CS431 WRITTEN ASSIGNMENT 3: QUARTENIONS

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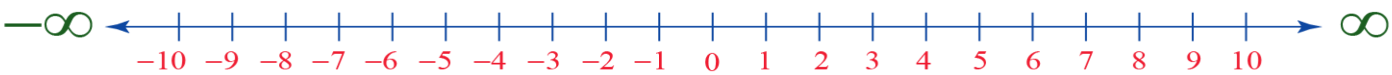
Before we dive into understanding Quaternions, it is important to get the fundamental understanding of number systems starting with **real numbers**. This may sound trivial but as we go along defining Quarternions, it becomes easier if we understand how the concepts build in lower dimensions first.

One Dimension: Real Line

Real numbers are a sequence of numbers that can be ordered from least to greatest. They include all the familiar characters we learn in school, like *–3.7, 3√5, and 42*. They consist of sets of rational and irrational numbers. All real numbers can be represented on a number line ranging from -∞ to +∞. Being one dimensional, it is easy to define the concept of stretching a real number which simply means multiplication. A real number *x*, when multiplied by a constant *c*, is said to become *(c*x)* represented again on the real number line.



Two Dimensions: Complex Plane

We then went ahead and introduced a term called iota (i) which is used to represent an imaginary number. Together with real numbers, they form complex numbers of the form $x + iy$ which can be represented in the 2D plane with the X-axis representing the real part while the Y-axis representing the imaginary part. Any point (x, y) in this 2D plane represents a complex number. With these complex numbers came the concept of addition, multiplication, and rotation in a 2D plane.

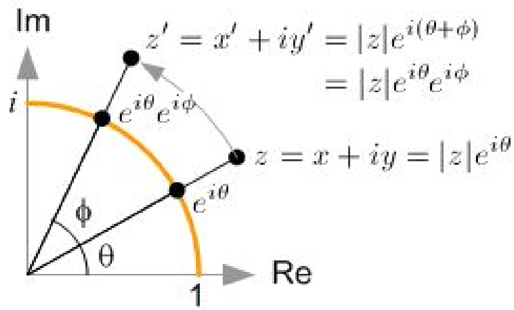
For two complex numbers
$z1 = a + ib$ and $z2 = c + id$

$z = z1 + z2$ is defined as:
z $= (a + c) + i(b + d)$

$z = z1 * z2$ is defined by:
$z = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$

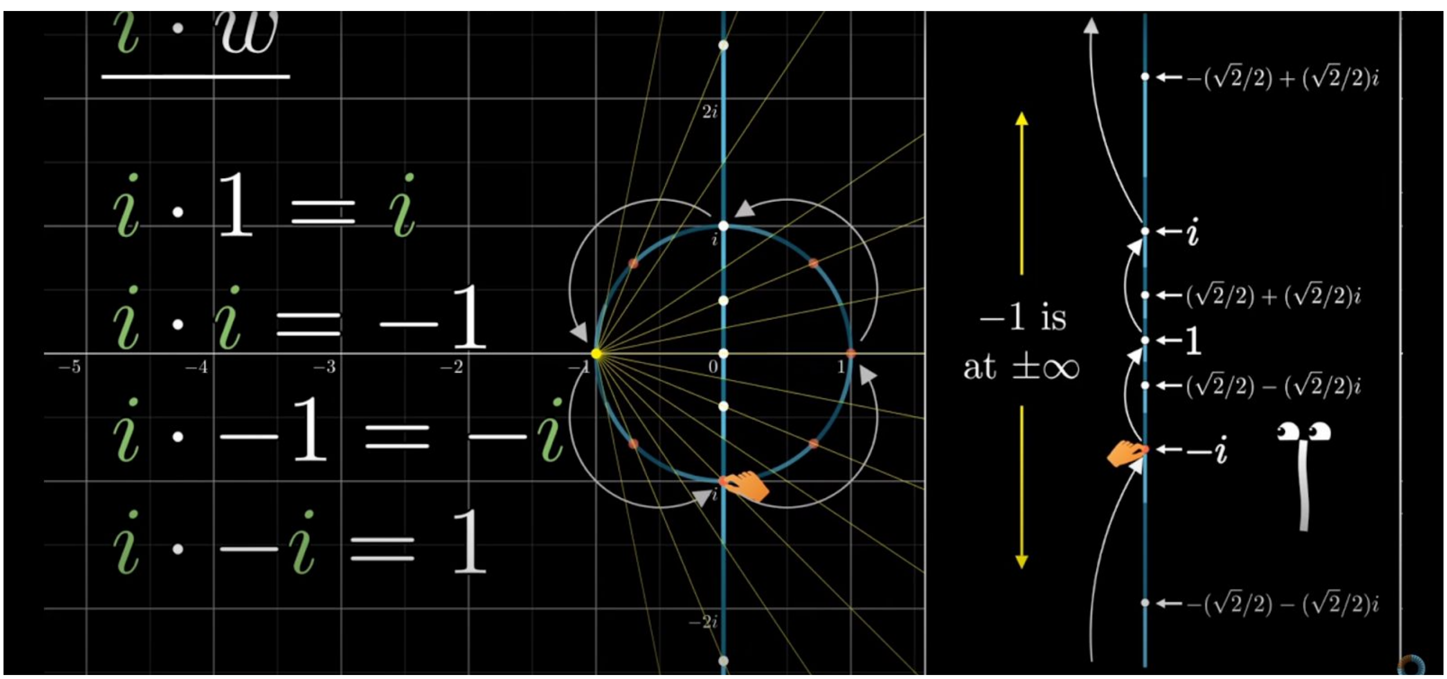
In polar form, a unit complex number can be represented as
$z(\theta) = cos\theta + i\, sin\theta$

