CS 754 : Advanced Image ProcessingAssignment 2

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Q1

A1.1

Need to show :

$$||h_{T_j}||_{l_2} \leq s^{1/2}||h_{T_j}||_{l_\infty} \tag{1}$$

Equivalently we need to show :

$$||A||_{l_2} \leq s^{1/2}||A||_{l_\infty}$$

where A is a s-sparse vector. Therefore

$$||A||_{l_2} = \sqrt{\sum_i a_i^2} \leq \sqrt{\sum_i max(a_i)^2} \leq s^{1/2}max(a_i) = ||A||_{l_\infty}$$

The $s^{1/2}$ term comes from the fact that A is s-sparse matrix, and hence there will be at most s non-zero elements.

A1.2

Need to show :

$$s^{1/2}||h_{T_j}||_{l_\infty} \leq s^{-1/2}||h_{T_{j-1}}||_{l_1} \tag{2}$$

for all $j \geq 2$

Equivalently we need to show :

$$s||h_{T_j}||_{l_\infty} \leq ||h_{T_{j-1}}||_{l_1}$$

From the definition of T_j it follows for $j \geq 2$ that all elements of h_{T_j} will be less than the smallest non-zero element of $h_{T_{j-1}}$. Also both h_{T_j} and $h_{T_{j-1}}$ are s-sparse matrix, hence it clearly follows that

$$s||h_{T_j}||_{l_\infty} = s * max(h_{T_j}) \leq \sum_i |h_{T_{j-1}}| = ||h_{T_{j-1}}||_{l_1}$$

A1.3

Need to show :

$$\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \tag{3}$$

This follows directly from 1 and 2.

$$||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_{j-1}}||_{l_1}$$

for all $j \geq 2$ Now summing over all $j \geq 2$ we get

$$\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...)$$

A1.4

Need to show:

$$s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \leq s^{-1/2}||h_{T_0{}^c}||_{l_1} \tag{4}$$

We note all of h_{T_1}, h_{T_2} all have disjoint support and therefore

$$||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ... \leq ||h_{T_0{}^c}||_{l_1}$$

A1.5

Need to show:

$$||h_{(T_0 \cup T_1)^c}||_{l_2} = ||\sum_{j \geq 2} h_{T_j}||_{l_2} \tag{5}$$

We note that

$$h_{(T_0 \cup T_1)^c} = h - h_{T_0} - h_{T_1} = h_{T_2} + h_{T_3} + ... = \sum_{j \geq 2} h_{T_j}$$

And hence 5 follows directly.

A1.6

Need to show:

$$||\sum_{j \geq 2} h_{T_j}||_{l_2} \leq \sum_{j \geq 2} ||h_{T_j}||_{l_2} \tag{6}$$

This is simple extension of triangle inequality, which states that

$$|a + b| \leq |a| + |b|$$

For n vectors it is simply

$$|a_1 + a_2 + a_3 + ... + a_n| \leq |a_1| + |a_2| + |a_3| + ... + |a_n|$$

And hence 6 follows directly

A1.7

Need to show:

$$\sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_0{}^c}||_{l_1} \tag{7}$$

This follows directly from 3 and 4.

$$\sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \leq s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

A1.8

Need to show:

$$||x||_{l_1} \geq ||x + h||_{l_1} \geq ||x_{T_0}||_{l_1} - ||h_{T_0}||_{l_1} + ||h_{T_0{}^c}||_{l_1} - ||x_{T_0^c}||_{l_1} \tag{8}$$

For first part we note that $x^* = x + h$ therefore, $||x^*||_{l_1} = ||x + h||_{l_1}$. According to our constraints,x^* has the minimum $||x^*||_{l_1}$ which also satisfies $||y - \Phi x||_{l_2} \leq \varepsilon$. Therefore if x is not s-sparse then

$$||x||_{l_1} > ||x^*||_{l_1}$$

And if x is s-sparse then

$$||x||_{l_1} = ||x^*||_{l_1}$$

Combining the two we can say

$$||x||_{l_1} \geq ||x^*||_{l_1}$$

From Triangle Inequality we know

$$||a+b||_{l_1} \geq |||a||_{l_1} - ||b||_{l_1}|$$

Therefore, we can also say

$$||a+b||_{l_1} \geq ||a||_{l_1} - ||b||_{l_1}$$

And

$$||a+b||_{l_1} \geq ||b||_{l_1} - ||a||_{l_1}$$

We note that

$$||x+h||_{l_1} = \sum_{i \in T_0} |x_i + h_i| + \sum_{i \in T_0^c} |x_i + h_i| \geq ||x_{T_0}||_{l_1} - ||h_{T_0}||_{l_1} + ||h_{T_0{}^c}||_{l_1} - ||x_{T_0^c}||_{l_1}$$

A1.9

Need to show:

$$||h_{T_0{}^c}||_{l_2} \leq ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1} \tag{9}$$

We can rearrange 8 to have

$$||h_{T_0{}^c}||_{l_1} \leq ||x||_{l_1} - ||x_{T_0}||_{l_1} + ||h_{T_0}||_{l_1} + ||x_{T_0^c}||_{l_1}$$

