ICTS Astrophysical SGWB Tutorial

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1 Day 4 (Part II) Problem

References : Section III A of [1]
We have two sets time series data

$$s_1(t) = n_1(t) + h_1(t) \tag{1}$$
$$s_2(t) = n_2(t) + h_2(t) \tag{2}$$

The cross correlation statistic is defined as

$$Y = \sum_t s_1(t)\, s_2(t) \tag{3}$$

I redefine, the time tag to indices $t \to i$ and assume, $i = 1, .., N$, where N is the total number of observations. Then

$$s_{1i} = n_{1i} + h_{1i} \tag{4}$$
$$s_{2i} = n_{2i} + h_{2i} \tag{5}$$
$$Y = \sum_{i=1}^{N} s_{1i}\, s_{2i} \tag{6}$$

The statistical properties of noise n are asuumed as

$$\langle n_{1i} \rangle = \langle n_{2i} \rangle = 0 \tag{7}$$
$$\mathrm{Var}\,(n_{1i}) = \sigma_{n_1}^2 \tag{8}$$
$$\mathrm{Var}\,(n_{2i}) = \sigma_{n_2}^2 \tag{9}$$

also the noise in two datasets and at two different time in same segment, is uncorrelated. The statistical properties of the source signal are

$$h_{1i} = h_{2i} = h \tag{10}$$
$$\langle h_{1i}^2 \rangle = \langle h_{2i}^2 \rangle = \langle h_{1i} h_{2i} \rangle = \langle h^2 \rangle = S_h \tag{11}$$

where S_h is variance of h_{Ii} in weak signal limit. The aim of problem is to find the required time to detect h . We can use SNR ρ of Y statistic to find the required time i.e. when $\rho > \rho_0$, and we can claim detection.

$$\boxed{\rho = \frac{\langle Y \rangle}{\sqrt{\mathrm{Var}\,(Y)}}} \tag{12}$$

First we calculate the mean of cross correlation statistic:

$$\begin{aligned}
\langle Y \rangle = \langle \sum_{i=1}^{N} s_{1i}\, s_{2i} \rangle = \sum_{i=1}^{N} \langle s_{1i}\, s_{2i} \rangle &= \sum_{i=1}^{N} \langle (n_{1i} + h_{1i})(n_{2i} + h_{2i}) \rangle & (13)\\
&= \sum_{i=1}^{N} \left(\langle n_{1i} n_{2i} \rangle + \langle h_{1i} h_{2i} \rangle + \langle h_{1i} n_{2i} \rangle + \langle n_{1i} h_{2i} \rangle \right) & (14)\\
&= \sum_{i=1}^{N} \left(\langle n_{1i} \rangle \langle n_{2i} \rangle + \langle h^2 \rangle + \langle h_{1i} \rangle \langle n_{2i} \rangle + \langle n_{1i} \rangle \langle h_{2i} \rangle \right) & (15)
\end{aligned}$$

$$\boxed{\langle Y \rangle = \sum_{i=1}^{N} S_h = N\, S_h} \tag{16}$$

Next we calculate the variance of the cross correlation statistic as : $\mathrm{Var}\,(Y) = \langle Y^2\rangle - (\langle Y\rangle)^2$. The second part of the variance , is already calculated. So we will calculate the first part of the Variance.

$$\begin{aligned}
\langle Y^2\rangle = \langle\sum_{i=1}^{N} s_{1i}\, s_{2i} \sum_{j=1}^{N} s_{1j}\, s_{2j}\rangle &= \langle\sum_{i=1}^{N}\sum_{j=1}^{N} s_{1i}\, s_{2i}\, s_{1j}\, s_{2j}\rangle && (17)\\
&= \langle\sum_{i=1}^{N}\sum_{j=1}^{N}(n_{1i}+h_{1i})(n_{2i}+h_{2i})(n_{1j}+h_{1j})(n_{2j}+h_{2j})\rangle && (18)\\
&= \langle\sum_{i=1}^{N}\sum_{j=1}^{N}(n_{1i}n_{2i}+h_{1i}h_{2i}+h_{1i}n_{2i}+n_{1i}h_{2i})(n_{1j}n_{2j}+h_{1j}h_{2j}+h_{1j}n_{2j}+n_{1j}h_{2j})\rangle\\
&= \langle\sum_{i=1}^{N}\sum_{j=1}^{N} n_{1i}n_{2i}n_{1j}n_{2j}+n_{1i}n_{2i}h_{1j}h_{2j}+n_{1i}n_{2i}h_{1j}n_{2j}+n_{1i}n_{2i}n_{1j}h_{2j}\\
&\quad +h_{1i}h_{2i}n_{1j}n_{2j}+h_{1i}h_{2i}h_{1j}h_{2j}+h_{1i}h_{2i}h_{1j}n_{2j}+h_{1i}h_{2i}n_{1j}h_{2j}\\
&\quad +h_{1i}n_{2i}n_{1j}n_{2j}+h_{1i}n_{2i}h_{1j}h_{2j}+h_{1i}n_{2i}h_{1j}n_{2j}+h_{1i}n_{2i}n_{1j}h_{2j}\\
&\quad +n_{1i}h_{2i}n_{1j}n_{2j}+n_{1i}h_{2i}h_{1j}h_{2j}+n_{1i}h_{2i}h_{1j}n_{2j}+n_{1i}h_{2i}n_{1j}h_{2j}\rangle && (19)\\
& && (20)
\end{aligned}$$

