ICTS Graduate Course PHY-404.5: Physics of Compact Objects
Tutorials

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Assignment 1: Stellar Structure

1. Assuming the equations of hydrostatic, radiative and thermal equilibria, derive the following equations describing the structure of homologous stars.

$$\frac{dp}{dm} = -\frac{m}{x^4}, \quad \frac{dx}{dm} = \frac{t^b}{x^2 p^a}, \quad \frac{dt}{dm} = -\frac{p^{an}\, l}{x^4 t^{3+s+bn}}, \quad \frac{dl}{dm} = A p^{a\lambda}\, t^{\nu - b\lambda}, \tag{0.1}$$

   where the dimensionless quantities p, x, t and l are defined in terms of the central pressure P_c, central temperature T_c and the scaling variables M, R and L.

$$P = P_c p, \quad T = T_c t, \quad r = Rx, \quad L_r := L(r) = Ll, M_r := M(r) = Mm, \tag{0.2}$$

   and assuming the following power-law approximations for the density ρ, opacity κ and energy density density ϵ:

$$\rho = \rho_0 P^a T^{-b}, \quad \kappa = \kappa_0 \rho^n T^{-s}, \quad \epsilon = \epsilon_0 \rho^\lambda T^\nu. \tag{0.3}$$

   The scaling variables M, R and L are defined by the following conditions

$$\frac{GM^2}{4\pi R^4 P_c} = 1, \quad \frac{M}{4\pi R^3 \rho_0} \frac{T_c^b}{P_c^a} = 1, \quad \frac{3\kappa_0 \rho_0^n}{64\pi^2 ac} \frac{P_c^{an} ML}{T_c^{4+s+bn} R^4} = 1, \tag{0.4}$$

   while A is given by

$$A = \frac{3\kappa_0 \epsilon_0 \rho_0^{n+\lambda}}{16\pi Gac} \frac{P_c^{a(n+\lambda)+1}}{T_c^{4+s+b(n+\lambda)-\nu}}. \tag{0.5}$$

2. Numerically solve the set of equations Eq. (0.1) assuming the boundary conditions: $p(0) = t(0) = 1, x(0) = l(0) = 0$ and the following power-law indices:

   - *Ideal gas equation of state*: $a = b = 1, \rho_0 = \mu/\mathcal{R}$ where $\mu \simeq 0.5X^{-0.57}$ is the mean molecular weight, $\mathcal{R} = 8.3 \times 10^7$ ergs/mol/K is the gas constant and X is the mass fraction of hydrogen (assume $X \simeq 0.75$).
   - *Electron scattering opacity*: $n = s = 0, \kappa_0 = 0.2(1 + X)\ \mathrm{cm}^2/\mathrm{g}$.
   - *Energy generation by $p - p$ chain*: $\lambda = 1, \nu = 4, \epsilon_0 \sim 10^{-30} X^2\ \mathrm{ergs\,cm^3\,g^{-2}\,s^{-1}\,K^{-4}}$.

   You can start by solving the dimensionless equations Eq.(0.1) by setting $A = A_0$, some constant. (Numerical experimentation shows that $p(m)$ goes to zero only for the choice $A_0 \lesssim 0.55$). Plot $p(m), x(m), l(m)$ and $t(m)$. Identify the value of m at which the $p(m)$ goes to zero, denoted by $m_\star$; i.e. $p(m_\star) = 0$. Physically we require that $t(m)$ should go to zero at the same value of m; i.e., $t(m_\star) = 0$. This is satisfied only for a specific value of A, say $A_\star$. Using $A = A_\star$, re-plot $p(m), x(m), l(m)$ and $t(m)$. These relations are independent of the mass of the star.

3. Setting $A = A_\star$ in Eq.(0.5), we can express P_c fully in terms of T_c. This allows us to express R, T_c and L in terms of M using Eqs.(0.4). Now, the only free parameter is $M := M_\star/m_\star$, where $M_\star$ is the actual mass of 4the star. Now plot the stellar structure quantities $M(r), \rho(r), P(r)$ and $L(r)$ for a star of mass $M_\star = 1 M_\odot$ in terms of the physical radial coordinate $r/R_\odot$.