Rotating a complex number $z1 = cos\theta + i\,sin\theta$ by an angle ϕ results in a complex number *z2* s.t. $z2 = z1 * (cos\phi + i\,sin\phi) = cos(\theta + \phi) + i\,sin(\theta + \phi)$



The rotation seems intuitive in the case of 2-dimensional space. However, later when we extend this to 4 dimensions, it becomes all the more difficult to visualize. Hence, I now introduce a concept of **Stereographic projection** which allows us to visualize a rotation of an N-dimensional object with a fixed magnitude as a morphing motion in (N-1) dimensions.

Consider complex numbers of unit magnitude, which are represented by a circle of the unit radius in a 2D complex plane. We map this circle to a 1-dimensional line using the below-described method. From the point (-1, 0), draw a line at each point of the circle. The point intersects of every such line with Y-axis represent the point in a 1-dimensional line (shown by the Y-axis). It can be observed point $-1 + i0$ is mapped to infinite while every other point is mapped to some point on the Y-axis as seen in the image below (reference is mentioned in the end). $1 + i0$ is mapped to 0, $0 + i1$ and $0 - i1$ are mapped to their original location and so on.



Consider rotation by 90 degrees in the anticlockwise direction, i.e, multiplying the complex number with i. The rotation of the circle and its mapping to the 1D line can be understood properly by looking at a few reference points as shown in the image above.

$1 \; moves \; to \; i, \;\; i \; moves \; to -1, \;\; -1 \; moves \; to \; -i, \;\; -i \; moves \; to \; 1$. This rotation of points is shown by arrows in the above image for both the 2D plane and its corresponding representation in 1 dimension.

Three Dimensions and its infeasibility

To extend this concept to 3 dimensions, William Rowan Hamilton, the Irish mathematician, hoped to climb out of the complex plane by adding an imaginary j axis. But there was something off about three dimensions and he realized extension to three dimensions is infeasible. The problem was multiplication. In the complex plane, multiplication produces rotations. No matter how Hamilton tried to define multiplication in 3-D, he couldn't find an opposing division that always returned meaningful results.
To understand how 3D rotations differ from 2D rotation, consider the rotation of a ring and that of a sphere. All the points in the ring move the same way at the same speed, implying they are being multiplied by the same 2D complex number. But for a sphere, points move at a higher speed near the equator compared to points as we move in either direction of the equator whiles poles don't change position at all. Hence 3D rotation is not trivially defined by just using 3 axes (one real and two imaginary).

