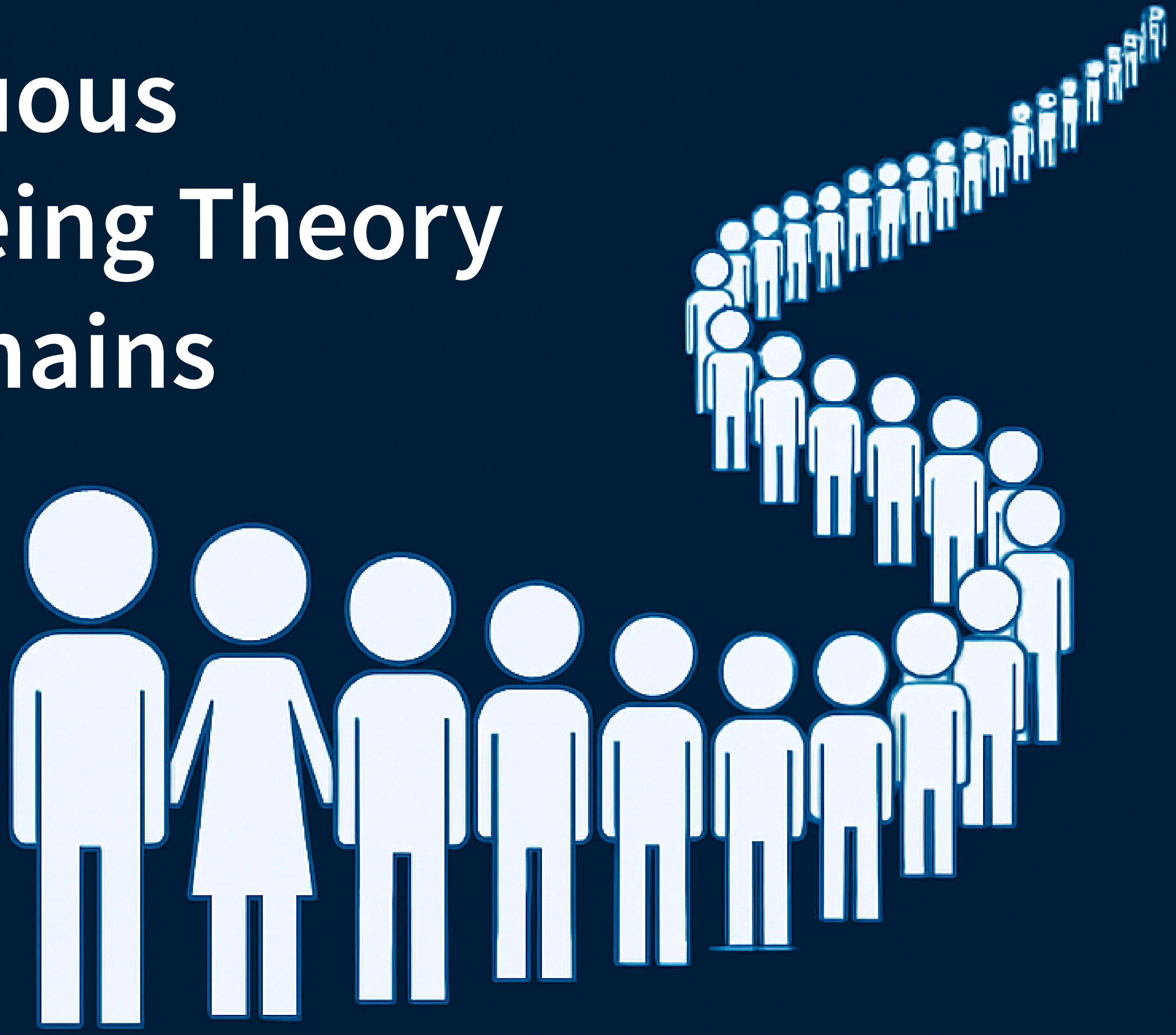
Introduction to Continuous Markov Chains & Queueing Theory in Healthcare Supply Chains



Udeshi Salgado

Data Lab for Social Good,

Cardiff University, UK

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Assumptions

- You should be familiar with basic probability and random variables.
- You are expected to be comfortable with R (or Python) for basic simulation and matrix operations.
- This is not a theory-heavy workshop—we will use simple examples to build intuition, not derive theorems.

What We Will Cover

- Key concepts and steady-state analysis of Continuous-Time Markov Chains (CTMCs)
- Modeling a vaccine observation room using CTMCs
- Introduction to Queueing Theory
- Introduction to M/M/1 queueing systems and performance metrics

What We Will *Not* Cover

- Theoretical proofs (e.g., Chapman-Kolmogorov, Little’s Law)
- Networks of queues
- Multiclass queues and priority scheduling
- Non-stationary arrivals and general service times
- Non-exponential queues
- Time-dependent queueing models

Materials

- You can find the workshop materials at here.
- Note: These materials are based on my learnings at NATCOR Taught Course Centre: Stochastic Modelling Course

