STELLAR STRUCTURE

Consider a spherically symmetric star in ~~equilibrium~~ steady state. The eqs of hydrostatic equilibrium are (Newtonian gravity)

$$\boxed{\frac{dP}{dr} = -G\frac{M(r)\rho(r)}{r^2}} \quad -(1)$$

$$\boxed{\frac{dM(r)}{dr} = \rho(r)4\pi r^2} \quad -(2)$$

We have three variables $P(r)$, $\rho(r)$ and $M(r)$. and 2 eqns. Need one more eqn to close the system – an EOS.

The EOS is given by

temperature

$$\boxed{P = n k_B T + \frac{1}{3} a T^4} \quad -(3)$$

gas pressure — radiation pressure

num density of gas particles

Stefan-Boltzmann const

IDEAL GAS EOS

{good approx. since the particle size (10^{-15} m) << inter-particle separation (10^{-10} m)}

The EOS is a fn of two variables $P(\rho, T)$. Thus we need one more eqn, giving the evolution of temperature with radius.

This is derived from the radiative equilibrium ~~assumptio~~ condition

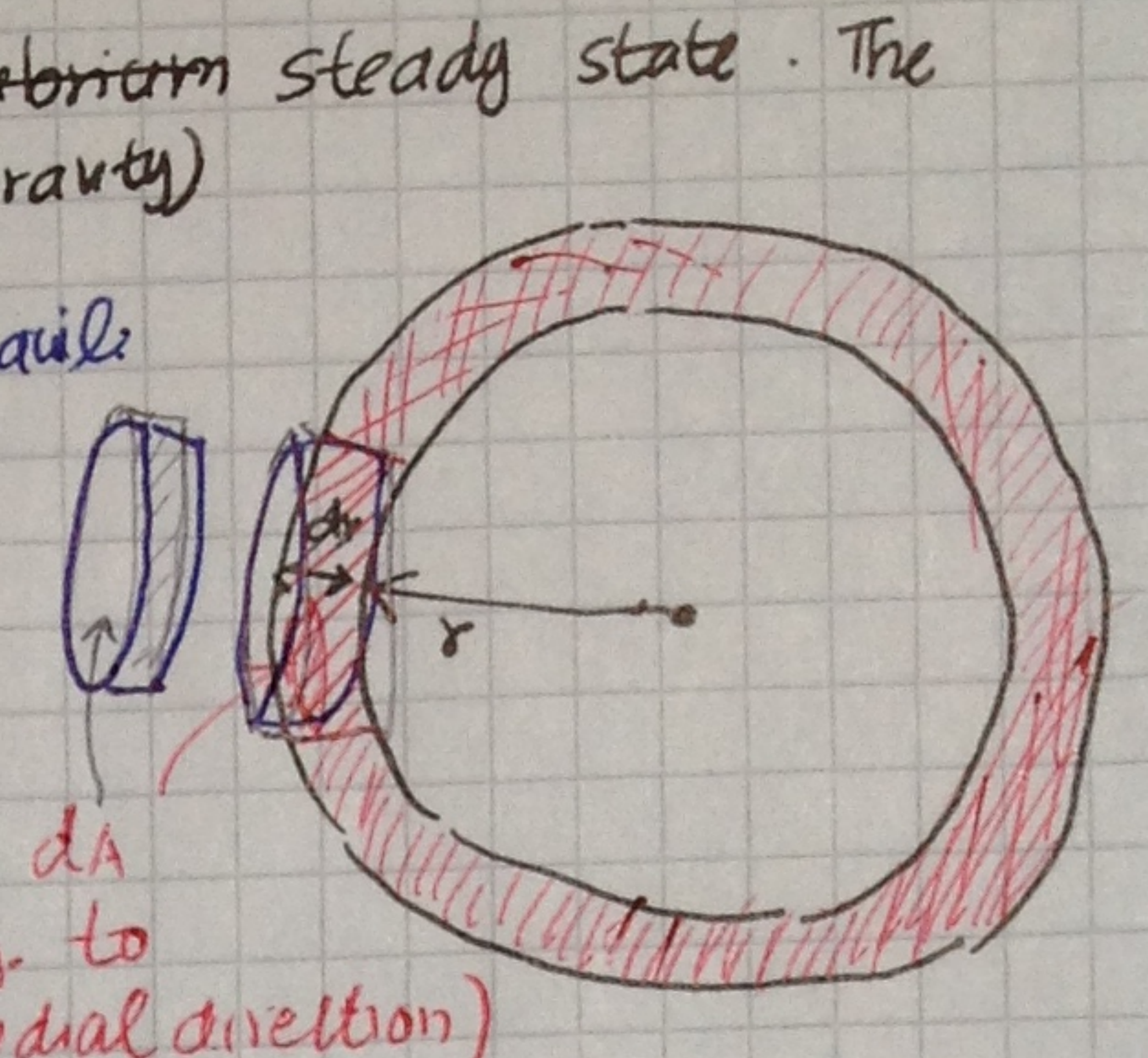
opacity of the material

$$\boxed{\frac{dT(r)}{dr} = -\frac{3}{4ac}\frac{\kappa\rho(r)}{T^3(r)}\frac{L(r)}{4\pi r^2}} \quad -(4)$$

Luminosity

Now we have 4 eqns to solve for 5 ~~four~~ unknowns $\rho(r)$, $P(r)$, $T(r)$, $M(r)$, $L(r)$! Need one more eqn to close the system.

Hydrostatic equil: Derivation:



