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ON THE BOND PRICING PARTIAL DIFFERENTIAL EQUATION IN A CONVERGENCE MODEL OF INTEREST RATES WITH STOCHASTIC CORRELATION

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ABSTRACT. Convergence models of interest rates are used to model a situation, where a country is going to enter a monetary union and its short rate is affected by the short rate in the monetary union. In addition, Wiener processes which model random shocks in the behaviour of the short rates can be correlated. In this paper we consider a stochastic correlation in a selected convergence model. A stochastic correlation has been already studied in different contexts in financial mathematics, therefore we distinguish differences which come from modelling interest rates by a convergence model. We provide meaningful properties which a correlation model should satisfy and afterwards we study the problem of solving the partial differential equation for the bond prices. We find its solution in a separable form, where the term coming from the stochastic correlation is given in its series expansion for a high value of the correlation.

1. Introduction

Short rate models of interest rates are formulated in terms of a stochastic differential equation (one factors models) or a system of them (multifactor models) which govern the behaviour of the instantaneous interest rate, so called short rate. The derivatives of the short rate, i.e. financial securities depending on the short rate, are then priced by a parabolic partial differential equation. Its terminal condition is the payoff of the derivative. A simple interest rate derivative is a discount bond, which pays a unit of currency at the specified time, called maturity of the bond. Bond prices are used in the construction of yield curves and they are necessary in discounting any future cash flows. Therefore it is important to be able to compute bond prices in various short rate models which have been proposed. The reader can find an overview of short rate models for example in [2] or [8].

In this paper we deal with convergence models of interest rates. These models capture the situation when a country is going to join a monetary union and its domestic short rate is affected by the short rate in the moneatary union. Furthermore, the Wiener processes which model random fluctuations and shocks can be correlated which adds another form of relation between the interest rates in the given country and the monetary union. In particular, we extend the model suggested by Corzo and Schwartz in [4] where the domestic short rate r_d and the union short rate r_u evolve

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according to the following stochastic differential equations:

$$\mathrm{d}r_d = (a + b(r_u - r_d))\mathrm{d}t + \sigma_d \mathrm{d}w_1,$$
$$\mathrm{d}r_u = c(d - r_u)\mathrm{d}t + \sigma_u \mathrm{d}w_2,$$

where $b, c, \sigma_d, \sigma_u > 0$ and $a, d \in \mathbb{R}$ are constants. Here, w_1, w_2 are Wiener processes and the correlation between their increments $\mathrm{d}w_1$ and $\mathrm{d}w_2$ is a constant $\rho \in (0, 1)$. Furthermore, the so called market prices of risk λ_d and λ_u are constant. Then, the price $P(r_d, r_u, \tau)$ of a domestic bond with maturity at time T (which is a parameter here) is a solution to the partial differential equation

$$\begin{aligned} -\frac{\partial P}{\partial \tau} + (a + b(r_u - r_d) - \lambda_d \sigma_d)\frac{\partial P}{\partial r_d} + (c(d - r_u) \\ - \lambda_u \sigma_u)\frac{\partial P}{\partial r_u} + \frac{1}{2}\sigma_d^2 \frac{\partial^2 P}{\partial r_d^2} + \frac{1}{2}\sigma_u^2 \frac{\partial^2 P}{\partial r_u^2} + \rho \sigma_d \sigma_u \frac{\partial^2 P}{\partial r_d \partial r_u} - r_d P = 0 \end{aligned} \tag{1.1}$$

for $r_d, r_u \in \mathbb{R}$ and time remaining to maturity $\tau \in (0, T)$, with terminal condition $P(r_d, r_u, 0) = 1$ for all $r_d, r_u \in \mathbb{R}$. The solution has the form

$$P(r_d, r_u, \tau) = \mathrm{e}^{A(\tau) - D(\tau) r_d - U(\tau) r_u},$$

where the functions $A(\tau), D(\tau), U(\tau)$ are solutions to a system of ordinary differential equations and can be expressed in a closed form, see [4].