Outline

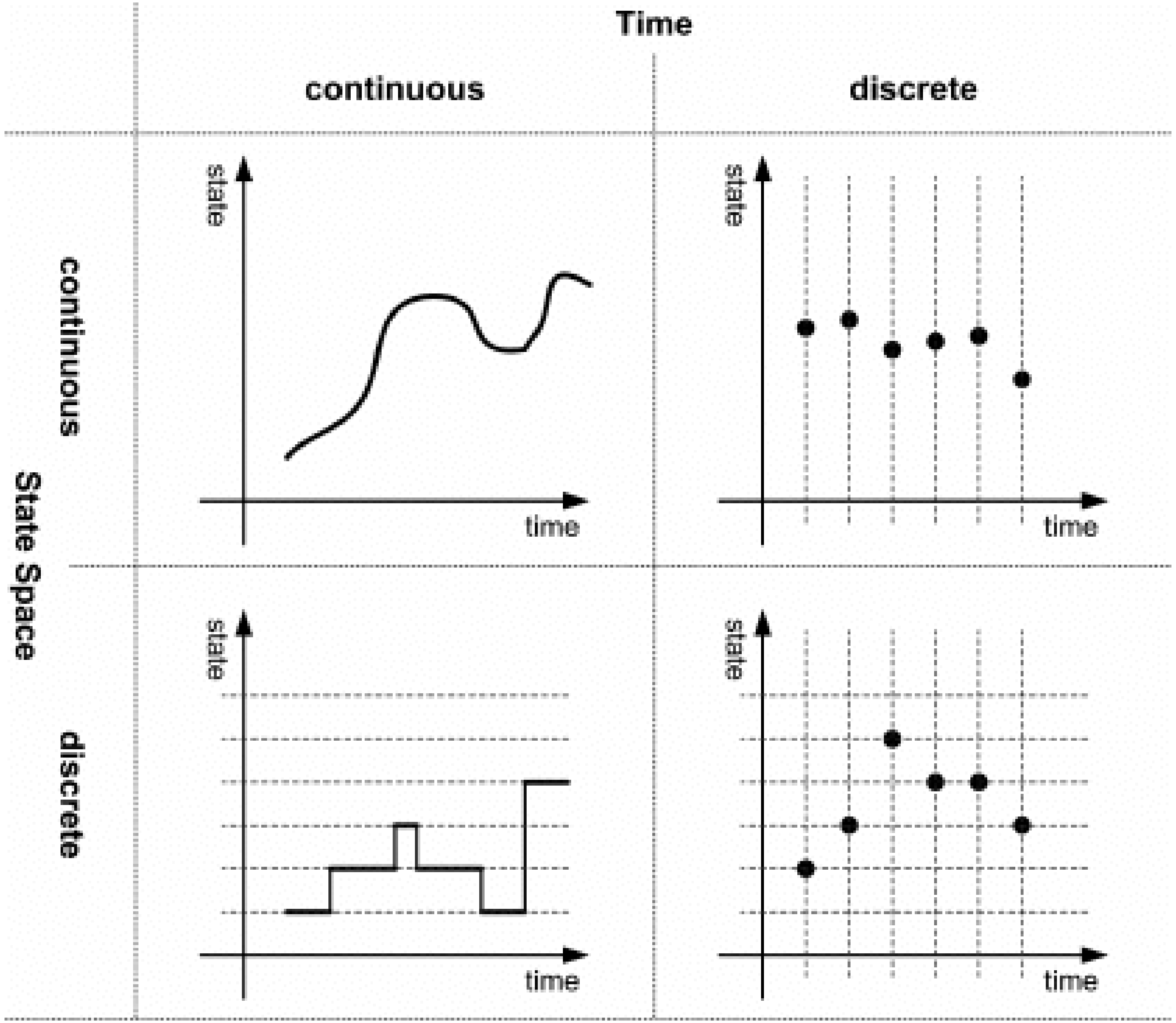
1. Recap
2. Continuous Time Markov Chains (CTMC)
3. Lab Activity: Part 1 – CTMC (Vaccine Observation Room)
4. Queueing Theory
5. Lab Activity: Part 2 – Queueing Theory (Check-In Desk)

Recap: Quiz Time!

