WHITE DWARFS

EOS of a degenerate electron gas

For a particle localized within a vol element ΔV, the localization within the 3D momentum space $\Delta^3 p$ is constrained by the uncertainty principle

$$\Delta V \, \Delta^3 p \geq h^3 \quad \text{— (1)}$$

quantum cell

vol of the particle in the 6D phase space

The num. of quantum states with momenta between p and $p+\Delta p$ in a spatial vol V is

$$g(p)\,dp = \frac{4\pi p^2 dp \; V \, g_s}{h^3} \quad \text{— (2)}$$

density of states

num of intrinsic quantum states of the particle (eg. spins or polarizations)

~~($g_s = 2$ for Fermions, due to Pauli's exclusion principle)~~

The relative occupation of ~~a~~ the available quantum states for particles in thermodynamic equilibrium at temp T

$$f(\epsilon_p) = \frac{1}{e^{(\epsilon_p - \mu)/kT} \pm 1} \quad \text{— (3)}$$

distribution fn.

energy of the ~~par~~ state

$$\epsilon_p^2 = p^2c^2 + m^2c^4$$

chemical potential (determined by the normalization $n = \int_0^\infty n(p)\,dp$)

+ for Fermions (Fermi-Dirac statistic)

− for Bosons (Bose-Einstein statistic)

The number ~~density~~ of electrons with momentum p is given by the product of $g(p)$ and $f(\epsilon_p)$. That is

$$N_e(p)\,dp = g(p)\,f(\epsilon_p)\,dp$$

The number density of electrons with momentum p (or, the momentum distribution of electrons is

for e^- (2 spin states) $g_s = 2$

$$n_e(p)\,dp = \frac{g_s}{h^3}\, 4\pi p^2 dp \, \frac{1}{e^{(\epsilon_p - \mu)/kT} + 1} \quad \text{— (4)}$$

