CS 754 : Advanced Image ProcessingAssignment 2

Meet Udeshi - 14D070007
Arka Sadhu - 140070011

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Q1

A1.1

$$y = \Phi x, y \in \mathbb{R}^m$$

If $m = 1$ then y is a single value. Assume $x_i \neq 0$ for some $i \in \{1, ..., n\}$ then $y = \Phi_i x_i$. Consider some other x' which has j^{th} element non-zero, $j \neq i$. If $\Phi_j x'_j = \Phi_i x_i$, then $y = \Phi x'$ is also satisfied. We can find such a x' for all $j \neq i$ where $x'_j = \Phi_i / \Phi_j x_i$. Hence there is no unique solution for this equation, and we cannot uniquely determine x from y.

For the case where we know the index of the non-zero element in x, we have been given i, and no other $j \neq i$ will satisfy the equation, leaving behind only one solution for x. Hence now we can uniquely determine x from y.

A1.2

If $m = 2$, then y is a 2D vector. Assuming i is the index of non-zero element in x

$$y = \begin{bmatrix} \Phi_{1i} x_i \\ \Phi_{2i} x_i \end{bmatrix}$$

We can say that y is the 2D column vector Φ_i scaled by x_i. Assume no two columns of Φ are parallel to each other in 2D space. Then we can say that we will find only one unique i for which the equation holds. This is because if $y \| \Phi_i$ and $y \| \Phi_j$, $i \neq j$ then $\Phi_i \| \Phi_j$, which is contrary to our assumption.

If the assumption holds for some Φ, we can obtain i by calculating normalised dot product of y with every column Φ_i, and whichever i gives $\frac{y \cdot \Phi_i}{|y||\Phi_i|} = 1$, we can then use it to calculate x_i by

$$x_i = |y| / |\Phi_i|$$

If there are two or more such i, we can say that our assumption doesn't hold on Φ and no unique solution can be found.

A1.3

For $m = 3$, y is a 3D vector which can be represented as the linear combination of two columns of Φ. Take i and j to be the two indices of x which are non-zero.

$$y = \Phi_i x_i + \Phi_j x_j$$

We can see that y in 3D space will lie in the 2D plane defined by Φ_i and Φ_j. So to find x given y, we need to find two columns of Φ which form $\{\Phi_i, \Phi_j, y\}$ as a set of coplanar 3D vectors. Thus we need to find i, j s.t.

$$\frac{y \times \Phi_i}{|y||\Phi_i|} = \frac{y \times \Phi_j}{|y||\Phi_j|}$$

We will be able to find a unique pair of i, j iff no three columns of Φ are coplanar in 3D.

Algorithm:

1. Create a binary search tree to add normalised cross products
2. Loop through the columns of Φ and for every Φ_i
   (a) Calculate normalised cross product $\hat{n_i} = \frac{y \times \Phi_i}{|y||\Phi_i|}$
   (b) Search for $\hat{n_i}$ in the tree and return both indices, current index and matched index if found. Break the loop.
   (c) If not found, add $\hat{n_i}$ to the tree.
3. Using the two indices we need to solve for x_i and x_j using

$$y = (\Phi_i \Phi_j) \begin{pmatrix} x_i \\ x_j \end{pmatrix}$$

   This is an over-determined system (three equations two variables) and we can use inverse to find a solution (by discarding one equation).

A1.4

For $m = 4$, we proceed in the same way as above. The only change is that now, y lies in the 2d column subspace of two columns of ϕ. So we will be able to find x uniquely iff no columns of ϕ are coplanar in a 2d plane defined in R^4 Since we. cannot take cross product in 4D space, we proceed differently. For every (i, j) pair, we make a matrix say B, whose columns are ϕ_i and ϕ_j. Now we equate $Bx_1 = y$ where $x_1 = [\alpha|\beta]^t$. Now we use gaussian elimination. If we get last two rows as 0 in both B as well as y, then we have obtained the solution required. x will have non-zero elements at indices i, j and the value would be α and β

Q2

A2.3

We have the relation :

$$E_u = \sum_{t=1}^{T} C_t \cdot F_t$$

Consider

$$E_1 = C_1 \cdot F_1$$

and suppose that we want to construct it as a matrix product, then we can write it as

$$E_1 = \phi_1 f_1$$

where $\phi_1 = diag(C_1)$ and $f_1 = vec(F_1)$ Hence

$$E_u = [\phi_1|\phi_2|...|\phi_T] \begin{bmatrix} f_1 \\ f_2 \\ .... \\ f_T \end{bmatrix}$$

