## The Chandrasekhar Limit : A simple understanding (Landau 1932)

Let there be N fermions in a star of radius R

Num. density of Fermions $n \sim N/R^3$

Vol per Fermion $\sim 1/n$ (Pauli's exclusion)

Momentum of Fermion $\sim \hbar n^{1/3} \sim \frac{\hbar N^{1/3}}{R}$ (from Heisenberg uncertainty)

Fermi energy of the particle in the relativistic limit $E_F \sim \hbar n^{1/3} c \sim \frac{\hbar c N^{1/3}}{R}$ — (28)

Grav binding energy per Fermion $E_G \sim -\frac{GMm_B}{R}$

$M = N m_B$ (even though the pressure comes from electrons most of the mass is in baryons)

Total energy

$$E = E_F + E_G = \frac{\hbar c N^{1/3}}{R} - \frac{G N m_B^2}{R} \quad \text{— (29)}$$

$\frac{\hbar c N^{1/3}}{R}$: dominates at low N

$\frac{G N m_B^2}{R}$: dominates at high N

Small mass

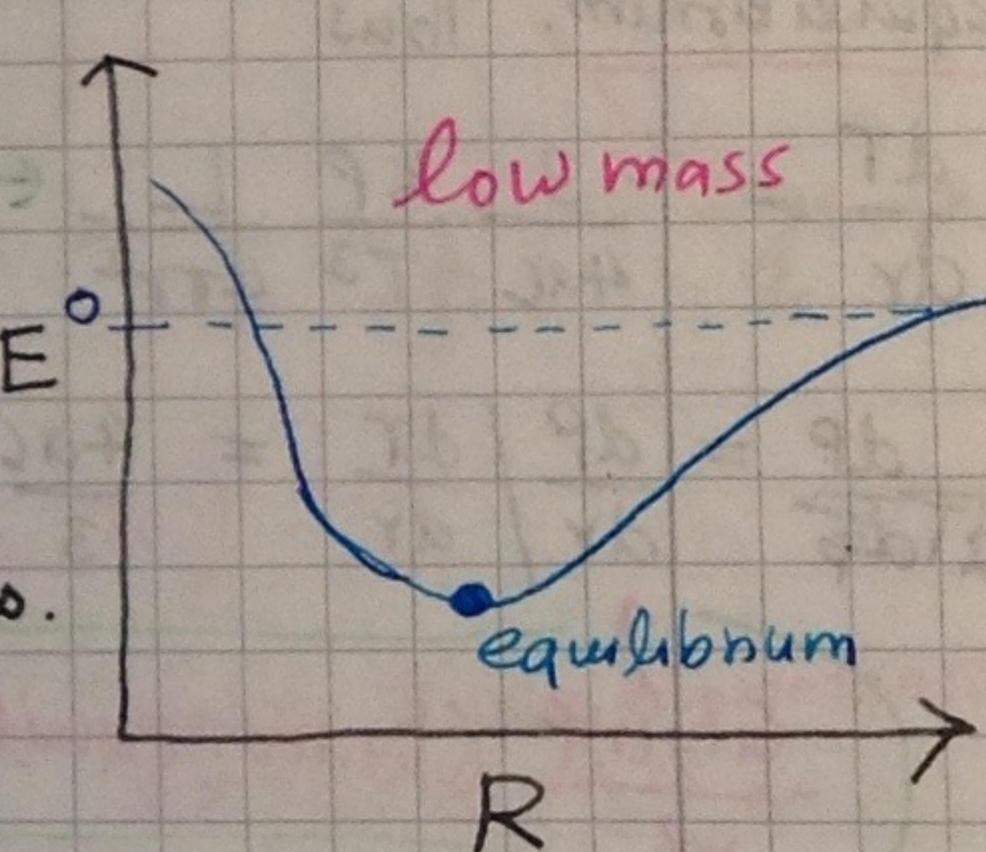
At low N : $E > 0$. E can be decreased by increasing R.

$\Rightarrow$ reduce density $\rightarrow$ Electrons become non relativistic: $E_F \sim p_F^2 \sim \frac{1}{R^2}$, $p_F \sim \frac{\hbar N^{1/3}}{R}$

$\rightarrow$ Eventually $E_G > E_F$ $(E<0)$.

