Some Topics in Ramsey Theory

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PROFILE

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Contents

Chapter 1

Introduction

Ramsey Theory, named after British mathematician Frank P. Ramsey, deals with the partioning of a set in r classes such that a given property holds. In Ramsey theory of integers, we typically colour the first n positive integers in r colors such that the property holds.

A few notations that we are going to use frequently:

$[n] = \{1, 2, 3, \ldots, n\}$

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$[n]^k = \{Y : Y \subseteq [n] \text{ and } |Y| = k\}$

$[n]^{\leq k} = \{Y : Y \subseteq [n] \text{ and } |Y| \leq k\}$

An r-coloring of a set S is a map

$$\chi : S \to [r]$$

Given $k \geq 1$ and integers $\ell_1, \ldots, \ell_r$, each at least k, we write

$$n \to \big(\ell_1, \ell_2, \ell_3, \ldots, \ell_r\big)^k$$

to denote that for every r colouring of $[n]^k$ there exist an i, where $1 \leq i \leq r$, and a set $S \subseteq [n]$ with cardinality ℓ_i such that $[S]^k$ is coloured i. We use the notation

$$n \to \big(\ell\big)_r^k$$

to denote the special case $\ell_i = \ell$ for $1 \leq i \leq r$.

The Ramsey function $\mathcal{R}\big(\ell_1, \ell_2, \ldots, \ell_r\big)$ is the smallest positive integer n such that

$$n \to \big(\ell_1, \ell_2, \ldots, \ell_r\big)^k.$$

We use $\mathcal{R}(\ell; r)$ to denote $\mathcal{R}\big(\ell_1, \ell_2, \ldots, \ell_r\big)$ in the special case $\ell_1 = \ell_2 = \cdots = \ell_r$, and $\mathcal{R}(\ell)$ for $\mathcal{R}(\ell; 2)$.

- **The Pigeon-Hole Principle:** *If m pigeons roost in n holes and $m > n$ then at least two pigeons share the same hole.*

- **Ramsey's Theorem [?], 1930**

  The function $\mathcal{R}$ is well defined for all values of $k, \ell_1, \ell_2, \ldots, \ell_r$ i.e. there exists n_0 such that for all $n \geq n_0$,

  $$n \to \big(\ell_1, \ell_2, \ldots, \ell_r\big)^k.$$

- **van der Waerden's Theorem [?], 1927**

  For all k, r there exists n_0, such that for all $n \geq n_0$, if $[n]$ is r-coloured there exist a monochromatic arithmetic progression $\big\{a, a + d, a + 2d, \ldots, a + (k-1)d\big\} \subseteq [n]$ of length k.

- **Schur's Theorem [?], 1916**

  For all r there exists n_0 such that for all $n \geq n_0$, if $[n]$ is r-coloured there exist $x, y, z \in [n]$ monochromatic, so that

  $$x + y = z.$$

- **Rado's Theorem [?], 1934**

  The single equation

  $$c_1 x_1 + c_2 x_2 + \ldots + c_m x_m = 0$$

  is regular if and only if some non empty subsets of c_i sums up to zero.

Throughout this report, we will focus on van der Waerden numbers and Rado numbers. In Chapter 2, we quote some main results on van der Waerden numbers and are going to be relevant to my research. In Chapter 3, some main results on Rado Numbers. A lot of research has been done in this field during the last two decades but the exact value is still hard to find. Improving the existing bounds for van der Waerden numbers and Rado numbers is what people mainly aim for.

Chapter 2

van der Waerden Numbers

Bartel Leendert van der Waerden (2 February 1903 – 12 January 1996) was a Dutch mathematician. He gave the proof of van der Waerden theorem in the year 1927.

Theorem 2.1. (van der Waerden [?])
For all $k, r \geq 1$ there exists a natural number n_0, such that for all $n \geq n_0$, if $[n]$ is $r-$coloured there exist a monochromatic arithmetic progression $\{a, a+d, a+2d, \ldots, a+(k-1)d\} \subseteq [n]$ of length k.

van der Waerden number is the least possible natural number $\mathtt{w}(k, r)$ such that for every r colouring of first $\mathtt{w}$ natural numbers we can always find a monochromatic arithmetic progression of length k.