Generalizations of the Corzo-Schwartz model with more general volatilities were studied in [13] by considering nonconstant volatilities and in [3] in nonparametric setting. Paper [10] uses an analogous model as in [13] for the domestic short rate, but a two factor model for the European short rate. Another approach to convergence models can be found in [1] which is based on Brownian bridge. In this paper we generalize the original model in a different way, by considering stochastic correlation.

The paper is organized as follows: In Section 2 we discuss stochastic correlation in financial models, we suggest general properties of a suitable correlation process for convergence models of interest rates and provide an example of such a process. Section 3 studies bond prices in this setting and we derive a series expansion for the correlation close to 1. We end the paper by concluding remarks and ideas for extending the results in Section 5.

2. Stochastic correlation in convergence models of interest rates

In many classical financial models, the correlation between Wiener processes arising in their stochastic differential equation formulation is assumed to be constant. However, there is an empirical evidence for a nonconstant correlation and models with stochastic correlation have been suggested for different settings for example in [11], [12]. In [11] the correlation refers to daily returns of the S&P 500 stock index and Euro/USD exchange rate, while in [12] it is related to the stock price and its volatility, thus generalising the classical Heston model [6]

In these cases, a suitable correlation process varies around a mean value and the probability mass tends to zero at the boundaries $+1$ and -1, see [11: p. 6]. The situation is different in our case of a convergence model of interest rate. Recall that a positive correlation describes similar reaction of the short rates to random shocks modelled by increments of Wiener processes and its value measures the strength of linear dependence of these increments. It is therefore reasonable to assume that the correlation process

$$\mathrm{d}\rho = \mu(\rho)\mathrm{d}t + \sigma(\rho)\mathrm{d}w \tag{2.1}$$

satisfies the following properties:

- *Property 1:* Its drift $\mu(\rho)$ is positive for all attainable values of the correlation ρ and equals zero at the boundary $\rho = 1$.
- *Property 2:* The expected value of the process $\mathbb{E}[\rho(t)]$ approaches 1 as time goes to infinity.

Continuation of the convergence process (here, the convergence is meant in a financial sense) motivates the requirement of a positive drift of $\mu(\rho)$. Similarly, with this financial background of the model in mind, we assume that the expected value of the stochastic correlation ρ has a limit of 1 (i.e., a perfect correlation) as time approaches infinity. Finally, the case of a perfect correlation $\rho = 1$ should be an absorbing state, instead of reflexing the process back to values $\rho < 1$. This implies that $\mu(1) = 0$, see [5] for a dicussion on absorbtion and reflexion.

Naturally, since we modell a correlation, necessarily we have:

- *Property 3:* The values of the process $\rho(t)$ are from the interval $(-1, 1)$. Eventually, having the interpretation of the convergence model in mind, we might want to restrict it to positive values, i.e., an interval $(0, 1)$.

***Example* 1.** An example of a process satisfying Properties 1–3 is

$$\rho = 1 - \frac{1}{1+x}, \qquad \text{where } \mathrm{d}x = \mu x \mathrm{d}t + \sigma x \mathrm{d}w, \tag{2.2}$$

where w is a Wiener process and $\mu > \sigma^2$ and $x(0) = x_0 > 0$ are constants.

It is straightforward to see that $\rho \in (0, 1)$. Furthermore, an application of the Itō lemma gives

$$\mathrm{d}\rho = \rho(1-\rho)(\mu - \rho\sigma^2)\mathrm{d}t + \sigma\rho(1-\rho)\mathrm{d}w,$$

from which we see the positivity of the drift for all $\rho \in (0, 1)$. Finally, we have

$$\rho_t > 1 - 1/x_t \tag{2.3}$$

and since x can be expressed in a closed form as

$$x(t) = x_0 \exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma w(t)\Big),$$

we have

$$\frac{1}{x_t} = \frac{1}{x_0} \exp\Big((-\mu + \frac{1}{2}\sigma^2)t - \sigma w(t)\Big).$$