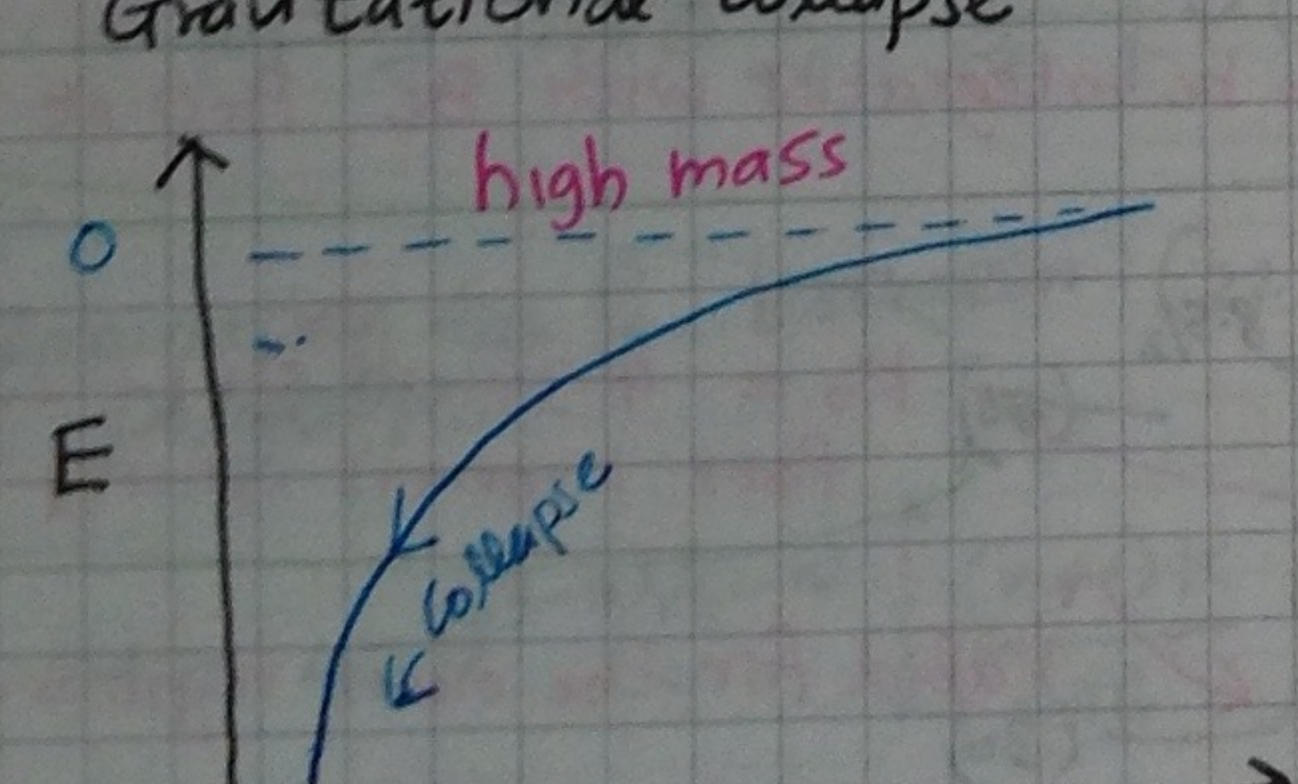
$\rightarrow$ As $R \rightarrow \infty$, $E \rightarrow 0$

Thus, there will be an equilibrium configuration at a finite value of R (~~that~~ minimizes E).




high mass

At high N: $E<0$, increasing to zero as $R \rightarrow \infty$.
E can be decreased without bound by decreasing $R \rightarrow$ No equilibrium exists.
Gravitational collapse