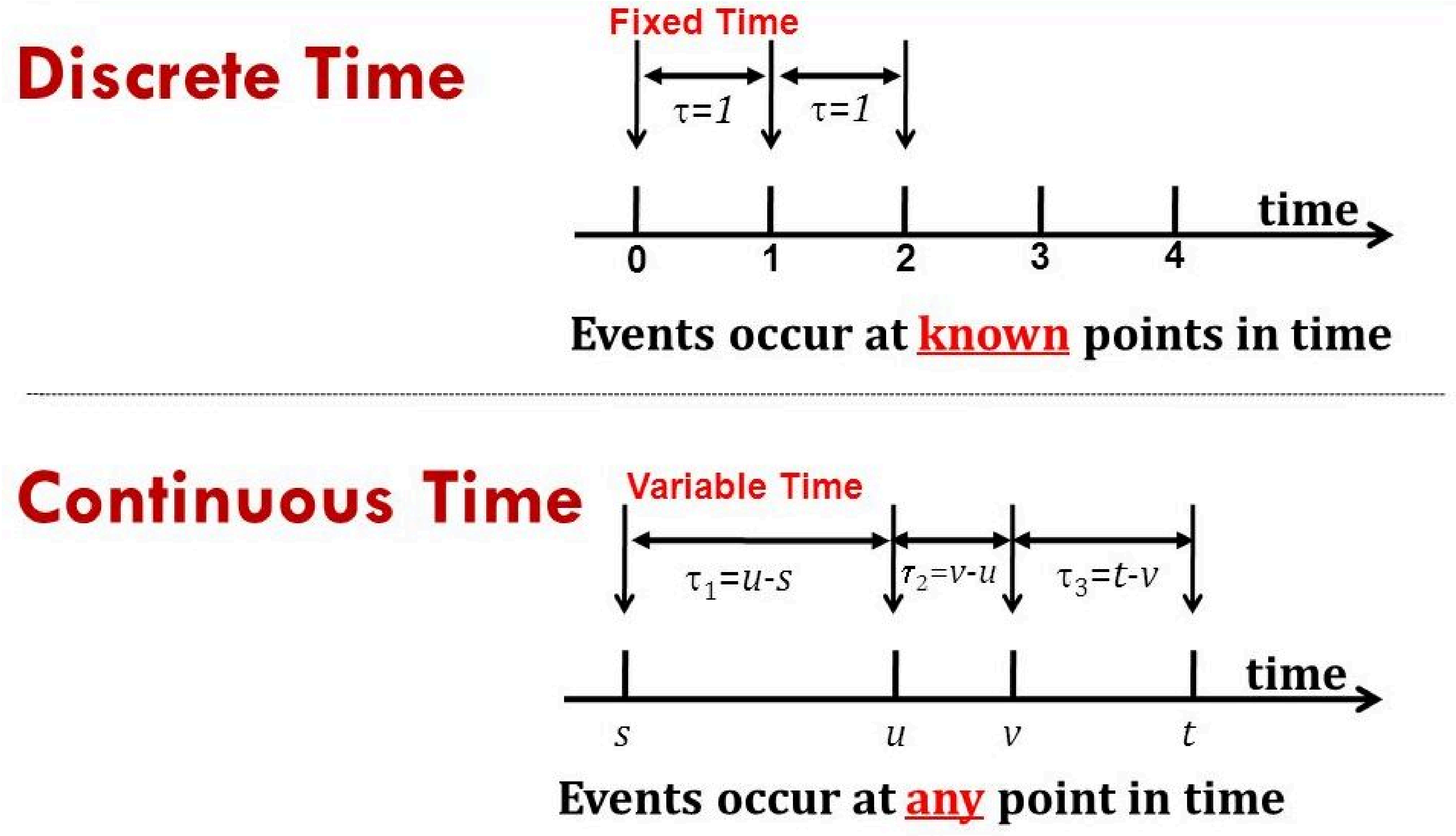
Recap: Discrete-Time Markov Chain (DTMC) Assumptions

- The system is always in exactly one state at any given time step.
- The next state depends only on the current state, not on the past (Markov property).
- Transitions happen at fixed, regular time steps.
- Transition probabilities stay the same over time.

Recap: State Space and Parameter Space



Recap: Discrete MC Vs Continuous MC



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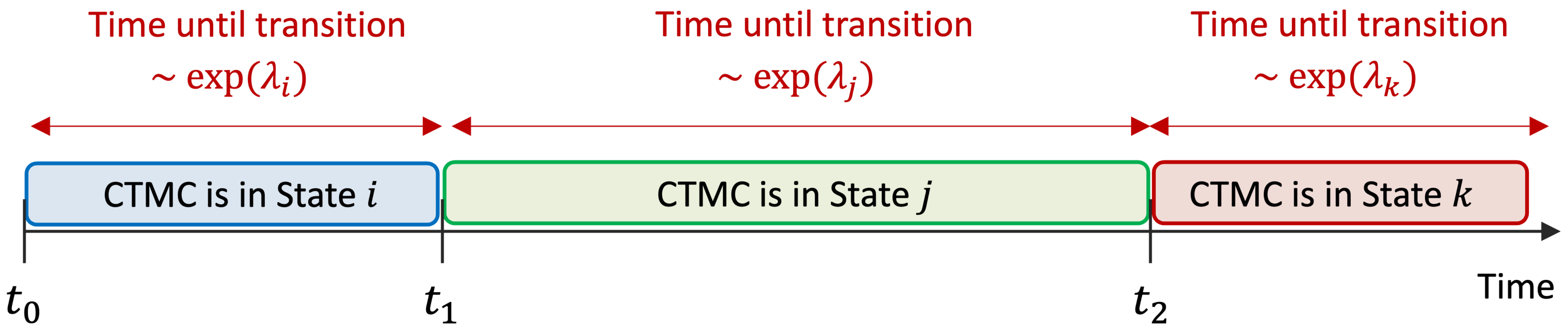
Continuous Time Markov Chains (CTMC)

- In a continuous-time Markov chain (CTMC), the system state can change at any point on the continuous time axis.



