CS 754 : Advanced Image ProcessingAssignment 3

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April 4, 2017

Q1

A1.1

Need to show :

$$||h_{T_j}||_{l_2} \leq s^{1/2}||h_{T_j}||_{l_\infty} \tag{1}$$

Equivalently we need to show :

$$||A||_{l_2} \leq s^{1/2}||A||_{l_\infty}$$

where A is a s-sparse vector. Therefore

$$||A||_{l_2} = \sqrt{\sum_i a_i^2} \leq \sqrt{\sum_i max(a_i)^2} \leq s^{1/2}max(a_i) = ||A||_{l_\infty}$$

The $s^{1/2}$ term comes from the fact that A is s-sparse matrix, and hence there will be at most s non-zero elements.

A1.2

Need to show :

$$s^{1/2}||h_{T_j}||_{l_\infty} \leq s^{-1/2}||h_{T_{j-1}}||_{l_1} \tag{2}$$

for all $j \geq 2$

Equivalently we need to show :

$$s||h_{T_j}||_{l_\infty} \leq ||h_{T_{j-1}}||_{l_1}$$

From the definition of T_j it follows for $j \geq 2$ that all elements of h_{T_j} will be less than the smallest non-zero element of $h_{T_{j-1}}$. Also both h_{T_j} and $h_{T_{j-1}}$ are s-sparse matrix, hence it clearly follows that

$$s||h_{T_j}||_{l_\infty} = s * max(h_{T_j}) \leq \sum_i |h_{T_{j-1}}| = ||h_{T_{j-1}}||_{l_1}$$

A1.3

Need to show :

$$\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \tag{3}$$

This follows directly from 1 and 2.

$$||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_{j-1}}||_{l_1}$$

for all $j \geq 2$ Now summing over all $j \geq 2$ we get

$$\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...)$$

A1.4

Need to show:

$$s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \leq s^{-1/2}||h_{T_0{}^c}||_{l_1} \tag{4}$$

We note all of h_{T_1}, h_{T_2} all have disjoint support and therefore

$$||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ... \leq ||h_{T_0{}^c}||_{l_1}$$

A1.5

Need to show:

$$||h_{(T_0 \cup T_1)^c}||_{l_2} = ||\sum_{j \geq 2} h_{T_j}||_{l_2} \tag{5}$$

We note that

$$h_{(T_0 \cup T_1)^c} = h - h_{T_0} - h_{T_1} = h_{T_2} + h_{T_3} + ... = \sum_{j \geq 2} h_{T_j}$$

And hence 5 follows directly.

A1.6

Need to show:

$$||\sum_{j \geq 2} h_{T_j}||_{l_2} \leq \sum_{j \geq 2} ||h_{T_j}||_{l_2} \tag{6}$$

This is simple extension of triangle inequality, which states that

$$|a + b| \leq |a| + |b|$$

For n vectors it is simply

$$|a_1 + a_2 + a_3 + ... + a_n| \leq |a_1| + |a_2| + |a_3| + ... + |a_n|$$

And hence 6 follows directly

A1.7

Need to show:

$$\sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_0{}^c}||_{l_1} \tag{7}$$

This follows directly from 3 and 4.

$$\sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \leq s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

A1.8

Need to show:

$$||x||_{l_1} \geq ||x + h||_{l_1} \geq ||x_{T_0}||_{l_1} - ||h_{T_0}||_{l_1} + ||h_{T_0{}^c}||_{l_1} - ||x_{T_0^c}||_{l_1} \tag{8}$$