Four-Dimensional Representation: Quaternions

With not two but three imaginary axes, i, j, and k, plus the real number line, Hamilton could define new numbers that are like vectors in 4-D space. He named them "**quaternions**." Now that we have a good understanding of multiplication and rotation fundamentals in lower dimensions, this concept of quaternions can be understood in a much better sense. Even though visualizing 4 dimensions seems tough for the human brain, quaternions have huge applications in Computer Graphics and Quantum Mechanics.

To understand quaternions better, I now define the mathematical operations on quaternions.
A quaternion is described as:

$$a \;+\; b\,\hat{\imath} \;+\; c\,j \;+\; d\,k \qquad or \qquad (a, b, c, d)$$

where a is the real part and b,c,d are the imaginary axes part.
The addition of two quaternions is similar to as defined for the complex plane:

$$q1 \;=\; a1 \;+\; b1i \;+\; c1j \;+ d1k$$

$$q2 \;=\; a2 \;+\; b2i \;+\; c2j \;+ d2k$$

$$q = q1 + q2 = (a1 + a2) + (b1 + b2)i + (c1 + c2)j + (d1 + d2)k$$

The multiplication between the unit vectors of imaginary axes *i,j,k* is defined as

$$i * i \;=\; j * j \;=\; k * k \;=\; -1$$

$$i * j \;=\; k, \quad j * i \;=\; -k$$

$$j * k \;=\; i, \quad k * j \;=\; -i$$

$$k * i \;=\; j, \quad i * k \;=\; -j$$

Using the above expressions, addition and multiplication can be formally defined as (here q1 and q2 are in 4 tuple form):

$$q \ = \ q1 \ + \ q2 \ = \ (a1 + a2,\ b1 + b2,\ c1 + c2,\ d1 + d2)$$

$$\begin{aligned} q \ &= \ q1 \ * \ q2 \ = \ (a1, b1, c1,\ d1) \ * \ (a2\ , b2, c2, d2) \\ &= (a1 * a2 \ - \ b1 * b2 \ - \ c1 * c2 \ - \ d1 * d2,\ a1 * b2 \ + \ b1 * a2 \ + \ c1 * d2 \ - \ d1 * c2, \\ &\quad\ a1 * c2 \ - \ b1 * d2 \ + \ c1 * a2 \ + \ d1 * b2,\ a1 * d2 \ + \ b1 * c2 \ - \ c1 * b2 \ + \ d1 * a2) \end{aligned}$$

Combining the three imaginary axes into a single imaginary axis, we can simplify this notation:

$$q1V \ = \ (b1,\ c1,\ d1) \qquad q1 \ = \ (a1,\ q1V)$$

$$q2V \ = \ (b2,\ c2,\ d2) \qquad q2 \ = \ (a2,\ q2V)$$

$$q \ = \ q1 \ * \ q2 \ = \ (a1 * a2 \ - \ q1V \ . \ q2V \ ,\ a1 * q1V \ + \ a2 * q2V \ + \ (q1V \ X \ q2V)\ )$$