The max ~~num~~ baryon number for equilibrium is determined by setting $E=0$ in (29)

$$N_{max} \sim \left(\frac{\hbar c}{G m_B^2}\right)^{3/2} \sim 2 \times 10^{37}$$

$$M_{max} \sim N_{max} m_B \sim 1.5 \, M_\odot$$

The equilibrium radius associated with mass $M$ approaching $M_{max}$ is determinded by the onset of relativistic degeneracy

$$E_F \approx mc^2$$

$$E_F \sim \frac{\hbar c N^{1/3}}{R}$$

mass of the electron (for WDs) or neutron (for NSs)

$$R \approx \frac{\hbar c N_{max}^{1/3}}{mc^2} = \frac{\hbar}{mc}\left(\frac{\hbar c}{G m_B^2}\right)^{1/2} \sim \begin{cases} 5\times 10^3 \text{ km}, & \text{for } m=m_e \quad \text{WDs} \\ 3 \text{ km}, & \text{for } m=m_n \quad \text{NSs} \end{cases}$$

$$\boxed{\text{Two distinct regimes of collapse. In both cases } M_{max} \sim M_\odot}$$

## Structure of the Surface Layers:

The interior of the WDs is completely degenerate. High thermal conductivity. This isothermal interior is covered by non degenerate surface layers that are in radiative equilibrium. Thus

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$ ← (eqn of radiative equilibrium – derived in the stellar structure part) (eq 4) — (29)

ideal gas EOS $\rho = \frac{P}{k_B T}\mu m_u$

$$\therefore \frac{dP}{dT} = \frac{dP}{dr}\Big/\frac{dT}{dr} = \frac{4ac}{3}\frac{4\pi GM}{L}\frac{T^3}{\kappa} = \frac{4ac}{3}\frac{4\pi GM}{K_0 L}\frac{T^{6.5}}{\rho}$$

$-\frac{Gm(r)\rho}{r^2}$ (hydro. equilibrium)

use Kramer's opacity $\kappa = K_0\rho T^{-3.5}$

$m(r) = M$ (surface layers)

dP should include a d\rho term too?

Can be integrated with BC. $P=0$ at $T=0$.

$$\therefore P\,dP = \frac{4ac}{3}\frac{4\pi GM}{K_0 L}\frac{k_B}{\mu m_u}T^{7.5}\,dT$$

replaced from the ideal gas EOS

$$P = \frac{\rho}{\mu m_u}kT = \left(\frac{2}{8.5}\frac{4ac}{3}\frac{4\pi GM}{K_0 L}\frac{k_B}{\mu m_u}\right)^{1/2} T^{8.5/2}$$ ~~(30)~~

$K_0 = 4.34\times10^{24} Z(1+X)\ \mathrm{cm^2 g^{-1}}$ for Kramer's opacity

$$\therefore \rho(r) = \left(\frac{2}{8.5}\frac{4ac}{3}\frac{4\pi GM}{K_0 L(r)}\frac{\mu m_u}{k_B}\right)^{1/2} T(r)^{3.25}$$ — (30)

Valid for the outer regions of WD

The 'boundary' between the degenerate and nondegenerate regions can be found by equating

ideal gas EOS = degenerate EOS

$$\frac{\rho_*}{\mu_e m_u}k_B T_* = K_{NR}\left(\frac{\rho_*}{\mu_e}\right)^{5/3} \Longrightarrow \rho_* = (2.4\times10^{-8}\ \mathrm{g/cm^3})\,\mu_e T_*^{3/2}$$

polytropic const for nonrelativistic EOS: $K_{NR} \simeq 10^{13}$ (cgs)

- Using (30), $\rho_*$ can be replaced by M, L and $K_0$. Thus, given M, L and the composition ($K_0$), we can determine the internal temperature of a WD.
- Given $T_*$, $\rho_*$ can be calculated using (30).

$T_*$ is a good approx of $T_c$ ~~the to~~ also due to the high thermal conductivity of $e^-$

Typically $L \sim 10^{-2} - 10^{-5} L_\odot \Rightarrow T_* \sim 10^6 - 10^7\,K \Rightarrow \rho_* \lesssim 10^3\ \mathrm{g/cm^3} \ll \rho_c$

surface layer does not affect the mass-radius relation (too thin)

~~Cooling of White Dwarfs~~

## Corrections to the polytropic model

- At low densities: Corrections due to Coulomb interactions.
- At high densities: 1) Nuclear interactions

  2) GR corrections.