We note that

$$||x||_{l_1} - ||x_{T_0}||_{l_1} \leq ||x - x_{T_0}||_{l_1} = ||x_{T_0^c}||_{l_1}$$

Therefore

$$||h_{T_0{}^c}||_{l_1} \leq ||x_{T_0^c}||_{l_1} + ||h_{T_0}||_{l_1} + ||x_{T_0^c}||_{l_1} = ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}$$

A1.10

Need to show:

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \leq ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2} \tag{10}$$

Combining 5 6 and 7 we get

$$||h_{(T_0 \cup T_1)^c}||_{l_2} = ||\sum_{j \geq 2} h_{T_j}||_{l_2} \leq \sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

From 9 we get

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2s^{-1/2}||x_{T_0^c}||_{l_1}$$

Also by definition

$$||x_{T_0^c}||_{l_1} = ||x - x_s||_{l_1}$$

This implies

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2s^{-1/2}||x - x_s||_{l_1}$$

Therefore

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

Now we also note, for any s-sparse vector A

$$||A||_{l_1} = \sum_i |a_i| = \sum_i |a_i| * 1 \leq \sqrt{s}\sqrt{\sum_i a_i^2} = s^{1/2}||A||_{l_2}$$

and here we have used Cauchy Schwartz Inequality. That is

$$s^{-1/2}||A||_{l_1} \leq ||A||_{l_2}$$

Thus it follows that

$$s^{-1/2}||h_{T_0}||_{l_1} \leq ||h_{T_0}||_{l_2}$$

And 10 directly follows

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \leq ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

A1.14

Need to show:

$$\Phi h_{(T_0 \cup T_1)} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j} \tag{11}$$

We know

$$h_{(T_0 \cup T_1)^c} = h - h_{(T_0 \cup T_1)} = h - h_{T_o} - h_{T_1} = \sum_{j \geq 2} h_{T_j}$$

Rearranging the equation

$$h_{(T_0 \cup T_1)} = h - \sum_{j \geq 2} h_{T_j}$$

Multiplying ϕ on both sides

$$\Phi h_{(T_0 \cup T_1)} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j}$$

A1.15

Need to show:

$$| < \Phi h_{(T_0 \cup T_1)}, \Phi h > | \leq ||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \tag{12}$$

This is simple application of Cauchy Schwartz Inequality which states that given two vectors a and b

$$< a, b > \leq ||a||_{l_2} ||b||_{l_2}$$

And therefore 12 directly follows from this.

A1.16

Need to show:

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \leq 2\varepsilon \sqrt{1 + \delta_{2s}} ||h_{(T_0 \cup T_1)}||_{l_2} \tag{13}$$

We note that

$$||\Phi(x^* - x)||_{l_2} \leq ||\Phi x^* - y||_{l_2} + ||y - \Phi x||_{l_2} \leq 2\varepsilon$$

The first part of the Inequality is a direct result of Traingle Inequality. The second part of the Inequality arises from the fact that $||y - \Phi x||_{l_2} \leq \varepsilon$ and both x and x^* are a solution. We have assumed $x^* = x + h$. Therefore

$$||\Phi h||_{l_2} \leq 2\varepsilon$$

Also from the definition of RIP

$$\sqrt{(1 - \delta_s)} ||x||_{l_2} \leq ||\Phi x||_{l_2} \leq \sqrt{(1 + \delta_s)} ||x||_{l_2}$$

where x is s-sparse vector. We know that $h_{(T_0 \cup T_1)}$ is a 2s-sparse vector. Therefore

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} \leq \sqrt{(1 + \delta_{2s})} ||h_{(T_0 \cup T_1)}||_{l_2}$$

Multiplying the inequalities directly gives 13

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \leq 2\varepsilon \sqrt{1 + \delta_{2s}} ||h_{(T_0 \cup T_1)}||_{l_2}$$

A1.17

Need to show:

$$||h_{T_0}||_{l_2} + ||h_{T_1}||_{l_2} \leq \sqrt{2} ||h_{(T_0 \cup T_1)}||_{l_2} \tag{14}$$

From AM-GM Inequality we know

$$||h_{T_0}||_{l_2} ||h_{T_1}||_{l_2} \leq \frac{||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}^2}{2}$$

Adding the RHS to both sides and Multiplying 2 on both sides

$$||h_{T_0} + h_{T_1}||_{l_2}^2 \leq 2||h_{(T_0 \cup T_1)}||_{l_2}^2$$

Taking square roots on both sides gives us 14

A1.18

Need to show:

$$(1-\delta_{2s})||h_{(T_0\cup T_1)}||_{l_2}^2 \le ||\Phi h_{(T_0\cup T_1)}||_{l_2}^2 \tag{15}$$

Since $h_{(T_0\cup T_1)}$ is a 2s-sparse vector, 15 follows from definition.