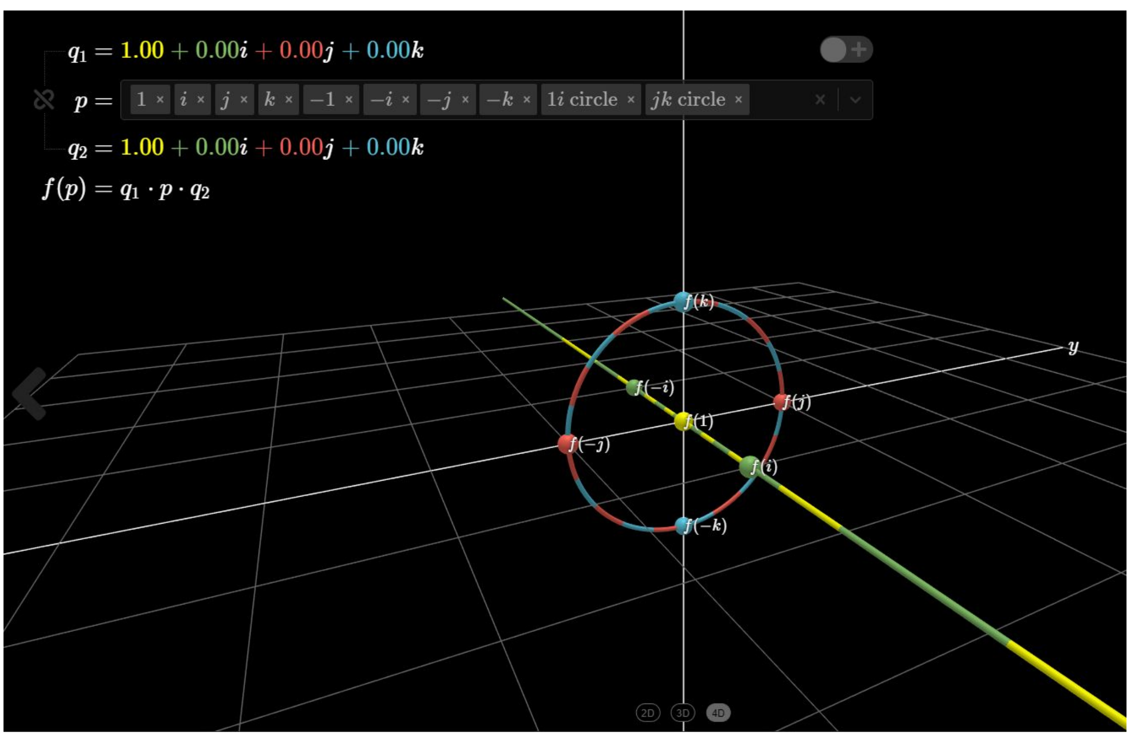
where . represents dot product and X represents the cross product.

Now having defined the basic operations on quaternions, I will try to describe their significance from a geometric (or graphical) point of view for better interpretation of quaternions. Quaternion multiplication of *q1 * p* can be thought of as quaternion *q1* manipulating quaternion *p*

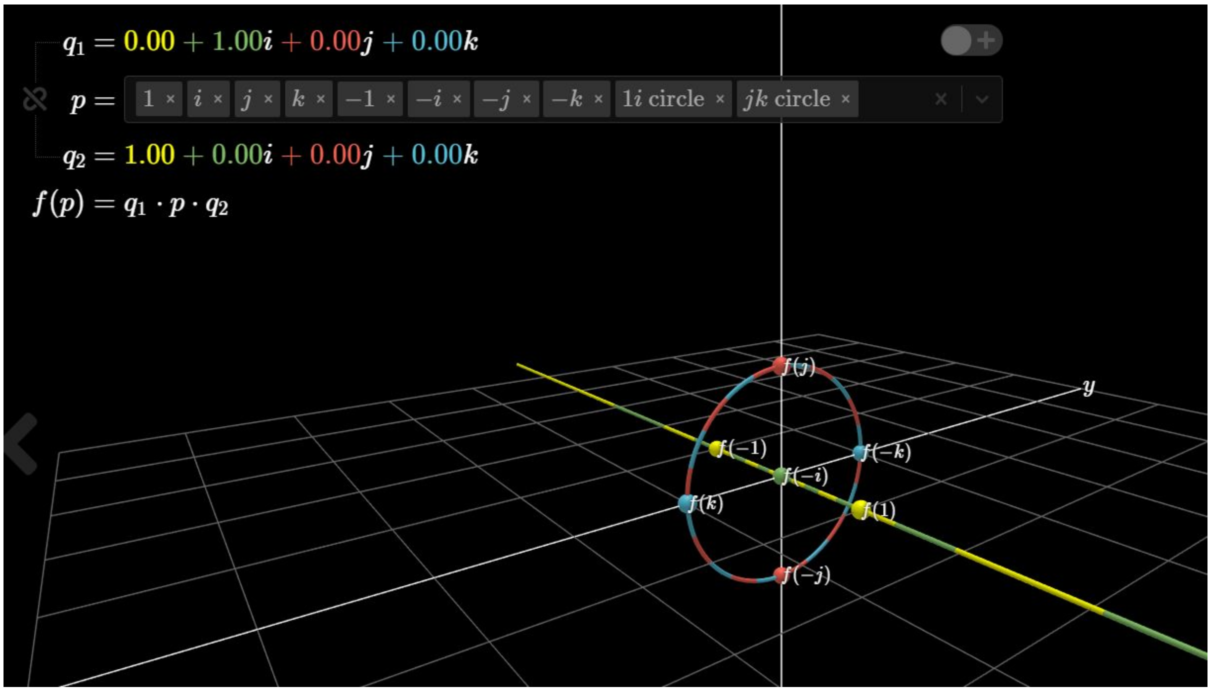
Just like in the complex plane, we saw the **Stereographic projection** of a unit circle to a 1-dimensional line, we will extend that concept to 4 dimensions. We project the quaternions of magnitude 1 onto a three-dimensional surface. Quaternions q has magnitude 1 implying