The only known exact values for nontrivial van der Waerden numbers are:

$$\mathtt{w}(3;2) = 9, \quad \mathtt{w}(4;2) = 35, \quad \mathtt{w}(5;2) = 178, \quad \mathtt{w}(3;3) = 27, \quad \mathtt{w}(3;4) = 76.$$

$r \rightarrow$ $k \downarrow$	2	3	4	5
3	9	27	76	?
4	35	≥ 292	≥ 1048	≥ 2254
5	178	≥ 1210	≥ 10437	≥ 24045
6	≥ 696	≥ 8886	≥ 90306	≥ 93456
7	≥ 3703	≥ 43855	≥ 119839	?

Theorem 2.2. (van der Waerden's Theorem – Generalized Version)

For positive integers $k_1, k_2, \ldots, k_r$, there exists n_0 such that whenever $n \geq n_0$ and $[n]$ is r-coloured, there exists a k_i-term monochromatic arithmetic progression coloured i, for some $i \in [r]$. For positive integers $k_1, k_2, \ldots, k_r$, the smallest positive integer $\mathtt{w}(k_1, k_2, \ldots, k_r; r) = \mathtt{w}$ for which every r-colouring of $[\mathtt{w}]$ contains k_i-term monochromatic arithmetic progression of colour i, for atleast one $i \in [r]$, is called a mixed van der Waerden number.

Some known bounds on van der Waerden number are listed below:

Theorem 2.3. (E. Berlekamp [?])
Let p be a prime. Then

$$\mathtt{w}(p+1; 2) \geq p2^p.$$

Theorem 2.4. *Let $\epsilon > 0$.There exist $k_0 = k(\epsilon)$ such that for all $k \geq k_0$*

$$\mathtt{w}(k; 2) \geq \frac{2^k}{k^\epsilon}.$$

Theorem 2.5. *Let $p \geq 5$ and q be primes. Then*

$$\mathtt{w}(p+1, q; 2) \geq p(q^p - 1) + 1.$$

Theorem 2.6. (W. T. Gowers [?])
For all $r \geq 2$,

$$\mathtt{w}(k, r) > \frac{r^k}{ekr}(1 + \circ(1)).$$

Theorem 2.7. (W. T. Gowers [?])
Let $f(k, r) = r^{2^{2^{k+9}}}$. Then

$$\mathtt{w}(k, r) \leq 2^{2^{f(k,r)}}.$$

2.1 Sequence of the type $\{x, ax + d, bx + 2d\}$

van der Waerden theorem focuses on the existence of an arithmetic progression,whereas we would like to focus on a more general progression of the form $\{x, ax+d, bx+2d\}$ namely (a, b) triple. Here a and b are fixed positive integers, $a \leq b$ and x and d are two positive integers.

Our aim is to determine that for what values of a, b and r, we have a number $T(a, b; r)$ such that when we r-colour the integers from 1 to T, we have a monochromatic (a, b) triple. There are a few known bounds for $T(a, b; r)$, and these are listed below.

Theorem 2.8. (Allen, Landman & Meeks [?])

Let a, b be two positive integers where $a \leq b$. When $b = 2a$, then $T(a, b; r)$ exist only if $r = 1$. Moreover

$$T(a, b; 2) \leq \begin{cases} 7b^2 - 6ab + 13b - 10a & \text{for } b \text{ is even, } b > 2a; \\ 14b^2 - 12ab + 26b - 20a & \text{for } b \text{ is odd, } b > 2a; \\ 3b^2 + 2ab + 16a & \text{for } b \text{ is even, } b < 2a; \\ 6b^2 + 4ab + 8b + 16a & \text{for } b \text{ is odd, } b < 2a. \end{cases}$$

Theorem 2.9. (Allen, Landman & Meeks [?])

