $\vee$)

$\equiv \sim p \vee q$ (Idempotent law)

This proves (i). Note that

$\sim(p \wedge q) \longrightarrow \big(\sim p \vee (\sim p \vee q)\big) \equiv (p \wedge q) \vee \big(\sim p \vee (\sim p \vee q)\big) \equiv \sim p \vee q$

Therefore $(p \wedge q) \vee \big(\sim p \vee (\sim p \vee q)\big) \equiv \sim p \vee q$

By the principle of duality

$$(p \vee q) \wedge \big(\sim p \wedge (\sim p \wedge q)\big) \equiv \sim p \wedge q$$

P1:

Show that $\sim(\boldsymbol{p} \vee (\sim\boldsymbol{p} \wedge \boldsymbol{q}))$ and $\sim\boldsymbol{p} \wedge \sim\boldsymbol{q}$ are logically equivalent.

Solution:

We prove this by using logical equivalences.

$$
\begin{aligned}
\sim(p \vee (\sim p \wedge q)) &\equiv \sim((p \vee \sim p) \wedge (p \vee q)) && \text{(Distributive law)} \\
&\equiv \sim(T_0 \wedge (p \vee q)) && (\because\ p \vee \sim p \text{ is a tautology}) \\
&\equiv \sim(p \vee q) && \text{(Identity law)} \\
&\equiv \sim p \wedge \sim q && \text{(De Morgan's law)}
\end{aligned}
$$

P2:

Show that $(\boldsymbol{p} \wedge \boldsymbol{q}) \longrightarrow (\boldsymbol{p} \vee \boldsymbol{q})$ is a tautology.

Solution:

We show this by using logical equivalences.

$(p \wedge q) \longrightarrow (p \vee q)$	$\equiv {\sim}(p \wedge q) \vee (p \vee q)$	$(\alpha \longrightarrow \beta \equiv {\sim}\alpha \vee \beta)$
	$\equiv ({\sim}p \vee {\sim}q) \vee (p \vee q)$	(De Morgans' law)
	$\equiv \big(({\sim}p \vee {\sim}q) \vee p\big) \vee q$	(Associative law for $\vee$)
	$\equiv \big({\sim}p \vee ({\sim}q \vee p)\big) \vee q$	(Associative law for $\vee$)
	$\equiv \big({\sim}p \vee (p \vee {\sim}q)\big) \vee q$	(Commutativity for $\vee$)
	$\equiv \big(({\sim}p \vee p) \vee {\sim}q\big) \vee q$	(Associative law for $\vee$)
	$\equiv ({\sim}p \vee p) \vee ({\sim}q \vee q)$	(Associative law for $\vee$)
	$\equiv T_0 \vee T_0 = T_0$	

P3:

Show that $\sim\left(\sim\left((\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r}\right) \vee \sim\boldsymbol{q}\right)$ and $\boldsymbol{q} \wedge \boldsymbol{r}$ are logical equivalent.

Solution:

We prove this by using logical equivalences.

$$
\begin{aligned}
\sim\left(\sim\left((p \vee q) \wedge r\right) \vee \sim q\right) &\equiv \sim\left(\sim\left((p \vee q) \wedge r\right)\right) \wedge \sim(\sim q) && \text{(De Morgan's law)} \\
&\equiv \left((p \vee q) \wedge r\right) \wedge q && \text{(Double negation law)} \\
&\equiv q \wedge \left((p \vee q) \wedge r\right) && \text{(Commutative law for } \wedge\text{)} \\
&\equiv \left(q \wedge (p \vee q)\right) \wedge r && \text{(Associative law for } \wedge\text{)} \\
&\equiv \left(q \wedge (q \vee p)\right) \wedge r && \text{(Commutative law for } \vee\text{)} \\
&\equiv q \wedge r && \text{(Absorption law)}
\end{aligned}
$$

P4:

Prove the following logical equivalence

$$\boldsymbol{p} \wedge \Big((\sim \boldsymbol{q} \longrightarrow (\boldsymbol{r} \wedge \boldsymbol{r})) \vee \sim \big(\boldsymbol{q} \vee ((\boldsymbol{r} \wedge \boldsymbol{s}) \vee (\boldsymbol{r} \wedge \sim \boldsymbol{s})) \big) \Big) \Leftrightarrow \boldsymbol{p}$$

Solution:

$$p \wedge \Big((\sim q \longrightarrow (r \wedge r)) \vee \sim \big(q \vee ((r \wedge s) \vee (r \wedge \sim s)) \big) \Big)$$

$$\equiv p \wedge \Big((\sim q \longrightarrow r) \vee \sim \big(q \vee \big(r \wedge (s \vee \sim s) \big) \big) \Big)$$

(Idempotent law and Distributive law)

$$\equiv p \wedge \big((\sim(\sim q) \vee r) \vee \big(q \vee (r \wedge T_0) \big) \big)$$

($\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta$, inverse law)

$$\equiv p \wedge \big((q \vee r) \vee \sim(q \vee r) \big)$$ (Double negation law, identity law)

$$\equiv p \wedge T_0$$ (Substitution instance of a tautology)

$$\equiv p$$ (Identity law)

P5:

Show that $\left(\sim\boldsymbol{p} \wedge (\sim\boldsymbol{q} \wedge \boldsymbol{r})\right) \vee (\boldsymbol{q} \wedge \boldsymbol{r}) \vee (\boldsymbol{p} \wedge \boldsymbol{r}) \Leftrightarrow \boldsymbol{r}$

Solution:

$\left(\sim p \wedge (\sim q \wedge r)\right) \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow \left(\sim p \wedge (\sim q \wedge r)\right) \vee \left((q \vee p) \wedge r\right)$ (Distributive law)

$\Leftrightarrow \left((\sim p \wedge \sim q) \wedge r\right) \vee \left((q \vee p) \wedge r\right)$ (Associative law for $\wedge$)

$\Leftrightarrow \left((\sim p \wedge \sim q) \vee (q \vee p)\right) \wedge r$ (Distributive law)

$\Leftrightarrow \left(\sim(p \vee q) \vee (p \vee q)\right) \wedge r$ (De Morgan's law and commutativity of $\vee$)

$\Leftrightarrow T_0 \wedge r$ (Substitution instance of a tautology and commutativity of $\vee$)

$\Leftrightarrow r$ (Identity law)

P6:

Prove the following without using truth tables

(a) $\sim(\boldsymbol{p} \longrightarrow \boldsymbol{q}) \equiv \boldsymbol{p} \wedge \sim\boldsymbol{q}$

(b) $\sim(\boldsymbol{p} \leftrightarrow \boldsymbol{q}) \equiv (\boldsymbol{p} \wedge \sim\boldsymbol{q}) \vee (\sim\boldsymbol{p} \wedge \boldsymbol{q})$

Solution:

(a) $\sim(p \longrightarrow q) \equiv \sim(\sim p \vee q)$ $\quad (\because p \longrightarrow q \equiv \sim p \vee q)$

$\equiv \big(\sim(\sim p)\big) \wedge \sim q$ (De Morgan's law)

$\equiv p \wedge \sim q$ (Double negation law)

(b) $\sim(p \leftrightarrow q) \equiv \sim\big((p \longrightarrow q) \wedge (q \longrightarrow p)\big)$

$\equiv \sim\big((\sim p \vee q) \wedge (\sim q \vee p)\big)$

$\equiv \big(\sim(\sim p \vee q)\big) \vee \big(\sim(\sim q \vee p)\big)$ (De Morgan's law)

$\equiv \big((\sim(\sim p) \wedge \sim q)\big) \vee \big((\sim(\sim q) \wedge \sim p)\big)$ (De Morgan's law)

