ASSIGNMENT-4

1.

Considering a surface of area ΔA with outward normal $\hat{n} = \hat{e}_z$
If a particle with velocity $\vec{v}$ strikes the surface, the momentum imparted to the surface assuming elastic collision is : $2mv_z$

Fraction of particles with velocities in the range $[v_z, v_z+dv_z]$ is $\mu(v_z)\,dv_z$

$\therefore$ Number of particles with velocity in the interval $[v_z, v_z+dv_z]$ that hit area ΔA in time Δt :

$$\Delta N(v_z) = \Delta A\, v_z\, \Delta t\, \mu(v_z)\,dv_z$$

Thus, the momentum imparted : $2mv_z^2\, \Delta A\, \Delta t\, \mu(v_z)\,dv_z$.

$\therefore$ Pressure, $dP = 2mv_z^2\, \mu(v_z)\,dv_z$

Total pressure, $P = \int_0^\infty 2mv_z^2\, \mu(v_z)\,dv_z$.

(The integration limits are 0 to ∞ since only those particles will strike the surface which have $v_z > 0$)

$\Rightarrow P = \int_{-\infty}^\infty mv_z^2\, \mu(v_z)\,dv_z$ (Assuming $\mu(v_z) = \mu(-v_z)$)

In 3-dimensions, $\mu(v_z) \rightarrow \bar{\mu}(\vec{v})$ and we get :

$$P = \int d^3v\; mv_z^2\; \bar{\mu}(\vec{v})$$

In the momentum space,

momentum distribution function in 3-D

$$P = \int d^3p\; mv_z^2\; \bar{n}(\vec{p}) \quad [\because \bar{n}(\vec{p})\,d^3p = \bar{\mu}(\vec{v})\,d^3v]$$

velocity distribution function in 3-D

Changing coordinates from Cartesian to polar :

$$\bar{n}(\vec{p})\,dp_x\,dp_y\,dp_z = \bar{n}(\vec{p})\,p^2\sin\theta\,dp\,d\theta\,d\phi$$

So, the 3-D momentum distribution in polar coordinates is

$$n(\vec{p}) \equiv n(p,\theta,\phi) = \bar{n}(\vec{p})\,p^2\sin\theta$$

for isotropic momentum distributions, $\bar{n}(\vec{p}) = \bar{n}(p)$

$$\therefore n(p) = \int_0^\pi d\theta \int_0^{2\pi} d\phi\; n(p,\theta,\phi) = 4\pi p^2\,\bar{n}(p)$$

$\therefore$ Pressure exterted by a gas of particles with isotropic momentum distribution $n(p)$ is given by :

$$P = \int d^3p\; mv_z^2\,\bar{n}(p) = \int_0^\infty dp\; 4\pi p^2\; mv_z^2\,\bar{n}(p) = \int_0^\infty dp\; mv_z^2\; n(p)$$

$$= \frac{1}{3}\int_0^\infty dp\; v_p\,(mv_p)\,n(p) \qquad [\because \text{Isotropy} \Rightarrow v_z^2 = \tfrac{1}{3}v_p^2]$$

$$\therefore \quad \boxed{P = \frac{1}{3}\int_0^\infty p\, v_p\, n(p)\, dp} \qquad [\because m v_p = p]$$

2. The degeneracy parameter: μ/kT, where μ is of the order of Fermi energy, E_F. Temperature, $T \sim 10^4$ K

Density of white dwarf $= 10^6 g/cm^3 = 10^9 kg/m^3$

Average number density of particles, $n \sim \frac{10^9}{10^{-27}} = 10^{36}$

$$E_F = \frac{h^2}{8m}\left(\frac{3}{\pi}n\right)^{2/3} \approx \left(\frac{6.6\times10^{-34}}{8\times 9.1\times10^{-31}}\right)^2 \left(\frac{3}{\pi}\times 10^{36}\right)^{2/3}$$

$$= 5.8\times10^{-14}\ J$$

$$\Rightarrow \mu \sim E_F \sim 10^{-13} J$$

Now, $\quad kT \sim 1.38\times10^{-23}\times10^4 = 1.38\times10^{-19}$

Since $\mu \gg kT$, we are justified in using a cold degenerate eqn of state

3.