For first part we note that $x^* = x + h$ therefore, $||x^*||_{l_1} = ||x + h||_{l_1}$. According to our constraints,x^* has the minimum $||x^*||_{l_1}$ which also satisfies $||y - \Phi x||_{l_2} \leq \varepsilon$. Therefore if x is not s-sparse then

$$||x||_{l_1} > ||x^*||_{l_1}$$

And if x is s-sparse then

$$||x||_{l_1} = ||x^*||_{l_1}$$

Combining the two we can say

$$||x||_{l_1} \geq ||x^*||_{l_1}$$

From Triangle Inequality we know

$$||a+b||_{l_1} \geq |||a||_{l_1} - ||b||_{l_1}|$$

Therefore, we can also say

$$||a+b||_{l_1} \geq ||a||_{l_1} - ||b||_{l_1}$$

And

$$||a+b||_{l_1} \geq ||b||_{l_1} - ||a||_{l_1}$$

We note that

$$||x+h||_{l_1} = \sum_{i \in T_0} |x_i + h_i| + \sum_{i \in T_0^c} |x_i + h_i| \geq ||x_{T_0}||_{l_1} - ||h_{T_0}||_{l_1} + ||h_{T_0{}^c}||_{l_1} - ||x_{T_0^c}||_{l_1}$$

A1.9

Need to show:

$$||h_{T_0{}^c}||_{l_2} \leq ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1} \tag{9}$$

We can rearrange 8 to have

$$||h_{T_0{}^c}||_{l_1} \leq ||x||_{l_1} - ||x_{T_0}||_{l_1} + ||h_{T_0}||_{l_1} + ||x_{T_0^c}||_{l_1}$$

We note that

$$||x||_{l_1} - ||x_{T_0}||_{l_1} \leq ||x - x_{T_0}||_{l_1} = ||x_{T_0^c}||_{l_1}$$

Therefore

$$||h_{T_0{}^c}||_{l_1} \leq ||x_{T_0^c}||_{l_1} + ||h_{T_0}||_{l_1} + ||x_{T_0^c}||_{l_1} = ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}$$

A1.10

Need to show:

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \leq ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2} \tag{10}$$

Combining 5 6 and 7 we get

$$||h_{(T_0 \cup T_1)^c}||_{l_2} = ||\sum_{j \geq 2} h_{T_j}||_{l_2} \leq \sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

From 9 we get

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2s^{-1/2}||x_{T_0^c}||_{l_1}$$

Also by definition

$$||x_{T_0^c}||_{l_1} = ||x - x_s||_{l_1}$$

This implies

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2s^{-1/2}||x - x_s||_{l_1}$$

Therefore

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

Now we also note, for any s-sparse vector A

$$||A||_{l_1} = \sum_i |a_i| = \sum_i |a_i| * 1 \leq \sqrt{s}\sqrt{\sum_i a_i^2} = s^{1/2}||A||_{l_2}$$

and here we have used Cauchy Schwartz Inequality. That is

$$s^{-1/2}||A||_{l_1} \leq ||A||_{l_2}$$

Thus it follows that

$$s^{-1/2}||h_{T_0}||_{l_1} \leq ||h_{T_0}||_{l_2}$$

And 10 directly follows

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \leq ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

A1.14

Need to show:

$$\Phi h_{(T_0 \cup T_1)} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j} \tag{11}$$

We know

$$h_{(T_0 \cup T_1)^c} = h - h_{(T_0 \cup T_1)} = h - h_{T_o} - h_{T_1} = \sum_{j \geq 2} h_{T_j}$$

Rearranging the equation

$$h_{(T_0 \cup T_1)} = h - \sum_{j \geq 2} h_{T_j}$$

Multiplying ϕ on both sides

$$\Phi h_{(T_0 \cup T_1)} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j}$$

A1.15

Need to show:

$$| < \Phi h_{(T_0 \cup T_1)}, \Phi h > | \leq ||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \tag{12}$$

This is simple application of Cauchy Schwartz Inequality which states that given two vectors a and b

$$< a, b > \leq ||a||_{l_2} ||b||_{l_2}$$

And therefore 12 directly follows from this.