Therefore, using the normal distribution $\mathcal{N}(0, t)$ of a Wiener process at time t (see e.g. [9])

$$\mathbb{E}\Big[\frac{1}{x_t}\Big] = \frac{1}{x_0} \exp\Big((-\mu + \frac{1}{2}\sigma^2)t + \frac{1}{2}\sigma^2 t\Big) = \frac{1}{x_0} \exp\big((-\mu + \sigma^2)t\big) \to 0$$

as $t \to \infty$ because of the assumption $\mu > \sigma^2$. Now, from (2.3) it follows that $\lim_{t\to\infty} \mathbb{E}(\rho_t) \geq 1$ as $t \to \infty$. However, recall that the process ρ has the values from the interval (0,1). Therefore

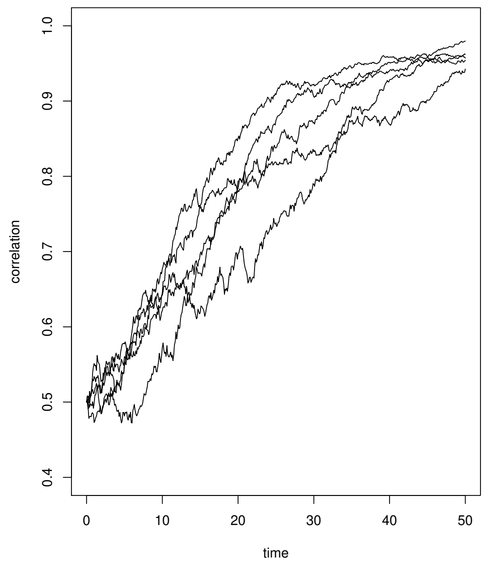
$$\lim_{t\to\infty} \mathbb{E}(\rho_t) = 1.$$

Figure 1 shows a simulation of the process (2.2) for parameter values $x_0 = 1, \mu = 0.1, \sigma = 0.25$, which agrees with our intuition about the behaviour of the correlation in a convergence model of interest rates.

***Example* 2.** A correlation process with values in $(-1, 1)$ or $(0, 1)$ does not necessarily satisfy Properties 1 and 2 which we postulated for a correlation in a convergence model. This is the case for example for Jacobi process, also known as Write-Fisher process (see [7]), where

$$\mathrm{d}\rho = \kappa(\theta - \rho)\mathrm{d}t + \sqrt{\sigma\rho(1-\rho)}\mathrm{d}w,$$

where w is a Wiener process, with parameters $\kappa, \sigma > 0$ and $\theta \in (0, 1)$.



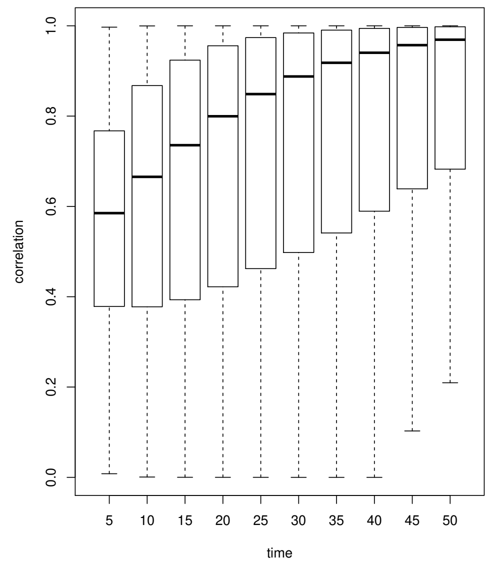


FIGURE 1. Simulated correlation process (2.2) for $x_0 = 1$, $\mu = 0.1$, $\sigma = 0.25$: five sample trajectories (left), boxplot from 10^5 trajectories (right)

In this model the drift is positive for $\rho < \theta$ and negative $\rho > \theta$ which models a mean-reversion to value θ. The expected value of the process

$$\mathbb{E}[\rho(t)] = \theta + \mathrm{e}^{-\kappa t}(\rho(0) - \theta) \to \theta$$

as $t \to \infty$ (see [5]). For $\rho = 1$ the volatility is zero, but the drift is strictly negative, which implies a reflection towards the interior, instead of a absorbtion, see again the discussion in [5].

