CS461 Assignment 1

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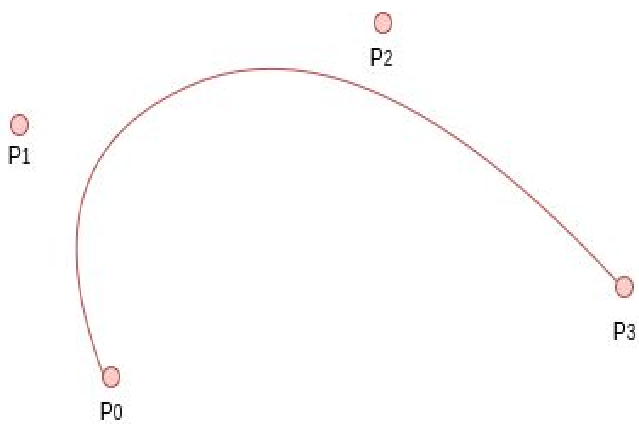
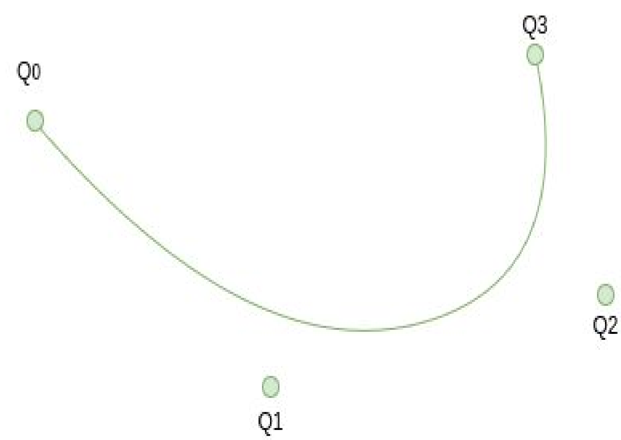
Continuity Conditions while Merging Cubic Bezier Curves

Consider two cubic Bezier curves S_0 and S_1:
S_0 has control points P_0 , P_1 , P_2 , P_3 .
$S_0(t) = (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3 , t \in [0,1]$

Q_0 has control points Q_0 , Q_1 , Q_2 , Q_3 .
$S_1(t) = (1-t)^3Q_0 + 3(1-t)^2tQ_1 + 3(1-t)t^2Q_2 + t^3Q_3 , t \in [0,1]$



S_0



S_1

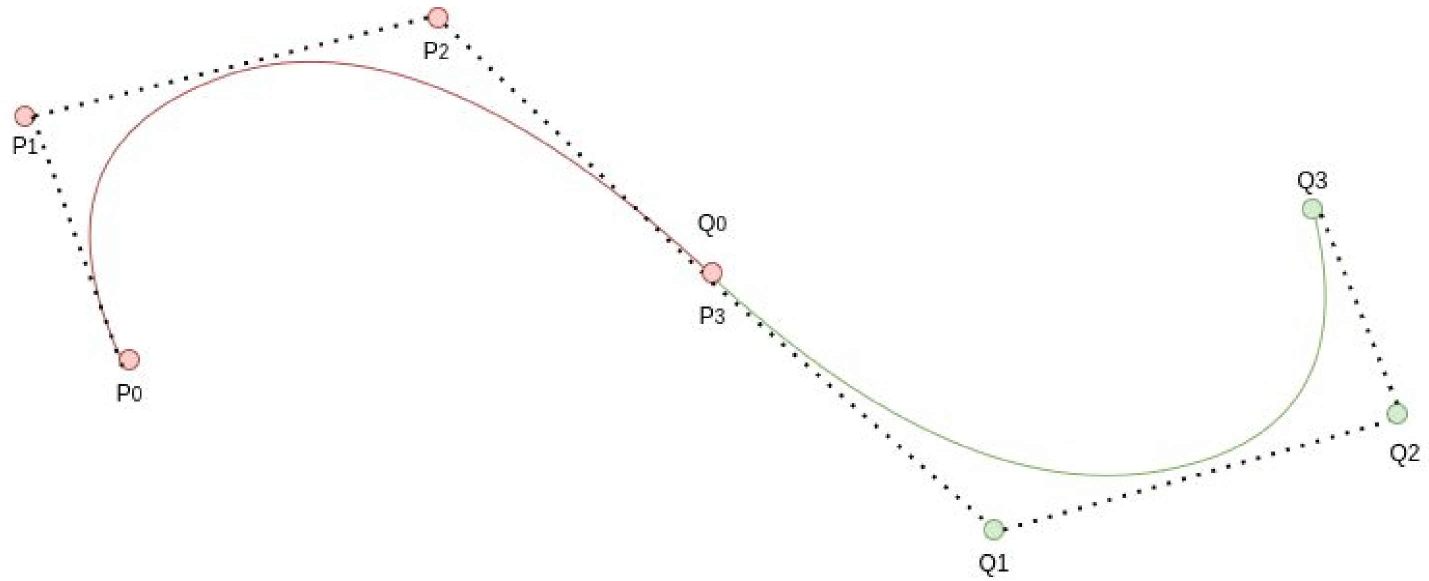
Continuity after merging two cubic Bezier curves can be defined in terms of Kth order continuity.