Therefore

$$x = \begin{bmatrix} f_1 \\ f_2 \\ .... \\ f_T \end{bmatrix}$$

$$y = vec(Eu)$$

$$A = [\phi_1|\phi_2|...|\phi_T]$$

Q3

A3

Coherence is defined as :

$$\mu(\phi,\psi) = \sqrt(n) * \max_{i,j \in 0,1,..,n-1} |\phi^{i^t}\psi_j|$$

(a) Upper Bound: Consider a row of ϕ matrix to be exactly same as one of the columns of ψ the inner product would be 1, because both matrices are unit normalized, and result in $\mu_{max}(\phi,\psi) = \sqrt(n)$.

(b) Lower Bound: Let g be a row of the ϕ matrix, which is unit normalized. g can be written with basis vectors as columns of ψ. That is

$$g = \sum_{k=1}^{n} \alpha_k \psi_k$$

Also since g is unit norm,

$$\sum_{k=1}^{n} \alpha_k^2 = 1$$

When we take the inner product of g with ψ_k, only one α will remain. Now consider the coherence of g and ψ. Clearly

$$\mu(g,\psi) = \sqrt{n} * max(\alpha_j)_{j=1}^n$$

where α_j is corresponds to the jth coefficient. Now we also know that

$$max(\alpha_j)_{j=1}^n \geq avg(\alpha_j)_{j=1}^n$$

and equality occurs when all α_j are equal. Hence to get the minimum coherence we need all α_j to be equal and using the previous constraint on unit norm we get

$$\alpha_j = \frac{1}{\sqrt{n}}$$

. This gives

$$\mu_{min}(g,\psi) = 1$$

This is the bound for all rows, and hence we have obtained

$$\mu_{min}(\phi,\psi) = 1$$

Q4

A4

We note that A is a matrix with unit-normalized columns. Now consider a vector θ which is S-sparse, and let s be the support of the vector θ. Clearly,

$$||A\theta||^2 = ||A_s\theta_s||^2$$

where A_s is constructed by taking only those columns of A corresponding to which θ is non-zero. The above statement is true because all other terms of $A\theta$ would be 0 and wouldn't contribute to the norm. The maximum and minimum value of $||A_s\theta_s||^2$ will be determined by the largest and smallest singular values. And singular values are basically eigenvalues of the matrix $A_s^tA_s$. Therefore,

$$\lambda_{min}||\theta_s||^2 \leq ||A_s\theta_s||^2 \leq \lambda_{max}||\theta_s||^2$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of the matix $A_s^tA_s$.
By Gershgorin's Theorem we have:

$$B_{ii} - r_i < \lambda < B_{ii} + r_i$$

where B_{ii} is the i^{th} diagonal element, and r_i is the sum of absolute values of the off-diagonal elements.
Here we consider $B = A_s^t A_s$. Since we have assumed columns of A are unit-normalized, hence the diagonal elements of the matrix B will be 1. Also we consider the definition of mutual coherence μ

$$\mu(A) = max_{i,j,i \neq j} |A_i \cdot A_j|$$

By this definition it is easy to see that

$$r_i \leq \mu * (S - 1)$$

because r_i is the sum of absolute off-diagonal elements of the matrix $A_s^t A_s$ and each such element would be less than μ and at most there can be $S - 1$ terms. Also by RIP definition we have

$$\delta_s = max(\lambda_{max} - 1, 1 - \lambda_{min})$$

If the first term is larger, we have $\delta_s = \lambda_{max} - 1$.

$$\delta_s < B_{ii} + r_i - 1$$

$$\delta_s < 1 + (S - 1)\mu - 1$$

$$\delta_s < (S - 1)\mu$$

If the second term is larger, we have $\delta_s = 1 - \lambda_{min}$

$$\delta_s < 1 - (B_{ii} - r_i)$$

$$\delta_s < 1 - (1 - (S - 1)\mu)$$

$$\delta_s < (S - 1)\mu$$

Hence in both cases we have proved that $\delta_s < (S - 1)\mu$