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$ (Double negation law)

$\equiv (p \wedge \sim q) \vee (\sim p \wedge q)$ (Commutative law for conjunction)

P7:

Show that

(a) $(\boldsymbol{p} \wedge \boldsymbol{q}) \longrightarrow \boldsymbol{r} \equiv \boldsymbol{p} \longrightarrow (\boldsymbol{q} \longrightarrow \boldsymbol{r})$
(b) $(\boldsymbol{p} \vee \boldsymbol{q}) \longrightarrow \boldsymbol{r} \equiv (\boldsymbol{p} \longrightarrow \boldsymbol{r}) \wedge (\boldsymbol{q} \longrightarrow \boldsymbol{r})$

Solution:

(a) $(p \wedge q) \longrightarrow r \equiv \big(\sim(p \wedge q)\big) \vee r$ $\quad(\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$
$\equiv (\sim p \vee \sim q) \vee r$ (De Morgan's law)
$\equiv \sim p \vee (\sim q \vee r)$ (Associative law for $\vee$)
$\equiv p \longrightarrow (\sim q \vee r)$ $\quad(\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$
$\equiv p \longrightarrow (q \longrightarrow r)$ $\quad(\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$

(b) $(p \vee q) \longrightarrow r \equiv \sim(p \vee q) \vee r$ $\quad(\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$
$\equiv (\sim p \wedge \sim q) \vee r$ (De Morgan's law)
$\equiv (\sim p \vee r) \wedge (\sim q \vee r)$ (Distributive law)
$\equiv (p \longrightarrow r) \wedge (q \longrightarrow r)$ $\quad(\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$

P8:

There are two restaurants next to each other. One has a sign that says, "*Good food is not cheap*", and the other has a sign that says, "*Cheap food is not good*". Are the signs saying the same thing?

Solution:

Let g and c respectively denote the propositions *The food is good* and *The food is cheap.*

The first sign *Good food is not cheap* can be written as $g \longrightarrow \sim c$.

The second sign *Cheap food is not good* can be written as $c \longrightarrow \sim g$.

Now, $g \longrightarrow \sim c \equiv \sim g \vee \sim c$ and $c \longrightarrow \sim g \equiv \sim c \vee \sim g$.

Then $g \longrightarrow \sim c \equiv \sim g \vee \sim c \equiv \sim c \vee g \equiv c \longrightarrow \sim g$.

$$\therefore g \longrightarrow \sim c \equiv c \longrightarrow \sim g$$

Thus, the two signs are equivalent. Therefore, they say the same thing.

1.3. Equivalence Formulas, Logical Equivalence and the Laws of Logic

Exercises:

I. Show the following equivalences:

1. $(p \vee q) \wedge \sim(\sim p \wedge q) \equiv p$
2. $p \vee q \vee \big(\sim p \wedge (\sim q \wedge r)\big) \Leftrightarrow p \vee q \vee r$
3. $\sim(p \vee q) \vee \big((\sim p \wedge q) \vee \sim q\big) \Leftrightarrow \sim(p \wedge q)$
4. $p \longrightarrow (q \vee r) \equiv (p \longrightarrow q) \vee (p \longrightarrow r)$
5. $\sim(p \leftrightarrow q) \equiv (p \vee q) \wedge \sim(p \wedge q)$
6. $\big((\sim p \vee \sim q) \longrightarrow (p \wedge q \wedge r)\big) \Leftrightarrow p \wedge q$
7. $\big((p \wedge q) \vee (p \wedge r)\big) \longrightarrow s \equiv \big(\sim p \vee (\sim q \wedge \sim r)\big) \vee s$

II. Show that the following are tautologies.

1. $(p \vee q) \longrightarrow (q \longrightarrow q)$
2. $p \longrightarrow \big(q \longrightarrow (p \wedge q)\big)$
3. $\big((p \vee q) \wedge (p \longrightarrow r) \wedge (q \longrightarrow r)\big) \longrightarrow r$

1.4. Normal Forms

Decision problem: The problem of determining, in a finite number of steps, whether a given formula is a tautology or a contradiction or at least satisfiable is known as a ***decision problem***.

Since the construction of truth tables involves a finite number of steps, a *decision problem* has a solution in the propositional calculus. The construction of truth tables may not be practical (even with the help of a computer). We therefore look for other methods known as reduction to normal forms.

We use the words "***product***" and "***sum***" in place of "*conjunction*" and "*disjunction*" respectively for convenience.

Elementary product and Elementary sum

A *product* of the variables and their negations is called an ***elementary product***.
A *sum* of the variables and their negations is called an ***elementary sum***.

If p and q are atomic variables then

- $p, \sim p$, $p \wedge \sim p$, $\sim p \wedge q$, $\sim p \wedge \sim q \wedge p \wedge \sim p$ are some examples of elementary products of two variables.
- $p, \sim p \vee q, \sim q \vee p \vee \sim p,\ p \vee \sim p,\ q \vee p \vee \sim q$ are some examples of elementary sums of two variables.

The following statements hold for elementary sums and products.

- A necessary and sufficient condition for an *elementary product* to be identically false (contradiction) is that it contains a variable and its negation.
- A necessary and sufficient condition for an *elementary sum* to be identically true (tautology) is that it contains a variable and its negation.

Disjunctive normal form of a formula

Let α be a formula. A formula which is equivalent to α and which consist of a sum of elementary products is called a ***disjunctive normal form*** (***DNF***) of α.

Consider the formula $p \vee (q \wedge r)$. Note that the formula is already in the disjunctive normal form. We see that

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \equiv \big((p \vee q) \wedge p\big) \vee \big((p \vee q) \wedge r\big)$$

$$\equiv (p \wedge p) \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r)$$

From this we say that $(p \wedge p) \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r)$ is also a disjunctive normal form of $p \vee (q \wedge r)$.Further,

$$p \wedge (q \vee r) \equiv (p \wedge p) \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r) \equiv p \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r)$$

Thus, $p \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r)$ is another disjunctive normal form of $p \vee (q \wedge r)$.

Example 1: Show that the formula $\sim\boldsymbol{q} \wedge (\boldsymbol{p} \longrightarrow \boldsymbol{q}) \wedge (\sim\boldsymbol{p} \longrightarrow \boldsymbol{q})$ is a contradiction.

Solution: We will first write a ***DNF*** of the formula

$$\sim q \wedge (p \longrightarrow q) \wedge (\sim p \longrightarrow q) \equiv \sim q \wedge (\sim p \vee q) \wedge (\sim(\sim p) \vee q)$$

$$\equiv \sim q \wedge (\sim p \vee q) \wedge (p \vee q) \equiv \sim q \wedge \big((\sim p \vee p) \wedge q\big)$$

$$\equiv \sim q \wedge (T_0 \wedge q) \equiv (\sim q \wedge q) \equiv F_0$$

Note:

(1) The disjunctive normal form of a given formula is not unique.
(2) Different disjunctive normal forms of a given formula are equivalent.
(3) To get a unique disjunctive normal form of a given formula, we will introduce the concept of *principle disjunctive normal form* (PDNF).
(4) A given formula is identically *false* (a contradiction) if every *elementary product* present in its disjunctive normal form is identically false (a contradiction). That is every *elementary product* has atleast a variable and its negation.

Example 2: Obtain disjunctive normal forms of

(a) $\boldsymbol{p} \wedge (\boldsymbol{p} \longrightarrow \boldsymbol{q})$

(b) $\sim(\boldsymbol{p} \vee \boldsymbol{q}) \longleftrightarrow (\boldsymbol{p} \wedge \boldsymbol{q})$.

