3.9 Matrix Multiplication

GTU : Winter-11, 14, Marks 4

Suppose we want to multiply two matrices A and B each of size n i.e.

$$C = A \times B \quad \text{then}$$

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

The multiplication gives

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21}$$

$$C_{12} = A_{11} \times B_{12} + A_{12} \times B_{22}$$

$$C_{21} = A_{21} \times B_{11} + A_{22} \times B_{21}$$

$$C_{22} = A_{21} \times B_{12} + A_{22} \times B_{22}$$

Thus to accomplish 2×2 matrix multiplication there are total 8 multiplications and 4 additions.

To accomplish this multiplication we can write the following algorithm for the same.

```
Algorithm Mat_Mul (A,B,C,n)
{
  for i:=1 to n do
      for j:=1 to n do
      C[i,j]:= 0;
          for k:=1 to n do
          C[i,j] := C[i,j] + A[i,j] × B[i,j] ;
}
```

The time complexity of above algorithm turns to be

$$O(n \times n \times n) = O(n^3)$$

Strassen showed that 2 × 2 matrix multiplication can be accomplished in 7 multiplication and 18 additions or subtractions.

The divide and conquer approach can be used for implementing Strassen's matrix multiplication.

- **Divide** : Divide matrices into sub-matrices : A0 , A1, A2 etc.
- **Conquer** : Use a group of matrix multiply equations.
- **Combine** : Recursively multiply sub-matrices and get the final result of multiplication after performing required additions or subtractions.

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned}
S1 &= (A_{11}+ A_{22})(B_{11} + B_{22}) \\
S2 &= (A_{21} + A_{22}) \times B_{11} \\
S3 &= A_{11} \times (B_{12} - B_{22}) \\
S4 &= A_{22} \times (B_{21} - B_{11}) \\
S5 &= (A_{11} + A_{12}) \times B_{22} \\
S6 &= (A_{21} - A_{11}) \times (B_{11} + B_{12}) \\
S7 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\
C_{11} &= S1 + S4 - S5 + S7 \\
C_{12} &= S3 + S5 \\
C_{21} &= S2 + S4 \\
C_{22} &= S1 + S3 - S2 + S6
\end{aligned}$$

Now we will compare the actual our traditional matrix multiplication procedure with Strassen's procedure. In Strassen's multiplication

$$\begin{aligned}
C_{11} &= S1 + S4 - S5 + S7 \\
&= (A_{11}+ A_{22})(B_{11}+B_{22}) + A_{22} \times (B_{21} - B_{11}) - (A_{11} + A_{12}) \times \\
&\qquad B_{22} + (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\
&= A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} + A_{22} B_{21} - A_{22} B_{11} \\
&\quad - A_{11} B_{22} - A_{12} B_{22} + A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
&= A_{11} B_{11} + A_{12} B_{21}
\end{aligned}$$

Example 3.9.1

$$If\ A = \begin{bmatrix} 5 & 3 & 0 & 2 \\ 4 & 3 & 2 & 6 \\ 7 & 8 & 1 & 4 \\ 9 & 4 & 6 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 4 & 7 \\ 2 & 5 & 2 & 9 \\ 3 & 9 & 0 & 3 \\ 7 & 6 & 2 & 1 \end{bmatrix}$$

Implement Strassen's matrix multiplication on A and B.

Solution : The given matrix is of order 4×4. Hence we will subdivide it in 2×2 submatrices. Hence we will compute $S_1, S_2, S_3, S_4, S_5, S_6$ and S_7.

Let,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} \begin{bmatrix} 5 & 3 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix} \\ \begin{bmatrix} 7 & 8 \\ 9 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \end{bmatrix} \qquad B = \begin{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} & \begin{bmatrix} 4 & 7 \\ 2 & 9 \end{bmatrix} \\ \begin{bmatrix} 3 & 9 \\ 7 & 6 \end{bmatrix} & \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \end{bmatrix}$$

Now, $S_1 = (A_{11} + A_{22})(B_{11} + B_{22})$

$$= \left[\begin{bmatrix} 5 & 3 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix}\right]\left[\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}\right] = \begin{bmatrix} 6 & 7 \\ 10 & 10 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 18+28 & 30+42 \\ 30+40 & 50+60 \end{bmatrix} = \begin{bmatrix} 46 & 72 \\ 70 & 110 \end{bmatrix}$$

$$S_2 = (A_{21} + A_{22})\, B_{11}$$

$$= \left[\begin{bmatrix} 7 & 8 \\ 9 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix}\right]\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 15 & 11 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 24+24 & 16+60 \\ 45+22 & 30+55 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 48 & 76 \\ 67 & 85 \end{bmatrix}$$

$$S_3 = A_{11}(B_{12} - B_{22})$$

$$= \begin{bmatrix} 5 & 3 \\ 4 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 7 \\ 2 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 4 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 4 \\ 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 20+0 & 20+24 \\ 16+0 & 16+24 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 20 & 44 \\ 16 & 40 \end{bmatrix}$$

$$S_4 = A_{22} \times (B_{21} - B_{11})$$

$$= \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 9 \\ 7 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 0 & 7 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+20 & 7+4 \\ 0+35 & 42+7 \end{bmatrix}$$

$$S_4 = \begin{bmatrix} 20 & 11 \\ 35 & 49 \end{bmatrix}$$

$$S_5 = (A_{11} + A_{12}) \times B_{22}$$

$$= \begin{bmatrix} 5 & 3 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 6 & 9 \end{bmatrix} \times \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+10 & 15+5 \\ 0+18 & 18+9 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 18 & 27 \end{bmatrix}$$