Exponential Timing in CTMCs

- **Assumption:** In a **Continuous-Time Markov Chain (CTMC)**, the time until the next state transition is modeled as a random variable

$$T \sim \text{Exponential}(\lambda)$$

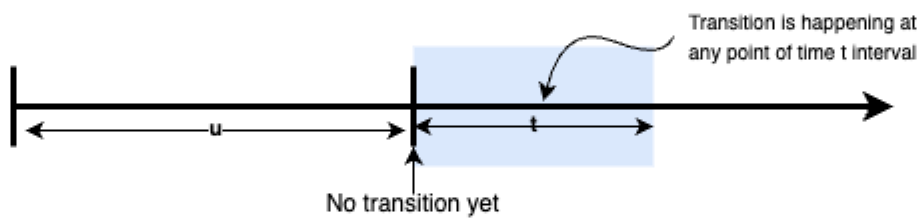
where:

- T = time until next transition
- $\lambda > 0$ = **transition rate** (higher $\lambda \rightarrow$ faster transitions)

Exponential Timing in CTMCs Cont.

- The **memoryless property** of the exponential distribution:

$$\Pr(T > t + u \mid T > u) = \Pr(T > t) \quad \text{for all } t, u > 0$$



- **Interpretation:** Time already spent in the current state does **not** influence the probability of future transitions.
- This is consistent with the **Markov property**: `The future depends only on the current state, not the past.`

Key Concepts: Stochastic Process, Time & State Spaces

- A **stochastic process** is a collection of random variables $\{X(t)\}$ indexed by time t.
- The **parameter space** defines the set of time points:
  - For continuous-time processes: $t \in [0, \infty)$.
- The **state space** S is the set of all possible values $X(t)$ can take:
  - Typically finite or countably infinite, e.g., $S = \{0, 1, 2, \ldots\}$.
- $X(t)$ represents the **state of the system at time** t.

CTMCs: Definition

A **continuous-time Markov chain (CTMC)** is a stochastic process:

$$\{X(t) \mid t \geq 0\}$$

where:

- $X(t)$ is a **random variable** representing the system's **state at time** t.
- t belongs to the **parameter space** $[0, \infty)$.
- The values of $X(t)$ come from a **state space** S, usually finite or countable.

CTMCs: Definition Cont.

The process satisfies the **Markov property**:

$$\Pr(A \mid X(t), 0 \leq t \leq T) = \Pr(A \mid X(T))$$

where:

- A is any **future event**.
- The future depends **only on the current state** $X(T)$, not on the history.

CTMC Example: Managing a Tiny Shop’s Flow

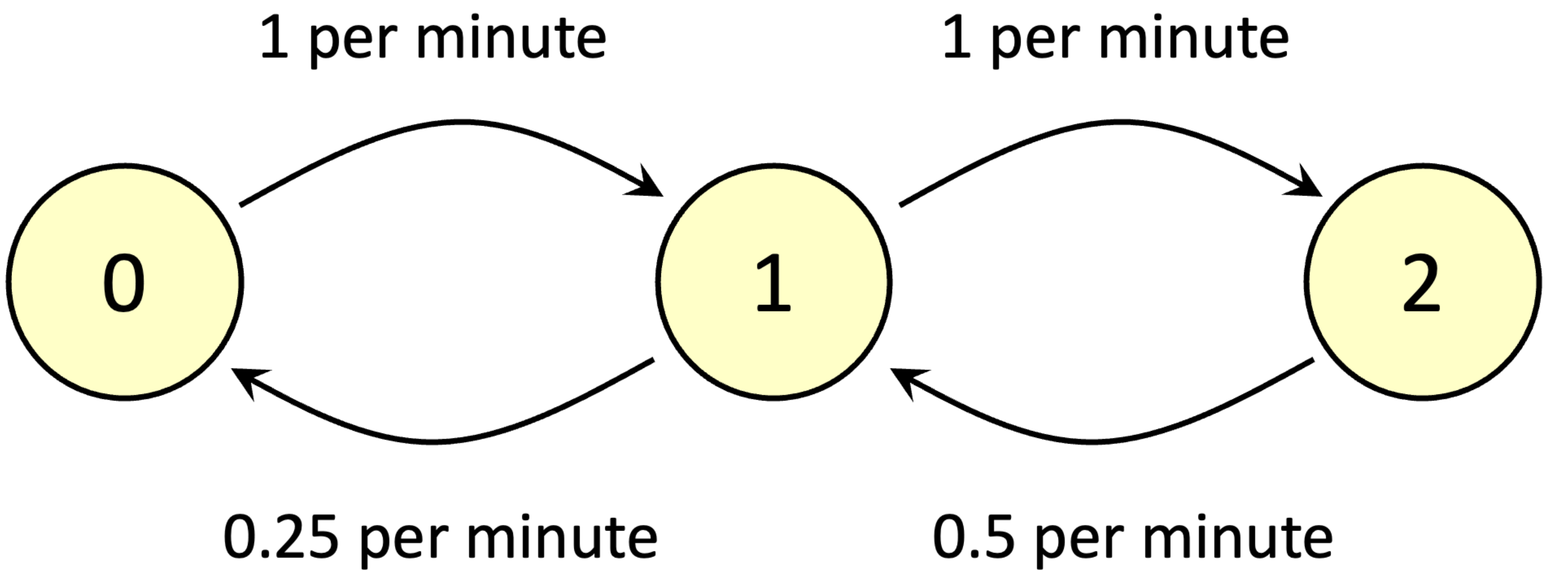
Imagine a **small neighborhood pharmacy** with space for **maximum of two customers at a time**. As the shop owner, you’re analyzing foot traffic to optimize customer experience without overcrowding or turning people away.