Solution:

(a) $p \wedge (p \longrightarrow q) \equiv p \wedge (\sim p \vee q) \equiv (p \wedge \sim p) \vee (p \wedge q)$

A disjunctive normal form of $p \wedge (p \longrightarrow q)$ is $(p \wedge \sim p) \vee (p \wedge q)$

(b) $\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv \big(\sim(p \vee q) \wedge (p \wedge q)\big) \vee \Big(\sim\big(\sim(p \vee q)\big) \wedge \sim(p \wedge q)\Big)$

$$\big(\because r \leftrightarrow s \equiv (r \wedge s) \vee (\sim r \wedge \sim s)\big)$$

$$\equiv \big(\sim(p \vee q) \wedge (p \wedge q)\big) \vee \big((p \vee q) \wedge \sim(p \wedge q)\big)$$

$$\equiv (\sim p \wedge \sim q \wedge p \wedge q) \vee \big((p \vee q) \wedge (\sim p \vee \sim q)\big)$$

$$\equiv (\sim p \wedge \sim q \wedge p \wedge q) \vee \big((p \vee q) \wedge \sim p\big) \vee \big((p \vee q) \wedge \sim q\big)$$

$$\equiv (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)$$

Conjunctive normal form of a formula:

Let α be a formula. A formula which is equivalent to α and which consists of a *product* of *elementary sums* is called a ***conjuctive normal form*** (***CNF***) of α.

Consider the formula $p \wedge (q \vee r)$. Notice that it is the dual of $p \vee (q \wedge r)$. We have already seen that

$$p \vee (q \wedge r) \equiv (p \wedge p) \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r)$$

$$p \vee (q \wedge r) \equiv p \vee (q \wedge p) \vee (p \wedge r) \vee (q \wedge r)$$

Applying the principle of duality to the above two, we get

$$p \wedge (q \vee r) \equiv (p \vee p) \wedge (q \vee p) \wedge (p \vee r) \wedge (q \vee r)$$

$$p \wedge (q \vee r) \equiv p \wedge (q \vee p) \wedge (p \vee r) \wedge (q \vee r)$$

Thus we see that

$p \wedge (q \vee r)$,$(p \vee p) \wedge (q \vee p) \wedge (p \vee r) \wedge (q \vee r)$ and $p \wedge (q \vee p) \wedge (p \vee r) \wedge (q \vee r)$
are *CNF*s of $p \wedge (q \vee r)$.

Example 3: Show that the formula $\boldsymbol{q} \vee (\boldsymbol{p} \wedge \sim\boldsymbol{q}) \vee (\sim\boldsymbol{p} \wedge \sim\boldsymbol{q})$ is a tautology.

Solution: We will first write a CNF of the formula.

$$q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q) \equiv q \vee \big((p \vee \sim p) \wedge \sim q\big)$$

$$\equiv (q \vee p \vee \sim p) \wedge (q \vee \sim q) \quad (CNF)$$

Note that each elementary sum is a tautology. Therefore the given formula is a tautology.

Note:

(1) The conjunctive normal form of a given formula is not unique
(2) Different conjunctive normal forms of a given formula are equivalent
(3) To obtain a unique *CNF* of a given formula, we will introduce the concept of a *principal conjunctive normal form* (*PCNF*)
(4) A given formula is identically *true* (a tautology) if every *elementary sum* present in its *CNF* is identically *true* (tautology). That is every *elementary sum* has atleast a variable and its negation.

Example 4: Obtain conjunctive normal forms of

(a) $\boldsymbol{p} \wedge (\boldsymbol{p} \longrightarrow \boldsymbol{q})$

(b) $\sim(\boldsymbol{p} \vee \boldsymbol{q}) \leftrightarrow (\boldsymbol{p} \wedge \boldsymbol{q})$.

Solution:

(a) $p \wedge (p \longrightarrow q) \equiv p \wedge (\sim p \vee q)$

$p \wedge (\sim p \vee q)$ is a conjunctive normal form of $p \wedge (p \longrightarrow q)$.

(b) $\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv \big(\sim(p \vee q) \longrightarrow (p \wedge q)\big) \wedge \big((p \wedge q) \longrightarrow \sim(p \vee q)\big)$

$$\big(\because r \leftrightarrow s \equiv (r \longrightarrow s) \wedge (s \longrightarrow r)\big)$$

$$\equiv \Big(\sim\big(\sim(p \vee q)\big) \vee (p \wedge q)\Big) \wedge \big(\sim(p \wedge q) \vee \sim(p \vee q)\big)$$

$$\equiv \big((p \vee q) \vee (p \wedge q)\big) \wedge \big((\sim p \vee \sim q) \vee (\sim p \wedge \sim q)\big)$$

$$\equiv (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$$

which is a *CNF* of $\sim(p \vee q) \leftrightarrow (p \wedge q)$.

Principal Disjunctive Normal Forms:

Minterm: A ***minterm*** in n propositional variables $p_1, p_2, \ldots, p_n$ is the formula $\boldsymbol{q_1} \wedge \boldsymbol{q_2} \wedge \ldots \wedge \boldsymbol{q_n}$ where each $\boldsymbol{q_i}$ is either $\boldsymbol{p_i}$ or $\sim\boldsymbol{p_i}$.

Since each q_i has two choices p_i or $\sim p_i$, *the number of minterms in* n *propositional variables is* 2^n.

The minterms in three propositional variables p, q and r are $p \wedge q \wedge r,\ p \wedge q \wedge \sim r, p \wedge \sim q \wedge r,\ p \wedge \sim q \wedge \sim r, \sim p \wedge q \wedge r, \sim p \wedge q \wedge \sim r,$ $\sim p \wedge \sim q \wedge r, \sim p \wedge \sim q \wedge \sim r.$

The minterms in two propositional variables p and q are

$$p \wedge q\ , p \wedge \sim q\ , \sim p \wedge q\ , \sim p \wedge \sim q$$

The following is the truth table for the above minterms.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{p \wedge \sim q}$	$\boldsymbol{\sim p \wedge q}$	$\boldsymbol{\sim p \wedge \sim q}$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

Note:

(1) Each minterm has truth value T for exactly one combination of the truth values of the variables.
(2) Different minterms have truth value T for different combinations of truth values of the variables.

Principal disjunctive normal form (Sum-of-products canonical form):

Let a be a formula in n propositional variables. An equivalent formula of a consisting of disjunctions of minterms (in n variables) only is called the ***principal disjunctive normal form (PDNF)*** of a. Such a normal form is also called the ***sum -of -products canonical form***.

Note:

(1) Every formula which is not a contradiction has a unique principal disjunctive normal form.
(2) Two formulas are equivalent if and only if their principal disjunctive normal forms are identical.
(3) A formula is a tautology if and only if all minterms appear in its *PDNF*.

Finding *PDNF* through Truth table: If the truth table of any given formula is known then for every truth value T in the truth table of the formula, select the minterm which also has the truth value T for the same combination of the truth values of the variables .The disjunction of these selected minterms will be equivalent to the given formula and it is the P*DNF* of the formula.

Example 5: Obtain the *PDNF* of the formula $\boldsymbol{q \vee (p \longrightarrow q)}$ through its truth table.

Solution: The truth table of the given formula $q \vee (p \longrightarrow q)$ is given below.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \longrightarrow q}$	$\boldsymbol{q \vee (p \longrightarrow q)}$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

The formula is T for the combination of truth values TT, FT and FF.The minterms which are True for the same combination of truth values are respectively $p \wedge q, \sim p \wedge q$ and $\sim p \wedge \sim q$. Note that the disjunction of these minterms $i.e., (p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$ is equivalent to the given formula and it is the *PDNF* of the given formula.

