Solution-3

(a) let $\alpha_i = 1 \quad \forall i = 1, 2, \dots m$ and $b = 0$.

for $\{x^i, y^i\} \in$ training set, we have

$$|f(x^i) - y^i| = \left| \sum_{j=1}^{m} y^j K(x^j, x^i) - y^i \right|$$

$$\left(\because K(x, z) = \exp(-\|x - z\|^2 / \tau^2) \right)$$

$$= \left| \sum_{j=1}^{m} y^j \exp(-\|x^j - x^i\|^2 / \tau^2) - y^i \right|$$

$$= \left| y^i + \sum_{j \neq i} y^j \exp(\|x^j - x^i\|^2 / \tau^2) - y^i \right|$$

$$= \left| \sum_{j \neq i} y^j \exp(-\|x^j - x^i\|^2 / \tau^2) \right|$$

$(\because |a+b| \leq |a| + |b|)$ $$\leq \sum_{j \neq i} | y^j \exp(-\|x^j - x^i\|^2 / \tau^2)$$

$$= \sum_{j \neq i} |y^j| . \exp(\|x^j - x^i\|^2 / \tau^2)$$

$$= \sum_{j \neq i} \exp(-\|x^j - x^i\|^2 / \tau^2)$$

$$\leq \sum_{j \neq i} \exp(-\varepsilon^2 / \tau^2) = (m-1) \exp(-\varepsilon^2 / \tau^2)$$

assuming $\|x^j - x^i\| \geq \varepsilon \quad \forall i \neq j$

$$(m-1) \exp(-\varepsilon^2 / \tau^2) < 1$$

$$\boxed{\tau < \frac{\varepsilon}{\log(m-1)}}$$

(b) Yes we will obtain zero training error.
Because if the SVM without slack variables is able to find a Solution it will always give zero training error.
We will now show that there exists at least one feasible point.

Let $y^i(w^T x^i + b)$ for some i and let $b=0$ then.

$$y^i(w^T x^i + b) = y^i(w^T x^i) = y^i f(x^i) > 0$$

Since, $y^i, f(x^i)$ have same sign. Choosing all α_i's which are large enough such that $y^i(w^T x^i + b) \geq 1$.
hence the opt. problem is feasible.

Solution-2

a) Consider a solution of the given Problem with $\mathcal{E} < 0$.

⇒ then $y^i(w^T x^i + b) \geq 1 - \mathcal{E}_i$ will be satisfied for $\mathcal{E}_i = 0$. and then the objective function will be lower

⇒ Which implies this can not be an optimal solution.

b)

$$L(w, b, \mathcal{E}, \alpha) = \frac{1}{2} w^T w + \frac{C}{2} \sum_{i=1}^{m} \mathcal{E}_i^2 - \sum_{i=1}^{m} \alpha_i \left(y^i (w^T x^i + b) - 1 + \mathcal{E}_i \right)$$

where $\alpha_i \geq 0$ $i = 1, 2, 3 \ldots m$.

c)

$$W(\alpha) = \min_{w, b, \mathcal{E}} L(w, b, \mathcal{E}, \alpha)$$

$$= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} (\alpha_i y^i x^i)^T (\alpha_j y^j x^j) + \frac{1}{2} \sum_{i=1}^{m} \frac{\alpha_i}{\mathcal{E}_i} \mathcal{E}_i^2$$

$$- \sum_{i=1}^{m} \alpha_i \left(y^i \left(\left(\sum_{j=1}^{m} \alpha_j y^j x^j \right)^T x^i + b \right) - 1 + \mathcal{E}_i \right)$$

$$= -\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y^i y^j (x^i)^T x^j + \frac{1}{2} \sum_{i=1}^{m} \alpha_i \mathcal{E}_i + \sum_{i=1}^{m} \alpha_i$$

$$- \left(\sum_{i=1}^{m} \alpha_i y^i \right) b - \sum_{i=1}^{m} \alpha_i \mathcal{E}_i$$

$$= \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y^i y^j (x^i)^T x^j - \frac{1}{2} \sum_{i=1}^{m} \alpha_i \mathcal{E}_i$$

$$= \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y^i y^j (x^i)^T x^j - \frac{1}{2} \sum_{i=1}^{m} \frac{\alpha_i^2}{C}.$$

Dual formulation

$$\max_{\alpha} \sum_{i=1}^{m} \alpha_i - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i \alpha_j y^i y^j (x^i)^T x^j - \frac{1}{2}\sum_{i=1}^{m} \frac{\alpha_i^2}{c}$$

such that $\alpha_i \geq 0, i = 1, 2, \dots m$

$$\sum_{i=1}^{m} \alpha_i y^i = 0.$$