NEUTRON STARS

Static, spherically symmetric stars in GR

(MTW)

Start with a general metric

(Using geometric units $G=c=1$)

$$ds^2 = -a^2dt^2 - 2ab\,dr\,dt + c^2dr^2 + R^2d\Omega^2 \quad \text{— (1)}$$

coefficients are fns of r only since the metric is static & spherically symmetric

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

Define new time coordinate: t' by

$$e^{\Phi}dt' = a\,dt + b\,dr \quad \text{—— (2)}$$

Using this, (1) $\Rightarrow$

$$ds^2 = -e^{2\Phi}dt'^2 + (b^2+c^2)\,dr^2 + R^2d\Omega^2$$

or

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + R^2d\Omega^2 \quad \text{— (3)}$$

dropped the prime

$e^{2\Lambda} \equiv b^2+c^2$

$\Phi(r)$, $\Lambda(r)$ and $R(r)$ are only fns of r.

We can also introduce a new radial coordinate $r' = R(r)$ so that

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2d\Omega^2 \quad \text{— (4)}$$

omitted the prime

In these Schwarzschild/curvature coordinates, the line element has only two unknown functions $\Phi(r)$ and $\Lambda(r)$.

Interpretation of the r coordinate = proper circumstance on a 2-sphere / 2π

We can see this by setting $t=\text{const}$, $r=\text{const}$, so that $ds = r\,d\Omega$

$$\oint_{\theta=\frac{\pi}{2}} ds = \int_0^{2\pi} r\,d\phi = 2\pi r$$

and $A = \int (r\,d\theta)(r\sin\theta\, d\phi) = 4\pi r^2$

Hence $r = \left[\dfrac{\text{proper area of a 2-sphere}}{4\pi}\right]^{1/2}$

The time coordinate is chosen such that the metric is static. (in variant under $t \to t+\Delta t$) and is normalized to be equal to Minkowski time at $r\to\infty$. ($\Phi(\infty)=0$)

To describe matter inside the star, perfect fluid is a good approximation.

characterized by the rest frame mass density ρ and isotropic pressure p. No shear stresses and energy transport during the hydrodynamical time scale

$$\underbrace{T^{\overset{0,1,2,3}{\mu\nu}}}_{\text{stress energy tensor}} = (\rho + p)\, u^\mu u^\nu + p\, g^{\mu\nu} \quad — (5)$$

$\rho(r)$ mass-energy density in the fluid rest frame

isotropic pressure

inverse of the metric $g_{\mu\nu}$

$$g^{\mu\nu} = \begin{bmatrix} -e^{-2\Phi} & & & \\ & e^{-2\Lambda} & & \\ & & r^{-2} & \\ & & & r^{-2}\sin^{-2}\theta \end{bmatrix}$$

4-velocity of the fluid

$$u^r = \frac{dr}{d\tau} = 0 \qquad u^\theta = \frac{d\theta}{d\tau} = 0 \qquad u^\phi = \frac{d\phi}{d\tau} = 0 \quad — (6) \quad \text{for the star to be static}$$

The normalization of the 4-velocity

$$-1 = \vec{u}\cdot\vec{u} = g_{\mu\nu} u^\mu u^\nu = g_{tt} u^t u^t = -e^{2\Phi} u^t u^t$$

all other ~~compo~~ u^μ are zero

$$\Rightarrow u^t = e^{-\Phi} \quad — (7)$$

Thus the non-zero components of the stress energy tensor become

$$T^{tt} = \rho e^{-2\phi}, \quad T^{rr} = p e^{-2\Lambda}, \quad T^{\theta\theta} = p r^{-2}, \quad T^{\phi\phi} = p r^{-2}\sin^{-2}\theta \quad — (8)$$

or

$$T_{tt} = \rho e^{2\Phi}, \quad T_{rr} = p e^{2\Lambda}, \quad T_{\theta\theta} = p r^2, \quad T_{\phi\phi} = p r^2 \sin^2\theta \quad — (9)$$

$$T_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} T^{\mu\nu}$$

We need to solve the Einsteins eqns

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} \rightarrow G_{\mu\nu} = 8\pi T_{\mu\nu} \quad — (10)$$