Let a, b be two positive integers such that $a \leq b$. Then

$$T(a, b; 2) \geq \begin{cases} 2b^2 + 5b - 2a + 4 & \text{if } b > 2a; \\ 3b^2 + 5b - 4a + 4 & \text{if } b < 2a. \end{cases}$$

Now we would like to focus on the case where $a = b$ and $r = 2$. There are a few known bounds for $T(a, a; 2)$. They are mentioned below.

Theorem 2.10. (Allen, Landman & Meeks [?])

For $a \geq 4$,

$$T(a, a; 2) \geq a^2 + 3a + 8.$$

Theorem 2.11. (Landman & Robertson [?])

$$T(a, a; 2) \leq \begin{cases} 3a^2 + a & \text{for } a \text{ even, } a \geq 4; \\ 8a^2 + a & \text{for } a \text{ is odd.} \end{cases}$$

Chapter 3

Rado Numbers

Richard Rado (28 April 1906 – 23 December 1989) was a German-born British mathematician. He was a doctoral student of Issai Schur and therefore extended his work. So, before talking about Rado numbers, one should be familiar with Schur numbers. Rado number is nothing but the generalization of Schur numbers.

3.1 Schur numbers

Schur numbers are the least positive number $s = \mathtt{s(r)}$ such that for every r-colouring of first s positive integers or $[1, s]$, we have a monochromatic solution to the equation $x + y = z$.

Theorem 3.1. (I. Schur [?])

For any $r \geq 1$, there exist a positive integer $\mathtt{s(r)}$ such that for every r colouring of $[1, \mathtt{s(r)}]$, we have a monochromatic solution to the equation $x + y = z$.

A triple $\big(x, y, z\big)$ that satisfies the equation $x+y=z$ is called a Schur triple. The only known exact values for Schur numbers are:

$$\mathtt{s}(1) = 2, \quad \mathtt{s}(2) = 5, \quad \mathtt{s}(3) = 14, \quad \mathtt{s}(4) = 45.$$

The colouring used in the proof of Schur's Theorem gives a bijection between edge colouring of $\mathcal{K}_n$ and colouring of $[n-1]$. The definition of this colouring implies that monochromatic triangles correspond to Schur triples. With $n = \mathcal{R}_r(3)$, this gives

$$\frac{1}{2}(3^r + 1) \leq \mathtt{s(r)} \leq \mathcal{R}_{\mathtt{r}}(3) - 1 \leq 3\mathtt{r}! - 1.$$

Let $L(t)$ represents the equation $x_1 + x_2 + \ldots + x_{t-1} = x_t$ where $x_1, x_2, \ldots, x_t$ are the unknown variables.

Theorem 3.2. (A. Robertson [?])

For $r \geq 1$ and, for $1 \leq i \leq r$, assume that $k_i \geq 3$. Then there exist a least positive integer $S = \mathcal{S}(k_1, k_2, \ldots, k_r)$, such that for every r-colouring of $[1, S]$, we have a monochromatic solution to $L(k_j)$ of colour j where $j \in \{1, 2, \ldots, r\}$.

The numbers $S = \mathcal{S}(k_1, k_2, \ldots, k_r)$ are called `generalized Schur numbers`. When $k_1 = k_2 = \ldots = k_r = k$, we denote it by $\mathcal{S}_r(k)$.

We have a theorem that gives us the exact values for all 2-coloured generalized Schur numbers.

Theorem 3.3. *Let $k, \ell \geq 3$. Then*

$$\mathcal{S}(k; \ell) = \begin{cases} 3\ell - 4 & \text{for } k = 3 \text{ and } \ell \text{ is odd;} \\ 3\ell - 5 & \text{if } k = 3 \text{ and } \ell \text{ is even;} \\ k\ell - \ell - 1 & \text{if } 4 \leq k \leq \ell. \end{cases}$$

We have an upper and lower bound for generalized Schur number $\mathcal{S}_r(k)$ too.