Procedure to obtain the PDNF of a given formula without constructing the truth table

1. Replace conditionals and biconditionals by their equivalent formulas containing $\wedge$,$\vee$ and $\sim$.
2. Apply laws of negations to the variables using De Morgan's laws and double negation law.
3. Apply distributive laws.
4. Obtain the minterms in the disjunctions by introducing the missing variables.

Example 6: Obtain the *PDNF* of the following formulas

(a) $\boldsymbol{p} \wedge (\boldsymbol{p} \longrightarrow \boldsymbol{q})$

(b) $\sim(\boldsymbol{p} \vee \boldsymbol{q}) \leftrightarrow (\boldsymbol{p} \wedge \boldsymbol{q})$.

Solution:

(a) $p \wedge (p \longrightarrow q) \equiv p \wedge (\sim p \vee q) \equiv (p \wedge \sim p) \vee (p \wedge q)$

$$\equiv F_0 \vee (p \wedge q) \equiv (p \wedge q)$$

$\therefore (p \wedge q)$ is the *PDNF* of $p \wedge (p \longrightarrow q)$

(b) $\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv \big(\sim(p \vee q) \wedge (p \wedge q)\big) \vee \Big(\sim\big(\sim(p \vee q)\big) \wedge \sim(p \wedge q)\Big)$

$$\equiv (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)$$

(See Example 2)

$$\equiv F_0 \vee F_0 \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee F_0 \equiv (\sim p \wedge q) \vee (p \wedge \sim q)$$

$\therefore (\sim p \wedge q) \vee (p \wedge \sim q)$ is the *PDNF* of $\sim(p \vee q) \leftrightarrow (p \wedge q)$

Example 7: Show that $\boldsymbol{p} \vee (\sim \boldsymbol{p} \wedge \boldsymbol{q})$ and $\boldsymbol{p} \vee \boldsymbol{q}$ are equivalent by obtaining their *PDNF*s.

Solution: We show that the two formulas have identical *PDNF*s.

We have $p \equiv p \wedge T_0 \equiv p \wedge (q \vee \sim q) \equiv (p \wedge q) \vee (p \wedge \sim q)$

Now, $p \vee (\sim p \wedge q) \equiv (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$

$$\equiv (\sim p \wedge q) \vee (p \wedge \sim q) \vee (p \wedge q)$$

On the other hand, $q \equiv T_0 \wedge q \equiv \binom{p \vee}{\sim p} \wedge q \equiv (p \wedge q) \vee (\sim p \wedge q)$

$$p \vee q \equiv (p \wedge q) \vee (p \wedge \sim q) \vee (p \wedge q) \vee (\sim p \wedge q)$$

$$\equiv (\sim p \wedge q) \vee (p \wedge \sim q) \vee (p \wedge q)$$

Thus, $p \vee (\sim p \wedge q) \equiv p \vee q$, since they have identical PDNFs.

Note:

The above procedure becomes *tedious* if the given formula is complicated and contains more than three variables.

Principal Conjunctive Normal Forms

Maxterm: A ***maxterm*** in n propositional variables $p_1, p_2, \ldots p_n$ is the formula $\boldsymbol{q_1} \vee \boldsymbol{q_2} \vee \ldots \vee \boldsymbol{q_n}$ where each $\boldsymbol{q_i}$ is either $\boldsymbol{p_i}$ or $\sim \boldsymbol{p_i}$.

Note that the maxterms are the duals of minterms and *the number of maxterms in n variables is* 2^n.

By the principle of duality, each of the maxterms has the truth value F for exactly one combination of the truth values of the variables.

Different maxterms have truth value F for different combinations of truth values of the variables.

Principal conjunctive normal form (Product-of-Sums canonical form)

Let a be a formula in n propositional variables. An equivalent formula of a consisting of conjunction of maxterms (in n variables) only is called the ***principal conjunctive normal form (PCNF)*** of a. Such a normal form is also called the **product-of- Sums canonical form**.

Note:

1. Every formula which is not a tautology has a unique principal conjunctive normal form
2. Two formulas are equivalent if and only if their principal conjunctive normal forms are identical.
3. A formula is a contradiction if and only if all max terms appear in its PCNF.
4. All assertions made for the PDNFs are also valid for the PCNFs by the principle of duality.

Example 8: Obtain the *PCNF* of the following formulas

(a) $\boldsymbol{p} \wedge (\boldsymbol{p} \longrightarrow \boldsymbol{q})$

(b) $\sim(\boldsymbol{p} \vee \boldsymbol{q}) \leftrightarrow (\boldsymbol{p} \wedge \boldsymbol{q})$**.**

Solution:

(a) $p \wedge (p \longrightarrow q) \equiv p \wedge (\sim p \vee q)$

Now $p \equiv p \vee F_0 \equiv p \vee (q \vee \sim q) \equiv (p \vee q) \wedge (p \vee \sim q)$

$\therefore\ p \wedge (p \longrightarrow q) \equiv p \wedge (\sim p \vee q) \equiv (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$

$\therefore$ $(p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$ is the *PCNF* of $p \wedge (p \longrightarrow q)$

(b) $\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv \big(\sim(p \vee q) \longrightarrow (p \wedge q)\big) \wedge \big((p \wedge q) \longrightarrow \sim(p \vee q)\big)$

$\equiv \big((p \vee q \vee p) \wedge (p \vee q \vee q)\big) \wedge \big((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)\big)$

(See Example 4)

$\equiv \big((p \vee q) \wedge (p \vee q)\big) \wedge \big((\sim p \vee \sim q) \wedge (\sim p \vee \sim q)\big) \equiv (p \vee q) \wedge (\sim p \vee \sim q)$

$\therefore$ $(p \vee q) \wedge (\sim p \vee \sim q)$ *is the PCNF* of $\sim(p \vee q) \leftrightarrow (p \wedge q)$.

A notation for the representation of minterms and maxterms:

Suppose that n variables $p_1, p_2, \dots, p_n$ are given and are arranged in a particular order say $p_1, p_2, \dots, p_n$.

The 2^n minterms and 2^n maxterms corresponding to the n variables are respectively denoted by

$$\boldsymbol{m_0, m_1, m_2 \dots, m_{(2^n-1)}} \text{ and } \boldsymbol{M_0, M_1, M_2 \dots, M_{(2^n-1)}}$$

The following procedure is followed to write m_j and M_j.
Write the subscript j in binary form and add appropriate number of zeros on the left (if necessary) so that the number of digits in the binary representation of j is exactly n.

Representation of the minterm $\boldsymbol{m_j}$: If there is a $\mathbf{1}$ (0) in the i^{th} place, from left, in the binary representation of j then the $\boldsymbol{i}^{\text{th}}$ **variable $\boldsymbol{i.e.}$, $\boldsymbol{p_i}$** (the negation of the i^{th} variable, $.e.$, $\sim p_i$) appears in the conjunction of the minterm m_j.

Representation of the maxterm $\boldsymbol{M_j}$: If there is a **1** (0) in the i^{th} place, from left, in the binary representation of j the **negation of the $\boldsymbol{i}^{\text{th}}$ variable, $\boldsymbol{i.e.}$, $\boldsymbol{\sim p_i}$** (the i^{th} variable $i.e.$, p_i) appears in the disjunction of the maxterm M_j.

By this representation, each of $m_0, m_1, \dots, m_{(2^n-1)}$ ($M_0, M_1, \dots, M_{(2^n-1)}$) corresponds to a unique minterm (maxterm), which can be determined by its subscript. Conversely, given any minterm (maxterm) we can find which of $m_0, m_1, \dots, m_{(2^n-1)}$ ($M_0, M_1, \dots, M_{(2^n-1)}$) denotes it.

