

CS 754 : Advanced Image Processing Assignment 2

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Q1

A1.1

Need to show :

$$\|h_{T_j}\|_{l_2} \leq s^{1/2} \|h_{T_j}\|_{l_\infty} \quad (1)$$

Equivalently we need to show :

$$\|A\|_{l_2} \leq s^{1/2} \|A\|_{l_\infty}$$

where A is a s -sparse vector. Therefore

$$\|A\|_{l_2} = \sqrt{\sum_i a_i^2} \leq \sqrt{\sum_i \max(a_i)^2} \leq s^{1/2} \max(a_i) = \|A\|_{l_\infty}$$

The $s^{1/2}$ term comes from the fact that A is s -sparse matrix, and hence there will be at most s non-zero elements.

A1.2

Need to show :

$$s^{1/2} \|h_{T_j}\|_{l_\infty} \leq s^{-1/2} \|h_{T_{j-1}}\|_{l_1} \quad (2)$$

for all $j \geq 2$

Equivalently we need to show :

$$s \|h_{T_j}\|_{l_\infty} \leq \|h_{T_{j-1}}\|_{l_1}$$

From the definition of T_j it follows for $j \geq 2$ that all elements of h_{T_j} will be less than the smallest non-zero element of $h_{T_{j-1}}$. Also both h_{T_j} and $h_{T_{j-1}}$ are s -sparse matrix, hence it clearly follows that

$$s \|h_{T_j}\|_{l_\infty} = s * \max(h_{T_j}) \leq \sum_i |h_{T_{j-1}}| = \|h_{T_{j-1}}\|_{l_1}$$

A1.3

Need to show :

$$\sum_{j \geq 2} \|h_{T_j}\|_{l_2} \leq s^{-1/2} (\|h_{T_1}\|_{l_1} + \|h_{T_2}\|_{l_1} + \dots) \quad (3)$$

This follows directly from 1 and 2.

$$\|h_{T_j}\|_{l_2} \leq s^{-1/2} \|h_{T_{j-1}}\|_{l_1}$$

for all $j \geq 2$ Now summing over all $j \geq 2$ we get

$$\sum_{j \geq 2} \|h_{T_j}\|_{l_2} \leq s^{-1/2} (\|h_{T_1}\|_{l_1} + \|h_{T_2}\|_{l_1} + \dots)$$

A1.4

?? Need to show:

$$s^{-1/2}(\|h_{T_1}\|_{l_1} + \|h_{T_2}\|_{l_1} + \dots) \leq s^{-1/2}\|h_{T_0^c}\|_{l_1} \quad (4)$$

A1.5

Need to show:

$$\|h_{(T_0 \cup T_1)^c}\|_{l_2} = \left\| \sum_{j \geq 2} h_{T_j} \right\|_{l_2} \quad (5)$$

We note that

$$h_{(T_0 \cup T_1)^c} = h - h_{T_0} - h_{T_1} = h_{T_2} + h_{T_3} + \dots = \sum_{j \geq 2} h_{T_j}$$

And hence 5 follows directly.

A1.6

Need to show:

$$\left\| \sum_{j \geq 2} h_{T_j} \right\|_{l_2} \leq \sum_{j \geq 2} \|h_{T_j}\|_{l_2} \quad (6)$$

This is simple extension of triangle inequality, which states that

$$|a + b| \leq |a| + |b|$$

For n vectors it is simply

$$|a_1 + a_2 + a_3 + \dots + a_n| \leq |a_1| + |a_2| + |a_3| + \dots + |a_n|$$

And hence 6 follows directly

A1.7

Need to show:

$$\sum_{j \geq 2} \|h_{T_j}\|_{l_2} \leq s^{-1/2} \|h_{T_0^c}\|_{l_1} \quad (7)$$

This follows directly from 3 and 4.

$$\sum_{j \geq 2} \|h_{T_j}\|_{l_2} \leq s^{-1/2}(\|h_{T_1}\|_{l_1} + \|h_{T_2}\|_{l_1} + \dots) \leq s^{-1/2} \|h_{T_0^c}\|_{l_1}$$

A1.8

??? Need to show:

$$\|x\|_{l_1} \geq \|x + h\|_{l_1} \geq \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1} + \|h_{T_0^c}\|_{l_1} - \|x_{T_0^c}\|_{l_1} \quad (8)$$