Theorem 3.4. *Let $r \geq 2$. If $k \geq 3$, then $\mathcal{S}_r(k) \leq \mathcal{R}_r(k) - 1$, i.e., for every r-colouring of $\mathcal{K}_{\mathcal{R}}$ we have monochromatic $\mathcal{K}_k$ in some colour $j \in \{1, 2, \ldots, \mathcal{R}\}$.*

Theorem 3.5. *Let $r \geq 2$. If $k \geq 3$, then*

$$\mathcal{S}_r(k) \geq \frac{k^{r+1} - 2k^r + 1}{k - 1}.$$

Now we will introduce the concept of `regularity`. Let S be a system of linear homogeneous equations. We say that S is r-`regular` if, for every r-colouring of positive integers, there is a monochromatic solution to S. If S is r-regular for all $r \geq 1$, then S is said to be `regular`.

3.2 Rado numbers

Theorem 3.6. (R. Rado [?])

Let $k \geq 2$. Let c_i be non zero integers, $1 \leq i \leq k$, be constants. Then

$$\sum_{i=1}^{k} c_i x_i = 0$$

is regular if and only if there exists a non-empty $D \subseteq \{c_i : 1 \leq i \leq k\}$ such that $\sum_{d \in D} d = 0$.

Theorem 3.7. (R. Rado [?])

Let $\epsilon(b)$ represent the linear equation

$$c_1 x_1 + c_2 x_2 + \ldots + c_k x_k = b,$$

where $k \geq 2$ and each c_i is a nonzero integer. Let $s = c_1 + \ldots + c_k$. Then the equation $\epsilon(b)$ is regular if and only if either

(i) *$\frac{b}{s}$ is a positive integer or*

(ii) *$\frac{b}{s}$ is a negative integer and $\epsilon(0)$ is regular.*

Theorem 3.8. (Rado's "Columns Condition")

Let $C = \left(\overrightarrow{c_1}, \ldots, \overrightarrow{c_n}\right)$ be a $k \times n$ matrix, where $\overrightarrow{c_i} \in \mathbb{Z}^k$ for $1 \leq i \leq n$. We say that C satisfies the "`Columns Condition`" if we can order the columns $\overrightarrow{c_i}$ with indices $1 = i_0 < i_1 < \ldots < i_s = n$ such that the following two conditions holds for $\overrightarrow{s_j} = \sum_{i_{j-1}+1}^{i_j} \overrightarrow{c_i}$ for $2 \leq j \leq t$.

(i) *$\overrightarrow{s_1} = \overrightarrow{0}$;*

(ii) *$\overrightarrow{s_j}$ can be expressed as a linear combination of $\overrightarrow{c_1}, \ldots, \overrightarrow{c_{j-1}}$ for $2 \leq j \leq t$.*

Theorem 3.9. (Rado's Theorem for a system of equations)(R. Rado, [?, ?, ?])

A system of linear homogeneous equations S denoted by $A\overrightarrow{x} = 0$ is regular if and only if A satisfies the "`Columns Condition`". Moreover S has a monochromatic solution of distinct positive integers if and only if S is regular and there exist distinct (not necessarily monochromatic integers) that satisfy S.

Theorem 3.10. (D. Schaal [?])

Let $b \geq 1$, $k \geq 3$, and let $\epsilon(b)$ represent the equation $x_1 + \ldots + x_{k-1} - x_k = -b$. Then Rado number $\mathtt{r}(\epsilon(b); 2)$ does not exist precisely when k is even and b is odd. Furthermore, we have

$$\mathtt{r}(\epsilon(b); 2) = k^2 + (b-1)(k+1)$$

whenever $\mathtt{r}(\epsilon(\mathtt{b}); 2)$ exists.

Theorem 3.11. (Burr & Loo [?])

For $b \geq 1$, Rado number $\mathtt{r}$(x+y-z=b;2) is

$$\mathtt{r}(x + y - z = b; 2) = b - \left\lceil \frac{b}{5} \right\rceil + 1.$$

Now some known Rado numbers for any given equation and the number of colours used is 2.