## Coulomb corrections at low densities:

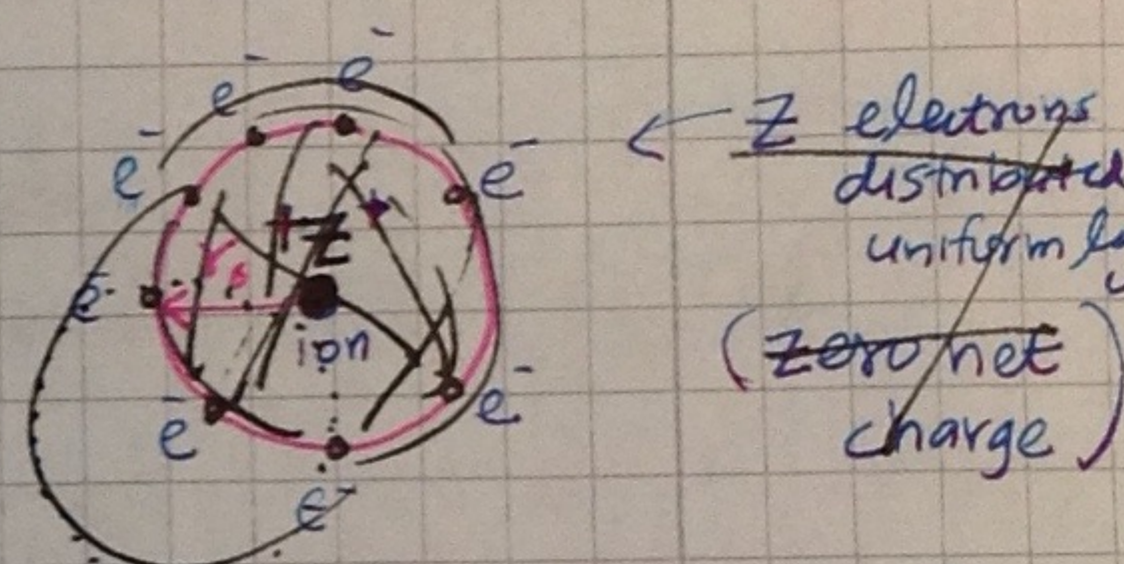
~~A F Using polytropic EOS, $R \propto \rho$~~

At low densities, ions can not be treated as ideal gas. They form solid lattice structure (due to Coulomb interaction).

Dominant correction comes from the Coulomb energy of the lattice.

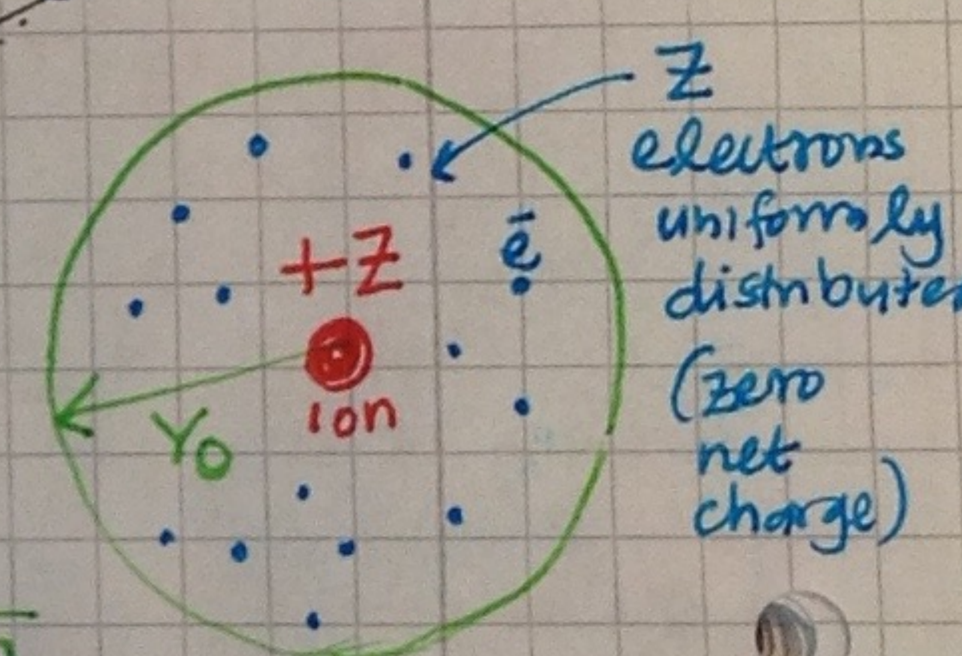
Divide the crystal lattice into spheres of radius $r_0$ ~~so that~~ such that