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \qquad \text{— (I)}$$

Equation of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{G m(r)\rho(r)}{r^2} \qquad \text{— (II)}$$

$$\Rightarrow \quad \frac{r^2}{\rho(r)}\frac{dP(r)}{dr} = -G m(r)$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -G\, 4\pi r^2 \rho(r) \qquad (\text{using (I)})$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP(r)}{dr}\right) = -4\pi G \rho(r) \qquad \text{— (III)}$$

Assuming a polytropic equation of state:

$$P = K\rho^{1+1/n}$$

$$\Rightarrow \quad \frac{dP}{dr} = \left(1+\frac{1}{n}\right) K \rho^{1/n} \frac{d\rho}{dr} \qquad \text{— (IV)}$$

Introducing dimensionless variable θ such that : $\rho = \rho_c \theta^n$

Plugging into eq^n (IV), we get

$$\frac{dP}{dr} = \left(1+\frac{1}{n}\right) K \rho_c^{1/n} \theta \; \rho_c \, n \frac{d\theta}{dr} \theta^{n-1}$$

$$\Rightarrow \frac{dP}{dr} = (1+n) \, \rho_c^{1+1/n} K \; \theta^n \frac{d\theta}{dr} \qquad \text{— (V)}$$

Plugging this into eq^n (III), we get :

$$\frac{1}{r^2} \frac{d}{dr} \left[\frac{r^2}{\rho_c \theta^n} (1+n) \, \rho_c^{1+1/n} K \theta^n \frac{d\theta}{dr} \right] = -4\pi G \rho_c \theta^n$$

$$\Rightarrow (1+n) K \rho_c^{1/n - 1} \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\theta}{dr} \right] = -4\pi G \theta^n \qquad \text{— (VI)}$$

Introducing another dimensionless variable ξ such that $r = a\xi$
equation (VI) becomes

$$(1+n) K \rho_c^{\frac{1}{n}-1} \frac{1}{a^2 \xi^2} \frac{d\xi}{dr} \frac{d}{d\xi}\left[a^2 \xi^2 \frac{d\xi}{dr} \frac{d\theta}{d\xi} \right] = -4\pi G \theta^n$$

$$\Rightarrow \frac{(1+n) K \rho_c^{1/n - 1}}{4\pi G} \frac{1}{a^2 \xi^2} \frac{d}{d\xi}\left[\xi^2 \frac{d\theta}{d\xi} \right] = -\theta^n \qquad \text{— (VII)}$$

If we fix 'a' such that

$$\frac{(1+n) K \rho_c^{1/n - 1}}{4\pi G a^2} = 1 \Rightarrow \boxed{a = \left[\frac{(1+n) K \rho_c^{\frac{1}{n}-1}}{4\pi G} \right]^{1/2}}$$

Thus, eq^n (VII) takes the form :

$$\boxed{\frac{1}{\xi^2} \frac{d}{d\xi}\left[\xi^2 \frac{d\theta}{d\xi} \right] = -\theta^n}$$

which is the Lane-Emden equation.

5. $r = a\xi \Rightarrow R_* = a\xi_* \Rightarrow \boxed{R_* = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{\frac{1-n}{2n}} \xi_*}$

$$\Rightarrow \rho_c = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{n}{n-1}} \left(\frac{R_*}{\xi_*}\right)^{\frac{2n}{1-n}} \quad \text{— (I)}$$

Total mass of the white dwarf:

$$M_* = \int_0^{R_*} 4\pi r^2 \rho(r)\, dr$$

$$= 4\pi a^3 \rho_c \int_0^{\xi_*} \xi^2 [\theta(\xi)]^n\, d\xi \quad \text{— (II)}$$

where I've replaced $r \to a\xi$ and $\rho \to \rho_c \theta^n$. Now using the Lane Emden equation to replace θ^n, we get:

$$M_* = 4\pi a^3 \rho_c \int_0^{\xi_*} \frac{d}{d\xi}\left[\xi^2 \frac{d\theta(\xi)}{d\xi}\right] d\xi.$$

$$= 4\pi a^3 \rho_c\, \xi_*^2 \left|\frac{d\theta(\xi_*)}{d\xi}\right| \quad \text{— (III)}$$

Substituting the value of 'a':

$$M_* = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-n}{2n}} \xi_*^2 \left|\frac{d\theta}{d\xi}(\xi_*)\right| \quad \text{— (IV)}$$

Replacing ρ_c using (I), we get:

$$M_* = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3-n}{2(n-1)}} \left(\frac{R_*}{\xi_*}\right)^{\frac{3-n}{1-n}} \xi_*^2 \left|\frac{d\theta}{d\xi}(\xi_*)\right|$$

$$\Rightarrow \boxed{M_* = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{n}{n-1}} \xi_*^{\frac{3-n}{n-1}} \xi_*^2 \left|\frac{d\theta(\xi_*)}{d\xi}\right| R_*^{\frac{3-n}{1-n}}}$$