Theorem 3.12. (Hopkins & Schaal [?])

Let $a_1, \ldots, a_{m-1}$ be positive integers, $m \geq 3$. Let $t = \min\{a_1, a_2, \ldots, a_{m-1}\}$ and $b = a_1 + a_2 + \ldots + a_{m-1} - t$. Then Rado number $\mathtt{r}(a_1x_1 + \ldots + a_{m-1}x_{m-1} = x_m; 2)$ is

$$\mathtt{r}(a_1x_1 + \ldots + a_{m-1}x_{m-1} = x_m; 2) \geq tb^2 + (2t^2 + 1)b + t^3.$$

Moreover, if $t = 2$,

$$\mathtt{r}(a_1x_1 + \ldots + a_{m-1}x_{m-1} = x_m; 2) = 2b^2 + 9b + 8.$$

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Theorem 3.13. (Burr & Loo) [?])

Let $a, b \geq 1$ with $(a, b) = 1$. Then Rado number in 2 colours $\mathtt{r}$(ax+by=bz;2) is

$$\mathtt{r}(ax + by = bz; 2) = \begin{cases} a^2 + 3a + 1 & \text{if } b = 1; \\ b^2 & \text{if } a < b; \\ a^2 + a + 1 & \text{if } 2 \leq b \leq a. \end{cases}$$

Theorem 3.14. (Grynkiewicz) [?])

Let us consider the equation $x_1 + x_2 - 2x_3 = c$ where c is any integer. Now a few restraints on the given equation.

$$\begin{aligned} L_1(c) = &\ \ x_1 + x_2 - 2x_3 = c, \\ L_2(c) = &\ \ x_1 + x_2 - 2x_3 = c, \quad x_i \neq x_j where i \neq j, \\ L_3(c) = &\ \ x_1 + x_2 - 2x_3 = c, \quad x_1 > x_2 > x_3, \\ L_4(c) = &\ \ x_1 + x_2 - 2x_3 = c, \quad x_3 > x_2 > x_1, \\ L_5(c) = &\ \ x_1 + x_2 - 2x_3 = c, \quad x_1 > x_3 > x_2. \end{aligned}$$

and $S_i(c)$ corresponds to $L_i(c)$. $S_i(c)$ denotes the minimum integer, if it exists, such that for every 2 colouring from $[S_i(c)] \to \{0, 1\}$, we have a monochromatic solution for the given equation and, otherwise $S_i(c) = \infty$. Then

(i) *For $i \in [5]$ and c odd, $S_i(c) = \infty$.*

(ii) *For c even, $S_1(c) = |c| + 1$.*

(iii) *For $c \geq 10$ and even, $S_3(c) = S_2(c) = c + 4$.*

(iv) *For $c \leq -10$ and even, $S_2(c) = S_4(c) = -c + 4$.*

(v) *For $c \leq 8$ and even, $S_3(c) = \infty$.*

(vi) *For $c \geq -8$ and even, $S_4(c) = \infty$.*

(vii) *For c even, $S_5(c) = 2|c| + 10$.*

Theorem 3.15. (Guo & Sun [?])

Let $a_1, \ldots, a_m$ be some positive integers and the equation under cosideration is $\sum_{i=1}^{m} a_i x_i = x_{m+1}$. Then the Rado number $\mathtt{r}(\sum_{i=1}^{m} a_i x_i = x_{m+1}; 2)$ is $av^2 + v - a$, where $a = \min(a_1, \ldots, a_m)$ and $v = \sum_1^m a_i$.

Theorem 3.16. (Schaal [?])

For every integer m and c, let $\mathtt{r}(m, c)$ denote the 2 colour Rado number for the equation $\sum_{i=1}^{m-1} x_i + c = x_m$. If m is odd or $c \geq 0$ and even, then $\mathtt{r}(m, c) = m^2 + (c - 1)(m + 1)$.

Theorem 3.17. (Beutelspacher & Brestovansky [?])