CTMC Example: Managing a Tiny Shop's Flow Cont.

- **Arrivals** follow an exponential distribution with **λ = 1 per minute**
- **Departures** depend on how many customers are in the shop:
  - **1 customer** → exponential(λ = 0.25)
  - **2 customers** → exponential(λ = 0.5)
- If a new customer arrives:
  - And there are **0 or 1 customers**, they enter the shop
  - If **2 customers** are already inside, they **leave without entering** (no queueing!)

CTMC Example: Transition Rate Diagram

- We can model this scenario as a **Continuous-Time Markov Chain (CTMC)**, where the **state** represents the **number of customers in the shop**.
- Thus, the possible states are: `0`, `1`, and `2`.
- We can visualize the **transition rates** between these states:



Time Until Next Transition

Let T_i be the time until the next transition from state $i \in \{0, 1, 2\}$

- $T_0 \sim \text{Exp}(1)$ *(arrival triggers transition)*
- $T_2 \sim \text{Exp}(0.5)$ *(departure triggers transition)*
- $T_1 = \min(A, D)$, where:
  - $A \sim \text{Exp}(1)$ *(a new customer arrives)*
  - $D \sim \text{Exp}(0.25)$ *(a customer leaves the shop)*

Exponential Minimum Property

Suppose T_1, T_2 are independent and

$$T_1 \sim \mathrm{Exp}(\lambda_1), \qquad T_2 \sim \mathrm{Exp}(\lambda_2)$$

Then:

$$\min(T_1, T_2) \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$$

Exponential Minimum Property Cont.

Application to our example:

- For state 1:

$$T_1 = \min(A, D); \quad A \sim \text{Exp}(1),\ D \sim \text{Exp}(0.25)$$

- So:

$$T_1 \sim \text{Exp}(1 + 0.25) = \text{Exp}(1.25)$$

Generator Matrix for a CTMC

The **generator matrix (Q)** summarizes the transition rates between states in a CTMC.

- Each element q_{ij} gives the **rate of moving from state i to state j**
- **Diagonal elements** are negative and equal to the **negative of the row sum**
- **Non-diagonal elements** are ≥ 0 and indicate direct transitions

Interpreting Key Entries in Generator Matrix

The rate of going from state 0 to state 1 is 1 per minute

The CTMC can't jump from state 0 to 2: only one customer arrives at a time. So the rate is 0.

$$\mathbf{Q} = \begin{pmatrix} \boxed{-1} & 1 & 0 \\ 0.25 & \boxed{-1.25} & 1 \\ 0 & 0.5 & \boxed{-0.5} \end{pmatrix}$$

Diagonal entries ensure rows sum to zero and show total exit rates.

The rate of moving from state 1 to state 0 is 0.25 per minute.

The rate of moving from state 2 to state 1 is 0.5 per minute.

Steady-State Probabilities

Steady-state probabilities represent the long-run behavior of the system. They tell us:

- The **proportion of time** the CTMC spends in each state in the long run.
- The **limiting probability** that the system is in a given state after a long time.

To find them, we solve:

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}, \quad \text{with} \quad \sum \pi_i = 1$$

Where: Q is the **generator matrix** of the CTMC, $\boldsymbol{\pi}$ is a **row vector** of steady-state probabilities.

Solving the System

From:

$$
\begin{aligned}
-\pi_0 + 0.25\pi_1 &= 0 \\
\pi_0 - 1.25\pi_1 + 0.5\pi_2 &= 0 \\
\pi_1 - 0.5\pi_2 &= 0 \\
\pi_0 + \pi_1 + \pi_2 &= 1
\end{aligned}
$$

Thus:

$$\pi_0 = \frac{1}{13}, \quad \pi_1 = \frac{4}{13}, \quad \pi_2 = \frac{8}{13}$$

Long-run Average Number of Customers

We calculate:

$$0 \cdot \frac{1}{13} + 1 \cdot \frac{4}{13} + 2 \cdot \frac{8}{13} = \frac{20}{13} \approx 1.54$$

This is the expected number of customers in the shop in the long run.

Summary of CTMCs

- Transition times are exponentially distributed.
- The **generator matrix** $\mathbf{Q}$ defines transition rates.
- Steady-state $\boldsymbol{\pi}$ solves:

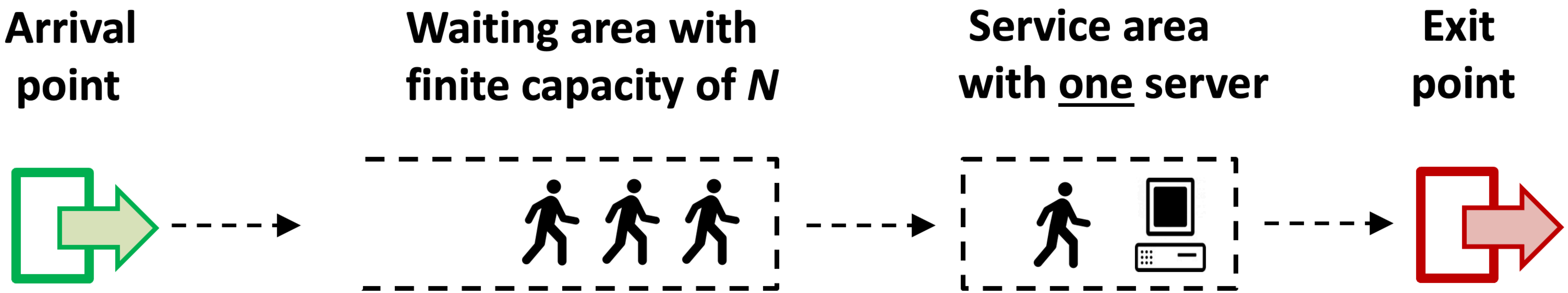
$$\boldsymbol{\pi}\mathbf{Q} = 0, \quad \sum \pi_i = 1$$

- π_i gives long-run proportions of time spent in each state.

Queueing Systems as CTMCs

A queueing system (e.g., customer service desk, call center, clinic) can often be modeled as a **continuous-time Markov chain**, under assumptions like:

- Exponentially-distributed inter-arrival times
- Exponentially-distributed service times
- Finite waiting area, single server, and one-at-a-time processing



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Lab Activity: Part 1 – CTMC (Vaccine Observation Room)

In this section, you'll model an **observation room** where recently vaccinated patients are monitored for adverse events. The room has limited capacity (e.g., 2 beds).

We will use a **Continuous-Time Markov Chain (CTMC)** to:

- Construct the **generator matrix** (Q)
- Write and solve the **steady-state equations**
- Interpret the **long-run distribution** across states

Make sure you load the required R packages: `Matrix`, `expm`

And now it's your turn!

⏳ 15:00

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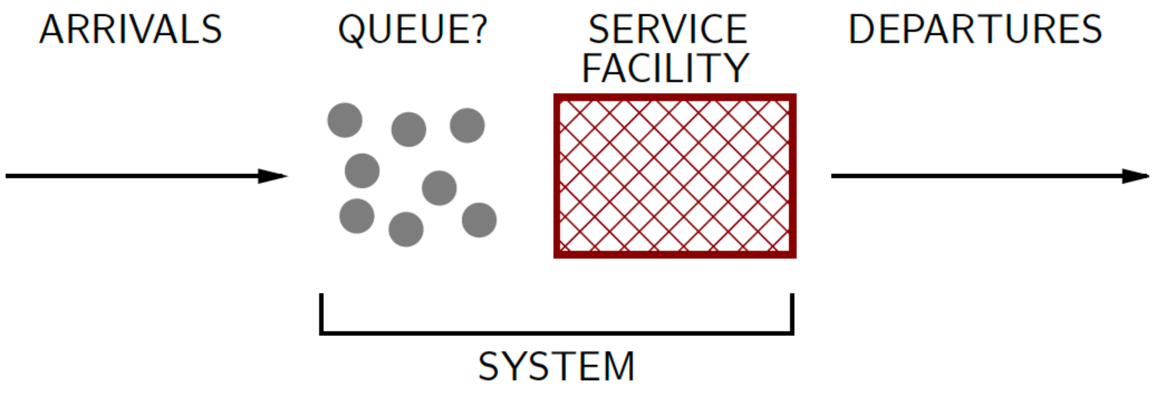
Queueing Theory

Welcome to Queueing Theory!
We’ll explore how to model and analyze systems where “waiting” happens — like hospitals, call centers, and shops.

What is a Queueing System?

A system with:

- **Arrivals** (e.g., customers, calls)
- **A queue** (optional)
- **Service process** (1 or more servers)
- **Departures**



Real-World Applications

- ATM and supermarket checkout lines
- Call centers and helpdesks
- Hospital patient flow
- Computer servers and networks

Kendall's Standard Notation

Queueing systems are described using: **A/B/S/d/e**

- **A** – Arrival distribution (e.g., M = Exponential)
- **B** – Service distribution (e.g., M, D = Deterministic, G = General)
- **S** – Number of servers
- **d** – System capacity (buffer size)
- **e** – Queue discipline (e.g., FIFO, LIFO)

