CS 754 : Advanced Image ProcessingAssignment 2

Meet Udeshi - 14D070007
Arka Sadhu - 140070011

March 12, 2017

Q1

A1.1

Need to show :

$$||h_{T_j}||_{l_2} \leq s^{1/2}||h_{T_j}||_{l_\infty} \tag{1}$$

Equivalently we need to show :

$$||A||_{l_2} \leq s^{1/2}||A||_{l_\infty}$$

where A is a s-sparse vector. Therefore

$$||A||_{l_2} = \sqrt{\sum_i a_i^2} \leq \sqrt{\sum_i max(a_i)^2} \leq s^{1/2}max(a_i) = ||A||_{l_\infty}$$

The $s^{1/2}$ term comes from the fact that A is s-sparse matrix, and hence there will be at most s non-zero elements.

A1.2

Need to show :

$$s^{1/2}||h_{T_j}||_{l_\infty} \leq s^{-1/2}||h_{T_{j-1}}||_{l_1} \tag{2}$$

for all $j \geq 2$

Equivalently we need to show :

$$s||h_{T_j}||_{l_\infty} \leq ||h_{T_{j-1}}||_{l_1}$$

From the definition of T_j it follows for $j \geq 2$ that all elements of h_{T_j} will be less than the smallest non-zero element of $h_{T_{j-1}}$. Also both h_{T_j} and $h_{T_{j-1}}$ are s-sparse matrix, hence it clearly follows that

$$s||h_{T_j}||_{l_\infty} = s * max(h_{T_j}) \leq \sum_i |h_{T_{j-1}}| = ||h_{T_{j-1}}||_{l_1}$$

A1.3

Need to show :

$$\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \tag{3}$$

This follows directly from 1 and 2.

$$||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_{j-1}}||_{l_1}$$

for all $j \geq 2$ Now summing over all $j \geq 2$ we get

$$\sum_{j\geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...)$$

A1.4

?? Need to show:

$$s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \leq s^{-1/2}||h_{T_0{}^c}||_{l_1} \tag{4}$$

A1.5

Need to show:

$$||h_{(T_0 \cup T_1)^c}||_{l_2} = ||\sum_{j \geq 2} h_{T_j}||_{l_2} \tag{5}$$

We note that

$$h_{(T_0 \cup T_1)^c} = h - h_{T_0} - h_{T_1} = h_{T_2} + h_{T_3} + ... = \sum_{j \geq 2} h_{T_j}$$

And hence 5 follows directly.

A1.6

Need to show:

$$||\sum_{j \geq 2} h_{T_j}||_{l_2} \leq \sum_{j \geq 2} ||h_{T_j}||_{l_2} \tag{6}$$

This is simple extension of triangle inequality, which states that

$$|a + b| \leq |a| + |b|$$

For n vectors it is simply

$$|a_1 + a_2 + a_3 + ... + a_n| \leq |a_1| + |a_2| + |a_3| + ... + |a_n|$$

And hence 6 follows directly

A1.7

Need to show:

$$\sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_0{}^c}||_{l_1} \tag{7}$$

This follows directly from 3 and 4.

$$\sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + ...) \leq s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

A1.8

??? Need to show:

$$||x||_{l_1} \geq ||x + h||_{l_1} \geq ||x_{T_0}||_{l_1} - ||h_{T_0}||_{l_1} + ||h_{T_0{}^c}||_{l_1} - ||x_{T_0^c}||_{l_1} \tag{8}$$

A1.9

Need to show:

$$||h_{T_0{}^c}||_{l_2} \leq ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1} \tag{9}$$

We can rearrange 8 to have

$$||h_{T_0{}^c}||_{l_1} \leq ||x||_{l_1} - ||x_{T_0}||_{l_1} + ||h_{T_0}||_{l_1} + ||x_{T_0^c}||_{l_1}$$

We note that

$$||x||_{l_1} - ||x_{T_0}||_{l_1} \leq ||x - x_{T_0}||_{l_1} = ||x_{T_0^c}||_{l_1}$$