Minterms and maxterms in two variables:

If p and q are two variables arranged in that order, then the corresponding $2^2 = 4$ minterms and maxterms respectively are denoted by m_0, m_1, m_2, m_3 and M_0, M_1, M_2, M_3. The binary representation of the indices 0,1,2 and 3 are respectively $00, 01, 10$ and 11.

Following the above procedure we write the minterms and maxterms in two variables p, q. They are

$$m_0: \sim p \wedge \sim q\ , \quad m_1: \sim p \wedge q\ , \quad m_2: p \wedge \sim q \text{ and } \quad m_3: p \wedge q$$

$$M_0: p \vee q\ , \quad M_1: p \vee \sim q\ , \quad M_2: \sim p \vee q \text{ and } \quad M_3: \sim p \vee \sim q$$

Minterms and Maxterms in three variables:

If p, q and r are three variables arranged in that order, then the corresponding $2^3 = 8$ minterms and maxterms respectively denoted by

$$m_0, m_1, m_2, \dots, m_6, m_7 \text{ and } M_0, M_1, M_2, \dots, M_6, M_7$$

The binary representation of the indices 0,1,2,3,4,5,6 and 7 are respectively

$$000, 001, 010, 011, 100, 101, 110 \text{ and } 111$$

The corresponding minterms and maxterms are written as

$m_0: \sim p \wedge \sim q \wedge \sim r\ ,\ m_1: \sim p \wedge \sim q \wedge r\ ,\ m_2: \sim p \wedge q \wedge \sim r\ ,\ m_3: \sim p \wedge q \wedge r\ ,$

$m_4: p \wedge \sim q \wedge \sim r\ , \quad m_5: p \wedge \sim q \wedge r\ , \quad m_6: p \wedge q \wedge \sim r\ , m_7: p \wedge q \wedge r$

and

$M_0: p \vee q \vee r\ ,\ M_1: p \vee q \vee \sim r\ ,\ M_2: p \vee \sim q \vee r\ ,\ M_3: p \vee \sim q \vee \sim r\ ,$

$M_4: \sim p \vee q \vee r\ ,\ M_5: \sim p \vee q \vee \sim r\ , M_6: \sim p \vee \sim q \vee r\ ,\ M_7: \sim p \vee \sim q \vee \sim r.$

Note 1: If we have six variables $p_1, p_2, \dots, p_5, p_6$ (arranged in this order) then there are $2^6 = 64$ minterms denoted by $m_0, m_1, m_2, \dots, m_{63}$ and $2^6 = 64$ maxterms denoted by $M_0, M_1, M_2, \dots, M_{63}$.

To write the minterm m_{26} , we write the binary representation of 26 and find it as 011010 .Then

$$m_{26}:\ \sim p_1 \wedge p_2 \wedge p_3 \wedge \sim p_4 \wedge p_5 \wedge \sim p_6$$

Further, the maxterm M_{26} is $p_1 \vee \sim p_2 \vee \sim p_3 \vee p_4 \vee \sim p_5 \vee p_6$.

Conversely, if a minterm say $p_1 \wedge \sim p_2 \wedge \sim p_3 \wedge p_4 \wedge p_5 \wedge \sim p_6$ is given ,then it is an m_j where j is the decimal equivalent of the corresponding binary number 100110. We see that $j = 32 + 0 + 0 + 4 + 2 + 0 = 38$ and the given minterm is m_{38}.

If a maxterm, say $\sim p_1 \vee \sim p_2 \vee p_3 \vee p_4 \vee \sim p_5 \vee p_6$ is given, then it is an M_j where j is the decimal equivalent of the corresponding binary number 110010.We see that $j = 32 + 16 + 0 + 0 + 2 + 0 = 50$ and the given maxterm is M_{50}.

Note 2: In the above discussion

$$m_{26} : \sim p_1 \wedge p_2 \wedge p_3 \wedge \sim p_4 \wedge p_5 \wedge \sim p_6$$

Notice that (i) $\sim m_{26} \equiv \sim(\sim p_1 \wedge p_2 \wedge p_3 \wedge \sim p_4 \wedge p_5 \wedge \sim p_6)$

$$\equiv p_1 \vee \sim p_2 \vee \sim p_3 \vee p_4 \vee \sim p_5 \vee p_6 \equiv M_{26}$$

(ii) The dual of m_{26} $i.e.,$ $(m_{26})^*$ is $\sim p_1 \vee p_2 \vee p_3 \vee \sim p_4 \vee p_5 \vee \sim p_6$ and it is M_{37}

In general in the case of n variables ,$\boldsymbol{\sim m_j \equiv M_j}$, $\boldsymbol{\sim M_j \equiv m_j}$ and

$$\left(\boldsymbol{m_j}\right)^* \equiv \boldsymbol{M_{(2^n-1)-j}} \ , \left(\boldsymbol{M_j}\right)^* \equiv \boldsymbol{m_{(2^n-1)-j}}$$

A representation of *PDNF* and *PCNF* of a formula:

If the *PDNF* of a given formula α is the disjunction of the minterms m_i, m_j, m_k(say), then we write the *PDNF* of α in a compact form as $\sum i, j, k$.

If the *PCNF* of a formula α is the conjunction of the maxterms M_k, M_l, M_m, M_n (say), then we write the *PCNF* of α in a compact form as $\prod k, l, m, n$.

Illustration:

The *PDNF* and *PCNF* of the formulas given in examples 6 and 8 are written in the compact form as

(a) $p \wedge (p \longrightarrow q) \equiv p \wedge q \equiv m_3 \equiv \sum 3$

and $p \wedge (p \longrightarrow q) \equiv (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q) \equiv M_0 \wedge M_1 \wedge M_2 \equiv \prod 0,1,2$

(b). $\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q) \equiv m_2 \vee m_1 \equiv \sum 1,2$

$\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv (p \vee q) \wedge (\sim p \vee \sim q) \equiv M_0 \wedge M_3 \equiv \prod 0,3$

Example 9: Obtain the principal disjunctive normal form of

$$\boldsymbol{a} : (\boldsymbol{p} \vee \sim\boldsymbol{q}) \longrightarrow (\sim\boldsymbol{p} \wedge \boldsymbol{r}).$$

Solution:

$$(p \vee \sim q) \longrightarrow (\sim p \wedge r) \equiv \sim(p \vee \sim q) \vee (\sim p \wedge r)$$

$$\equiv \big(\sim p \wedge \sim(\sim q)\big) \vee (\sim p \wedge r) \equiv (\sim p \wedge q) \vee (\sim p \wedge r)$$

$(\sim p \wedge q) \vee (\sim p \wedge r)$ is a *DNF* of a.

$$\equiv \big((\sim p \wedge q) \wedge T_0\big) \vee \big((\sim p \wedge r) \wedge T_0\big)$$

$$\equiv \big((\sim p \wedge q) \wedge (r \vee \sim r)\big) \vee \big((\sim p \wedge r) \wedge (q \vee \sim q)\big)$$

$$\equiv (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$$

The *PDNF* (or sum -of -products canonical form) of a is

$$(\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r).$$

$$i.e, m_1 \vee m_2 \vee m_3 \equiv \sum 1,2,3,$$

Therefore, *PDNF* of a in compact form is given by

$$(\sim p \vee \sim q) \longrightarrow (\sim p \wedge r) \equiv \sum 1,2,3$$

Example 10: Obtain the *PCNF* of the formula $\boldsymbol{a}$ given by $(\sim\boldsymbol{p} \longrightarrow \boldsymbol{r}) \wedge (\boldsymbol{q} \leftrightarrow \boldsymbol{p})$