Example: M/M/2/5/FIFO

→ Exponential arrivals, exponential service, 2 servers,

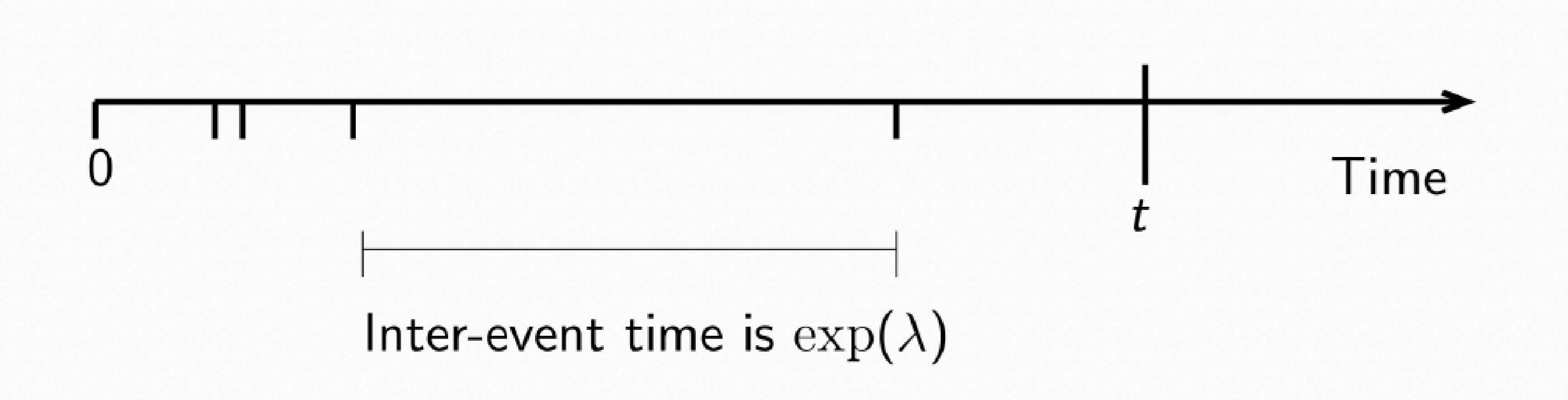
→ Max 5 customers in system, served in order of arrival

Poisson Process

- Many queueing systems assume **Poisson arrivals**, a key foundation for M/M/1 and related models.
- This process is simple but powerful and forms the basis for most queueing models.

Definition:

A Poisson process with rate λ models **random arrivals over time**, where:



Poisson Process Cont.

- The number of arrivals in time t follows:

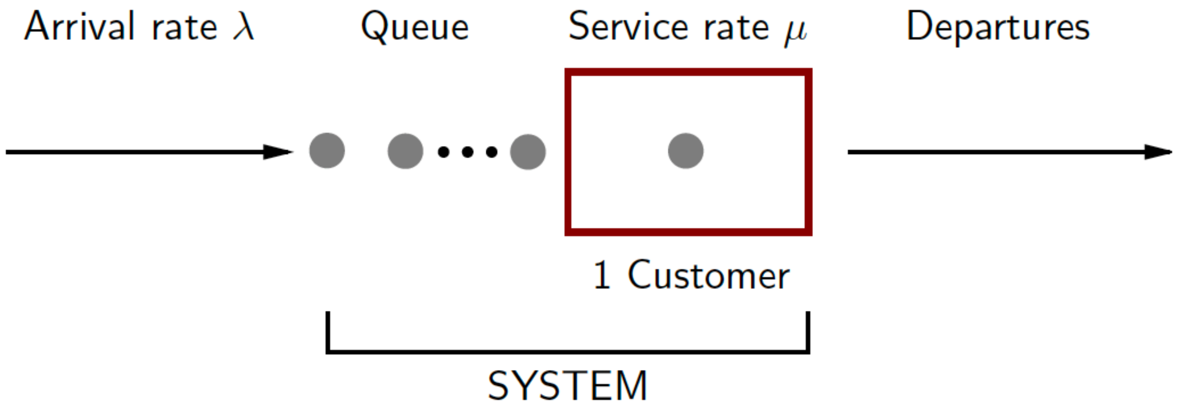
$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

- **Interarrival times** are exponentially distributed:

$$T \sim \mathrm{Exp}(\lambda)$$

- **Memoryless**: the probability of arrival doesn't depend on the past

The M/M/1 Queue



- A single-server queue with:
  - **Exponential interarrival times** (rate λ)
  - **Exponential service times** (rate μ)
- The number of customers in the system defines the **state**
- The queue has **infinite capacity**

The M/M/1 Queue Cont.

Traffic intensity: `How busy the system is?`

$$\rho = \frac{\lambda}{\mu}$$

- If $\rho < 1$, the system reaches a **steady-state**
- If $\rho \geq 1$, the queue grows indefinitely
- λ = arrival rate, μ = service rate

💡 Think of ρ as the **load** on the server.
If arrivals outpace service, the system can’t cope.

M/M/1 as a CTMC

- The **M/M/1 queue** can be modeled as a **Continuous-Time Markov Chain (CTMC)**.
- **States** represent the number of customers in the system: $0, 1, 2, \ldots$
- The process is a **birth–death process**:
  - **Births (arrivals)** occur at rate λ
  - **Deaths (departures)** occur at rate μ

M/M/1 as a CTMC Cont.

Generator matrix (Q):

$$
\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\ 0 & \mu & -(\lambda+\mu) & \lambda & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}
$$

- This structure defines the dynamics of the queue over time.

M/M/1: Stationary Distribution & Performance

Assume the system reaches **steady state** ($\rho < 1$), with probabilities $\{p_n,\ n \geq 0\}$.