3. Partial differential equation for the bond price and series expansion of its solution

The partial differential equation for the bond price $P = P(r_d, r_u, \rho, \tau)$, assuming that $\mathrm{d}w$ is independent of $\mathrm{d}w_1$ and $\mathrm{d}w_2$, reads as (cf. for example [8] for a derivation of a partial differential equation in multifactor models)

$$\begin{aligned} -\frac{\partial P}{\partial \tau} + ((a - \lambda_d \sigma_d) + b r_u - b r_d)\frac{\partial P}{\partial r_d} + ((cd - \lambda_u \sigma_u) - c r_u)\frac{\partial P}{\partial r_u} + \tilde{\mu}(\rho)\frac{\partial P}{\partial \rho} \\ + \frac{1}{2}\sigma_d^2 \frac{\partial^2 P}{\partial r_d^2} + \frac{1}{2}\sigma_u^2 \frac{\partial^2 P}{\partial r_u^2} + \frac{1}{2}\sigma^2(\rho)\frac{\partial^2 P}{\partial \rho^2} + \sigma_d \sigma_u \rho \frac{\partial^2 P}{\partial r_d \partial r_u} - r_d P \ = 0, \end{aligned} \tag{3.1}$$

where we denoted by $\tilde{\mu}$ the so called risk neutral drift of the correlation process

$$\tilde{\mu}(\rho) = \mu(\rho) - \lambda(\rho)\sigma(\rho). \tag{3.2}$$

The functions μ and σ come from the correlation process (2.1), while λ is the market price of risk of correlation which needs to be specified in addition to the parameters already given above.

More details about market prices of risk related to factors of interest rate models can be found for example in [8]. Here we only recall that in general, λ can be a function of time t, the domestic and union short rates r_d, r_u and the correlation ρ; however, it can not depend on maturity of the bond. We consider the case when the market price of risk of correlation depends only on ρ. Furthermore, we assume that it is defined also for the boundary values of the correlation process even thought they may be unattainable. This is satisfied for example if it is taken to be constant, similarly to the market prices of risk of domestic and union short rates in the original model [4], as well as in our extension of this model.

In what follows, we consider a general model with

$$\widetilde{\mu}(\rho) = O(1-\rho), \sigma(\rho) = O(1-\rho) \tag{3.3}$$

for $\rho \to 1$, which can be expanded into Taylor series around $\rho = 1$. This holds for example for a process (2.2) considered in Section 2 with a constant market price of risk of correlation discussed above.

Now, we look for the solution of (3.1) in the form

$$P(r_d, r_u, \rho, \tau) = \exp\left(A(\rho,\tau) - D(\tau)r_d - U(\tau)r_u\right). \tag{3.4}$$

Substituting (3.4) into the partial differential equation (3.1) yields the following system of equations for the functions $A(\rho,\tau)$, $D(\tau)$, $U(\tau)$:

$$\begin{aligned}
\dot{D} &= -bD + 1 \\
\dot{U} &= -cU + bD \\
\frac{\partial A}{\partial \tau} &= \widetilde{\mu}(\rho)\frac{\partial A}{\partial \rho} + \frac{1}{2}\sigma^2(\rho)\Big(\frac{\partial^2 A}{\partial \rho^2} + \Big(\frac{\partial A}{\partial \rho}\Big)^2\Big) - (a - \lambda_d\sigma_d)D \\
&\quad - (cd - \lambda_u\sigma_u)U + \frac{1}{2}\sigma_d^2 D^2 + \frac{1}{2}\sigma_u^2 U^2 + \sigma_d\sigma_u\rho DU
\end{aligned} \tag{3.5}$$

with initial conditions $A(\rho, 0) = 0$ for all $\rho \in (0,1)$ and $D(0) = 0$, $U(0) = 0$. To simplify further computations, we introduce substitutions