- Zeroth order continuity ensures **positional continuity**.
- First order continuity ensures **tangential continuity (or continuity in velocity)**.
- Second order continuity ensures **curvature continuity (or continuity in acceleration)**.

There are two major notions of what is meant by curve continuity of kth order: **Parametric Continuity** **(**C^0 , C^1 , C^2 ...**)** and **Geometric Continuity (**G^0, G^1 , G^2 ...**)**.

- **Parametric** continuity means smoothness both of the curve and of its parameterization. Hence it deals with both magnitude and direction of kth derivative at the intersection point.
- **Geometric** continuity means simply the smoothness of the track. Hence it deals with only the direction of kth derivative at the intersection point.

We will now look at continuity conditions for the above two cubic Bezier curves S_0 and S_1 in terms of positional, tangential and curvature continuity.



Merging cubic Beizner curves S_0 and S_1

1) Zeroth Order Continuity (Positional Continuity):

The two curves are given by:

$S_0(t) = (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3\,,\ t \in [0,1]$

$S_1(t) = (1-t)^3Q_0 + 3(1-t)^2tQ_1 + 3(1-t)t^2Q_2 + t^3Q_3\,,\ t \in [0,1]$

For C^0 continuity, the value of the curves must agree at the junction.

$S_0(\ t = 1\ ) = S_1(\ t = 0\ )$

Now,

$S_0(\ t = 1\ ) = P_3$ *and* $S_1(\ t = 0\ ) = Q_0$

Hence, $\boldsymbol{P_3 = Q_0}$ or P_3 and Q_0 must be the same point. Considering points in 3D space we can say, $(P_{3x} = Q_{0x}, P_{3y} = Q_{0y}, P_{3z} = Q_{0z})$

For zeroth order continuity, C^0 and G^0 have the same condition.

2) First Order Continuity (Tangential Continuity or Continuity in Velocity):

Now the positional continuity is ensured, so in total we have 7 points since P_3 and Q_0 are the same point.

The two curves are given by:

$S_0(t) = (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3\,,\ t \in [0,1]$

$S_1(t) = (1-t)^3P_3 + 3(1-t)^2tQ_1 + 3(1-t)t^2Q_2 + t^3Q_3\,,\ t \in [0,1]$

For C^1 continuity, the curve must be C^0 continuous and the all the first derivative must agree at the junction (to ensure same direction and magnitude).
$S_0(t = 1) = S_1(t = 0)$ ***and*** $(dS_0/dt)(t = 1) = (dS_1/dt)(t = 0)$

Differentiating the two curves to get the first derivatives (or velocity)

$V_0(t) = S_0'(t) = 3\,((1-t)^2(P_1-P_0) + 2(1-t)t(P_2-P_1) + t^2(P_3-P_2))\ ,\ t \in [0,1]$
$V_1(t) = S_1'(t) = 3\,((1-t)^2(Q_1-P_3) + 2(1-t)t(Q_2-Q_1) + t^2(Q_3-Q_2)),\ t \in [0,1]$

For C^1 continuity, apart from C^0 continuity,
$V_0(t = 1) = V_1(t = 0)$
Now,
$V_0(t = 1) = P_3 - P_2$
$V_1(t = 0) = Q_1 - P_3$

Hence,
$\boldsymbol{P_3 - P_2 = Q_1 - P_3}$ *or* $\boldsymbol{2P_3 = Q_1 + P_2}$
or $P_{3x} - P_{2x} = Q_{1x} - P_{3x}$ $\quad P_{3y} - P_{2y} = Q_{1y} - P_{3y}$ $\quad P_{3z} - P_{2z} = Q_{1z} - P_{3z}$