Solution:

$$(\sim p \longrightarrow r) \wedge (q \leftrightarrow p) \equiv (\sim(\sim p) \vee r) \wedge \big((q \longrightarrow p) \wedge (p \longrightarrow q)\big)$$

$$\equiv (p \vee r) \wedge (\sim q \vee p) \wedge (\sim p \vee q)$$

Now, $p \vee r \equiv p \vee r \vee F_0 \equiv p \vee r \vee (q \wedge \sim q)$

$$\equiv (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \equiv (p \vee q \vee r) \wedge (p \vee \sim q \vee r)$$

$$\sim q \vee p \equiv \sim q \vee p \vee F_0 \equiv \sim q \vee p \vee (r \wedge \sim r)$$

$$\equiv (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r) \equiv (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r)$$

Similarly, $\sim p \vee q = (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

Therefore, $(\sim p \longrightarrow r) \wedge (q \leftrightarrow p)$

$\equiv (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

$\equiv (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

The *PCNF* of a is

$(p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

$$i.e., M_0 \wedge M_2 \wedge M_3 \wedge M_4 \wedge M_5$$

Therefore, $(\sim p \longrightarrow r) \wedge (q \leftrightarrow p) \equiv \prod 0,2,3,4,5$

Theorem 1:

1. ***If the PDNF (PCNF) of a given formula $\boldsymbol{a}$ containing $\boldsymbol{n}$ variables is known, then the PDNF (PCNF) of $\sim\boldsymbol{a}$ is the disjunction (conjunction) of the remaining minterms (maxterms) which do not appear in the PDNF (PCNF) of $\boldsymbol{a}$.***
2. ***Since $\boldsymbol{a} \equiv \sim(\sim\boldsymbol{a})$, the PCNF (PDNF) of $\boldsymbol{a}$ can be obtained by applying De Morgan's laws to PDNF (PCNF) of $\sim\boldsymbol{a}$.***

Illustration:

We have, the *PCNF* of the formula

$$a: (\sim p \longrightarrow r) \wedge (q \leftrightarrow p)$$

is $\prod 0,2,3,4,5$. The *PCNF* of $\sim a$ is the conjunction of the remaining maxterms. Therefore, the *PCNF* of $\sim a$ is $\prod 1,6,7$

That is, $\sim a \equiv (p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$

Now, $a \equiv \sim(\sim a) \equiv \sim[(p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)]$

$\equiv \sim(p \vee q \vee \sim r) \vee \sim(\sim p \vee \sim q \vee r) \wedge \sim(\sim p \vee \sim q \vee \sim r)$

$\equiv (\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) \wedge (p \wedge q \wedge r)$ (By De Morgan's law)

Thus, the *PDNF* of a is $(\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)$

$$i.e, \ \ m_1 \vee m_6 \vee m_7 \equiv \sum 1,6,7$$

Note:

1. In the above problem, we have $\sim a \equiv \prod 1,6,7$

   Now, $a \equiv \sim(\sim a) \equiv \sim(\prod 1,6,7) \equiv \sim(M_1 \wedge M_6 \wedge M_7) \equiv \sim M_1 \vee \sim M_6 \vee \sim M_7$

$$\equiv m_1 \vee m_6 \vee m_7 \equiv \sum 1,6,7$$

Thus, the *PDNF* of a is $\sum 1,6,7$

- ***In general given any formula a containing n variables and using the compact forms to represent the equivalent PDNF and PCNF, we see that all numbers between 0 and $2^n - 1$ which do not appear in one principal normal form will appear in other principal normal form***

Example 11:

The truth table for a formula a is given in the following table. Find the *PDNF* and *PCNF* of a.

Truth Table

p	q	r	a
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

Solution: The PDNF of a is the disjunction of the minterms corresponding to each truth value T of a.

The minterms corresponding to each truth value T of a are

$$p \wedge \sim q \wedge r,\ \sim p \wedge q \wedge r,\ \sim p \wedge q \wedge \sim r \text{ and } \sim p \wedge \sim q \wedge \sim r$$

Therefore, the PDNF of a is

$$(p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r)$$

$$i.e., m_5 \vee m_3 \vee m_2 \vee m_0 \equiv \sum 0,2,3,5$$

The *PCNF* of a is the conjunction of the maxterms corresponding to each truth value F of a.

The maxterms corresponding to each truth value F of a are

$$\sim p \vee \sim q \vee \sim r, \sim p \vee \sim q \vee r, \sim p \vee q \vee r, p \vee q \vee \sim r$$

Here a maxterms is written by including the variable if its truth value is F and its negation if its truth value is T.Therefore the *PCNF of* a is

$$(\sim p \vee \sim q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (\sim p \vee q \vee r) \wedge (p \vee q \vee \sim r)$$

$$i.e, M_7 \wedge M_6 \wedge M_4 \wedge M_1 \equiv \prod 1,4,6,7$$

P1:

Obtain a DNF of $\boldsymbol{p} \vee \left(\sim\boldsymbol{p} \longrightarrow (\boldsymbol{q} \vee (\boldsymbol{q} \longrightarrow \sim\boldsymbol{r}))\right)$

Solution: $p \vee \left(\sim p \longrightarrow (q \vee (q \longrightarrow \sim r))\right)$

$$\equiv p \vee \left(\sim p \longrightarrow (q \vee (\sim q \vee \sim r))\right) \qquad (\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$$

$$\equiv p \vee \left(\sim p \longrightarrow (q \vee \sim q \vee \sim r)\right)$$

$$\equiv p \vee \left(\sim(\sim p) \vee (q \vee \sim q \vee \sim r)\right) \qquad (\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta)$$

$$\equiv p \vee p \vee q \vee \sim q \vee \sim r \qquad \text{(Double negation law)}$$

Thus, $p \vee p \vee q \vee \sim q \vee \sim r$ is a DNF of the given formula

Note:

1. $p \vee p \vee q \vee \sim q \vee \sim r$ is also a *CNF* of the given formula
2. $p \vee \left(\sim p \longrightarrow (q \vee (q \longrightarrow \sim r))\right) \equiv p \vee p \vee q \vee \sim q \vee \sim r \equiv T_0 (\because\ q \vee \sim q \equiv T_0)$
3. Since the given formula is a tautology, all minterms will appear in its *PDNF*.
   Thus $p \vee \left(\sim p \longrightarrow (q \vee (q \longrightarrow \sim r))\right) \equiv \sum 0,1,2,3,4,5,6,7$

P2:

Obtain *CNF* of $(\boldsymbol{p} \wedge \sim(\boldsymbol{q} \wedge \boldsymbol{r})) \vee (\boldsymbol{p} \leftrightarrow \boldsymbol{q})$

Solution: $(p \wedge \sim(q \wedge r)) \vee (p \leftrightarrow q)$

$$\equiv (p \wedge (\sim q \vee \sim r)) \vee ((\sim p \vee q) \wedge (p \vee \sim q))$$

$$(r \leftrightarrow s \equiv (\sim r \vee s) \wedge (r \vee \sim s))$$

$$\equiv \left(p \vee ((\sim p \vee q) \wedge (p \vee \sim q))\right) \wedge \left((\sim q \vee \sim r) \vee ((\sim p \vee q) \wedge (p \vee \sim q))\right)$$

$$\equiv \left((p \vee (\sim p \vee q)) \wedge (p \vee (p \vee \sim q))\right) \wedge ((\sim q \vee \sim r \vee \sim p \vee q) \wedge (\sim q \vee \sim r \vee p \vee \sim q))$$

$$\equiv (p \vee \sim p \vee q) \wedge \ (p \vee p \vee \sim q) \wedge (\sim q \vee \sim r \vee \sim p \vee q) \wedge (\sim q \vee \sim r \vee p \vee \sim q)$$