~~In the non relativistic limit, $\epsilon_p = p^2/2m_e$~~

~~$n_e(p)\,dp$~~

max value for $n_e(p)\,dp$ is when $f(\epsilon_p) = 1$. That is

$$\boxed{n_{max}(p)\,dp = \frac{8\pi}{h^3}\,p^2 dp}$$

with the following normalization, which fixes μ

$$n_e = \int_0^\infty n_e(p)\,dp \quad \text{— (5)}$$

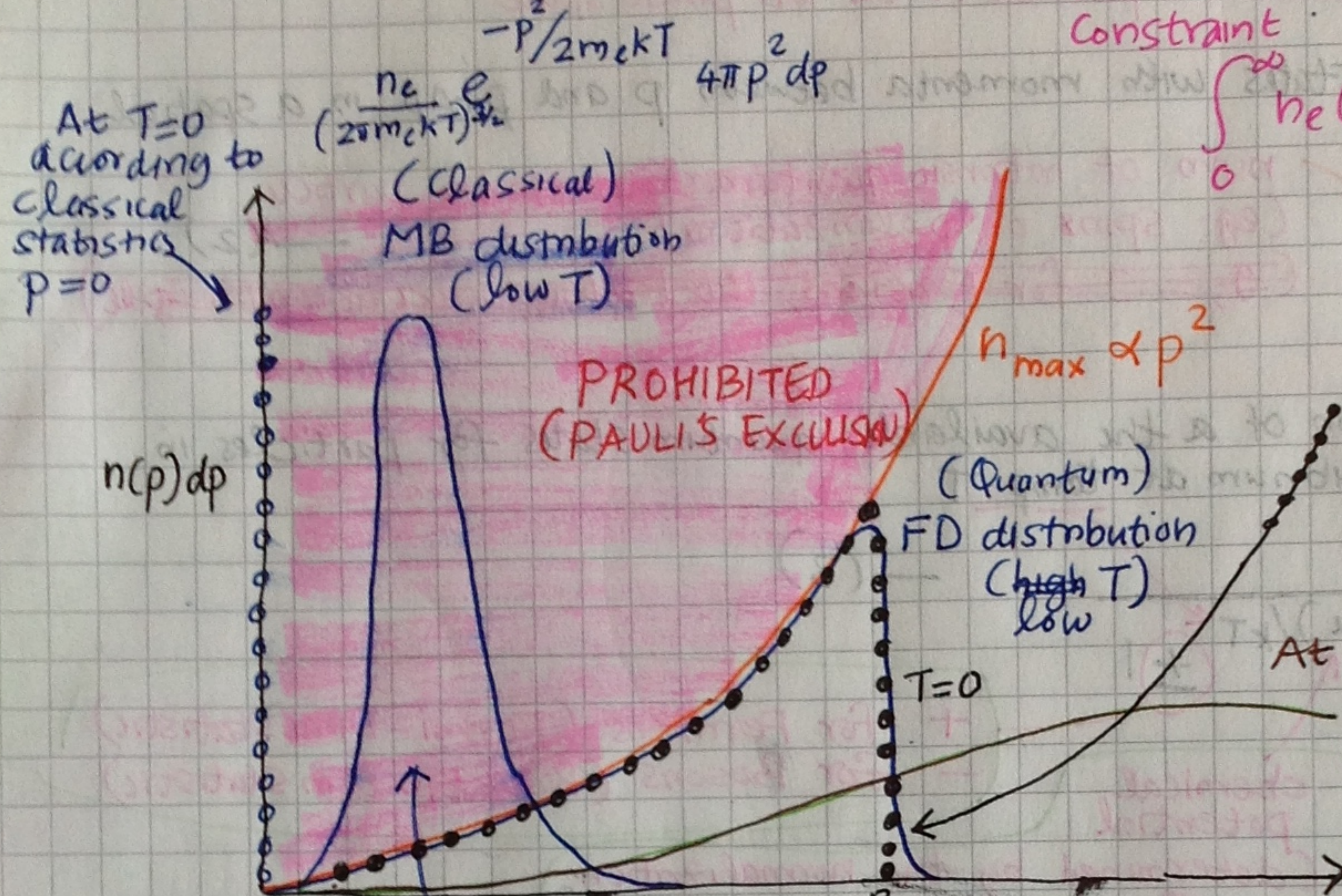
avg num density of electrons

In the non relativistic limit $\epsilon_p = p^2/2m$

$$\therefore\ n_e(p)\,dp = \frac{2}{h^3}\,\frac{1}{e^{(p^2/2m_ekT)-\psi}+1}\,4\pi p^2\,dp \qquad \text{— (6)}$$

degeneracy parameter $\psi = \frac{\mu}{kT}$

Determined by the Constraint $\int_0^\infty n_e(p)\,dp = n_e$



At ~~T=0~~ T→0 all available momentum states are occupied upto a max value, while all higher states are empty. Complete degeneracy.

At high temp FD & MB distributions become similar

number of electrons with small p expected from classical statistics exceed the limit imposed by Pauli's exclusion principle. These electrons are forced to occupy states with high p → peak of the distribution shifts to high p.

Due to this high momenta and velocities, the electron gas excerts a higher pressure, than expected from classical physics → degeneracy pressure.

At $T \to 0$

$$n_e(p) = \begin{cases} \dfrac{8\pi p^2}{h^3} & \text{for } p \leq p_F \\ 0 & \text{for } p > p_F \end{cases} \qquad \text{—(7)}$$

Fermi momentum (Determined by $\int_0^{p_F} n_e(p)\,dp = n_e$)

$$\boxed{p_F = h\left(\frac{3n_e}{8\pi}\right)^{1/3}} \qquad (8)$$

The pressure of a completely degenerate electron gas can be computed from the pressure integral (pressure is the momentum flux)

$$P = \frac{1}{3}\int_0^\infty p\, v_p\, n_e(p)\,dp \qquad \text{—— (9)}$$

comes from the assumption that p are isotropically distributed

HW: Derive this

In the non relativistic limit, ~~we~~ $v = p/m$. Hence

$$P_e = \frac{1}{3}\int_0^{p_F} \frac{8\pi p^2}{h^3}\, p\, \frac{p}{m_e}\, dp = \frac{1}{3}\int_0^{p_F} \frac{8\pi p^4}{h^3 m_e}\, dp = \frac{8\pi}{15 h^3 m_e} p_F^5$$

$$P_e = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \quad \text{— (10)}$$

