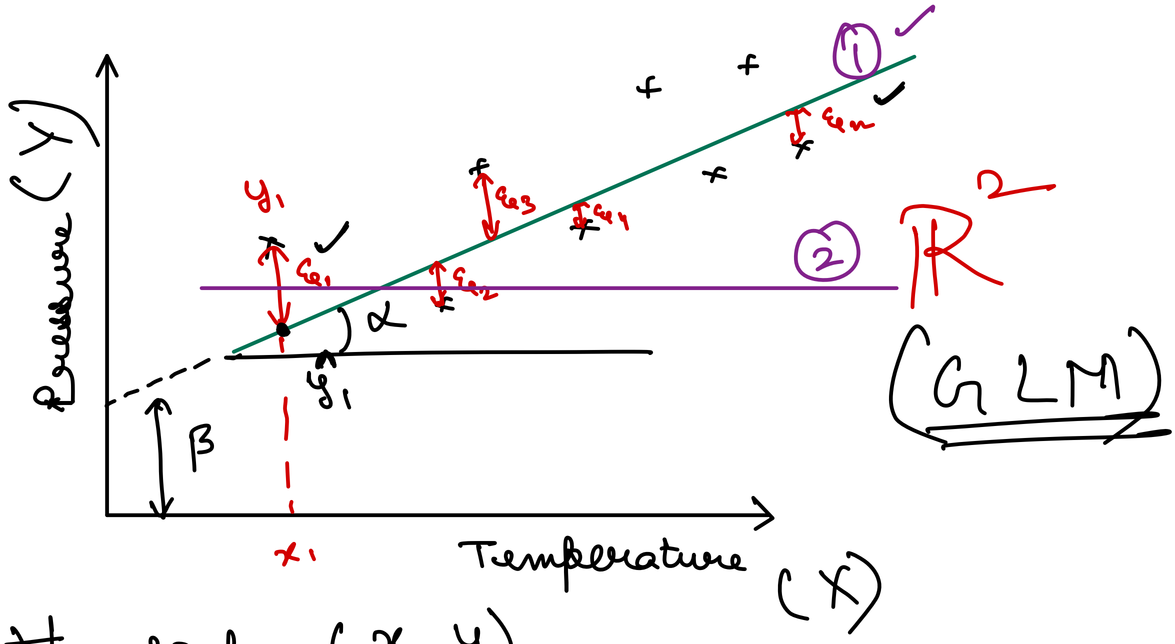
Linear Regression

Finding parameters

β



$\forall$ each (x_i, y_i)

$$\epsilon_i = y_i - \hat{y}_i$$

$$= y_i - (\alpha x_i + \beta)$$

$$\epsilon_i = y_i - \alpha x_i - \beta$$

$$\epsilon = \sum_{i=1}^{n} \epsilon_i = \sum_{i=1}^{n} (y_i - \alpha x_i - \beta)$$

$$J = \sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - \alpha x_i - \beta)^2$$

$\hat{\alpha}, \hat{\beta}$

$$\hat{\alpha} = \underset{\alpha}{\operatorname{argmin}} J$$

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} J$$

$$\frac{\partial J}{\partial \alpha} = 0 \quad ; \quad \frac{\partial J}{\partial \beta} = 0$$

$$\frac{\partial J}{\partial \alpha} = \sum_{i=1}^{n} \cancel{2}(y_i - \alpha x_i - \beta)(+x_i)$$

$$\boxed{0 = \sum_{i=1}^{n} (x_i y_i - \alpha x_i^2 - \beta x_i)} \quad (1)$$

$$\frac{\partial J}{\partial \beta} = 0$$

$$\sum_{i=1}^{n} 2(y_i - \alpha x_i - \beta)(-1) = 0$$

$$\sum_{i=1}^{n} y_i - \alpha \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \beta$$

$$\sum_{i=1}^{n} y_i - \alpha \sum_{i=1}^{n} x_i = n\beta$$

$$\Rightarrow \beta = \frac{\sum_{i=1}^{n} y_i}{n} - \alpha \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\boxed{\hat{\beta} = \bar{y} - \alpha \bar{x}} \quad (2)$$

$$(2) \rightarrow (1)$$

$$-\sum_{i=1}^{n} (\bar{y} - \alpha \bar{x})(x_i) + \alpha\left(\sum_{i=1}^{n} x_i^2\right) = \sum_{i=1}^{n} x_i y_i$$