$$a^2 + b^2 + c^2 + d^2 = 1 \ where \ q \ = \ (a, b, c, d)$$

I will try to explain using the example shown in the image below (reference is mentioned in the end) Notice the projected line created by the real axis and i vector (1i) shown by the yellow-green line in the image below, and the j,k vector (the jk circle) shown by the red-blue circle in the image below.



Through the image above and its transformation below, we see multiplication causing two separate 2D rotations happening in sync with one another. In the images, q2 is set to 1, hence does not create any rotation in multiplication.
Extending the notion of multiplication in complex numbers where multiplication results in the 1D line projection moving forward or backward. In the image example, q1 is (0,1,0,0) and it is changing the two circles (projected as lines). In the image, the yellow-green line moves forward and the red-blue line rotates counter-clockwise by 90 degrees as can be seen below.



One important thing to note is that unlike complex multiplication, quaternions multiplication is not commutative, i.e.,

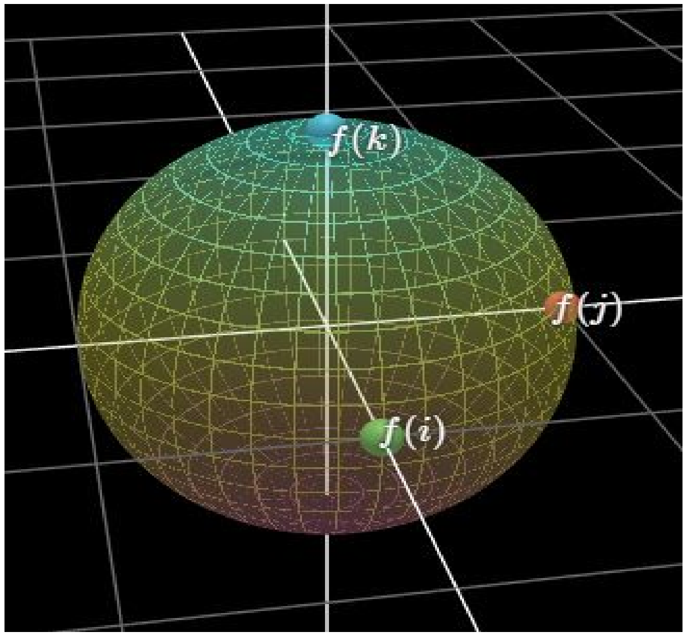
$$q1 * q2 \neq q2 * q1$$

In the given image if *q1 * p* is changed to *p * q1*, the yellow-green will show similar behavior as described above but the red-blue line to move in the opposite direction.