A1.16

Need to show:

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \leq 2\varepsilon \sqrt{1 + \delta_{2s}} ||h_{(T_0 \cup T_1)}||_{l_2} \tag{13}$$

We note that

$$||\Phi(x^* - x)||_{l_2} \leq ||\Phi x^* - y||_{l_2} + ||y - \Phi x||_{l_2} \leq 2\varepsilon$$

The first part of the Inequality is a direct result of Traingle Inequality. The second part of the Inequality arises from the fact that $||y - \Phi x||_{l_2} \leq \varepsilon$ and both x and x^* are a solution. We have assumed $x^* = x + h$. Therefore

$$||\Phi h||_{l_2} \leq 2\varepsilon$$

Also from the definition of RIP

$$\sqrt{(1 - \delta_s)} ||x||_{l_2} \leq ||\Phi x||_{l_2} \leq \sqrt{(1 + \delta_s)} ||x||_{l_2}$$

where x is s-sparse vector. We know that $h_{(T_0 \cup T_1)}$ is a 2s-sparse vector. Therefore

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} \leq \sqrt{(1 + \delta_{2s})} ||h_{(T_0 \cup T_1)}||_{l_2}$$

Multiplying the inequalities directly gives 13

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \leq 2\varepsilon \sqrt{1 + \delta_{2s}} ||h_{(T_0 \cup T_1)}||_{l_2}$$

A1.17

Need to show:

$$||h_{T_0}||_{l_2} + ||h_{T_1}||_{l_2} \leq \sqrt{2} ||h_{(T_0 \cup T_1)}||_{l_2} \tag{14}$$

From AM-GM Inequality we know

$$||h_{T_0}||_{l_2} ||h_{T_1}||_{l_2} \leq \frac{||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}^2}{2}$$

Adding the RHS to both sides and Multiplying 2 on both sides

$$||h_{T_0} + h_{T_1}||_{l_2}^2 \leq 2||h_{(T_0 \cup T_1)}||_{l_2}^2$$

Taking square roots on both sides gives us 14

A1.18

Need to show:

$$(1-\delta_{2s})||h_{(T_0\cup T_1)}||_{l_2}^2 \le ||\Phi h_{(T_0\cup T_1)}||_{l_2}^2 \tag{15}$$

Since $h_{(T_0\cup T_1)}$ is a 2s-sparse vector, 15 follows from definition.

A1.19

Need to show:

$$||\Phi h_{(T_0\cup T_1)}||_{l_2}^2 \le ||h_{(T_0\cup T_1)}||_{l_2}(2\varepsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s}\sum_{j\ge 2}||h_{T_j}||_{l_2}) \tag{16}$$

Clearly

$$||\Phi h_{(T_0\cup T_1)}||_{l_2}^2 =< \Phi h_{(T_0\cup T_1)}, \Phi h > - < \Phi h_{(T_0\cup T_1)}, \sum_{j\ge 2} h_{T_j} >$$

To get the maximum we want to maximize the first term and minimize the second term. As such we want to consider both the absolute values. From 13

$$||\Phi h_{(T_0\cup T_1)}||_{l_2}||\Phi h||_{l_2} \le 2\varepsilon\sqrt{1+\delta_{2s}}||h_{(T_0\cup T_1)}||_{l_2}$$

Also we note:

$$| < \Phi h_{(T_0\cup T_1)}, \Phi h_{T_j} > | \le | < \Phi h_{T_0}, \Phi h_{T_j} > | + | < \Phi h_{T_1}, \Phi h_{T_j} > | \le \delta_{2s}||h_{T_j}||_{l_2}(||h_{T_0}||_{l_2} + ||h_{T_1}||_{l_2})$$

From 14

$$| < \Phi h_{(T_0\cup T_1)}, \Phi h_{T_j} > | \le \sqrt{2}\delta_{2s}||h_{T_j}||_{l_2}||h_{(T_0\cup T_1)}||_{l_2}$$