Thus, $(p \vee \sim p \vee q) \wedge \ (p \vee p \vee \sim q) \wedge (\sim q \vee \sim r \vee \sim p \vee q) \wedge (\sim q \vee \sim r \vee p \vee \sim q)$

is the *CNF* of $(p \wedge \sim(q \wedge r)) \vee (p \leftrightarrow q)$

Note: The *PCNF* of the given formula is

$$(p \vee \sim p \vee q) \wedge \ (p \vee p \vee \sim q) \wedge (\sim q \vee \sim r \vee \sim p \vee q) \wedge (\sim q \vee \sim r \vee p \vee \sim q)$$

$$\equiv T_0 \wedge \ (p \vee \sim q) \wedge T_0 \wedge (p \vee \sim q \vee \sim r)$$

$$\equiv (p \vee \sim q) \wedge (p \vee \sim q \vee \sim r)$$

$$\equiv (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee \sim r) \equiv (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r)$$

$i.e., M_2 \wedge M_3 \equiv \prod 2,3$

P3:

Obtain the principal disjunctive normal forms of

(a) $\sim\boldsymbol{p} \vee \boldsymbol{q}$

(b) $(\boldsymbol{p} \wedge \boldsymbol{q}) \vee (\sim\boldsymbol{p} \wedge \boldsymbol{r}) \vee (\boldsymbol{q} \wedge \boldsymbol{r})$ **.**

Solution:

(a) $\sim p \equiv \sim p \wedge T_0 \equiv \sim p \wedge (q \vee \sim q) \equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q)$ (Distributive law)

$q \equiv T_0 \wedge q \equiv (p \vee \sim p) \wedge q \equiv (p \wedge q) \vee (\sim p \wedge q)$ (Distributive law)

Now, $\sim p \vee q \equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q) \vee (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge q)$

Thus the *PDNF* of $\sim p \vee q \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge q)$

The *PDNF of* $\sim p \vee q$ in compact form *is* $m_0 \vee m_1 \vee m_3 \equiv \sum 0,1,3$

(b) $p \wedge q \equiv p \wedge q \wedge T_0 \equiv p \wedge q \wedge (r \vee \sim r) \equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r)$

$\sim p \wedge r \equiv \sim p \wedge (q \vee \sim q) \wedge r \equiv (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

$q \wedge r \equiv (p \vee \sim p) \wedge q \wedge r \equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$

Now,

$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$

$\equiv (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)$

Thus, the *PDNF* of the given formula is

$$(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r) \equiv (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)$$

$$i.e., m_1 \vee m_3 \vee m_6 \vee m_7$$

The *PDNF* of the given formula in compact form is $\sum 1,3,6,7$

P4:

Obtain the *PDNF* of $\boldsymbol{p \longrightarrow \big((p \longrightarrow q) \wedge \sim(\sim q \vee \sim p)\big)}$

Solution: $p \longrightarrow \big((p \longrightarrow q) \wedge \sim(\sim q \vee \sim p)\big)$

$$\equiv p \longrightarrow \Big((\sim p \vee q) \wedge \big(\sim(\sim q) \wedge \sim(\sim p)\big)\Big)$$

($\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta$ and De Morgan's law)

$$\equiv \sim p \vee \big((\sim p \vee q) \wedge (q \wedge p)\big)$$

($\alpha \longrightarrow \beta \equiv \sim\alpha \vee \beta$ and Double negation formula)

$$\equiv \sim p \vee \big((\sim p \wedge q \wedge p) \vee (q \wedge q \wedge p)\big)$$ (Distributive law)

$$\equiv \sim p \vee F_0 \vee (p \wedge q) \equiv \sim p \vee (p \wedge q) \equiv \big(\sim p \wedge (q \vee \sim q)\big) \vee (p \wedge q)$$

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$$

$$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge q)$$

$$\equiv m_0 \vee m_1 \vee m_3 \equiv \sum 0,1,3$$

P5:

Obtain the product-of-sums canonical form (*PCNF*) of

$$\boldsymbol{a} : (\boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r}) \vee (\sim\boldsymbol{p} \wedge \boldsymbol{r} \wedge \boldsymbol{q}) \vee (\sim\boldsymbol{p} \wedge \sim\boldsymbol{q} \wedge \sim\boldsymbol{r}).$$

Solution:

$(p \wedge q \wedge r) \vee (\sim p \wedge r \wedge q) \vee (\sim p \wedge \sim q \wedge \sim r)$

$\equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

$\equiv \big((p \vee \sim p) \wedge (q \wedge r)\big) \vee (\sim p \wedge \sim q \wedge \sim r)$

$\equiv \big(T_0 \wedge (q \wedge r)\big) \vee (\sim p \wedge \sim q \wedge \sim r)$

$\equiv (q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

$\equiv \big(q \vee (\sim p \wedge \sim q \wedge \sim r)\big) \wedge \big(r \vee (\sim p \wedge \sim q \wedge \sim r)\big)$

$\equiv (q \vee \sim p) \wedge (q \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p) \wedge (r \vee \sim q) \wedge (r \vee \sim r)$

$\equiv (\sim p \vee q) \wedge T_0 \wedge (q \vee \sim r) \wedge (\sim p \vee r) \wedge (\sim q \vee r) \wedge T_0$

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee \sim r) \wedge$
$(\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee r) \wedge (\sim p \vee \sim q \vee r)$

$\equiv (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee r)$

The product-of-sums canonical form of a is

$(\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee r)$

$$i.e., M_4 \wedge M_5 \wedge M_1 \wedge M_6 \wedge M_2 \equiv \prod 1,2,4,5,6$$

Aliter:

Notice that a itself is in the *PDNF*, and the *PDNF* of a is $\sum 0,3,7$

Then the *PDNF* of $\sim a$ is the disjunction of the remaining minterms, therefore the *PDNF* of $\sim a$ is $\sum 1,2,4,5,6$

Now, $a \equiv \sim(\sim a) \equiv \sim(\sum 1,2,4,5,6) \equiv \sim(m_1 \vee m_2 \vee m_4 \vee m_5 \vee m_6)$

$$\equiv \sim m_1 \wedge \sim m_2 \wedge \sim m_4 \wedge \sim m_5 \wedge \sim m_6$$

$$\equiv M_1 \wedge M_2 \wedge M_4 \wedge M_5 \wedge M_6$$

$$\equiv \prod 1,2,4,5,6$$

Thus the *PCNF* of a is $\prod 1,2,4,5,6$

P6:

Obtain the sum -of- products canonical form of

$$\boldsymbol{a} : \boldsymbol{p} \vee (\sim\boldsymbol{p} \wedge \sim\boldsymbol{q} \wedge \boldsymbol{r})$$

Solution: Notice that a is in the *DNF* and $\sim p \wedge \sim q \wedge r$ is a minterm. We will introduce the missing variables

$$p \equiv p \wedge T_0 \equiv p \wedge (q \vee \sim q) \equiv (p \wedge q) \vee (p \wedge \sim q)$$

$$\equiv \big((p \wedge q) \wedge T_0\big) \vee \big((p \wedge \sim q) \wedge T_0\big)$$

$$\equiv \big((p \wedge q) \wedge (r \vee \sim r)\big) \vee \big((p \wedge \sim q) \wedge (r \vee \sim r)\big)$$

$$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$$

Therefore

$$p \vee (\sim p \wedge \sim q \wedge r) \equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$$

Thus, the sum -of- products canonical form (or *PDNF*) of a is

$$(\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r)$$

$$i.e., m_1 \vee m_4 \vee m_5 \vee m_6 \vee m_7 \equiv \sum 1,4,5,6,7$$

P7:

Find the *PDNF* and *PCNF* of $\boldsymbol{a} : (\boldsymbol{p} \longrightarrow (\boldsymbol{q} \longrightarrow \boldsymbol{r})) \longrightarrow ((\boldsymbol{p} \longrightarrow \boldsymbol{q}) \longrightarrow \boldsymbol{r})$ through its Truth table.