For G^1 continuity, the curve must be G^0 continuous and the first derivatives must be proportional at the junction (to ensure same tangential direction but not magnitude).
Hence,
$P_3 - P_2 = k\,(Q_1 - P_3) \qquad k>0$
or $P_{3x} - P_{2x} = k\,(Q_{1x} - P_{3x})$ $\quad P_{3y} - P_{2y} = k\,(Q_{1y} - P_{3y})$ $\quad P_{3z} - P_{2z} = k\,(Q_{1z} - P_{3z})$

3) Second Order Continuity (Curvature Continuity or Continuity in Acceleration):

Now the zeroth and first order continuity is ensured, so in total we have 7 points since P_3 and Q_0 are the same point.

The two curves are given by:
$S_0(t) = (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3\ ,\ t \in [0,1]$
$S_1(t) = (1-t)^3P_3 + 3(1-t)^2tQ_1 + 3(1-t)t^2Q_2 + t^3Q_3\ ,\ t \in [0,1]$

For C^2 continuity, the curve must be C^0 and C^1 continuous (conditions are mentioned before) and also the second derivative must be equal at the junction. Hence both direction and magnitude of the second derivative are the same.
$S_0(t=1) = S_1(t=0)$ ***and*** $(dS_0/dt)(t=1) = (dS_1/dt)(t=0)$ ***and*** $(d^2S_0/dt^2)(t=1) = (d^2S_1/dt^2)(t=0)$

Differentiating the two curves to get the first derivatives (or velocity)

$V_0(t) = S_0'(t) = 3\ (\ (1-t)^2(P_1\text{-}P_0) + 2(1-t)t(P_2\text{-}P_1) + t^2(P_3\text{-}P_2)\ )\ ,\ t \in [0,1]$

$V_1(t) = S_1'(t) = 3\ (\ (1-t)^2(Q_1\text{-}P_3) + 2(1-t)t(Q_2\text{-}Q_1) + t^2(Q_3\text{-}Q_2)\ ),\ t \in [0,1]$

Differentiating the two curves to get the second derivatives (or acceleration)

$A_0(t) = S_0''(t) = 6\ (\ (1-t)(P_0 + P_2 - 2P_1) + t(P_1 + P_3 - 2P_2)\ )\ ,\ t \in [0,1]$

$A_1(t) = S_1''(t) = 6\ (\ (1-t)(P_3 + Q_2 - 2Q_1) + t(Q_1 + Q_3 - 2Q_2)\ )\ ,\ t \in [0,1]$

For C^2 continuity, apart from C^0 and C^1 continuity,

$A_0\ (\ t = 1\ ) = A_1\ (\ t = 0\ )$

Now,

$A_0\ (\ t = 1\ ) = P_1 + P_3 - 2P_2$

$A_1\ (\ t = 0\ ) = P_3 + Q_2 - 2Q_1$

Hence,

$P_1 + P_3 - 2P_2 = P_3 + Q_2 - 2Q_1$

or $\mathbf{P_1 - 2P_2 = Q_2 - 2Q_1}$ or $\mathbf{2P_2 - P_1 = 2Q_1 - Q_2}$

or $P_{1x} - 2P_{2x} = Q_{2x} - 2Q_{1x}$ $P_{1y} - 2P_{2y} = Q_{2y} - 2Q_{1y}$ $P_{1z} - 2P_{2z} = Q_{2z} - 2Q_{1z}$

For G^2 continuity, the curve must be G^0 and G^1 continuous (conditions are mentioned before) and the second derivatives must be proportional at the junction.

Hence, the direction of the second derivatives are identical.

$P_1 - 2P_2 = k\ (\ Q_2 - 2Q_1\ )\quad k>0$

or $P_{1x} - 2P_{2x} = k\ (Q_{2x} - 2Q_{1x})$ $P_{1y} - 2P_{2y} = k\ (Q_{2y} - 2Q_{1y})$ $P_{1z} - 2P_{2z} = k\ (Q_{2z} - 2Q_{1z})$

The above text explains conditions for zeroth order(postional), first order(tangential), and second order(curvature) continuity for merging of Bezier curves.

Since S_0 and S_1 have cubic equations, higher order continuity (from third order onwards) will be too constrained and there is no application or significance of higher order continuity (from third order onwards) for Cubic Bezier curves.