mass of electron

avg num density of electrons

$$n_e = \frac{1}{\mu_e}\frac{\rho}{m_u}$$

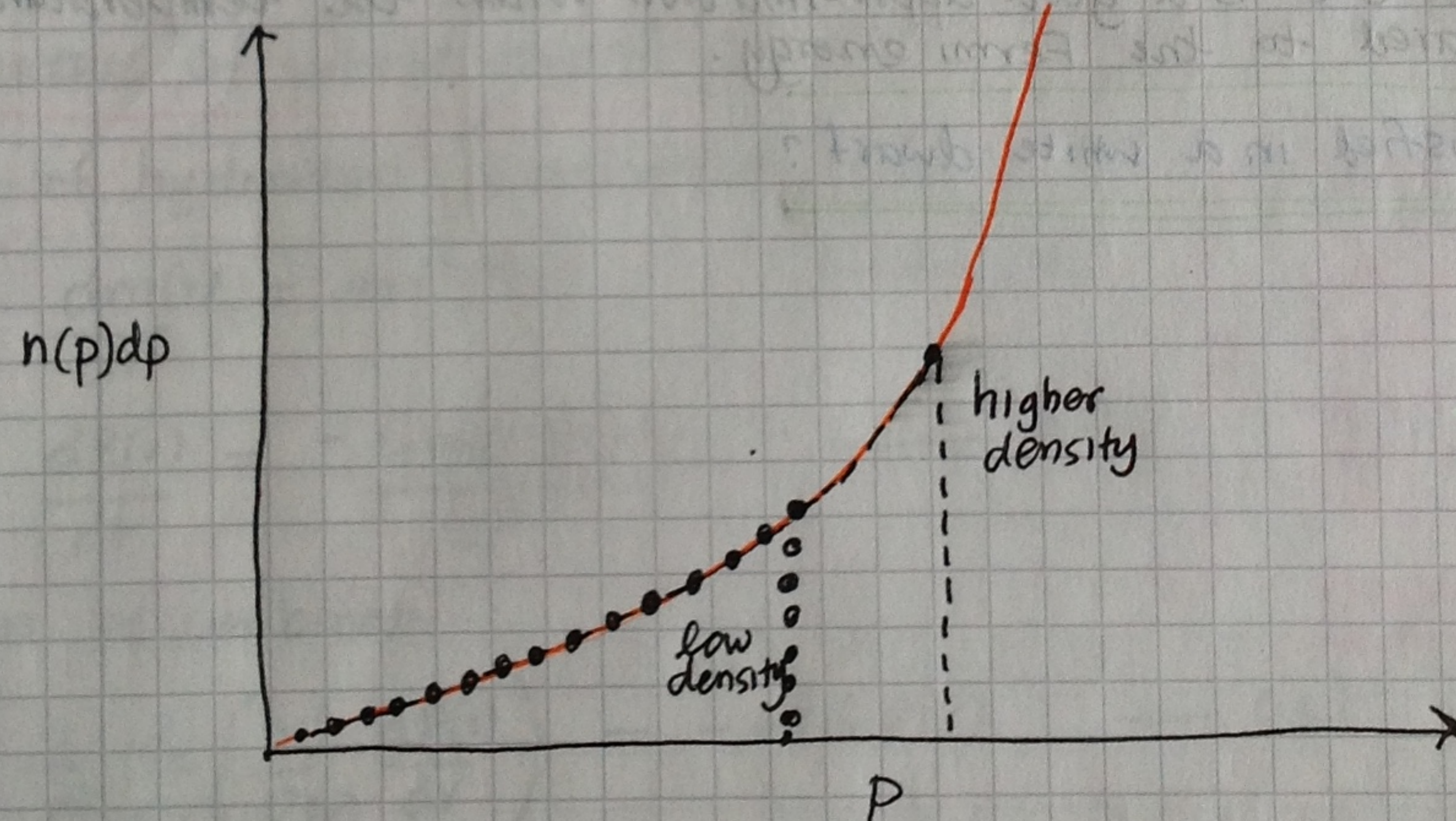
AMU

mean molecular weight per free electron

$$P_e = K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3} \quad \text{— (11)}$$

$$K_{NR} = \frac{h^2}{20 m_e m_u^{5/3}}\left(\frac{3}{\pi}\right)^{2/3} = 1.0036 \times 10^{13} \text{ (cgs)}$$

Higher density → more electrons occupy states with larger momenta → $P_e \propto \rho^{5/3}$