4. Repeat the calculation using stars of different masses. Plot the radius $r_\star$, central temperature T_c, and surface luminosity $L_\star$ of the stars as a function of their masses (hint: It would be useful to plot them in log-log scale so that the power-law relationships become apparent).

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Assignment 2: Stellar Evolution

1. Derive the homology relations for the pressure P, temperature T and luminosity L (express them in terms of the mass M and radius R, as well as the mean molecular weight μ of the gas particles). Compare these with the numerical results that you obtained in the previous section (Problem 4).
2. Using the homologous relations derived above, show that, for a star of mass M in quasi-equilibrium,
$$P_c = CGM^{2/3}\rho_c^{4/3}. \tag{0.6}$$
3. Consider a slowly contracting star in quasi-hydrostatic equilibrium for which the pressure is given by a combination of ideal gas and electron degeneracy : $P = (\mathcal{R}/\mu)\rho T + K(\rho/\mu_e)^\gamma$, where γ varies between 5/3 (non-relativistic) and 4/3 (extreme relativistic). Plot the variation of T_c with ρ_c.
4. Derive an expression for the maximum central temperature reached by a star of mass M.

Assignment 3: Stellar Collapse

1. Neutron stars have a radius of $\simeq$ 10 km. Use this to estimate the energy generated during a core collapse supernova (Hint: assume that before the collapse the core is like a white dwarf with Chandrasekhar mass, and that it suffers no significant mass loss after the collapse).
2. Compute the kinetic energy of the ejecta as well as the energy lost into photons and neutrinos. Assume that the progenitor star has an original mass of 10 km, the measured velocity of the ejecta is $\sim 10^4$ km/s, and the supernova shines with an (electromagnetic) luminosity of $2 \times 10^8 L_\odot$ for $\sim$ two months.
3. Assuming that these neutrinos have an average energy of $\simeq$ 5 MeV, how many neutrinos are produced in the collapse? Estimate the expected flux of neutrinos from a galactic supernova ($d_L \sim 10$ kpc) here on earth. Compare it with he neutrino flux from the Sun.

Assignment 4: Electron Degeneracy and White Dwarfs

1. Show that the pressure exerted by a gas of particle with isotropic momentum distribution $n(p)$ is given by
$$P = \frac{1}{3}\int_0^\infty p\, v_p\, n(p)\, dp, \tag{0.7}$$
where v_p is the velocity associated with momentum p.
2. Argue why we are justified in using a "cold" degenerate equation of state to describe a white dwarf with a temperature $T \sim 10^4$ K (Hint: Show that the degeneracy parameter $\mu/kT \gg 0$, where μ is the chemical potential and k the Boltzmann constant. The density of the white dwarf is $\sim 10^6$ g/cm^3 and the chemical potential $\sim$ the Fermi energy. Assume that $\mu_e = 2$).
3. Starting from the equations of the hydrostatic equilibrium and a polytropic equation of state $P = K\rho^{1+1/n}$, derive the Lane-Emden equation.
$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0, \tag{0.8}$$
where ξ and θ are related to the density ρ and radial coordinate r by $\rho = \rho_c\,\theta^n$ and $r = a\,\xi$, where ρ_c is the central density. Show that a takes the form
$$a = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{1/2}. \tag{0.9}$$
4. Numerically solve the Lane-Emden equation assuming a polytropic index of $n = 3/2$, with the following boundary conditions: $\theta(0) = 1$, $\frac{d\theta}{d\xi}(0) = 0$. Plot $\theta(\xi)$ and determine the surface $\xi_\star$ of the star, where $\theta(\xi_\star) = 0$.