Combining the two inequalities we directly get 16

A1.20

Need to show:

$$||h_{(T_0\cup T_1)}||_{l_2} \le \alpha\varepsilon + \rho s^{-1/2}||h_{T_0{}^c}||_{l_1}, \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}} \tag{17}$$

From 15 and 16 we get

$$(1-\delta_{2s})||h_{(T_0\cup T_1)}||_{l_2}^2 \le ||h_{(T_0\cup T_1)}||_{l_2}(2\varepsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s}\sum_{j\ge 2}||h_{T_j}||_{l_2})$$

From 7 and then dividing by $(1-\delta_{2s})||h_{(T_0\cup T_1)}||_{l_2}$ we get

$$||h_{(T_0\cup T_1)}||_{l_2} \le \frac{2\varepsilon\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\varepsilon + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

Which is exactly 17

A1.21

Need to show:

$$||h_{(T_0\cup T_1)}||_{l_2} \le \alpha\varepsilon + \rho||h_{(T_0\cup T_1)}||_{l_2} + 2\rho e_0 \Rightarrow ||h_{(T_0\cup T_1)}||_{l_2} \le (1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0) \tag{18}$$

From 17 we know

$$||h_{(T_0\cup T_1)}||_{l_2} \le \alpha\varepsilon + \rho s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

And from 9 we get

$$s^{-1/2}||h_{T_0{}^c}||_{l_2} \le s^{-1/2}(||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}) \le ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

Also

$$||h_{T_0}||_{l_2} \le ||h_{(T_0\cup T_1)}||_{l_2}$$

Therefore we can conclude

$$||h_{(T_0\cup T_1)}||_{l_2} \le \alpha\varepsilon + \rho||h_{(T_0\cup T_1)}||_{l_2} + 2\rho e_0$$

Rearranging the equation we directly get

$$||h_{(T_0\cup T_1)}||_{l_2} \le (1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0)$$

A1.22

Need to show:

$$||h||_{l_2} \le ||h_{(T_0 \cup T_1)}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2} \le 2||h_{(T_0 \cup T_1)}||_{l_2} + 2e_0 \le 2(1-\rho)^{-1}(\alpha\varepsilon + (1+\rho)e_0) \quad (19)$$

The first part is a direct result of Triangle Inequality.

$$||h||_{l_2} = ||h_{(T_0 \cup T_1)} + h_{(T_0 \cup T_1)^c}||_{l_2} \le ||h_{(T_0 \cup T_1)}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2}$$

From 10 we have a bound on $||h_{(T_0 \cup T_1)^c}||_{l_2}$

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \le ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

Since $||h_{(T_0 \cup T_1)}||_{l_2} \ge ||h_{T_0}||_{l_2}$

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \le ||h_{(T_0 \cup T_1)}||_{l_2} + 2e_0$$

Therefore

$$||h_{(T_0 \cup T_1)}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2} \le 2||h_{(T_0 \cup T_1)}||_{l_2} + 2e_0$$

From 18 we directly get the bound on $||h_{(T_0 \cup T_1)}||_{l_2}$. Thus

$$2||h_{(T_0 \cup T_1)}||_{l_2} + 2e_0 \le 2(1-\rho)^{-1}(\alpha\varepsilon + (1+\rho)e_0)$$

Hence we have proved all the inequalities and can directly get 19

A1.23

If we had $y = \Phi x + \eta$ where η represents (bounded) impulse noise, one would want to solve the following problem: $min\|x\|_1$ such that $\|y - \Phi x\|_1 \le \epsilon$ where ϵ is suitably picked (a conservative bound is mR if R is the maximum value of the impulse). What part of this proof do you think will need to modified to derive error bounds for this case?
Soln: We note that the only place where we have used $||y - \Phi x||_{l_2} \le \varepsilon$ is in 13. And this arises from the Lemma 2.1 in the paper. We know that the triangle Inequality is followed by any p-norm matrix. Therefore

$$||\Phi(x^* - x)||_{l_1} \le ||\Phi x^* - y||_{l_1} + ||y - \Phi x||_{l_1} \le 2\varepsilon$$