For every integer m and c, let $\mathtt{r}(m,c)$ denote the 2 colour Rado number for the equation $\sum_{i=1}^{m-1} x_i + c = x_m$. For $m \geq 3$ and $c = 0$, $\mathtt{r}(m,0) = m^2 - m - 1$.

Theorem 3.18. (Kosek & Schaal [?])

For every integer $m \geq 3$ and every integer c, let $\mathtt{r}(m,c)$ denote the 2 colour Rado number for the equation $\sum_{i=1}^{m-1} x_i + c = x_m$. Here m is even or c is odd as $\mathtt{r}(m,c) = \infty$ when m is even and c is odd. Then for $c < 0$, we have the following cases

$$\mathtt{r}(m,c) = \begin{cases} m^2 - (c-1)(m+1) & \text{when } -m+2 < c < 0; \\ 2 & \text{when } c = -2(m-2); \\ 3 & \text{when } c = -2(m-2)-1; \\ j(m-1)+c & \text{where } -j(m-2) \leq c \leq -(j-1)(m-1), \\ & j = 3, \ldots, m-1; \\ \left\lceil \frac{1-(m+1)c}{m^2-m-1} \right\rceil & \text{when } c < -(m-1)(m-2). \end{cases}$$

Moreover $\mathtt{r}(m,c) \leq m$ when $-2(m-2)+1 \leq c \leq -m+2$.

Theorem 3.19. (Gupta, Thulasirangan & Tripathi [?])

For the equation of the form $ax + by = (a+b)z$ where a and b are integers, Rado number $\mathtt{r}(ax+by=(a+b)z;2)$ is given by

$$\mathtt{r}(ax+by=(a+b)z;2) = \begin{cases} 4(a+b)-1 & \text{if } a = 1 \text{ or } 4 \mid b \text{ or } (a,b) = (3,4); \\ 4(a+b)+1 & \text{otherwise.} \end{cases}$$

Theorem 3.20. (Burr & Loo [?])

For $a \geq 1$,

$$\mathtt{r}(ax+ay=z;2) = a(4a^2+1).$$

Chapter 4

Proposed Work

I propose to investigate aspects of Ramsey theory related to van der Waerden's theorem and to Rado's theorem. For every pair of positive integers k and r, van der Waerden's theorem gives us the existence of a monochromatic k-term arithmetic progression for every r colouring of the set of integers in $[1, n]$, for all sufficiently large values of n. The set $\{x, ax + d, bx + 2d\}$ is a generalization of an arithmetic progression since $a = b = 1$ makes the elements in one. Moreover, for any a, b, $(2a - b)x - 2(ax + d) + (bx + 2d) = 0$, making the elements of the set fall under the category of Rado's theorem as well.

- Existence of $T(a, b; r)$ depends on a, b and r, but is not guaranteed. One has to first to determine the degree of regularity before trying to determine $T(a, b; r)$. For $(a, b) \neq (1, 1)$, it has been shown that degree of regularity is always less than or equal to 23 by Fox and Radoičić [**?**]. For every pair of integers a, b, we propose to determine the degree of regularity, and determine or estimate $T(a, b; r)$.

- We would like to focus on the case where $a = b$. We know that for $r = 2$, $T(a, a; 2)$ exists. Whereas bounds for $T(a, a; 2)$ exist, the gap between the upper and lower bounds is quite large. The exact value where $a = b$ has been calculated upto 7 with the help of a programme in the paper by Landman and Robertson [**?**]. We propose to improve the upper and lower bounds, thereby decreasing the gap between them, and if possible, to find the exact value for $T(a, a; 2)$. We also hope to find bounds for $T(a, a; r)$ for $r > 2$.

- A linear equation of the form $\sum_{i=1}^{k-1} a_i x_i = a_k x_k$ is regular if it satisfies Rado's theorem. The Rado numbers for the 2-colour case corresponding to $k = 3$ has been completely resolved. We hope to look at the 2-colour case for $k > 3$, and also hope to give some bounds for the general case with r colours for $k = 3$.

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