5. Show that the mass $M_\star$ and radius $R_\star$ of the white dwarf are given by

$$R_\star = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n}\, \xi_\star, \qquad M_\star = 4\pi R_\star^{(3-n)/(1-n)} \left[\frac{(n+1)K}{4\pi G}\right]^{n/(n-1)} \xi_\star^{(3-n)/(n-1)}\, \xi_\star^2 \left|\frac{d\theta}{d\xi}(\xi_\star)\right| \tag{0.10}$$

6. Compute the mass and radius of the white dwarfs with central densities $\rho_c = 10^6 - 10^9$ g/cm^3, assuming that they are made of non-relativistic electron gas (polytropic constant $K \simeq 10^{13}\mu_e^{-5/3}$ in cgs units, where the μ_e is the mean molecular weight per free electron. Assume $\mu_e = 2$). Compare the numerically obtained mass-radius relation with the analytical relation given by Eq. (0.10)

Assignment 5: Corrections to the White Dwarf Structure

1. Show that, in the white dwarf interior, the Coulomb energy per free electron is

$$\epsilon_{\rm C} = \frac{-9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{1/3}, \tag{0.11}$$

where Z is the atomic number, e is the elementary charge and n_e is the number density of free electrons.

2. Numerically solve the stellar structure equations

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad \frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \tag{0.12}$$

assuming the equation of state of non-relativistic degenerate electron gas *including Coulomb corrections*: $P(\rho) = K\rho^{5/3}\left[1 - (\rho_0/\rho)^{1/3}\right]$, where $K \simeq 10^{13}\mu_e^{-5/3}$ (cgs units) and $\rho_0 \simeq 0.4\,Z^2\mu_e$ g/cm^3. Assume the atomic number to be $Z = 6$ (Carbon) an the mean molecular weight per free electron to be $\mu_e = 2$. The obvious boundary conditions are $m(r=0) = 0$ and $P(r=0) = P(\rho_c)$, where ρ_c is the central density. Compute the mass-radius of white dwarfs for central densities ranging $\rho_c = 10^6 - 10^9$ g/cm^3. Compare the mass-radius relation with the same computed earlier neglecting Coulomb corrections.

Assignment 6: Stellar Structure in General Relativity

1. In this section, we will use geometrized units ($G = c = 1$). Starting from the following metric of a static, spherically symmetric spacetime

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2, \tag{0.13}$$

where $x^\mu \equiv \{t, r, \theta, \phi\}$ are the Schwarzschild coordinates, and assuming a stress-energy tensor of a perfect fluid

$$T^{\mu\nu} = (\rho + P)\, u^\mu u^\nu + P\, g^{\mu\nu}, \tag{0.14}$$

where ρ is the mass-energy density in the rest frame of the fluid, P is the isotropic pressure and u^μ are components of the four-velocity of the fluid, derive the Tolman-Oppenheimer-Volkoff (TOV) equations describing the structure of a static, spherically symmetric star in General Relativity.

$$\frac{dm}{dr} = 4\pi r^2 \rho, \quad \frac{dP}{dr} = -\frac{m\rho}{r^2}\left[1 + \frac{P}{\rho}\right]\left[1 + \frac{4\pi r^3 P}{m}\right]\left[1 - \frac{2m}{r}\right]^{-1}, \quad \frac{d\Phi}{dr} = \frac{4\pi P r^3 + m}{r(r-2m)}. \tag{0.15}$$

2. Show that the exterior solution is given by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{0.16}$$

3. Numerically solve the TOV equations using the same equation of state used in the problem 2 of Assignment 5. The obvious boundary conditions are $m(r=0) = 0, P(r=0) = P(\rho_c), \Phi(r=R) = \frac{1}{2}\ln(1 - 2M/R)$ where ρ_c is the central density, R is the surface of the star, determined by the condition $P(r=R) = 0$, and $M \equiv m(r=R)$.

4. Plot $m(r)$, $P(r)$ and $\rho(r)$ on top of the same obtained using Newtonian gravity (problem 2 of Assignment 5). What is the effect of GR corrections?

5. Plot the mass of the star obtained using different central densities against the corresponding radius (same as problem 2 of Assignment 5). Compare this with the same obtained using Newtonian gravity.