$$\frac{4}{3}\pi r_0^3 \, \underbrace{n_e}_{\text{num density of electrons}} = \underbrace{Z}_{\text{atomic number}} \qquad — (35)$$




Coulomb energy ~~per electron~~ of a cell of radius $r_0$

$$E_c = \underbrace{E_{ee}}_{\substack{\text{self energy of a -vely charged sphere of rad. } r_0 \\ \left(\frac{3}{5}\frac{Z^2e^2}{r_0}\right)}} + \underbrace{E_{ei}}_{\substack{\text{electrostatic interaction energy } \text{~~of~~ between the nucleus and the electron cloud.} \\ \left(\frac{-3}{2}\frac{Z^2e^2}{r_0}\right)}} = \frac{-9}{10}\frac{Z^2e^2}{r_0} \qquad — (36)$$




$\boxed{\text{HW: Derive this}}$

Coulomb energy per electron

$$\epsilon(n_e) \equiv \frac{E_c}{Z} = \frac{-9}{10}\frac{Ze^2}{r_0} = \frac{-9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{1/3} \qquad — (37)$$

$$r_0 = \left(\frac{3}{4\pi}\frac{Z}{n_e}\right)^{1/3}$$

The corresponding pressure

$$P = -\frac{dE}{dV} = -\frac{dE}{d(N/n)} = -\frac{d\epsilon}{d(1/n)} = n^2\frac{d\epsilon}{dn} \qquad — (38)$$

($\epsilon$ = E/N)

$$\therefore \; P_c = \cancel{n_e^2} \; \frac{-3}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{4/3} \qquad — (39)$$

← Pressure due to electrostatic interactions

The total pressure $P = P_o + P_c$

Pressure of the Ideal Fermi gas

Pressure due to electrostatic interactions

$$P_o^{NR} \simeq \frac{(3\pi^2)^{2/3}}{5} \frac{\hbar^2 n_e^{5/3}}{m_e}$$

(in the NR limit)

$$P_o^{ER} \simeq \frac{(3\pi^2)^{1/3}}{4} \hbar c n_e^{4/3}$$

(In the Extr Rel limit)

- The Coulomb correction to the degeneracy pressure (ER limit)

$$\frac{P}{P_o} \equiv \frac{P_o^{ER} + P_c}{P_o^{ER}} = 1 - \frac{2^{5/3}}{5}\left(\frac{3}{\pi}\right)^{1/3} \alpha Z^{2/3} \quad \text{— (40)}$$

$$\boxed{\alpha = \frac{e^2}{\hbar c} \simeq 10^{-2}} \Rightarrow \boxed{\text{Small}}$$

Fine structure const

- Coulomb correction in the NR limit

$$\frac{P}{P_o} \equiv \frac{P_o^{NR} + P_c}{P_o^{NR}} = 1 - \frac{Z^{2/3}}{2^{1/3}\pi a_o n_e^{1/3}} \quad \text{— (41)}$$

$$\boxed{\text{Bohr radius: } a_o = \frac{\hbar^2}{e^2 m_e}}$$

atomic number

Thus, the pressure will go to zero when $n_e = n_{e_o} \equiv \dfrac{Z^2}{2\pi^3 a_o^3} \equiv n_{e_o}$

number density of $e^-$ at which $P \to 0$

The corresponding density

$$\rho_o \simeq \mu_e m_u n_e \simeq 0.4 Z^2 \mu_e \ \text{g/cm}^3$$

$\Rightarrow$ A solid state structure with finite density and zero pressure can arise because of the electrostatic correction.