One could have ignored all h dependent terms here in weak signal limit and keep only first term. But to get a generic solution of the problem, I have derived full expression here.

$$\begin{aligned}
\langle Y^2\rangle &= \sum_{i=j=1}^{N}\sigma_{n_1}^2\sigma_{n_2}^2+0+0+0+0+\sum_{i=1}^{N}\sum_{j=1}^{N}\langle h_{1i}h_{2i}h_{1j}h_{2j}\rangle+0+0+0+0+\sum_{i=j=1}^{N}\sigma_{n_2}^2 S_h\\
&\quad +0+0+0+0+\sum_{i=j=1}^{N}\sigma_{n_1}^2 S_h && (21)\\
&= \sum_{i=j=1}^{N}\sigma_{n_1}^2\sigma_{n_2}^2+\sum_{i=1}^{N}\sum_{j=1}^{N}\langle h_{1i}h_{2i}h_{1j}h_{2j}\rangle+\sum_{i=j=1}^{N}\sigma_{n_2}^2\sigma_h^2+\sum_{i=j=1}^{N}\sigma_{n_1}^2\sigma_h^2 && (22)\\
&= N\sigma_{n_1}^2\sigma_{n_2}^2+\sum_{i=1}^{N}\sum_{j=1}^{N}\langle h_{1i}h_{2i}h_{1j}h_{2j}\rangle+N\sigma_{n_2}^2 S_h+N\sigma_{n_1}^2 S_h && (23)
\end{aligned}$$

The term $\sum_{i=1}^{N}\sum_{j=1}^{N}\langle h_{1i}h_{2i}h_{1j}h_{2j}\rangle$ is a bit complicated. In that case,

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N}\langle h_{1i}h_{2i}h_{1j}h_{2j}\rangle &= \sum_{i=1}^{N}\sum_{j=1}^{N}\langle h_{1i}h_{2i}\rangle\langle h_{1j}h_{2j}\rangle+\sum_{i=1}^{N}\sum_{j=1}^{N}\langle h_{1i}h_{1j}\rangle\langle h_{2i}h_{2j}\rangle+\sum_{i=1}^{N}\sum_{j=1}^{N}\langle h_{1i}h_{2j}\rangle\langle h_{2i}h_{1j}\rangle && (24)\\
&= (N^2+2N)S_h^2 && (25)
\end{aligned}$$

Then variance of cross correlation statistic is

$$\mathrm{Var}\,(Y) = N\sigma_{n_1}^2\sigma_{n_2}^2+(N^2+2N)S_h^2+N\sigma_{n_2}^2 S_h+N\sigma_{n_1}^2 S_h-N^2 S_h^2 \tag{26}$$

$$\boxed{\mathrm{Var}\,(Y) = N\sigma_{n_1}^2\sigma_{n_2}^2+2NS_h^2+N\sigma_{n_2}^2 S_h+N\sigma_{n_1}^2 S_h} \tag{27}$$

Case I When h is constant then $S_h = h^2$ and in weak signal limit (h«1) higher order terms having h in the variance can be neglected. Then

$$\mathrm{Var}\,(Y) = N\sigma_{n_1}^2\sigma_{n_2}^2 \tag{28}$$

and SNR is given by,

$$\boxed{\rho_1 = \frac{\langle Y\rangle}{\sqrt{\mathrm{Var}\,(Y)}} = \frac{Nh^2}{\sqrt{N}\sigma_{n_1}\sigma_{n_2}}} \tag{29}$$

also given $\sigma_{n_1} = \sigma_{n_2} = 1$ then SNR $\rho_1 = \sqrt{N}h^2$. The SNR threshold is ρ_0. Hence we want $\sqrt{N}h^2 \geq \rho_0 \implies N \geq \rho_0^2/h^4$. If one data sample is of δT duration, then required time should be $N\delta T$. In LIGO detectors, typical sampling rate used is 4096Hz i.e. $\delta T \sim 2\times 10^{-4}s$. If $\rho_0 = 1$ and $h = 10^{-3}$, then required time $T \sim 2\times 10^{8}s \sim 6.34y$. As h will decrease further, required time will increase.