$$dm = \rho\, dA\, dr$$

Mass interior to a radius r

$$m(r) = \int_0^r \rho(r) 4\pi r^2 dr$$

The net outward pressure on dm is

$$-[P(r+dr) - P(r)]\, dA$$

(Convention: Direction towards gravity is +ve)

So, in equilibrium

$$-\frac{dP}{dr} dr\, dA = \frac{G m(r) dm}{r^2}$$

$$\frac{dP}{dr} = -\frac{G m(r)}{r^2}\frac{dm}{dr\, dA}$$

$$= -\frac{G m(r)}{r^2}\frac{\cancel{\rho(r)4\pi r^2}}{\cancel{dA}}\frac{\rho(r) dA dr}{dr dA}$$

$$\frac{dP}{dr} = -\frac{G m(r)\rho(r)}{r^2}$$

Thermal equilibrium: Derivation

The total luminosity crossing a surface of rad. r is

energy generated per unit mass per unit time

$$L(r) = \int_0^r \varepsilon\rho\, 4\pi r^2 dr$$

$$\therefore \frac{dL(r)}{dr} = \varepsilon\rho(r)\, 4\pi r^2$$

Let $\epsilon(\rho, T)$ denotes the energy generated per unit mass of the star at density ρ and temp T. From the defn of the luminosity

$$\boxed{\frac{dL(r)}{dr} = 4\pi r^2 \rho \epsilon} \quad -(5)$$

thermal equilibrium

Note that ϵ is a fn of ρ and T

~~Ef~~ The form of $\epsilon(\rho, T)$ ~~depends on the dominant~~

~~nuclear reactions providing the energy in the star,~~

~~A power law approximation is~~

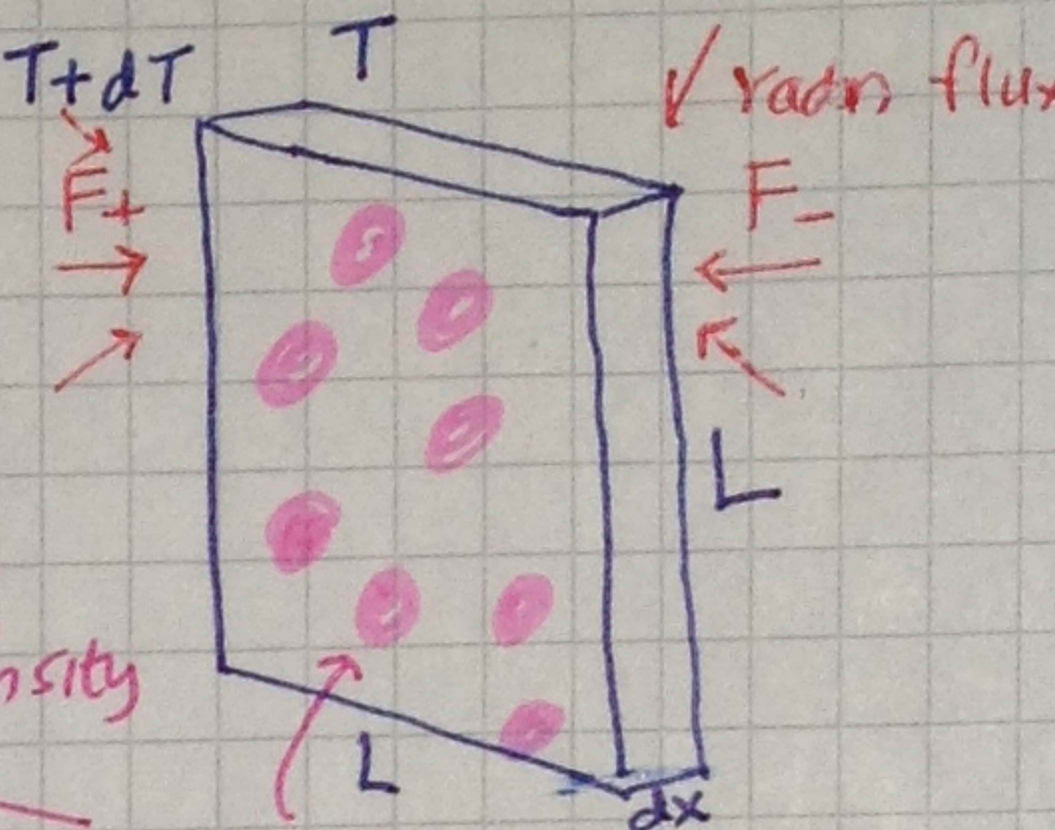
~~$\epsilon = \epsilon_0 \rho^{\lambda} T^{\nu}$~~