Because of the lowering of pressure for a given density, the tot pressure

$$P = P_o\left[1 - \left(\frac{n_{e_o}}{n_e}\right)^{1/3}\right] = P_o\left[1 - \left(\rho_o/\rho\right)^{1/3}\right]$$

$$P = K_{NR}\left(\rho/\mu_e\right)^{5/3}\left[1 - \underbrace{\left(\rho_o/\rho\right)^{1/3}}_{\text{dominates when } \rho < \rho_o}\right] \quad \text{— (42)}$$

$\rho_o \simeq 0.4 Z^2 \mu_e \ \text{g/cm}^3$

$$K_{NR} = \frac{h^2}{20 m_e m_u^{5/3}}\left(\frac{3}{\pi}\right)^{2/3}$$

HW: Repeat the WD structure calculation with the corrected EoS.

We can understand the qualitative effect of this in the M–R relationship ~~by~~ using the approximate forms of the hydrostatic equilibrium condition

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \Rightarrow \frac{P}{R} \approx \frac{GM}{R^2}\left(\frac{M}{R^3}\right)$$

$$P \approx \frac{GM^2}{R^4} = K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3}\left[1-\left(\rho_0/\rho\right)^{1/3}\right]$$

$$\rho = \frac{3M}{4\pi R^3}$$

$$R = \left[\frac{G\mu_e^{5/3}}{K_{NR}}M^{1/3} + \frac{\rho_0^{1/3}}{M^{1/3}}\right]^{-1}$$