A1.9

Need to show:

$$\|h_{T_0^c}\|_{l_2} \leq \|h_{T_0}\|_{l_1} + 2\|x_{T_0^c}\|_{l_1} \quad (9)$$

We can rearrange 8 to have

$$\|h_{T_0^c}\|_{l_1} \leq \|x\|_{l_1} - \|x_{T_0}\|_{l_1} + \|h_{T_0}\|_{l_1} + \|x_{T_0^c}\|_{l_1}$$

We note that

$$\|x\|_{l_1} - \|x_{T_0}\|_{l_1} \leq \|x - x_{T_0}\|_{l_1} = \|x_{T_0^c}\|_{l_1}$$

Therefore

$$\|h_{T_0^c}\|_{l_1} \leq \|x_{T_0^c}\|_{l_1} + \|h_{T_0}\|_{l_1} + \|x_{T_0^c}\|_{l_1} = \|h_{T_0}\|_{l_1} + 2\|x_{T_0^c}\|_{l_1}$$

A1.10

Need to show:

$$\|h_{(T_0 \cup T_1)^c}\|_{l_2} \leq \|h_{T_0}\|_{l_2} + 2e_0, e_0 \equiv s^{-1/2}\|x - x_s\|_{l_2} \quad (10)$$

Combining 5 6 and 7 we get

$$\|h_{(T_0 \cup T_1)^c}\|_{l_2} = \left\| \sum_{j \geq 2} h_{T_j} \right\|_{l_2} \leq \sum_{j \geq 2} \|h_{T_j}\|_{l_2} \leq s^{-1/2} \|h_{T_0^c}\|_{l_1}$$

From 9 we get

$$s^{-1/2} \|h_{T_0^c}\|_{l_1} \leq s^{-1/2} \|h_{T_0}\|_{l_1} + 2s^{-1/2} \|x_{T_0^c}\|_{l_1}$$

Also by definition

$$\|x_{T_0^c}\|_{l_1} = \|x - x_s\|_{l_1}$$

This implies

$$s^{-1/2} \|h_{T_0^c}\|_{l_1} \leq s^{-1/2} \|h_{T_0}\|_{l_1} + 2s^{-1/2} \|x - x_s\|_{l_1}$$

Therefore

$$s^{-1/2} \|h_{T_0^c}\|_{l_1} \leq s^{-1/2} \|h_{T_0}\|_{l_1} + 2e_0, e_0 \equiv s^{-1/2} \|x - x_s\|_{l_2}$$

Now we also note, for any s-sparse vector A

$$\|A\|_{l_1} = \sum_i |a_i| = \sum_i |a_i| * 1 \leq \sqrt{s} \sqrt{\sum_i a_i^2} = s^{1/2} \|A\|_{l_2}$$

and here we have used Cauchy Schwartz Inequality. That is

$$s^{-1/2} \|A\|_{l_1} \leq \|A\|_{l_2}$$

Thus it follows that

$$s^{-1/2} \|h_{T_0}\|_{l_1} \leq \|h_{T_0}\|_{l_2}$$

And 10 directly follows

$$\|h_{(T_0 \cup T_1)^c}\|_{l_2} \leq \|h_{T_0}\|_{l_2} + 2e_0, e_0 \equiv s^{-1/2} \|x - x_s\|_{l_2}$$

A1.14

Need to show:

$$\Phi h_{(T_0 \cup T_1)} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j} \quad (11)$$

We know

$$h_{(T_0 \cup T_1)^c} = h - h_{(T_0 \cup T_1)} = h - h_{T_0} - h_{T_1} = \sum_{j \geq 2} h_{T_j}$$

Rearranging the equation

$$h_{(T_0 \cup T_1)} = h - \sum_{j \geq 2} h_{T_j}$$

Multiplying ϕ on both sides

$$\Phi h_{(T_0 \cup T_1)} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j}$$

A1.15

Need to show:

$$| \langle \Phi h_{(T_0 \cup T_1)}, \Phi h \rangle | \leq \| \Phi h_{(T_0 \cup T_1)} \|_{l_2} \| \Phi h \|_{l_2} \quad (12)$$

This is simple application of Cauchy Schwartz Inequality which states that given two vectors a and b

$$\langle a, b \rangle \leq \|a\|_{l_2} \|b\|_{l_2}$$

And therefore 12 directly follows from this.