$$S_6 = (A_{21} - A_{11}) \times (B_{11} + B_{12})$$

$$= \begin{bmatrix} 7 & 8 \\ 9 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 3 \\ 4 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 2 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 5 & 1 \end{bmatrix} \times \begin{bmatrix} 7 & 9 \\ 4 & 14 \end{bmatrix} = \begin{bmatrix} 14+20 & 18+70 \\ 35+4 & 45+14 \end{bmatrix}$$

$$S_6 = \begin{bmatrix} 34 & 88 \\ 39 & 49 \end{bmatrix}$$

$$S_7 = (A_{12} - A_{22}) \times (B_{21} + B_{22})$$

$$= \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 9 \\ 7 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ -4 & -1 \end{bmatrix} \times \begin{bmatrix} 3 & 12 \\ 9 & 7 \end{bmatrix}$$

$$S_7 = \begin{bmatrix} -21 & -26 \\ -21 & -55 \end{bmatrix}$$

$$C_{11} = S_1 + S_4 - S_5 + S_7$$

$$= \begin{bmatrix} 46 & 72 \\ 70 & 110 \end{bmatrix} + \begin{bmatrix} 20 & 11 \\ 35 & 49 \end{bmatrix} - \begin{bmatrix} 10 & 20 \\ 18 & 27 \end{bmatrix} + \begin{bmatrix} -21 & -26 \\ -21 & -55 \end{bmatrix}$$

$$C_{11} = \begin{bmatrix} 35 & 37 \\ 66 & 77 \end{bmatrix}$$

$$C_{12} = S_2 + S_4$$

$$= \begin{bmatrix} 20 & 44 \\ 16 & 40 \end{bmatrix} + \begin{bmatrix} 10 & 20 \\ 18 & 27 \end{bmatrix}$$

$$C_{12} = \begin{bmatrix} 30 & 64 \\ 34 & 67 \end{bmatrix}$$

$$C_{21} = S_2 + S_4$$

$$= \begin{bmatrix} 48 & 76 \\ 67 & 85 \end{bmatrix} + \begin{bmatrix} 20 & 11 \\ 35 & 49 \end{bmatrix} = \begin{bmatrix} 68 & 87 \\ 102 & 134 \end{bmatrix}$$

$$C_{22} = S_1 + S_3 - S_2 + S_6$$

$$\begin{bmatrix} 46 & 72 \\ 70 & 110 \end{bmatrix} + \begin{bmatrix} 20 & 44 \\ 16 & 40 \end{bmatrix} - \begin{bmatrix} 48 & 76 \\ 67 & 85 \end{bmatrix} + \begin{bmatrix} 34 & 88 \\ 39 & 49 \end{bmatrix}$$

$$= \begin{bmatrix} 66 & 116 \\ 86 & 150 \end{bmatrix} - \begin{bmatrix} 48 & 76 \\ 67 & 85 \end{bmatrix} - \begin{bmatrix} 34 & 88 \\ 39 & 49 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 128 \\ 58 & 124 \end{bmatrix}$$

Thus the final product matrix C will be -

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 35 & 37 \\ 66 & 77 \end{bmatrix} & \begin{bmatrix} 30 & 64 \\ 34 & 67 \end{bmatrix} \\ \begin{bmatrix} 68 & 87 \\ 102 & 134 \end{bmatrix} & \begin{bmatrix} 52 & 128 \\ 58 & 124 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathbf{35} & \mathbf{37} & \mathbf{30} & \mathbf{64} \\ \mathbf{66} & \mathbf{77} & \mathbf{34} & \mathbf{67} \\ \mathbf{68} & \mathbf{87} & \mathbf{52} & \mathbf{128} \\ \mathbf{102} & \mathbf{134} & \mathbf{58} & \mathbf{124} \end{bmatrix}$$

3.9.1 Algorithm

Here we are dividing matrices in sub-matrices and recursively multiplying sub-matrices.

```
Algorithm St_Mul(int *A, int *B, int *C, int n)
{
 if (n == 1)then
 {
      (*C) =(*C)+ (*A) * (*B);
 }
 else
 {
      St_Mul(A,B,C,n/4);
      St_Mul(A, B+(n/4), C+(n/4), n/4);
      St_Mul(A+2 *(n/4), B, C+2 *(n/4), n/4);
      St_Mul(A+2 *(n/4), B+(n/4), C+3 *(n/4), n/4);
      St_Mul(A+(n/4), B+2*(n/4), C, n/4);
      St_Mul(A+(n/4), B+3*(n/4), C+(n/4), n/4);
      St_Mul(A+3*(n/4), B+2*(n/4), C+2*(n/4), n/4);
      St_Mul(A+3*(n/4), B+3*(n/4), C+3*(n/4), n/4);
 }
}
```

3.9.2 Analysis of Algorithm

$$T(1) = 1$$

$$T(n) = 7\,T(n/2) \qquad \text{assume } n = 2^k$$

$$T(n) = 7^k\,T(n/2^k)$$

$$T(n) = 7^{\log n}$$

$$T(n) = n^{\log 7} = n^{2.81}$$

Thus divide and conquer is an algorithmic strategy having with the principle idea of dividing the problem into subproblems. Then solution to these subproblems is obtained in order to get the final solution for the given problem.

Review Question

1. *Explain how divide and conquer method help multiplying two square matrices.* **GTU : Winter-11, Marks 4**