Also we know

$$||a||_{l_2} \le ||a||_{l_1}$$

for all vectors a. And therefore from 12 and 13we can write it as

$$| < \Phi h_{(T_0 \cup T_1)}, \Phi h > | \le ||\Phi h_{(T_0 \cup T_1)}||_{l_2}||\Phi h||_{l_2} \le ||\Phi h_{(T_0 \cup T_1)}||_{l_2}||\Phi h||_{l_1} \le 2\varepsilon\sqrt{1+\delta_{2s}}||h_{(T_0 \cup T_1)}||_{l_2}$$

And the rest of the proof follows as is.

Q2

The given algorithm fails to outperform the original Φ Matrix. We have tried another (much simpler) algorithm which gives a much better reconstruction error. Here we show both the results. We henceforth call the algorithm in the paper as Algo1 and our Algorithm as Algo2.
A brief description of our algorithm :

- The Λ that we get corresponds to the eigen values of the matrix $\Psi\Psi^{'}$.
- We are choosing Ψ at random, and therefore $\Psi\Psi^{'}$ is almost surely full rank matrix, and hence Λ is also full rank matrix.
- If the Φ matrix were to have $m = n$, then we would have $\Gamma = \Lambda^{-1/2}$ as an exact solution.
- If we were to take $\Gamma = \Lambda^{-1/2}$ and simply drop off the last few rows, to get the desired m, we get extremely good sensing matrix. This is further shown in our results.

A2.1

The given Algorithm gives the following :

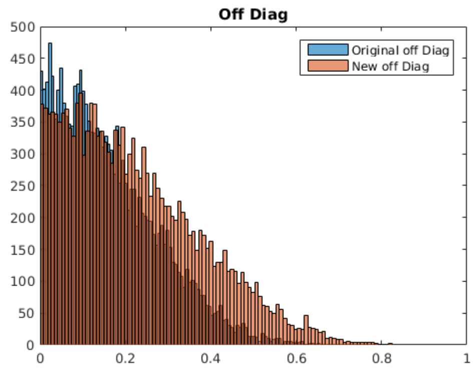

Figure 1: Algo1 Off Diagonal Matrix Comparison

The average relative error in Algo1 case is : 1.163337
The Algo2 gives the following Histogram :

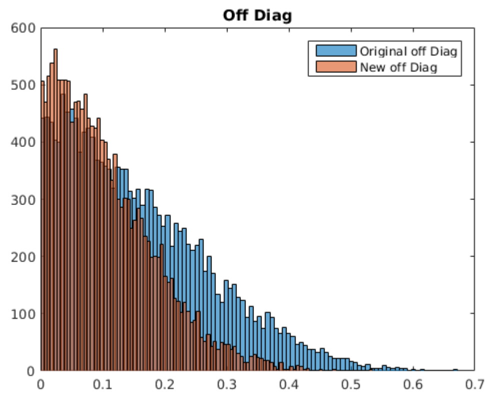

Figure 2: Algo2 Off Diagonal Element Histogram

The average relative error in Algo2 case is : 0.147862

A2.2

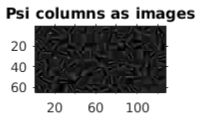

Figure 3:
The columns of the matrix ψ rearranged as an image.