In the extreme relativistic limit $(v \to c)$, the pressure becomes

$$P_e = \frac{1}{3}\int_0^{p_F} pc\, \frac{8\pi p^2}{h^3}\, dp = \frac{8\pi c}{12 h^3} p_F^4 = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} n_e^{4/3}$$

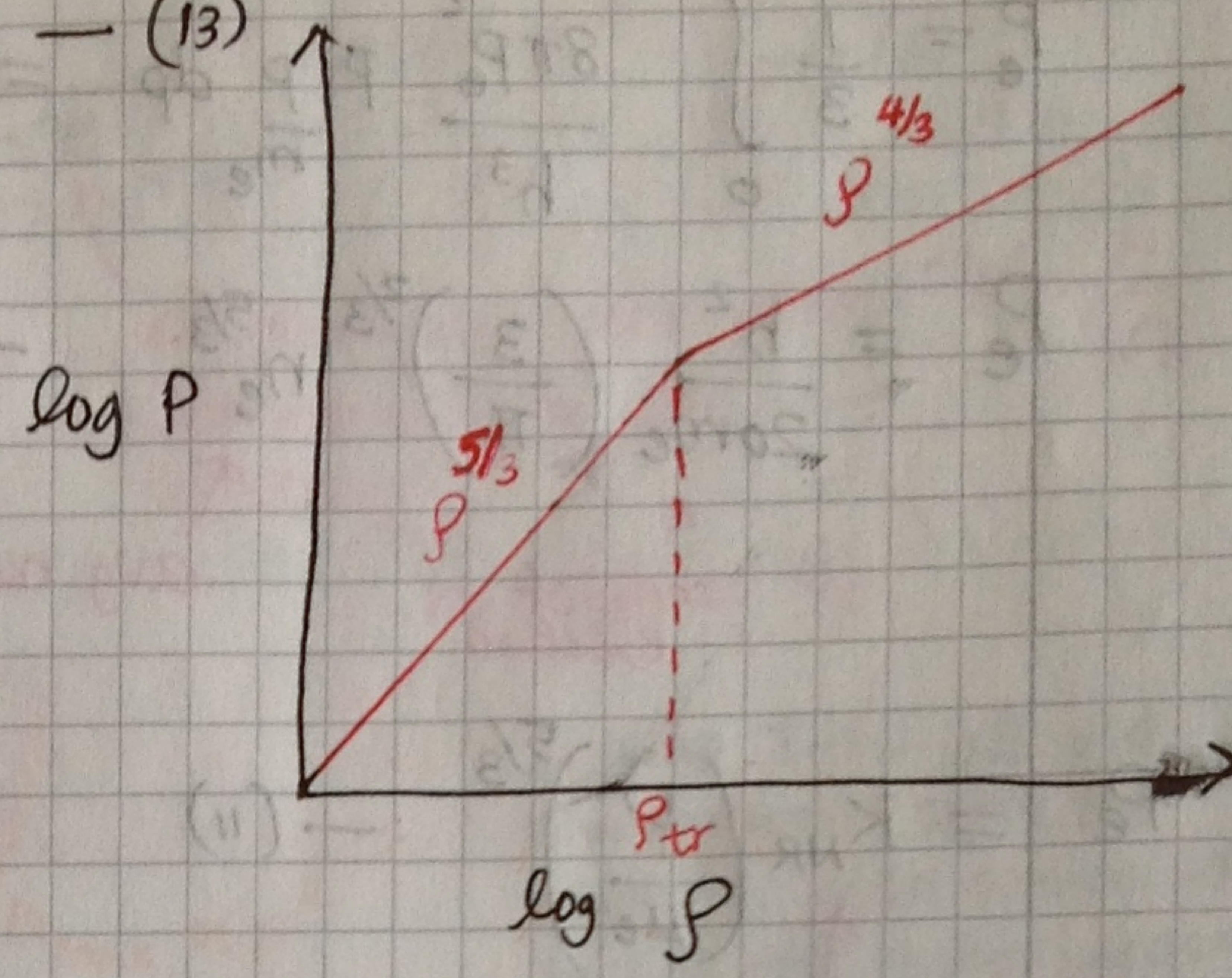
$$P_e = K_{ER}\left(\frac{\rho}{\mu_e}\right)^{4/3} \quad \text{— (12)}$$

$$K_{ER} = \frac{hc}{8 m_u^{4/3}}\left(\frac{3}{\pi}\right)^{1/3} = 1.2435 \times 10^{15} \text{ (cgs)}$$