Case II When h is Gaussian random variable and in weak signal limit (h«1) i.e. noise intrinsic in detectors is much larger than signal then the term S_h^2 can be neglected. Then

$$\mathrm{Var}\,(Y) = N\sigma_{n_1}^2\sigma_{n_2}^2 + N\sigma_{n_2}^2 S_h + N\sigma_{n_1}^2 S_h = N\sigma_{n_1}^2\sigma_{n_2}^2\left(1 + \frac{S_h}{\sigma_{n_1}^2} + \frac{S_h}{\sigma_{n_2}^2}\right) \tag{30}$$

and observed SNR

$$\rho_2 = \frac{\sqrt{N}S_h}{\sigma_{n_1}\sigma_{n_2}}\left(1 + \frac{S_h}{\sigma_{n_1}^2} + \frac{S_h}{\sigma_{n_2}^2}\right)^{-1/2} \tag{31}$$

$$\boxed{\rho_2 = \rho_1\left(1 - \frac{1}{2}\left(\frac{S_h}{\sigma_{n_1}^2} + \frac{S_h}{\sigma_{n_2}^2}\right)\right)} \tag{32}$$

Given $\sigma_{n_1} = \sigma_{n_2} = 1 \implies \rho_2 = \rho_1(1 - S_h)$. To detect h, $\rho_2 \geq \rho_0 \implies \rho_1(1 - S_h) \geq \rho_0 \implies \sqrt{N}S_h(1 - S_h) \geq \rho_0$

$$N \geq \frac{\rho_0^2}{S_h^2(1 - S_h)^2} \tag{33}$$

Considering similar values as previous case, but including uncertainty $S_h \sim 10^{-6}$, then required time $T \sim 2 \times 10^8 s \sim 6.34y$.

2 Part I of Day 5 Problem

Eq.(93) in lecture notes: Assuming the Gaussian additive noise, the C_{ft}'s are Gaussian distributed . Hence likelihood for the CSD C_{ft}'s obtained from one time segment t and one frequency bin having frequency f and $f + df$ is given as

$$L \propto \exp\biggl(-\frac{1}{2}(C_{ft} - H(f)\mathcal{P}_\alpha\gamma_{\alpha,ft})^* \frac{1}{P_1(f,t)P_2(f,t)}(C_{ft} - H(f)\mathcal{P}_\beta\gamma_{\beta,ft})\biggr) \tag{34}$$

The combined likelihood for a estimator obtained from combining multiple independent time segments and frequency bins can be written as

$$L \propto \prod_{tf}\exp\biggl(-\frac{1}{2}(C_{ft} - H(f)\mathcal{P}_\alpha\gamma_{\alpha,ft})^* \frac{1}{P_1(f,t)P_2(f,t)}(C_{ft} - H(f)\mathcal{P}_\beta\gamma_{\beta,ft})\biggr) \tag{35}$$

The log likelihood is given as

$$\mathcal{L} := ln(L) \propto \sum_{tf} -\frac{1}{2}(C_{ft} - H(f)\mathcal{P}_\alpha\gamma_{\alpha,ft})^* \frac{1}{P_1(f,t)P_2(f,t)}(C_{ft} - H(f)\mathcal{P}_\beta\gamma_{\beta,ft}) \tag{36}$$

The estimator $\hat{\mathcal{P}}_\alpha$ which maximizes the log likelihood $\mathcal{L}$ is called, the maximum likelihood estimator i.e.

$$\frac{\partial}{\partial\mathcal{P}_\alpha}\mathcal{L}|_{\mathcal{P}_\alpha = \hat{\mathcal{P}}_\alpha} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial\mathcal{P}_\alpha^*}\mathcal{L}|_{\mathcal{P}_\alpha^* = \hat{\mathcal{P}}_\alpha^*} = 0 \tag{37}$$

$$\frac{\partial}{\partial\mathcal{P}_\beta^*}\biggl(\sum_{tf}(C_{ft} - H(f)\mathcal{P}_\alpha\gamma_{\alpha,ft})^* \frac{1}{P_1(f,t)P_2(f,t)}(C_{ft} - H(f)\mathcal{P}_\beta\gamma_{\beta,ft})\biggr) = 0 \tag{38}$$