Solution:

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$(\boldsymbol{p}$	$\longrightarrow$	$(\boldsymbol{q}$	$\longrightarrow$	$\boldsymbol{r}))$	$\longrightarrow$	$((\boldsymbol{p}$	$\longrightarrow$	$\boldsymbol{q})$	$\longrightarrow$	$\boldsymbol{r})$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	T	F	F
T	F	T	T	T	F	T	T	T	T	F	F	T	T
T	F	F	T	T	F	T	F	T	T	F	F	T	F
F	T	T	F	T	T	T	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F	F	F	T	T	F	F
F	F	T	F	T	F	T	T	T	F	T	F	T	T
F	F	F	F	T	F	T	F	F	F	T	F	F	F
1	1	1	1	3	1	2	1	6	1	4	1	5	1

Notice that the given formula a is T when the combination of truth values TTT, TTF, TFT, TFF, FTT and FFT for pqr.

The minterms corresponding to each truth value T of a are

$p \wedge q \wedge r, p \wedge q \wedge \sim r, p \wedge \sim q \wedge r, p \wedge \sim q \wedge \sim r, \sim p \wedge q \wedge r$ and $\sim p \wedge \sim q \wedge r$

The *PDNF* of a is the disjunction of minterms corresponding to each truth value T of a. Therefore, the PDNF of a is

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$$

$$i.e., m_7 \vee m_6 \vee m_5 \vee m_4 \vee m_3 \vee m_1 \equiv \sum 1,3,4,5,6,7$$

Notice that the given formula a is F when the combination of truth values FTF and FFF for pqr.

The maxterms corresponding to each truth value F of a are $p \vee \sim q \vee r$ and $p \vee q \vee r$. (A maxterm is written by including the variable if its truth value is F and its negation if its truth value is T).

The *PCNF* of a is the conjunction of maxterms corresponding to each truth value F of a. Therefore the *PCNF* of a is $(p \vee \sim q \vee r) \wedge (p \vee q \vee r)$.

$$i.e., M_2 \wedge M_0 \equiv \prod 0,2$$

P8:

Find the *PDNF* and *PCNF* of $\boldsymbol{a} : (\boldsymbol{p} \longrightarrow (\boldsymbol{q} \longrightarrow \boldsymbol{r})) \longrightarrow ((\boldsymbol{p} \longrightarrow \boldsymbol{q}) \longrightarrow \boldsymbol{r})$ without constructing its truth table.

***Solution*:** We obtain first the PDNF of a.

$(p \longrightarrow (q \longrightarrow r)) \longrightarrow ((p \longrightarrow q) \longrightarrow r)$

$\equiv (p \longrightarrow (\sim q \vee r)) \longrightarrow ((\sim p \vee q) \longrightarrow r)$

$\equiv (\sim p \vee (\sim q \vee r)) \longrightarrow (\sim(\sim p \vee q) \vee r)$

$\equiv \sim(\sim p \vee (\sim q \vee r)) \vee (\sim(\sim p \vee q) \vee r)$

$\equiv (\sim(\sim p) \wedge (\sim(\sim q) \wedge \sim r)) \vee ((\sim(\sim p) \wedge \sim q) \vee r)$

$\equiv (p \wedge (q \wedge \sim r)) \vee ((p \wedge \sim q) \vee r)$

Now $p \wedge \sim q \equiv (p \wedge \sim q) \wedge T_0 \equiv (p \wedge \sim q) \wedge (r \vee \sim r)$

$\equiv (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$

$r \equiv T_0 \wedge r \equiv (q \vee \sim q) \wedge r \equiv (q \wedge r) \vee (\sim q \wedge r)$

$\equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

Therefore,

$(p \longrightarrow (q \longrightarrow r)) \longrightarrow ((p \longrightarrow q) \longrightarrow r)$

$\equiv (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

$\equiv (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \vee \sim q \vee \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

$$i.e., \ m_7 \vee m_6 \vee m_5 \vee m_4 \vee m_3 \vee m_1 \equiv \sum 1,3,4,5,6,7$$

The above is the *PDNF* of a.

Now, the *PDNF* of $\sim a$ is the disjunction of the remaining minterms, namely $\sum 0,2$

Now, $a \equiv \sim(\sim a) \equiv \sim(\sum 0,2) \equiv \prod 0,2$ (substantiate!)

1.4 Normal forms

EXERCISES

1. Obtain *DNF* of the following formulas:
   i. $(\sim p \vee q) \leftrightarrow (p \wedge q)$
   ii. $p \longrightarrow \big((p \longrightarrow q) \wedge \sim(\sim q \vee \sim p)\big)$
2. Obtain *CNF* of the following formulas:
   i. $\big((p \longrightarrow q) \wedge \sim q\big) \longrightarrow \sim p$
   ii. $\big((p \longrightarrow q) \wedge \sim p\big) \longrightarrow \sim q$
3. Obtain *PDNF* of the formulas: $\sim\big((p \vee q) \wedge r\big) \wedge (p \vee r)$
4. Obtain *PCNF* of the formulas:
   i. $(p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$
   ii. $(p \wedge q) \vee (\sim p \wedge \sim q \wedge r)$
5. Obtain the *PCNF* and *PDNF* of the following formulas:
   i. $p \vee \Big(\sim p \longrightarrow \big(q \vee (\sim q \longrightarrow r)\big)\Big)$
   ii. $(\sim p \vee \sim q) \longrightarrow (p \leftrightarrow \sim q)$
   iii. $\big(p \longrightarrow (q \wedge r)\big) \wedge \big(\sim p \longrightarrow (\sim q \wedge \sim r)\big)$
   iv. $q \wedge (p \vee \sim q)$
   v. $q \vee (p \wedge \sim q)$
   vi. $(q \longrightarrow p) \wedge (\sim p \wedge q)$
   vii. $p \longrightarrow \big(p \wedge (q \longrightarrow p)\big)$
6. Obtain the product of sums canonical form (*PCNF*) of the formula

   $(\sim p \wedge q \wedge r \wedge \sim s) \vee (p \wedge \sim q \wedge \sim r \wedge s) \vee (p \wedge \sim q \wedge r \wedge \sim s) \vee$
   $(\sim p \wedge q \wedge \sim r \wedge s) \vee (p \wedge q \wedge \sim r \wedge \sim s)$

1.4. Normal forms

Answers:

1.

   i. $(\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim p) \vee (q \wedge \sim q)$

   ii. $\sim p \vee (\sim p \wedge p \wedge q) \vee (p \wedge q)$

2.

   i. $(p \vee q \vee \sim p) \wedge (\sim q \vee q \vee \sim p)$

   ii. $(p \vee p \vee \sim q) \wedge (\sim q \vee p \vee \sim q)$

3. $\sum 1,4,6$

4.

   i. $\prod 0$

   ii. $\prod 0,2,4,5$

5. *PDNF* ; *PCNF*

	PDNF		*PCNF*
i.	$\sum 1,2,34,5,6,7$	;	$\prod 0$
ii.	$\sum 1,2,3$	;	$\prod 0$
iii.	$\sum 0,7$	;	$\prod 1,2,3,4,5,6$
iv.	$\sum 3$	;	$\prod 0,1,2$
v.	$\sum 1,2,3$	;	$\prod 0$
vi.	No *PDNF*	;	$\prod 0,1,2,3$
vii.	$\sum 0,1,2,3$	;	No *PDNF*

6. $\prod 0,1,2,3,4,7,8,11,13,14,15$