~~Where~~

~~$\lambda = 1$, $\nu = 4$ for p-p chain~~

~~$\lambda = 1$, $\nu = 15-18$ for CNO cycle, etc.~~

Radiative equilibrium: Derivation



Mean free path

$$l = \frac{1}{n\sigma} = \frac{1}{\kappa\rho}$$

Satterers per unit vol

density

opasity (abs. coeff per unit mass) $\left(\frac{\text{Area}}{\text{mass}}\right)$

effective scattering cross section σ (dimension ~ Area)

Probability of absorption by the slab of area L^2 and thickness dx

$$P = \frac{\text{Tot. area. of absorbers}}{\text{Tot. are of the slab}} = \frac{\sigma n L^2 dx}{L^2}$$

$$P = n\sigma dx = \kappa\rho dx = dx/l$$

The corresponding decrease in the beam intensity

$$dI = -I\frac{dx}{l} \quad \text{or} \quad \frac{dI}{dx} = -I/l$$

$$\therefore \boxed{I = I_0 e^{-x/l}}$$

The incident intensity decreases ~~with~~ exponentially with characteristic length l

Temp. on the two sides is different → ~~differen~~ pressure gradient → net force on the slab. → imparts a momentum on the slab. For the slab to remain in equilibrium it has to use up this momentum, The slab absorbs this momentum and use it to supplement the gas pressure. in its effort to support itself against gravity.

The ~~fraction of the~~ radiation flux absorbed by the slab will be (per unit area)

$$F \underbrace{\kappa}_{\text{area/mass}} \underbrace{\rho dx}_{\text{mass/area}}$$

Incident radn flux (Energ/time/area)

~~Momentum absorbed by unit surface area of the slab per unit time~~ $= F\kappa\rho dx/c$

SOLVING THE EQNS OF STELLAR STRUCTURE

Given the functions $\kappa(\rho, T)$ and $\epsilon(\rho, T)$ we can integrate eqs (1)-(5) to determine the equilibrium structure of stars.

Boundary condns:

- $M(r),\ L(r) \to 0$ as $r \to 0$
- $P(r),\ \rho(r) \to$ as $r \to R$ (surface)
- ~~Also, we use an~~ approximation $T(r) \to 0$ as $r \to R$

(Actual treatment of the surface boundary condns is much more complicated. But the above conditions provide a reasonable approx.)

It is often more convenient to rewrite these eqns (1-5) in terms of $M_r := M(r)$ and to treat $r(M_r)$ as a dependent relation.

Dividing (1), (4), (5) by (2) ⇒

$M_r = M(r)$

$$\frac{dP}{dM_r} = -\frac{G M_r}{4\pi r^4} \quad \text{— (6)}$$

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho} \quad \text{— (7)}$$

$L_r := L(r)$

$$\frac{dT}{dM_r} = -\frac{3\kappa}{64\pi^2 ac}\,\frac{1}{T^3}\left(\frac{L_r}{r^4}\right) \quad \text{— (8)}$$

$$\frac{dL_r}{dM_r} = \epsilon \quad \text{— (9)}$$

Momentum absorbed by the slab per unit volume per unit time is $\mathcal{F}\,\kappa\rho / c$.