Transition from NR to ER regime happens when $P_F \approx m_e c$, or

$$\rho_{tr} \approx \mu_e m_u \frac{8\pi}{3}\left(\frac{m_e c}{h^3}\right)^3 \quad -(13)$$

$\sim 10^6 \mu_e \ g/cm^3$



Even though (12) and (13) are computed in the $T \to 0$ limit, they provide an excellent approx for the pressure at low temp. That is, when the degeneracy parameter $\psi \gg 0$

$$\frac{\mu}{KT} \gg 0$$

chemical potential.

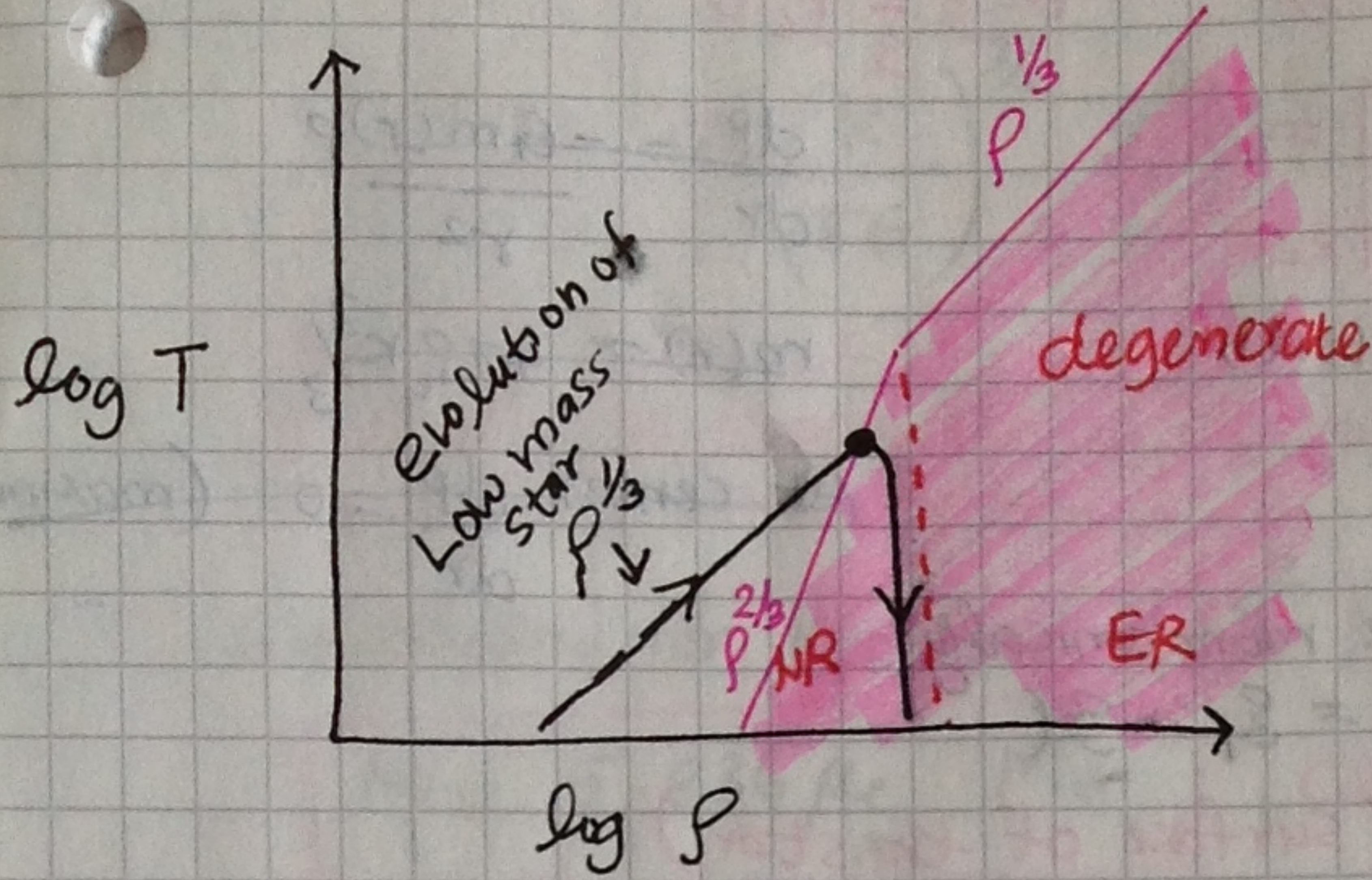
(typical energy required to add a new particle to the system) which is so of the order of the Fermi energy of the system

$$E_F = \left(p_F^2 c^2 + m_e^2 c^4\right)^{1/2}$$

That is, the cold EOS is a good approximation when the temperatures are small compared to the Fermi energy.