A1.19

Need to show:

$$||\Phi h_{(T_0\cup T_1)}||_{l_2}^2 \le ||h_{(T_0\cup T_1)}||_{l_2}(2\varepsilon\sqrt{1+\delta_{2s}}+\sqrt{2}\delta_{2s}\sum_{j\ge 2}||h_{T_j}||_{l_2}) \tag{16}$$

Clearly

$$||\Phi h_{(T_0\cup T_1)}||_{l_2}^2 =< \Phi h_{(T_0\cup T_1)}, \Phi h > - < \Phi h_{(T_0\cup T_1)}, \sum_{j\ge 2} h_{T_j} >$$

To get the maximum we want to maximize the first term and minimize the second term. As such we want to consider both the absolute values. From 13

$$||\Phi h_{(T_0\cup T_1)}||_{l_2}||\Phi h||_{l_2} \le 2\varepsilon\sqrt{1+\delta_{2s}}||h_{(T_0\cup T_1)}||_{l_2}$$

Also we note:

$$| < \Phi h_{(T_0\cup T_1)}, \Phi h_{T_j} > | \le | < \Phi h_{T_0}, \Phi h_{T_j} > |+| < \Phi h_{T_1}, \Phi h_{T_j} > | \le \delta_{2s}||h_{T_j}||_{l_2}(||h_{T_0}||_{l_2}+||h_{T_1}||_{l_2})$$

From 14

$$| < \Phi h_{(T_0\cup T_1)}, \Phi h_{T_j} > | \le \sqrt{2}\delta_{2s}||h_{T_j}||_{l_2}||h_{(T_0\cup T_1)}||_{l_2}$$

Combining the two inequalities we directly get 16

A1.20

Need to show:

$$||h_{(T_0\cup T_1)}||_{l_2} \le \alpha\varepsilon + \rho s^{-1/2}||h_{T_0{}^c}||_{l_1}, \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}} \tag{17}$$

From 15 and 16 we get

$$(1-\delta_{2s})||h_{(T_0\cup T_1)}||_{l_2}^2 \le ||h_{(T_0\cup T_1)}||_{l_2}(2\varepsilon\sqrt{1+\delta_{2s}}+\sqrt{2}\delta_{2s}\sum_{j\ge 2}||h_{T_j}||_{l_2})$$

From 7 and then dividing by $(1-\delta_{2s})||h_{(T_0\cup T_1)}||_{l_2}$ we get

$$||h_{(T_0\cup T_1)}||_{l_2} \le \frac{2\varepsilon\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\varepsilon + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

Which is exactly 17

A1.21

Need to show:

$$||h_{(T_0\cup T_1)}||_{l_2} \le \alpha\varepsilon + \rho||h_{(T_0\cup T_1)}||_{l_2} + 2\rho e_0 \Rightarrow ||h_{(T_0\cup T_1)}||_{l_2} \le (1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0) \tag{18}$$

From 17 we know

$$||h_{(T_0\cup T_1)}||_{l_2} \le \alpha\varepsilon + \rho s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

And from 9 we get

$$s^{-1/2}||h_{T_0{}^c}||_{l_2} \le s^{-1/2}(||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}) \le ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x-x_s||_{l_2}$$

Also

$$||h_{T_0}||_{l_2} \le ||h_{(T_0\cup T_1)}||_{l_2}$$

Therefore we can conclude

$$||h_{(T_0\cup T_1)}||_{l_2} \le \alpha\varepsilon + \rho||h_{(T_0\cup T_1)}||_{l_2} + 2\rho e_0$$

Rearranging the equation we directly get

$$||h_{(T_0\cup T_1)}||_{l_2} \le (1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0)$$

A1.22

Need to show:

$$||h||_{l_2} \le ||h_{(T_0 \cup T_1)}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2} \le 2||h_{(T_0 \cup T_1)}||_{l_2} + 2e_0 \le 2(1-\rho)^{-1}(\alpha\varepsilon + (1+\rho)e_0) \quad (19)$$

The first part is a direct result of Triangle Inequality.

$$||h||_{l_2} = ||h_{(T_0 \cup T_1)} + h_{(T_0 \cup T_1)^c}||_{l_2} \le ||h_{(T_0 \cup T_1)}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2}$$

From 10 we have a bound on $||h_{(T_0 \cup T_1)^c}||_{l_2}$

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \le ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

Since $||h_{(T_0 \cup T_1)}||_{l_2} \ge ||h_{T_0}||_{l_2}$

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \le ||h_{(T_0 \cup T_1)}||_{l_2} + 2e_0$$

Therefore

$$||h_{(T_0 \cup T_1)}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2} \le 2||h_{(T_0 \cup T_1)}||_{l_2} + 2e_0$$

From 18 we directly get the bound on $||h_{(T_0 \cup T_1)}||_{l_2}$. Thus

$$2||h_{(T_0 \cup T_1)}||_{l_2} + 2e_0 \le 2(1-\rho)^{-1}(\alpha\varepsilon + (1+\rho)e_0)$$

Hence we have proved all the inequalities and can directly get 19

A1.23