TUTORIAL 7

The Schwarzchild metric for the spacetime outside a spherically symmetric non-rotating black hole is given by:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Determine the geodesic equations of motion by first determining the Christoffel symbols

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

And then plugging them into the following:

$$\frac{du^{\alpha}}{d\tau} = -\Gamma^{\alpha}_{\beta\gamma}u^{\beta}u^{\gamma}$$

Where u is the four-velocity defined as $u^{\alpha} = \dfrac{dx^{\alpha}}{d\tau}$

You can do this one of two ways. If you're feeling particularly industrious, you can compute each of the non-vanishing Christoffel symbols by hand and plug them into the geodesic equation. Alternatively, you could use Mathematica to determine the Christoffel symbols. This is the recommended way.

TUTORIAL 8

Another way to set up the radial geodesic equation is to determine the conserved quantities in the Schwarzschild metric, and then normalise the four-velocity.

A) Determine the Killing vectors that correspond to the time-independence and the azimuthal-angle independence of the metric.
B) Determine the conserved quantities corresponding to these Killing vectors. To do so, project the four velocity onto the Killing vectors. Let e and l be the conserved quantities that correspond to the time-invariance and azimuthal-angle invariance of the metric.
C) Write down the radial equation in terms of these conserved quantities by normalising the four velocity for timelike geodesics.
D) Recast the radial equation so that there is a direct correspondence with the Newtonian case. Specifically, determine $\mathscr{E}$ and V_{eff} in the radial equation so that the radial equation becomes:

$$\mathscr{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}$$

E) Find the maximum and minimum radius that corresponds to the maximum and minimum in the effective potential. The minimum corresponds to stable circular orbits.

F) Arbitrarily small stable circular orbits are not allowed in this metric. The radius that corresponds to the smallest allowed is the radius of the Innermost Stable Circular Orbit (ISCO). Determine this radius.

G) Plot V_{eff} as a function of r/M, for different values of l/M, say 3, 4, 5.

H) Determine the angular velocity $\Omega \equiv \dfrac{d\phi}{dt}$ in terms of M and r only, for circular orbits. You should find an exact correspondence with the non-relativistic Kepler case. Hint: The radial velocity for circular orbits is zero, and the effective potential is at its minimum.

I) Write down the radial equation for null geodesics in terms of these conserved quantities by normalising the four velocity for null geodesics. Write this equation in terms of l and $b \equiv \frac{l^2}{e^2}$
Determine the effective potential for null geodesics, and the radius of the maximum of the effective potential.