HW: Is this satisfied in a white dwarf?

White Dwarfs



- End products of low mass stars.
- Early observational evidence: Sirius B
  - * $T_{eff} \sim 8000K$ (from color 'white') — Fitting the planck distribution
  - * $M \sim 1 M_\odot$ (from binary orbit)
  - * $R \sim 1 R_\oplus$ (from flux) — $L = 4\pi R^2 \sigma T_{eff}^4$

$\Rightarrow \rho \sim 10^6\ g/cm^3$!

~~Structur~~ At these densities, electrons are degenerate. The EOS can be written in the polytropic form

$$P = K\rho^{\Gamma}, \quad (15)$$

where $\Gamma = \frac{5}{3}$ for non relativistic electrons ~~and~~ (low ρ)

$\Gamma = \frac{4}{3}$ for relativistic electrons (very high density)

K: polytropic const

Structure of white dwarfs:

Eqns of hydrostatic equilibrium:

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \quad (16)$$

$$\frac{dP(r)}{dr} = -\frac{G m(r)\rho(r)}{r^2} \quad (17)$$

These eqns can be numerically integrated along with an EOS ~~and~~ with BCs

$m(0) = 0$

$\rho(0) = \rho_c$

$P(0) = K\rho_c^{\Gamma}$

However it is useful to solve them in the dimensionless form.

can be combined:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G\rho(r) \quad (18)$$

This eqn can be written in a dimensionless form by introducing the following variables,

$$\rho = \rho_c \theta^n \qquad r = a\xi \qquad \Gamma = 1 + \frac{1}{n} \quad (19)$$

ρ_c: central density; n: polytropic index

$$a = \left[\frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}\right]^{1/2}$$

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n} \quad (20)$$

Lane-Emden Eqn for a polytropic index n

The boundary condns are

$$\theta(0) = \cancel{0}1 \quad \leftarrow \text{follows from (19)} \qquad \cancel{\theta}\ \rho = \rho_c \theta^n \qquad \text{— (21)}$$

$$\theta'(0) = 0 \quad \leftarrow \text{follows from } \left.\frac{dP}{dr}\right|_{r=0} = 0$$

$$\frac{dP}{dr} = \frac{-Gm(r)\rho}{r^2}$$

$$\cancel{m(r) = 4\pi\rho_c r^3/3}$$

~~At center $\frac{dP}{dr} = 0$ (maxima)~~

Eq (20) can be integrated numerically with the boundary condns (21), starting from ~~ξ_0~~ $\xi = 0$.

For $n<5$ $\left(\Gamma > \frac{6}{5}\right)$, the solns decrease monotonically and have a zero at a finite value $\xi = \xi_1$; ~~ξ_1~~

$$\theta(\xi_1) = 0 \quad \Rightarrow \quad P = \rho = 0 \quad \text{(surface of the star)}$$

The radius of the star is

$$\cancel{r}\ R_* = a\xi_1 = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n} \xi_1 \qquad \text{— (22)}$$