$$-\frac{n}{n}\sum_{i=1}^{n} \bar{y}\, x_i - \alpha\, \bar{x} \sum_{i=1}^{n} x_i + \alpha\left(\sum_{i=1}^{n} x_i^2\right) = \sum_{i=1}^{n} x_i y_i$$

$$\Rightarrow -n\bar{y} \boxed{\sum_{i=1}^{n} \frac{x_i}{n}} - \frac{\alpha\, \bar{x}\, n}{n} \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

$$\Rightarrow -n\, \bar{y}\, \bar{x} - \alpha\, \bar{x}\, \bar{x} + \alpha \sum_{i=1}^{n} x_i^2 = \frac{n}{n} \sum_{i=1}^{n} x_i y_i$$

$$\alpha = \frac{\frac{n}{n}\sum_{i=1}^{n} \boxed{x_i y_i} - n\bar{x}\bar{y}}{\frac{n\sum_{i=1}^{n} x_i^2}{n} - \frac{n\bar{x}^2}{n}}$$

$$\frac{\cancel{\frac{1}{n}}\left[E(XY) - E(X)E(Y)\right]}{\frac{\text{var}(X)}{\cancel{n}}}$$

$$\boxed{\alpha = \frac{\text{cov}(XY)}{\text{var}(X)}}$$

$$\hat{\beta} =$$

Case with multiple variables

$$y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 \dots + \beta_k X_k$$

y ↑ temperature; β_0 ↑ Intercept; X_1 ↑ CO_2; X_2 ↑ CFC; X_3 ↑ O_3; X_k ↑ CH_4

"n samples"

$$\left\{\begin{array}{l} Y_1 = \beta_0 + \beta_1 X_{11} + \beta_2 X_{21} + \dots + \beta_k X_{k1} \\ \vdots \\ Y_n = \beta_0 + \beta_1 X_{1n} + \beta_2 X_{2n} + \dots + \beta_k X_{kn} \end{array}\right\}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \underset{n \times k}{\begin{bmatrix} 1 & x_{11} & x_{21} & x_{k1} \\ \vdots & \ddots & & \vdots \\ 1 & X_{1n} & \dots & X_{kn} \end{bmatrix}} \underset{k \times 1}{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n \times 1}{\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}}$$

$$\vec{Y} = [X]_{n \times k} \vec{\beta}_{k \times 1} + \vec{e}_{n \times 1}$$

$$\vec{e} = \boxed{\vec{Y} - [X]\vec{\beta}}$$

$$e_1^2 + e_2^2 + e_3^3 + \dots + e_n^2$$

$$\min \sum_{i=1}^{n} e_i^2$$

$$\vec{e}^T \vec{e} = [e_1 \; e_2 \dots e_n] \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

$$= \boxed{e_1^2 + e_2^2 + \dots + e_n^2}$$

$$\vec{e}^T e = (\vec{Y} - X\beta)^T (Y - X\beta)$$

$$= (\vec{Y}^T - (X\beta)^T (Y - X\beta))$$

$$\boxed{(AB)^T = B^T A^T}$$

$$= (\vec{Y}^T - \beta^T X^T)(Y - X\beta)$$

$$= \boxed{Y^T Y - \beta^T X^T Y - \boxed{Y^T X \beta} + \beta^T X^T X \beta}$$

$Y : n \times 1 \; ; \; X = n \times k$

$Y^T : (1 \times n) \times (n \times k) \times (k \times 1)$

$: (1 \times k) \times (k \times 1)$

$: \boxed{(1 \times 1)}$

$$3^T = 3$$

$$(Y^T X \beta)^T = (Y^T X \beta) = \beta^T X^T Y$$

$$e^T e = \boxed{Y^T Y - 2\beta^T X^T Y + \beta^T X^T X \beta}$$

$$\frac{\partial \vec{e}^{\,T} \vec{e}}{\partial \beta} = 0$$

$$\Rightarrow \quad -2X^T Y + 2X^T X \beta = 0$$

$$\Rightarrow \quad X^T Y = X^T X \beta$$

$$\Rightarrow \quad (X^T X)^{-1} X^T Y = (X^T X^{-1})(X^T X)\beta$$

$$\Rightarrow \quad \beta = (X^T X^{-1}) X^T Y$$