$$\frac{\partial}{\partial\mathcal{P}_\beta^*}\biggl(\sum_{tf}\biggl(\frac{C_{ft}^*C_{ft}}{P_1(f,t)P_2(f,t)} - \frac{H(f)\mathcal{P}_\alpha^*\gamma_{\alpha,ft}^*C_{ft}}{P_1(f,t)P_2(f,t)} - \frac{C_{ft}^*H(f)\mathcal{P}_\alpha\gamma_{\alpha,ft}}{P_1(f,t)P_2(f,t)} + \frac{H^2(f)\mathcal{P}_\alpha\mathcal{P}_\beta^*\gamma_{\beta,ft}^*\gamma_{\alpha,ft}}{P_1(f,t)P_2(f,t)}\biggr)\biggr) = 0 \tag{39}$$

$$2\sum_{ft}\frac{H^2(f)\hat{\mathcal{P}}_\alpha\gamma_{\beta,ft}^*\gamma_{\alpha,ft}}{P_1(f,t)P_2(f,t)} - 2\sum_{ft}\frac{H(f)\gamma_{\beta,ft}^*C_{ft}}{P_1(f,t)P_2(f,t)} = 0 \tag{40}$$

$$\hat{\mathcal{P}}_\alpha\biggl[\sum_{ft}\frac{H^2(f)\gamma_{\beta,ft}^*\gamma_{\alpha,ft}}{P_1(f,t)P_2(f,t)}\biggr] = \sum_{ft}\frac{H(f)\gamma_{\beta,ft}^*C_{ft}}{P_1(f,t)P_2(f,t)} \tag{41}$$

The dirty map can be defined as (Eq.(94)) in lecture notes

$$X_\beta = \sum_{ft} \frac{H(f)\gamma^*_{\beta,ft} C_{ft}}{P_1(f,t)P_2(f,t)} \tag{42}$$

and the fisher matrix (which is noise covariance matrix of the dirty map) is defined as

$$\Gamma_{\alpha\beta} = \sum_{ft} \frac{H^2(f)\gamma^*_{\beta,ft}\gamma_{\alpha,ft}}{P_1(f,t)P_2(f,t)} \tag{43}$$

Hnece the clean map estimators are given by,

$$\hat{\mathcal{P}}_\alpha = (\Gamma^{-1})_{\alpha\beta} X_\beta \tag{44}$$

3 Part II of Day 5 Problem

It is implied that the dirty map is resultant of the convolution of the fisher with true map i.e.

$$\mathbf{X} = \boldsymbol{\Gamma}.\boldsymbol{\mathcal{P}} + \mathbf{n} \tag{45}$$

The clean map estimator in Eq.44 are unbiased estimator of the true estimators i.e.

$$\mathcal{P}_\alpha = \langle \hat{\mathcal{P}}_\alpha \rangle \tag{46}$$

Now the true cross power spectrum C_l in the spherical basis is defined as

$$C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} \mathcal{P}_{lm} \mathcal{P}^*_{lm} = \frac{1}{2l+1} \sum_{m=-l}^{l} \langle \hat{\mathcal{P}}_{lm} \rangle \langle \hat{\mathcal{P}}^*_{lm} \rangle \tag{47}$$

The expectation value of the estimated cross power spectrum $\hat{C}_l$ is defined as

$$\langle \hat{C}_l \rangle = \frac{1}{2l+1} \sum_{m=-l}^{l} \langle \hat{\mathcal{P}}_{lm} \hat{\mathcal{P}}^*_{lm} \rangle \tag{48}$$

Now,

$$\langle \hat{C}_l \rangle - C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} \langle \hat{\mathcal{P}}_{lm} \hat{\mathcal{P}}^*_{lm} \rangle - \langle \hat{\mathcal{P}}_{lm} \rangle \langle \hat{\mathcal{P}}^*_{lm} \rangle \tag{49}$$

Using Eq. (98) in lecture notes

$$\langle \hat{C}_l \rangle - C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} (\Gamma^{-1})_{lm,lm} \tag{50}$$

So after correcting for the bias , the correct estimator will be

$$\hat{C}_l^{reg.} = \hat{C}_l - \frac{1}{2l+1} \sum_{m=-l}^{l} (\Gamma^{-1})_{lm,lm} \tag{51}$$

References

[1] Bruce Allen and Joseph D. Romano. "Detecting a stochastic background of gravitational radiation: Signal processing strategies and sensitivities". In: *Phys. Rev. D* 59 (10 Mar. 1999), p. 102001. DOI: 10.1103/PhysRevD.59.102001. URL: https://link.aps.org/doi/10.1103/PhysRevD.59.102001.