The components of the Einstein tensor can be calculated as

$$\left.\begin{aligned} G_{tt} &= \frac{1}{r^2} e^{2(\Phi - \Lambda)} \left(2r\Lambda' - 1 + e^{2\Lambda}\right) \\ G_{rr} &= \frac{1}{r^2}\left(2r\Phi' + 1 - e^{2\Lambda}\right) \end{aligned}\right\} \quad — (11)$$

$$\phi' = \frac{\partial\phi}{\partial r} \qquad \Lambda' = \frac{\partial\Lambda}{\partial r}$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}\left[\Phi'' + \Phi'^2 - \Phi'\Lambda' + \frac{1}{r}(\phi' - \Lambda')\right], \quad G_{\phi\phi} = \sin^2\theta\, G_{\theta\theta}$$

$$\phi'' = \partial^2\phi/\partial r^2$$

From the Einsteins eqns

mass-energy density

$$\frac{1}{r^2} e^{-2\Lambda}\left(2r\Lambda' - 1 + e^{2\Lambda}\right) = 8\pi\rho \quad \leftarrow \text{tt component}$$

$$\frac{1}{r^2} e^{-2\Lambda}\left(2r\Phi' + 1 - e^{2\Lambda}\right) = 8\pi p \quad \leftarrow \text{rr component} \qquad (12)$$

$$e^{-2\Lambda}\left(\phi'' + \phi'^2 - \phi'\Lambda' + \frac{1}{r}(\phi' - \Lambda')\right) = 8\pi p \quad \leftarrow \theta\theta \text{ and } \phi\phi \text{ components (give the same eqn)}$$

From the tt eqn

$$\frac{d}{dr}\left[r\left(e^{-2\Lambda} - 1\right)\right] = -8\pi\rho r^2$$

$$d\left[r\left(e^{-2\Lambda} - 1\right)\right] = -8\pi\rho r^2 dr \quad — (13)$$

$$\frac{d}{dr}\left(r\left(e^{-2\Lambda} - 1\right)\right) = e^{-2\Lambda} - 1 + r\left[-2e^{-2\Lambda}\Lambda'\right]$$
$$= e^{-2\Lambda} - 1 - 2r\Lambda' e^{-2\Lambda}$$
$$= e^{-2\Lambda}\left(1 - e^{2\Lambda} - 2r\Lambda'\right)$$
$$= -e^{-2\Lambda}\left(2r\Lambda' - 1 + e^{2\Lambda}\right)$$

This can be integrated from $r' = 0$ to ~~R~~ r

$$\int_0^r d\left[r'\left(e^{-2\Lambda} - 1\right)\right] = -2\int_0^r 4\pi r'^2 \rho \, dr'$$

$$r\left(e^{-2\Lambda} - 1\right) = -2m(r)$$

$$\boxed{m(r) = \int_0^r 4\pi r'^2 \rho(r') dr'}$$

mass enclosed in the radius r

note that $\rho(r)$ is the mass-energy density (unlike the Newtonian defn)

$$\therefore e^{2\Lambda} = \left[1 - \frac{2m(r)}{r}\right]^{-1} \quad — (14)$$

~~We can also also define~~ anot

The total mass inside a radius R

$$\boxed{M = \int_0^R 4\pi r^2 \rho(r) dr} \quad — (15)$$

Difference in M and M_G is the gravitational binding energy

We can also define the gravitational mass

$$\boxed{M_G = \int_0^R \cancel{4\pi r^2}\, \rho(r) \underbrace{4\pi r^2 \left(e^{\Lambda} dr\right)}_{\substack{dV \\ \text{(volume element)} \\ \sqrt{\det(g)}}} = 4\pi \int_0^R \frac{\rho(r) r^2 dr}{\left[1 - \frac{2m(r)}{r}\right]^{1/2}}} \quad — (16)$$

