CORRECTIONS TO THE WHITE DWARF STRUCTURE

1. Considering the white dwarf interior to be in crystalline form. The crystal lattice is divided into spheres of radius r_0 such that

$$\frac{4}{3}\pi r_0^3 n_e = Z$$

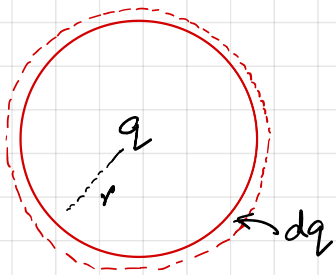
where, $n_e \to$ number density of free electrons.
$Z \to$ atomic number.

Coulomb energy of such a cell of radius r_0 consists of contribution from electron-electron and electron-ion interactions, i.e.

$$E_C = E_{ee} + E_{ei} \qquad \text{—— (1)}$$

where E_{ee} := Self energy of a negatively charged sphere of radius r_0.
E_{ei} := electrostatic interaction energy between the nucleus and the electron cloud.

Calculating E_{ee}:



Consider 'q' charge being distributed uniformly with charge density ρ_e, within a spherical volume of radius r. Let 'dq' amount of charge be added such that

$$dq = 4\pi r^2 \rho_e \, dr$$

The energy of the configuration is:

$$dE_{ee} = \frac{q\,dq}{r} = \frac{1}{r}\left(\rho_e \cdot \frac{4}{3}\pi r^3\right)\left(\rho_e 4\pi r^2 dr\right)$$

$$= \frac{16\pi^2}{3}\rho_e^2 r^4 dr$$

$$\Rightarrow E_{ee} = \frac{16\pi^2}{3}\rho_e^2 \int_0^{r_0} r^4 dr$$

$$\Rightarrow E_{ee} = \frac{16\pi^2}{3} \rho_e^2 \frac{r_0^5}{5}$$

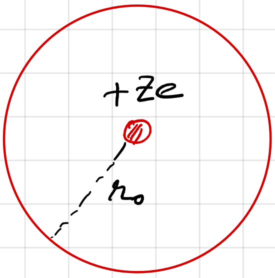
Replacing, $\rho_e = Ze \Big/ \frac{4}{3}\pi r_0^3$

$$\Rightarrow E_{ee} = \frac{16\pi^2}{3} \times \frac{9Z^2e^2}{16\pi^2 r_0} \times \frac{1}{5}$$

$$E_{ee} = \frac{3}{5}\frac{Z^2e^2}{r_0} \quad \text{—— (2)}$$

Calculating E_{ei}:

Let the +ve ion be concentrated at the origin surrounded by an electron cloud upto a radius r_0.



Considering a shell at distance r from the origin.

The electrostatic energy between the ion and this shell is:

$$dE_{ei} = \frac{-(Ze)\cdot \rho_e \cdot 4\pi r^2 dr}{r}$$

Total energy of interaction between the ion and the electron cloud:

$$E_{ei} = \int dE_{ei} = -4\pi \rho_e \cdot Ze \int_0^{r_0} r\,dr = 2\pi \rho_e Ze r_0^2$$

$$= -2\pi \frac{Ze}{\frac{4}{3}\pi r_0^3} Ze\, r_0^2$$

$$E_{ei} = \frac{-3}{2}\frac{Z^2e^2}{r_0} \quad \text{—— (3)}$$

Plugging (2) and (3) into (1), we get the Coulomb energy of a cell,

$$E_c = \frac{3}{5}\frac{Z^2e^2}{r_0} - \frac{3}{2}\frac{Z^2e^2}{r_0}$$

$$E_c = \frac{-9}{10}\frac{Z^2e^2}{r_0} \quad \text{—— (4)}$$

n_e by definition is :

$$n_e = \frac{Z}{\frac{4}{3}\pi r_0^3} \Rightarrow r_0 = \left(\frac{3Z}{4\pi n_e}\right)^{1/3}$$

Plugging this into (4), we get :

$$E_c = \frac{-9}{10} Z^2 e^2 \frac{(4\pi n_e)^{1/3}}{(3Z)^{1/3}}$$

$$= \frac{-9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{5/3} e^2 n_e^{1/3}$$

Since there are Z electrons, so Coulomb energy per free electron is

$$\boxed{\epsilon(n_e) \equiv \frac{E_c}{Z} = \frac{-9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{1/3}}$$