For the star to be in equilibrium, the momentum should ~~by~~ be absorbed by the gas, increasing its pressure. ~~This is the force per unit volume.~~

$$\frac{dP_r}{dr} = -\mathcal{F}\,\frac{\kappa\rho}{c}, \quad \text{or} \quad \boxed{\mathcal{F} = -\frac{c}{\kappa\rho}\frac{dP_r}{dr}}$$

gradient of radn pressure (force)

energy flux

Net flux is prop. to pressure gradient & inv. prop. to κ

From Stefan's law $P_r = \frac{1}{3} a T^4$

$$\therefore \mathcal{F}(r) = -\frac{4acT^3}{3\rho\kappa}\left(\frac{dT}{dr}\right)$$

$$\boxed{\therefore \frac{dT}{dr} = -\frac{3}{4ac}\,\frac{\kappa\rho(r)}{T^3}\,\mathcal{F}(r)}$$

$\mathcal{F}(r) = \frac{L(r)}{4\pi r^2}$

Eq. Radiative equilibrium

With the following boundary condns (4 first order ODEs. Hence 4 BCs)

- At the center ($M_r = 0$), $r = L_r = 0$ —— (10)
- At the surface ($M_r = M$), $\rho = T = 0$ —— (11)

Surface BCs are approx. (for eg $T \neq 0$ actually). Accurate modelling of the stellar atmosphere needs semiemperical relationships and matching with interior solns. However the above BCs provide reasonable approx. (for the interior parts).

7 variables: $r(M_r)$, $\rho(M_r)$, ~~r(Mr)~~, $P(M_r)$, $L_r(M_r)$, T ~~(Mr)~~ ϵ ~~(Mr)~~, $\kappa(M_r)$

4 ODEs + 4 BCs for r, ρ, L, T
(6), (7), (8), (9) (10), (11)

3 auxiliary eqns for P, ϵ, κ

System is fully determined.

Strictly speaking this description is only valid for timescales that are small compared to the nuclear evolution time scales of the star (zero age main sequence - ZAMS - star). [time independent].

We have also assumed that

- The star is spherically symmetric. (ignore rotation and magnetic fields).
- Simple boundary condns at the surface; ignore stellar winds.
- Newtonian theory of gravity.
- The energy transport is entirely ~~due~~ radiative. (~~good~~ reasonable approx for low mass stars $M < M_\odot$)

~~Attx~~

Solving the stellar sturcuture eqns numerically:

- Note the BCs are prescribed partly at the center and partly at the surface. Not straightforward to integrate outward from center. Need to use ~~th~~ numerical techniques to solve boundary value problems (eg. shooting method)
- The pressure P, opacity κ and energy generation ϵ have very complicated relations with the internal composition as well as on ρ and T
  - * P depends on n (number density of particles (ideal gas)), which depends on the mass fraction of various elements in the star: $X, Y, Z := 1-(X+Y)$

mass fraction of hydrogen — helium — heavy elements (metalicity)

* κ depends on the processes that absorbs/ scatter photons inside the star. Eg.

  - ~~electr~~ electron scattering.
  - bound-free absorption
  - free-free absorption, etc.

  In general $\kappa = \kappa(\rho, T, \text{chemical composition})$

* $\epsilon(\rho, T)$ depends on the dominant nuclear reactions, eg

  - p-p chain
  - CNO cycle, etc.

However, one can use some ~~eq~~ ~~a~~ simple forms of ~~$P\epsilon\rho$~~ P, κ, ϵ to construct simple stellar models.

o Here we have assumed that the energy transport is entirely radiative. This is not true actually. For eg. high-mass stars have a convective core. Approximate treatment of convective transfer can be formulated. This will change the ~~dt~~ $\frac{dT}{dM_r}$ eqn.

TOY STELLAR MODELS

The stellar structure eqns require input functions $P(\rho, T)$, $\kappa(\rho, T)$ and $\epsilon(\rho, T)$.

Power-law approximations:

$$\rho = \rho_0 P^a T^{-b} \qquad \kappa = \kappa_0 \rho^n T^{-s} \qquad \epsilon = \epsilon_0 \rho^\lambda T^\nu \quad \text{— (12)}$$

valid when a single process makes the dominant contribution to pressure & opacity

valid in limited ranges (depending on which nuclear reactions dominate the energy production)

When this is true, the stellar structure eqns involve only powers of the variables.