GRAVITATIONAL MASS

In BNS systems we measure the gravitational mass

From the rr component of (12)

$$2r\Phi' + 1 - e^{2\Lambda} = 8\pi p r^2 e^{2\Lambda} \qquad \left(1-\frac{2m}{r}\right)^{-1}$$

$$\frac{d\Phi}{dr} = \frac{8\pi p r^2 e^{2\Lambda} + e^{2\Lambda} - 1}{2r} = \left(1 - \frac{2m(r)}{r}\right)^{-1} \frac{(8\pi p r^2 + 1)}{2r} - \frac{1}{2r}$$

$$\boxed{\frac{d\Phi}{dr} = \frac{4\pi p r^3 + m}{r(r-2m)}} \quad \text{— (17)}$$

TOV eqn 1

from the r-derivative of $\frac{d\Phi}{dt}$

To solve the $\theta\theta$ component of (12) we need Φ', Φ'' and Λ'.
To get Λ', we use the 'tt' eqn (12).

$$2r\Lambda' - 1 + e^{2\Lambda} = 8\pi\rho r^2 e^{2\Lambda} \qquad \cancel{= e^{2\Lambda}(8\pi\rho r^2 - 1)}$$

$$\Lambda' = \frac{(8\pi\rho r^2 - 1)e^{2\Lambda} + 1}{2r}$$

Plugging in Φ', Φ'' and Λ' in the '$\theta\theta$' eqn of (12)

$$\boxed{\frac{dp}{dr} = -\frac{(\rho+p)(m + 4\pi r^3 p)}{r(r-2m)}} \quad \text{— (18)}$$

TOV eqn 2

The differential form of the mass eqn

$$\boxed{\frac{dm}{dr} = 4\pi r^2 \rho.} \quad \text{— (19)}$$

TOV eqn 3

~~Along t with~~

These eqns (TOV eqns) along with an eqn of state constitute the relativistic eqns of stellar structure. (inside the star).

Exterior soln:

~~Mass~~ Density and pressure vanish. No need of eqns (18) and (19)

Eqn (17) becomes

$$\frac{d\Phi}{dr} = \frac{m}{r(r-2m)}$$

With the boundary condn $\phi(r) \to 0$ as $r \to \infty$, this can be solved

$$e^{2\phi} = \left(1 - \frac{2GM}{r}\right) \quad —(20) \qquad \text{for } r > R_*$$

This gives the Schwarzschild metric as the external soln

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2} \quad —(21)$$

Numerically solving the TOV eqns:

Boundary conds

$$m(r=0) = 0$$
$$\rho(r=0) = \rho_c$$
$$P(r=0) = P(\rho_c)$$
$$\Phi(r=R_*) = \frac{1}{2}\ln\left(1 - \frac{2GM}{Rc^2}\right)$$

- First solve $\frac{dp}{dr}$, $\frac{dm}{dr}$, $P(\rho)$ from $r=0$ to R.
- Then solve $\frac{d\Phi}{dr}$ from $r=R$ to 0.

Comparing with the hydro. equilibrium in Newtonian gravity:

$$\frac{dp}{dr} = \underbrace{\frac{-m\rho}{r^2}}_{\text{Newtonian part}} \underbrace{\left[1 + \frac{p}{\rho}\right]\left[\frac{1+4\pi r^3 p}{m}\right]\left[1 - \frac{2m}{r}\right]^{-1}}_{\text{GR corrections}}$$

GR corrections increases the pressure gradient. (larger numerator and smaller denominator)

⇓

GR predicts stronger grav. forces in a stationary body than Newtonian gravity → more compact stars

→ Some stars can also be unstable due to GR corrections (while Newtonian gravity predicts hydro. equilibrium).

Obvious example: No star can have $\frac{2m(r)}{r} \geq 1$. $\left(\frac{dp}{dr} = \infty\right)$