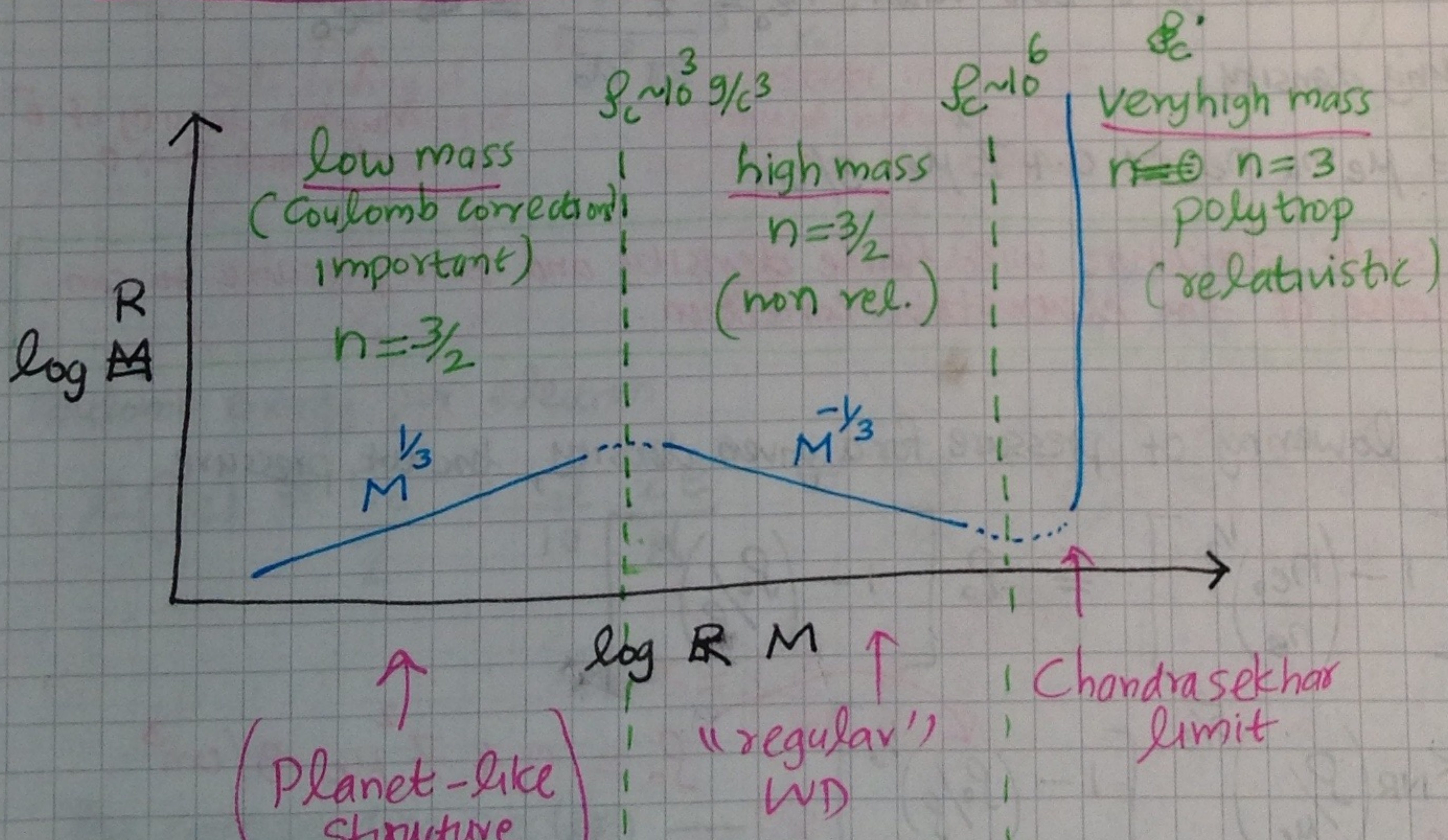
Hence

$$\frac{R}{R_\odot} = \left[\frac{G\mu_e^{5/3}R_\odot}{K_{NR}}M^{1/3} + \frac{\left(R_\odot^3\rho_0\right)^{1/3}}{M^{1/3}}\right]^{-1}$$

dominates at large M
$R \propto M^{-1/3}$

dominates at small M
$R \propto M^{1/3}$ like (regular solids) of const density

$\boxed{\text{At low masses WDs approach a planet-like structure.}}$



Corrections at higher densities:

1. Neutronization:

At high densities ( $\rho_c \gg 10^6 \; g/cm^3$ ) , the $\underbrace{\text{Fermi energy}}$ of electrons will become high enough to induce inverse $\beta$-decay

$$E_F = \hbar c \left(\frac{3 n_e}{8\pi}\right)^{1/3} = \hbar c \left(\frac{3 \rho}{8\pi \mu_e m_u}\right)^{1/3}$$

$$p + e \rightarrow n + \nu$$

Reduced num. of electrons → reduce $\underbrace{\text{degeneracy pressure}}_{P_e \propto n_e^{4/3}}$ → $\boxed{\text{instability}}$

'neutronization'

If the difference in the bind. energy of the two nuclei (A, Z) and (A, Z-1) is ~~s~~lower than the Fermi energy of the electron, this neutronization happens.

Typical densities:

$${}^1_1H \rightarrow n : (\rho_0 \sim 10^7 \; g/cm^3)$$

Heavier ~~elemen~~ nuclei( ~~eg.~~ ${}^4_2He \rightarrow {}^3_1H + n \rightarrow 4n$ ) : $\rho_0 \sim 10^{10} - 10^{11} \; g/cm^3$

2. ~~Poly~~ Pycnonuclear reactions:

Nuclear reactions can take place even at zero temp in a crystal lattice at sufficient high densities. (zero point oscillations of the nuclei allow them to tunnel through the Coulomb potential ~~barien~~ barrier to the neighboring site and induce nuclear reactions).

At densities above $\sim 10^4$ ($H \rightarrow {}^4He$) $- 10^9 \; g/cm^3$ (${}^{12}C \rightarrow {}^{24}Mg$) ~~polyn~~ pycnonuclear reactions can ~~convert~~ ~~$H \rightarrow {}^4He$~~ synthesize heavier elements at the timescale of $\sim 10^5$ yrs.

This might affect some observable properties of WDs (especially when binary systems are involved).

3. GR corrections:

So far gravity was treated using Newtonian theory. We obtain the equilibrium configuration by balancing the gravitational field of a mass density by pressure. In principle we can arrange an EOS such that the pressure is arbitrarily high for a given mass density; hence balancing any gravitational field.

However, in GR, pressure pressure also contributes an effective mass (part of the stress energy tensor). Increasing pressure → increasing grav. force. We cannot ensure ~~that~~ stability by increasing the pressure arbitrarily

→ ~~1st~~ $\boxed{\text{GR instability}}$