Therefore

$$||h_{T_0{}^c}||_{l_1} \leq ||x_{T_0^c}||_{l_1} + ||h_{T_0}||_{l_1} + ||x_{T_0^c}||_{l_1} = ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}$$

A1.10

Need to show:

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \leq ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2} \tag{10}$$

Combining 5 6 and 7 we get

$$||h_{(T_0 \cup T_1)^c}||_{l_2} = ||\sum_{j \geq 2} h_{T_j}||_{l_2} \leq \sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq s^{-1/2}||h_{T_0{}^c}||_{l_1}$$

From 9 we get

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2s^{-1/2}||x_{T_0^c}||_{l_1}$$

Also by definition

$$||x_{T_0^c}||_{l_1} = ||x - x_s||_{l_1}$$

This implies

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2s^{-1/2}||x - x_s||_{l_1}$$

Therefore

$$s^{-1/2}||h_{T_0{}^c}||_{l_1} \leq s^{-1/2}||h_{T_0}||_{l_1} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

Now we also note, for any s-sparse vector A

$$||A||_{l_1} = \sum_i |a_i| = \sum_i |a_i| * 1 \leq \sqrt{s}\sqrt{\sum_i a_i^2} = s^{1/2}||A||_{l_2}$$

and here we have used Cauchy Schwartz Inequality. That is

$$s^{-1/2}||A||_{l_1} \leq ||A||_{l_2}$$

Thus it follows that

$$s^{-1/2}||h_{T_0}||_{l_1} \leq ||h_{T_0}||_{l_2}$$

And 10 directly follows

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \leq ||h_{T_0}||_{l_2} + 2e_0, e_0 \equiv s^{-1/2}||x - x_s||_{l_2}$$

A1.14

Need to show:

$$\Phi h_{(T_0 \cup T_1)} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j} \tag{11}$$

We know

$$h_{(T_0 \cup T_1)^c} = h - h_{(T_0 \cup T_1)} = h - h_{T_o} - h_{T_1} = \sum_{j \geq 2} h_{T_j}$$

Rearranging the equation

$$h_{(T_0 \cup T_1)} = h - \sum_{j \geq 2} h_{T_j}$$

Multiplying ϕ on both sides

$$\Phi h_{(T_0 \cup T_1)} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j}$$

A1.15

Need to show:

$$| < \Phi h_{(T_0 \cup T_1)}, \Phi h > | \leq ||\Phi h_{(T_0 \cup T_1)}||_{l_2}||\Phi h||_{l_2} \tag{12}$$

This is simple application of Cauchy Schwartz Inequality which states that given two vectors a and b

$$< a, b > \leq ||a||_{l_2}||b||_{l_2}$$

And therefore 12 directly follows from this.

A1.16

Need to show:

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \leq 2\varepsilon \sqrt{1+\delta_{2s}} ||h_{(T_0 \cup T_1)}||_{l_2} \tag{13}$$

We note that

$$||\Phi(x^* - x)||_{l_2} \leq ||\Phi x^* - y||_{l_2} + ||y - \Phi x||_{l_2} \leq 2\varepsilon$$

The first part of the Inequality is a direct result of Traingle Inequality. The second part of the Inequality arises from the fact that $||y - \Phi x||_{l_2} \leq \varepsilon$ and both x and x^* are a solution. We have assumed $x^* = x + h$. Therefore

$$||\Phi h||_{l_2} \leq 2\varepsilon$$

Also from the definition of RIP

$$\sqrt{(1-\delta_s)} ||x||_{l_2} \leq ||\Phi x||_{l_2} \leq \sqrt{(1+\delta_s)} ||x||_{l_2}$$

where x is s-sparse vector. We know that $h_{(T_0 \cup T_1)}$ is a 2s-sparse vector. Therefore

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} \leq \sqrt{(1+\delta_{2s})} ||h_{(T_0 \cup T_1)}||_{l_2}$$

Multiplying the inequalities directly gives 13

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \leq 2\varepsilon \sqrt{1+\delta_{2s}} ||h_{(T_0 \cup T_1)}||_{l_2}$$