A1.16

Need to show:

$$\|\Phi h_{(T_0 \cup T_1)}\|_{l_2} \|\Phi h\|_{l_2} \leq 2\varepsilon \sqrt{1 + \delta_{2s}} \|h_{(T_0 \cup T_1)}\|_{l_2} \quad (13)$$

We note that

$$\|\Phi(x^* - x)\|_{l_2} \leq \|\Phi x^* - y\|_{l_2} + \|y - \Phi x\|_{l_2} \leq 2\varepsilon$$

The first part of the Inequality is a direct result of Triangle Inequality. The second part of the Inequality arises from the fact that $\|y - \Phi x\|_{l_2} \leq \varepsilon$ and both x and x^* are a solution. We have assumed $x^* = x + h$. Therefore

$$\|\Phi h\|_{l_2} \leq 2\varepsilon$$

Also from the definition of RIP

$$\sqrt{(1 - \delta_s)} \|x\|_{l_2} \leq \|\Phi x\|_{l_2} \leq \sqrt{(1 + \delta_s)} \|x\|_{l_2}$$

where x is s -sparse vector. We know that $h_{(T_0 \cup T_1)}$ is a $2s$ -sparse vector. Therefore

$$\|\Phi h_{(T_0 \cup T_1)}\|_{l_2} \leq \sqrt{(1 + \delta_{2s})} \|h_{(T_0 \cup T_1)}\|_{l_2}$$

Multiplying the inequalities directly gives 13

$$\|\Phi h_{(T_0 \cup T_1)}\|_{l_2} \|\Phi h\|_{l_2} \leq 2\varepsilon \sqrt{1 + \delta_{2s}} \|h_{(T_0 \cup T_1)}\|_{l_2}$$

A1.17

Need to show:

$$\|h_{T_0}\|_{l_2} + \|h_{T_1}\|_{l_2} \leq \sqrt{2} \|h_{(T_0 \cup T_1)}\|_{l_2} \quad (14)$$

From AM-GM Inequality we know

$$\|h_{T_0}\|_{l_2} \|h_{T_1}\|_{l_2} \leq \frac{\|h_{T_0}\|_{l_2}^2 + \|h_{T_1}\|_{l_2}^2}{2}$$

Adding the RHS to both sides and Multiplying 2 on both sides

$$\|h_{T_0} + h_{T_1}\|_{l_2}^2 \leq 2 \|h_{(T_0 \cup T_1)}\|_{l_2}^2$$

Taking square roots on both sides gives us 14

A1.18

Need to show:

$$(1 - \delta_{2s}) \|h_{(T_0 \cup T_1)}\|_{l_2}^2 \leq \|\Phi h_{(T_0 \cup T_1)}\|_{l_2}^2 \quad (15)$$

Since $h_{(T_0 \cup T_1)}$ is a $2s$ -sparse vector, 15 follows from definition.

A1.19

Need to show:

$$\|\Phi h_{(T_0 \cup T_1)}\|_{l_2}^2 \leq \|h_{(T_0 \cup T_1)}\|_{l_2} (2\varepsilon \sqrt{1 + \delta_{2s}} + \sqrt{2}\delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_{l_2}) \quad (16)$$

Clearly

$$\|\Phi h_{(T_0 \cup T_1)}\|_{l_2}^2 = \langle \Phi h_{(T_0 \cup T_1)}, \Phi h \rangle - \langle \Phi h_{(T_0 \cup T_1)}, \sum_{j \geq 2} h_{T_j} \rangle$$

To get the maximum we want to maximize the first term and minimize the second term. As such we want to consider both the absolute values. From 13

$$\|\Phi h_{(T_0 \cup T_1)}\|_{l_2} \|\Phi h\|_{l_2} \leq 2\varepsilon \sqrt{1 + \delta_{2s}} \|h_{(T_0 \cup T_1)}\|_{l_2}$$

Also we note:

$$|\langle \Phi h_{(T_0 \cup T_1)}, \Phi h_{T_j} \rangle| \leq |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| + |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \delta_{2s} \|h_{T_j}\|_{l_2} (\|h_{T_0}\|_{l_2} + \|h_{T_1}\|_{l_2})$$

From 14

$$|\langle \Phi h_{(T_0 \cup T_1)}, \Phi h_{T_j} \rangle| \leq \sqrt{2}\delta_{2s} \|h_{T_j}\|_{l_2} \|h_{(T_0 \cup T_1)}\|_{l_2}$$