The mass is

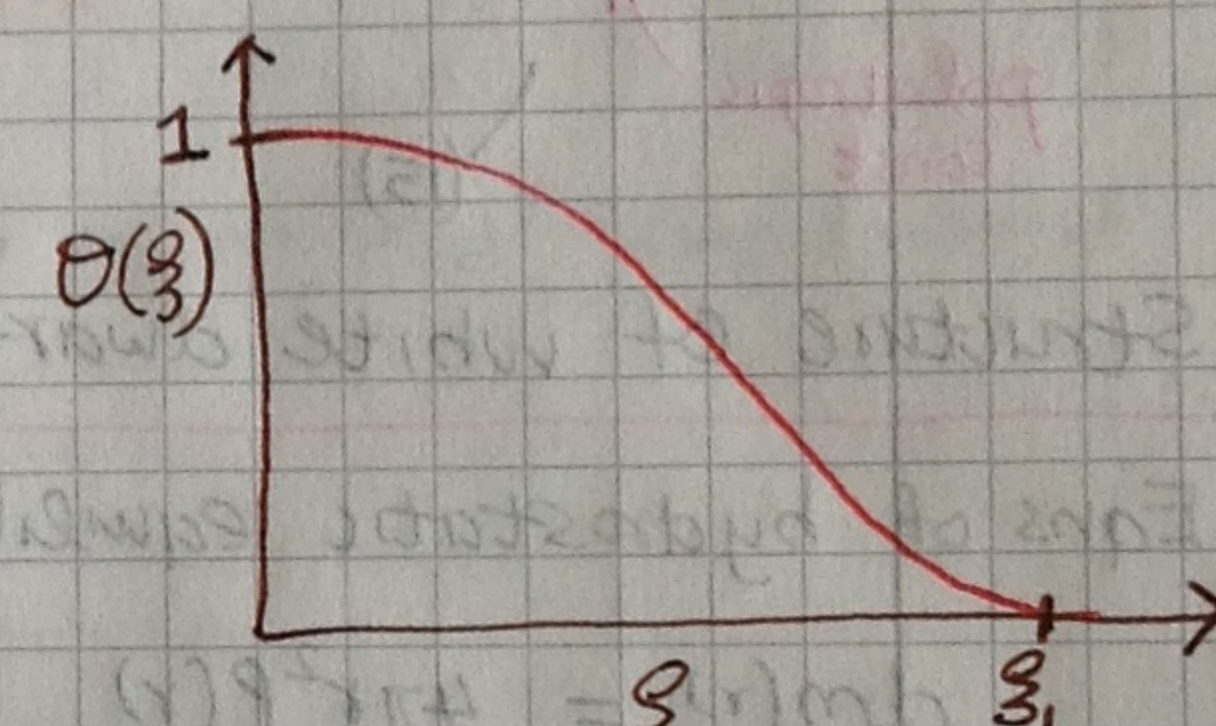
$$M = \int_0^R 4\pi r^2 \rho(r)\, dr$$

$$= 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n \, d\xi$$

$$= -4\pi a^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi \qquad \text{(from 20)}$$

$\theta'(\xi) \leq 0$ since θ is a monotonically decreasing fn of ξ

$$M = 4\pi a^3 \rho_c\, \xi_1^2\, |\theta'(\xi_1)| \qquad \text{— (23)}$$



Eliminating ~~the~~ ρ_c between (22) and (23), we get the mass-radius relation of polytropes

$$M = 4\pi R^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{n}{n-1}} \xi_1^{\frac{3-n}{1-n}}\, \xi_1^2\, |\theta'(\xi_1)| \qquad \text{— (24)}$$

For $n = \frac{3}{2}$ $\left(\Gamma = \frac{5}{3}\right.$, non relativistic limit)

$$M \propto R^{-3} \qquad \text{— (25)}$$

For $n = 3$ ($\Gamma = 4/3$, extreme relativistic limit)

$$M \propto R^0 \qquad \text{— (26)}$$

$\rightarrow$ mass is independent of the radius uniquely determined by K

For a given K, there is only one value of M for which the hydro. equilibrium can be satisfied if $n=3$.

$$K_{ER} \propto \mu_e^{-4/3}$$

$$M = 4\pi \left(\frac{K}{\pi G}\right)^{3/2} \underbrace{\xi_1^2 \left|\theta'(\xi_1)\right|}_{2.01 \text{ for } n=3}$$

The max mass of a gas sphere in hydrostatic equilibrium that can be supported by degenerate electrons → max mass of a White Dwarf

Substituting for K_{ER} and $\xi_1^2 \left|\theta'(\xi_1)\right|$

$$\boxed{M = 5.836\, \mu_e^{-2}\; M_\odot}$$

CHANDRASEKHAR MASS

For fully ionized gas $\mu \approx \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)^{-1}$

For CO white dwarfs

$X = Y = 0$ and $Z = 1 \Rightarrow \mu \simeq 2$

$M_{Ch} \simeq 1.4\, M_\odot$