$$f_1 = -(a - \lambda_d\sigma_d)D - (cd - \lambda_u\sigma_u)U + \frac{1}{2}\sigma_d^2 D^2 + \frac{1}{2}\sigma_u^2 U^2, \tag{3.6}$$

$$f_2 = \sigma_d\sigma_u DU. \tag{3.7}$$

The ordinary differential equations for D and U are the same as in the case of the original convergence model and therefore we are able to express them in the closed form. For the function A we derive its series expansion. We will be interested in the regime with a high correlation, i.e., close to 1. This corresponds to a situation when domestic short rate is already highly correlated with the short rate in the monetary union. In addition to being pushed towards the monetary union rate by the drift, it also has very similar reaction to external shocks modelled by Wiener processes.

Therefore we make a substitution $x = 1 - \rho$ and consider the series expansion of $A(x,\tau)$ around $x = 0$. Using this substitution, the partial differential equation (3.5) can be written in the form

$$\frac{\partial A}{\partial \tau} + \widetilde{\mu}(x)\frac{\partial A}{\partial x} - \frac{1}{2}\sigma^2(x)\Big(\frac{\partial^2 A}{\partial x^2} + \Big(\frac{\partial A}{\partial x}\Big)^2\Big) = (f_1(\tau) + f_2(\tau)) - f_2(\tau)x, \tag{3.8}$$

where f_1 and f_2 are given by (3.6) and (3.7).

The assumption (3.3) implies expansions

$$\widetilde{\mu}(x) = \mu_1 x + \mu_2 x^2 + \dots, \quad \sigma^2(x) = \sigma_2 x^2 + \sigma_3 x^3 + \dots \tag{3.9}$$

for certain constants μ_i and σ_i. We expand the unknown function $A(x,\tau)$ in the same way

$$A(x,\tau) = A_0(\tau) + A_1(\tau)x + A_2(\tau)x^2 + \dots \tag{3.10}$$

and note that from its initial condition it follows that $A_i(0) = 0$ for all i. Substituting (3.10) into (3.8) leads to a system of ordinary differential equations, from which the first ones read as

$$\begin{aligned}
\dot{A}_0 &= f_1 + f_2, \\
\dot{A}_1 + \mu_1 A_1 &= -f_2, \\
\dot{A}_2 + (2\mu_1 - \sigma_2)A_2 &= -\mu_2 A_1 + \frac{1}{2}\sigma_2 A_1^2, \\
\dot{A}_3 + (3\mu_1 - 3\sigma_2)A_3 &= (\sigma_3 - 2\mu_2)A_2 + 2\sigma_2 A_1 A_2 + \frac{1}{2}\sigma_3 A_1^2 - \mu_3 A_1.
\end{aligned}$$

In general, for $i \geq 1$, the i-th equation has the form

$$\dot{A}_i + k_i A_i = F_i, \tag{3.11}$$

where k_i is a constant and $F_i = F_i(\tau)$ depends on model parameters and already computed A_j for $j < i$, and therefore can be easily computed.

If we use the terms $A_i(\tau)$ for $i = 0, 1, \dots, k$, to construct the approximation

$$A^{ap}(\rho,\tau) = A_0(\tau) + A_1(\tau)(1-\rho) + \dots + A_k(\tau)(1-\rho)^k,$$

then the resulting appoximation

$$P^{ap}(r_d, r_u, \rho, \tau) = \exp(A^{ap}(\rho,\tau) - D(\tau)r_d - U(\tau)r_u)$$

satisfies

$$\log P(r_d, r_u, \rho, \tau) - \log P^{ap}(r_d, r_u, \rho, \tau) = O((1-\rho)^{k+1})$$

for $\rho \to 1$.