Combining the two inequalities we directly get 16

A1.20

Need to show:

$$\|h_{(T_0 \cup T_1)}\|_{l_2} \leq \alpha\varepsilon + \rho s^{-1/2} \|h_{T_0^c}\|_{l_1}, \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}} \quad (17)$$

From 15 and 16 we get

$$(1-\delta_{2s})\|h_{(T_0 \cup T_1)}\|_{l_2}^2 \leq \|h_{(T_0 \cup T_1)}\|_{l_2} (2\varepsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_{l_2})$$

From 7 and then dividing by $(1-\delta_{2s})\|h_{(T_0 \cup T_1)}\|_{l_2}$ we get

$$\|h_{(T_0 \cup T_1)}\|_{l_2} \leq \frac{2\varepsilon\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\varepsilon + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}s^{-1/2}\|h_{T_0^c}\|_{l_1}$$

Which is exactly 17

A1.21

Need to show:

$$\|h_{(T_0 \cup T_1)}\|_{l_2} \leq \alpha\varepsilon + \rho\|h_{(T_0 \cup T_1)}\|_{l_2} + 2\rho e_0 \Rightarrow \|h_{(T_0 \cup T_1)}\|_{l_2} \leq (1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0) \quad (18)$$

From 17 we know

$$\|h_{(T_0 \cup T_1)}\|_{l_2} \leq \alpha\varepsilon + \rho s^{-1/2}\|h_{T_0^c}\|_{l_1}$$

And from 9 we get

$$s^{-1/2}\|h_{T_0^c}\|_{l_2} \leq s^{-1/2}(\|h_{T_0}\|_{l_1} + 2\|x_{T_0^c}\|_{l_1}) \leq \|h_{T_0}\|_{l_2} + 2e_0, e_0 \equiv s^{-1/2}\|x - x_s\|_{l_2}$$

Also

$$\|h_{T_0}\|_{l_2} \leq \|h_{(T_0 \cup T_1)}\|_{l_2}$$

Therefore we can conclude

$$\|h_{(T_0 \cup T_1)}\|_{l_2} \leq \alpha\varepsilon + \rho\|h_{(T_0 \cup T_1)}\|_{l_2} + 2\rho e_0$$

Rearranging the equation we directly get

$$\|h_{(T_0 \cup T_1)}\|_{l_2} \leq (1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0)$$

A1.22

Need to show:

$$\|h\|_{l_2} \leq \|h_{(T_0 \cup T_1)}\|_{l_2} + \|h_{(T_0 \cup T_1)^c}\|_{l_2} \leq 2\|h_{(T_0 \cup T_1)}\|_{l_2} + 2e_0 \leq 2(1-\rho)^{-1}(\alpha\varepsilon + (1+\rho)e_0) \quad (19)$$

The first part is a direct result of Triangle Inequality.

$$\|h\|_{l_2} = \|h_{(T_0 \cup T_1)} + h_{(T_0 \cup T_1)^c}\|_{l_2} \leq \|h_{(T_0 \cup T_1)}\|_{l_2} + \|h_{(T_0 \cup T_1)^c}\|_{l_2}$$

From 10 we have a bound on $\|h_{(T_0 \cup T_1)^c}\|_{l_2}$

$$\|h_{(T_0 \cup T_1)^c}\|_{l_2} \leq \|h_{T_0}\|_{l_2} + 2e_0, e_0 \equiv s^{-1/2}\|x - x_s\|_{l_2}$$

Since $\|h_{(T_0 \cup T_1)}\|_{l_2} \geq \|h_{T_0}\|_{l_2}$

$$\|h_{(T_0 \cup T_1)^c}\|_{l_2} \leq \|h_{(T_0 \cup T_1)}\|_{l_2} + 2e_0$$

Therefore

$$\|h_{(T_0 \cup T_1)}\|_{l_2} + \|h_{(T_0 \cup T_1)^c}\|_{l_2} \leq 2\|h_{(T_0 \cup T_1)}\|_{l_2} + 2e_0$$

From 18 we directly get the bound on $\|h_{(T_0 \cup T_1)}\|_{l_2}$. Thus

$$2\|h_{(T_0 \cup T_1)}\|_{l_2} + 2e_0 \leq 2(1-\rho)^{-1}(\alpha\varepsilon + (1+\rho)e_0)$$