These come from solving the **balance equations** of the CTMC:

State 0 balance:

$$\lambda p_0 = \mu p_1$$

For $n \geq 1$:

$$(\lambda + \mu) p_n = \lambda p_{n-1} + \mu p_{n+1}$$

Solving recursively gives:

$$p_n = \rho^n p_0 \quad \text{where} \quad \rho = \frac{\lambda}{\mu}$$

Then apply the **normalization condition**:

$$\sum_{n=0}^{\infty} p_n = 1 \quad \Rightarrow \quad p_0 = 1 - \rho$$

So we get:

- Probability the system is **empty**:

$$p_0 = 1 - \rho$$

- Probability the **server is busy**:

$$1 - p_0 = \rho$$

- Probability of n **customers** in the system:

$$p_n = \rho^n(1 - \rho)$$

🧠 These describe the **stationary distribution** of the M/M/1 queue.

M/M/1: Mean Number in System (L)

Expected number of customers in the system:

$$L = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho) = \rho \sum_{n=1}^{\infty} n\rho^{n-1}(1-\rho) = \frac{\rho}{1-\rho}$$

- This is one of the **key performance metrics** in queueing analysis.
- L increases sharply as $\rho \to 1$, showing how queues explode near capacity.

💡 Use this to inform decisions on service rates (μ) to keep queues manageable.

Performance Measures in Queueing

Four key summary performance measures:

- L: Mean number of customers **in the system**
- L_q: Mean number of customers **in the queue**
- W: Mean time a customer spends **in the system**
- W_q: Mean time a customer spends **waiting in the queue**

Little's Law

A fundamental relationship linking **arrival rate**, **waiting time**, and **number in the system**.

Mean number of customers in the system:

$$L = \lambda W \qquad (\lambda = \text{arrival rate}, W = \text{mean time in system})$$

Also:

$$L = L_q + \frac{\lambda}{\mu} \qquad (\text{total in system} = \text{queue} + \text{in service})$$

Mean waiting time:

$$W = W_q + \frac{1}{\mu} \qquad \left(W_q = \text{time in queue}, \frac{1}{\mu} = \text{service time}\right)$$

Mean number of customers in the queue:

$$L_q = \lambda W_q \qquad (\lambda = \text{arrival rate}, W_q = \text{mean time in queue})$$

M/M/1 Queue: Performance Formulas

Using $\rho = \frac{\lambda}{\mu}$ and Little's Law:

- Mean number in the system:

$$L = \frac{\rho}{1 - \rho}$$

- Mean number in queue:

$$L_q = \frac{\rho^2}{1 - \rho}$$

M/M/1 Queue: Performance Formulas Cont.

- Mean time in the system:

$$W = \frac{1}{\mu(1-\rho)}$$

- Mean waiting time in queue:

$$W_q = \frac{\rho}{\mu(1-\rho)}$$

Comparing M/M/1, M/M/s, and M/M/s/b

Feature	M/M/1	M/M/s	M/M/s/b
Servers	1	s	s
System Capacity	Infinite	Infinite	b (finite)
Arrival Rate (λ)	Poisson	Poisson	Poisson
Service Time	Exponential	Exponential	Exponential
Queue Capacity	Infinite	Infinite	$b - s$
Blocking?	No	No	Yes (if system full)
Common Use	Basic single-server queue	Multi-server (e.g., call center)	Limited capacity (e.g., hospital beds)

Let’s Practice: Model Matching

You’ll see 3 real-world situations.
For each one, pick the **most suitable queueing model**:

- M/M/1
- M/M/s
- M/M/s/b

You’ll have **30 seconds** to discuss or decide for each!

Scenario 1

An emergency care unit in a district hospital has five treatment beds. Once all beds are occupied, incoming patients must be transferred to a different facility, as there is no space for waiting or holding.

Which queueing model applies?

- M/M/1
- M/M/s
- M/M/s/b

⌛ 00:30

Scenario 1 – Answer

M/M/s/b Queueing Model

- Multiple beds = multiple servers
- No waiting space = limited capacity
- Incoming patients are blocked if full

Scenario 2

A mobile vaccination clinic is staffed by a single nurse. Patients from a rural community arrive randomly and are vaccinated one at a time. If the nurse is busy, others wait in line without any restrictions on queue length.

Which model applies?

- M/M/1
- M/M/s
- M/M/s/b

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Scenario 2 – Answer

M/M/1 Queueing Model

- One nurse = one server
- Unlimited queue = no blocking
- Standard basic queueing setup

Scenario 3

A city hospital's diagnostic laboratory operates with four technicians. Incoming test samples are processed as soon as a technician is free. If all are busy, samples wait in a queue with no limit.

Which model applies?

- M/M/1
- M/M/s
- M/M/s/b

⏳ 00:30

Scenario 3 – Answer

M/M/s Queueing Model

- Four technicians = multiple servers
- Queue allowed = no blocking
- Standard multi-server queue

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Lab Activity: Part 2 – Queueing Theory (Check-In Desk)

In this section, you'll model the **check-in desk** where patients arrive and wait to be registered by a nurse.

We will use an **M/M/1 queue** to:

- Calculate **traffic intensity** ($\rho = \lambda/\mu$)
- Compute steady-state metrics:
  - Average number in system (L) and queue (L_q)
  - Waiting times (W, W_q)
- Reflect on how arrival/service rates affect congestion

You’ll be guided with real-world tasks followed by code. Let’s begin!

And now it’s your turn!

⏳ 15:00

Thank You!

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