We introduce dimensionless variables p, t, x, l

$$P = P_c\, p \qquad T = T_c\, t \qquad r = R x \qquad L_r = L\, l \qquad M_r = M m \quad \text{— (13)}$$

central press.   central temp   ~~Stellar rad.~~   ~~Surface luminosity~~   ~~mass of the star~~

With these, the stellar structure eqns reduce to

$$\frac{dp}{dm} = -\left(\frac{GM^2}{4\pi R^4 P_c}\right)\frac{m}{x^4} \quad \cancel{= -\frac{m}{x^4}}, \text{ ~~where we choose~~ } \cancel{M} \quad \text{— (14)}$$

$$\frac{dx}{dm} = \frac{M}{4\pi R^3 \rho}\frac{1}{x^2} = \frac{M}{4\pi R^3 \rho_0 P_c^a p^a T_c^{-b} t^{-b}}\frac{1}{x^2} = \left(\frac{M}{4\pi R^3 \rho_0}\frac{T_c^b}{P_c^a}\right)\frac{t^b}{p^a x^2} \quad \text{(15)}$$

$$\frac{dt}{dm} = -\left(\frac{3\kappa_0 \rho_0^n}{64\pi^2 ac}\frac{P_c^{an} ML}{T_c^{4+s+nb} R^4}\right)\frac{p^{an} l}{x^4 t^{3+bn+s}} \quad \text{— (16)}$$

$$\frac{dl}{dm} = \frac{M}{L}\epsilon_0 \rho^\lambda T^\nu = \frac{M}{L}\epsilon_0 \rho_0^\lambda p^{a\lambda} T^{-b\lambda+\nu} = \left(\frac{M}{L}\epsilon_0 \rho_0^\lambda P_c^{a\lambda} T_c^{\nu-b\lambda}\right) p^{a\lambda} t^{\nu-b\lambda} \quad \text{(17)}$$

We # choose M, R, L from the following conditions

$$\frac{GM^2}{4\pi R^4 P_c} = 1, \quad \frac{M}{4\pi R^3 \rho_0}\frac{T_c^b}{P_c^a} = 1, \quad \frac{3\kappa_0 \rho_0^n}{64\pi^2 ac}\frac{P_c^{an} ML}{T_c^{4+s+nb} R^4} = 1 \quad \text{— (18)}$$

Using these substitutions (14) – (17) becomes

$$\boxed{\frac{dp}{dm} = \frac{-m}{x^4}, \quad \frac{dx}{dm} = \frac{t^b}{x^2 p^a}, \quad \frac{dt}{dm} = \frac{-p^{an}\,\ell}{x^4 t^{3+s+bn}}, \quad \frac{d\ell}{dm} = A p^{a\lambda} t^{\nu - b\lambda}} \quad \text{— (19)}$$

where

$$A = \left(\frac{M}{L}\right) \epsilon_0 \rho_0^{\lambda} P_c^{a\lambda} T_c^{\nu - b\lambda}$$

$$A = \frac{3\kappa_0 \rho_0^{n+\lambda}}{16\pi G a c} \, \epsilon_0 \, \frac{P_c^{an + a\lambda + 1}}{T_c^{4+s+nb+b\lambda-\nu}}$$

(substituting for $\frac{M}{L}$ from 18)

The 4 first order ODEs in (19) can be solved for the 4 variables $p(m)$, $x(m)$, $t(m)$, $\ell(m)$, with the four following BCs.

$$p(0) = 1, \quad t(0) = 1 \qquad \text{[from defn (13)]}$$
$$x(0) = 0, \quad \ell(0) = 0 \qquad \text{— (20)}$$

These ODEs can be integrated forward. From the signs of (19), it is clear that $p(m)$ and $t(m)$ are decreasing fns of m, while ~~x an~~ $x(m)$ and $\ell(m)$ are increasing fns.

Some appoximate choices of the power law indices for low-mass stars (assuming energy generation by p-p chain and ideal gas EoS)

$a = b = 1$

$\lambda = 1, \ \nu = 4$ — (21)

$n = 1, \ s = 3.5$ (Kramer's opacity)

$n = 0, \ s = 0$ (electron scattering opacity).

These are called <u>homologous</u> (self-similar) models, since the behavior of various physical quantities depend only on dimensional parameters (like, m and x), except for a scaling (by M, ~~and~~ L, etc).

For eg, the curve L_r/L vs M_r/m is the same for all stars with the same ~~of~~ laws of opacity and energy generation.