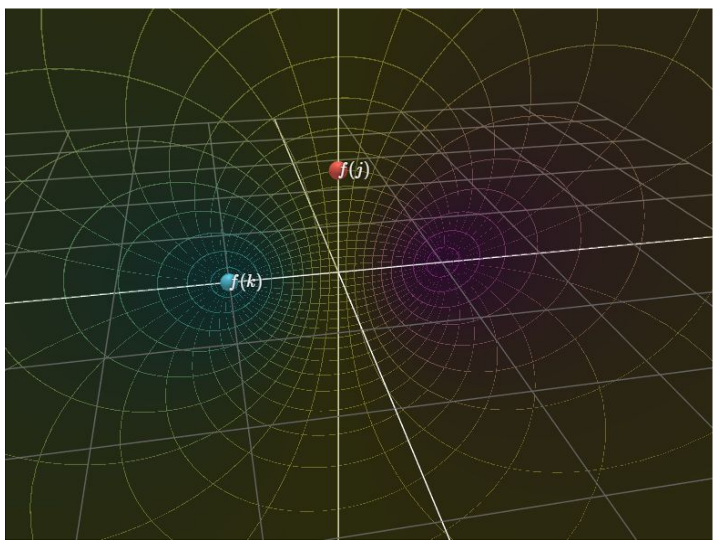
To have a similar extension to a sphere and understand the effect of multiplication, consider the following polar notation of a unit quaternion q1 (magnitude 1).

$$q1 = cos(\theta) + sin(\theta)\,[b\,i + c\,j + d\,k]\ \ (where\ \ b^2 + c^2 + d^2 = 1\ as\ |q| = 1).$$

This is just an extension of complex plane analogy, where θ has the same meaning (rotation angle) and $bi + cj + dk$ represents the axis around which to rotate. For the complex plane, we saw to rotate by angle ϕ, the complex number is multiplied with $cos(\phi) + i\,sin(\phi)$. There is, however, a certain complication in extending this notion to quaternions. Consider the following sphere representation of a 3D projection of a unit quaternion.



Rotating by $q1 = cos(\phi) + sin(\phi)\,[b\,i + c\,j + d\,k]$ causes the sphere to change its shape and transform into a plane as shown below:



To adjust for this transformation, we need a **cover**. We multiply p with $q1^{-1}$ called the inverse or conjugate on the right side. The rotating angle is then adjusted such that the overall effect of rotation equals ϕ. The lack of multiplication commutativity is used to reverse the ball exploding while keeping the rotation.

Hence, to rotate the point p by ϕ degrees about the axis $b\,i + c\,j + d\,k$, the angle is changed to $\phi/2$ and p is multiplied before and after by $q1$ *and* $q1^{-1}$. Since both $q1$ *and* $q1^{-1}$ rotate the object by $\phi/2$, the overall rotation is by an angle ϕ. The rotation is written as:

$$q_{fin} = q1 * p * q2 = q1 * p * q1^{-1}$$

$$where\ q1 = cos(\phi/2) + sin(\phi/2)\,[b\,i + c\,j + d\,k].$$

Quaternions overcome the issue of numerical inaccuracy which can arise in matrix based transformations. The above explanation should allow one to understand the concepts of quaternions. I will now mention a few real-world applications of quaternions to highlight their importance.

Applications of Quaternions in Real Word

- Quaternions are vital for the control systems that guide aircraft and rockets. For an aircraft in flight, the changes in orientation are given by three rotations called pitch, roll, and yaw. When the two rotation axes align, it can give rise to a situation known as gimbal lock.
  The mathematical properties of unit quaternions make them ideal for representing rotations in three dimensions. Each quaternion has an axis giving its direction and a magnitude giving the size of the rotation and uses just one rotation to represent a change of orientation. This saves time and storage and also solves the problem of gimbal lock.
- Quaternions have been used in the design of a system that tracks the position of an electric toothbrush in the mouth relative to the user's teeth. It automatically compensates for movements of the head during brushing.
- Quaternions are much more space-efficient than rotation matrices and are easy to interpolate than Euler angle rotations. Their rotation mechanism is similar to as seen in the spin -½ particles and hence find varied usage in quantum physics as well. They also solve the problem of gimbal lock as explained above. Hence engineers and scientists find quaternions very useful in the domain of computer graphics.

References Used to Understand the Concept

- https://eater.net/quaternions/
- https://www.quantamagazine.org/the-strange-numbers-that-birthed-modern-algebra-20180906/
- https://m.youtube.com/playlist?list=PL4jF-nm_eudMNkEqLVbi0vznveXqpDF3P
- https://www.irishtimes.com/news/science/the-many-modern-uses-of-quaternions-1.3642385
- https://math.stackexchange.com/questions/71/real-world-uses-of-quaternions