4. Application of the series expansion for a particular correlation process

We consider the parameteres of the convergence model from the paper [4]: $a = 0.0938$, $b = 3.67$, $\sigma_d = 0.032$, $c = 0.2087$, $d = 0.035$, $\sigma_u = 0.016$, $\rho = 0.219$, $\lambda_u = -0.655$, $\lambda_d = 3.315$ However, we replace their constant correlation with a stochastic one given by the process considered in Example 1. We note that in this case the expansions (3.9) have only three nonzero terms; in particular we have

$$\mu_1 = \mu - \lambda\sigma - \sigma^2, \quad \mu_2 = -(\mu - \lambda\sigma) + 2\sigma^2, \quad \mu_3 = -\sigma^2$$

and

$$\sigma_2 = \sigma^2, \quad \sigma_3 = -2\sigma^2, \quad \sigma_4 = \sigma^2.$$

For an illustration we use the parameters $x_0 = 1, \mu = 0.1, \sigma = 0.25$ from Figure 1. Finally, we take the market price of correlation risk to be constant and equal to $\lambda = -5$. Our aim is to show the performance of the approximation suggested in the previous section, when applied to a moderately high correlation.

Table 1 shows approxination of the bond prices for $r_d = 0.05$, $r_u = 0.03$ and $x = 0.2$ (i.e., correlation $\rho = 0.8$), when using the expansion (3.10) up to A_k term for $k = 0, 1, 2, 3$. Table 2 shows the convergence of implied interest rates, which are given by $-\log P/\tau$, where P is the corresponding bond price and τ is time to maturity, cf. [2]. For a better visual comparison we write them in percentages, i.e., we multiply the values defined above by 100.

TABLE 1. Approximate bond prices obtained by the first terms of the expansion

time to maturity	term A_0	terms up to A_1	terms up to A_2	terms up to A_3
5	0.793508	0.793462	0.793458	0.793457
10	0.568282	0.568232	0.568228	0.568227
15	0.392083	0.392045	0.392041	0.392040
20	0.267031	0.267004	0.267001	0.267001
25	0.181039	0.181020	0.181018	0.181018
30	0.122542	0.122530	0.122528	0.122528
35	0.082900	0.082892	0.082891	0.082891
40	0.056071	0.056066	0.056065	0.056065
45	0.037922	0.037919	0.037918	0.037918
50	0.025647	0.025645	0.025644	0.025644

TABLE 2. Approximate interest rates obtained by the first terms of the bond price expansion

time to maturity	term A_0	terms up to A_1	terms up to A_2	terms up to A_3
5	4.62584	4.62699	4.62710	4.62711
10	5.65137	5.65225	5.65233	5.65234
15	6.24188	6.24253	6.24259	6.24260
20	6.60196	6.60246	6.60251	6.60252
25	6.83618	6.83659	6.83663	6.83664
30	6.99766	6.99801	6.99804	6.99805
35	7.11461	7.11491	7.11494	7.11494
40	7.20282	7.20308	7.20311	7.20311
45	7.27158	7.27181	7.27184	7.27184
50	7.32664	7.32685	7.32687	7.32687

5. Conclusions

In this paper we introduced a stochastic correlation to a convergence model of interest rates. We discussed the properties of a reasonable stochastic correlation process in this context and compared them with properties of correlation appearing in modelling stock prices, their volatility and the exchange rates, which have been already considered in literature. Afterwards, we introduced such a correlation process into a known convergence model of interest rates.

For this model we studied the problem of pricing bonds. We have shown that the bond price can be written in a separated form, similarly as in the original model. For the term involving the value of the correlation we have derived a series expansion for high values of correlation. The approximation was subsequently tested on a model example and we observed the convergence of bond prices and term structures of interest rates. Since the computation involves only solving linear ordinary differential equations, it provides a simple and convenient way of computation of the approximative bond prices and interest rates. This feature is important for implementation of the model and its further use.

Further research in this area includes studying the properties of this expansion and its comparison with a numerical solution of the corresponding partial differential equation for the function $A(\rho, \tau)$. Another path is empirical and consists of finding a calibration method to see how the model fits the real data, as well as to determine what kind of correlation process is the most consistent with the data observed in the financial market.

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