# Assignment - 2

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**1.** Given:

$n = 200$

$\bar{X} = 16$ ; $\sigma = 2.9$

95% CI for population mean $\mu$ :

$$C.I. = \bar{x} \pm \frac{\sigma}{\sqrt{n}} \cdot z$$

$$= 16 \pm \frac{2.9}{\sqrt{200}} \cdot (1.96)$$

[ $z = 1.96$ for 95% CI ]
$(2 * 47.5)$

$$= 16 \pm 0.401$$

$\therefore$ C.I. lies b/w

$\boxed{15.598 \text{ to } 16.401}$

**2.** $\sigma = 4.6$ min

a) $n = 220$

$\bar{x} = 16.2$ min

92% CI :

$$C.I. = 16.2 \pm \frac{4.6}{\sqrt{220}} \cdot (1.75)$$

$$= 16.2 \pm 0.542.$$

$\therefore$ C.I. lies b/w

$\boxed{15.658 \text{ to } 16.742}$

b) $n = ?$ $\left(\text{Margin of error} = \frac{\sigma}{\sqrt{n}} \cdot z\right)$

$$\frac{10}{60} = \frac{4.6}{\sqrt{n}} \cdot (1.75)$$

$$\Rightarrow \sqrt{n} = 48.3$$

$\boxed{n \simeq 2333}$

**3.** Error $= 0.02$.

a) $\alpha = 1 - C.I.$

$= 1 - 0.8$ $\quad (80\% = 0.8)$

$= 0.2$

Rearranging the formula for $n$,

$$n = \left[ p(1-p) \left( z_{\alpha/2} / E \right)^2 \right]$$

Assume, $p = 0.5$

$$n = \left[ 0.5(1-0.5) (1.29/0.02)^2 \right]$$

$$= 1040$$

(b) $n = 10000$

happy $= 400$

$$p = \frac{400}{10000} = 0.04$$

C.I. :

$$p \pm z_{\alpha/2} \sqrt{p(1-p)/n}$$

$$= 0.04 \pm 1.96 \sqrt{0.04(0.96)/10000}$$

$$= [0.0361, 0.0438]$$

4. $n = 20$

| $X$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X)$ | $2\theta/3$ | $\theta/3$ | $2(1-\theta)/3$ | $(1-\theta)/3$ |

Sol$^n$: Since the given observations are independent

$$L(\theta) = \prod_{i=1}^{20} P(x_i|\theta)$$

$$= \left(\frac{2\theta}{3}\right)^4 \cdot \left(\frac{\theta}{3}\right)^6 \cdot \left(\frac{2(1-\theta)}{3}\right)^6 \cdot \left(\frac{(1-\theta)}{3}\right)^4$$

Taking log,

$$L'(\theta) = \log\left[\left(\frac{2\theta}{3}\right)^4 \left(\frac{\theta}{3}\right)^6 \left(\frac{2(1-\theta)}{3}\right)^6 \left(\frac{(1-\theta)}{3}\right)^4\right]$$

$$= \log\left(\frac{2\theta}{3}\right)^4 + \log\left(\frac{\theta}{3}\right)^6 + \log\left(\frac{2(1-\theta)}{3}\right)^6 + \log\left(\frac{1-\theta}{3}\right)^4$$

For calculating max likelihood, diffrentiating w.r.t $\theta$,

$$\frac{d\,L'(\theta)}{d\theta} = 4 \cancel{\log}\left(\frac{\cancel{3}}{\cancel{2}\theta}\right)\cdot\frac{\cancel{2}}{\cancel{3}} + 6\cancel{\log}\left(\frac{\cancel{3}}{\theta}\right)\frac{1}{\cancel{3}} + 6\cancel{\log}\left(\frac{\cancel{3}}{\cancel{2}(1-\theta)}\right)\left[-\frac{\cancel{2}}{\cancel{3}}\right] + 4\cdot\frac{\cancel{3}}{(1-\theta)}\cdot\left[-\frac{1}{\cancel{3}}\right]$$

$$\frac{d\,L'(\theta)}{d\theta} = \frac{4}{\theta} + \frac{6}{\theta} - \frac{6}{1-\theta} - \frac{4}{1-\theta}$$