A2.3

The plots for Q2c part are as follows: For Algo1:

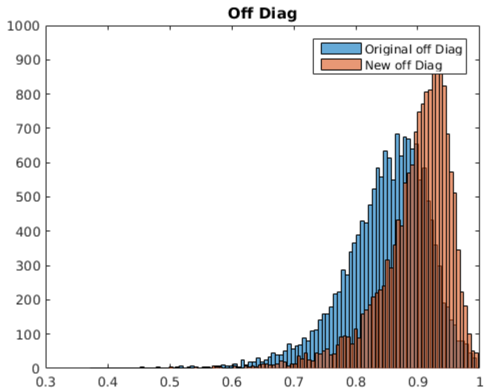

Figure 4: Algo1 plot Hist comparison

The relative error in this case : 1.421040 For Algo2

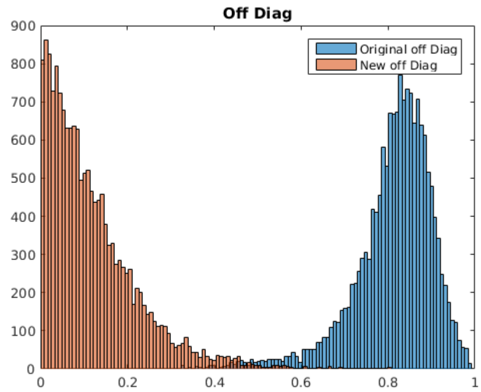

Figure 5: Algo2 plot Hist Comparison

The relative error in this case: 0.161012

A2.4

The plots for Q2d part are as follows: For Algo1:

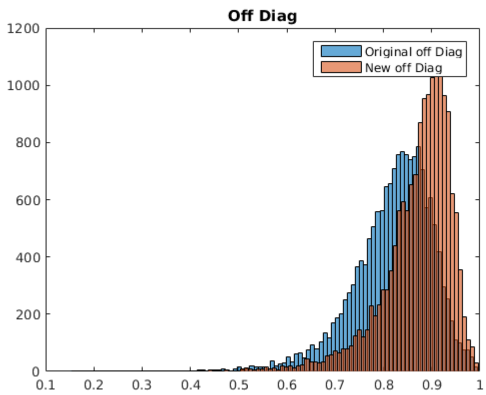

Figure 6: Algo1 plot Hist comparison

The relative error in this case : 1.395116
For Algo2

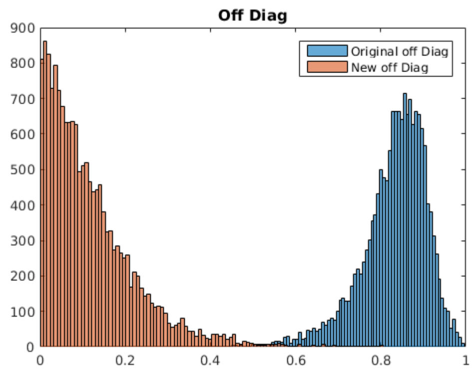

Figure 7: Algo2 plot Hist Comparison

The relative error in this case is 0.377541

A2.E

The algorithm we have implemented generates a CS matrix with arbitrary real values. There are multiple problems in the realisation of such a CS matrix

- It is difficult to assign different magnitude of weights to each pixel. The accuracy of the hardware will limit the real values to a few quantized levels and hence we will have to use a matrix which performs worse but which can be created by the hardware.

  In Hitomi camera, the CS matrix contained only 0s and 1s hence the hardware was a simple cardboard sheet with each pixel having a hole(representating 1s) or no hole(representing 0s). In our case, we will have to make holes with varying diameters to assign different weight to each pixel. These diameters can be created only to a certain degree of accuracy and hence we will have to reduce our matrix to that accuracy. The error in these diameters will also be higher than for a simple CS matrix with 1s and 0s.

- The CS matrix we have created will require some way to capture negative wieghted pixels also. This will require creating a separate hardware matrix for positive and negative values and capturing two compressed measurements and subtracting them (The subtraction can be done either before measurement, which doubles the amount of hardware used, or can be done after measurement in software, which doubles the number of measurements hence reduces compression of the setup).

Observations

We note that this algo is not giving us better results, in almost all cases it is giving us worse relative probability error (≥ 1) which is an absurd result. Our algorithm on the other hand is giving much better relative error(0.15)