Equating to zero,

$$\frac{10}{\theta} = \frac{10}{(1-\theta)}$$

$$\frac{(1-\theta)}{\theta} = 1$$

$$2\theta = 1$$

$$\boxed{\theta = 0.5}$$

5. $f(x|\sigma) = \frac{1}{2\sigma} \cdot e^{\left(\frac{-|x|}{\sigma}\right)}$

Sol$^n$:

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{1}{2\sigma} \cdot e^{\left(\frac{-|x_i|}{\sigma}\right)}\right)$$

Taking log,

$$L'(\theta) = \sum_{i=1}^{n} \left[-\log 2 - \log\sigma - \frac{|x_i|}{\sigma}\right]$$

Equating to zero,

$$\frac{d\,L'(\theta)}{d\theta} = \sum_{i=1}^{n}\left[\frac{-1}{\sigma} + \frac{|x_i|}{\sigma^2}\right] = 0$$

$$\Rightarrow -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}|x_i|}{\sigma^2} = 0$$

$$\Rightarrow \boxed{\sigma = \frac{\sum_{i=1}^{n}|x_i|}{n}}$$

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6. Uniform distribution's probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

Given interval $(0, \theta)$, pdf becomes

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & , 0 \le x \le \theta \\ 0 & , \text{else.} \end{cases}$$

Then, likelihood $L(\theta)$ can be expressed as,

$$L(\theta) = \begin{cases} \frac{1}{\theta^n} & , 0 \le x_i \le \theta \\ 0 & , \text{else} \end{cases}$$

where $i = 1, 2, \ldots n$

Since $\theta$ must be smallest possible value $> \forall x_i$ for $L(\theta)$ to be maximum,

We pick $\theta$ s.t. $\theta = \max(x_1, x_2 \ldots x_n)$

$\therefore$ MLE $\boxed{L(\theta) = \max(x_1, x_2 \ldots x_n)}$

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9. $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

Sol$^n$: $|A-\lambda I| = \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$

$\Rightarrow (1-\lambda)\left[(-5-\lambda)(4-\lambda)+18\right]+3\left[3.(4-\lambda)-18\right]$
$+\ 3\left[-18 - 6.(-5-\lambda)\right]$

$\Rightarrow (1-\lambda)\left[-20+5\lambda-4\lambda+\lambda^2+18\right]+3\left[12-3\lambda-18\right]$
$+\ 3\left[-18+30+6\lambda\right]$

$\Rightarrow (1-\lambda)(\lambda^2+\lambda-2) + (-18-9\lambda)+(36+18\lambda)$

$\Rightarrow (\cancel{\lambda^2}+\lambda-2-\lambda^3-\cancel{\lambda^2}+2\lambda) + (9\lambda+18)$

$\Rightarrow -\lambda^3+12\lambda+16=0$

$\Rightarrow \lambda^3-12\lambda-16=0$

$\Rightarrow$ Eigen values: $\lambda = -2, -2, 4$

For each eigen value $\lambda$,

Eigen vector $\Rightarrow (A-\lambda I).\vec{x} = 0$

For $\lambda = -2$.

$A+2I \Rightarrow \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Augmented matrix:

$$\left[\begin{array}{ccc:c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right]$$

① $R_1 \leftarrow \frac{1}{3}R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 - x_2 + x_3 = 0$

$\Rightarrow \boxed{x_1 = x_2 - x_3}$

$\vec{x}$ can be rewritten as:

$$\vec{x} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $x_2 = 0$, $x_3 = 1$

$\Rightarrow x_1 = -1$

$$V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $x_2 = 1$, $x_3 = 0$

$\Rightarrow x_1 = 1$

$$V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 4$:

$(A - 4I)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right]$$

$R_1 \leftarrow -\frac{1}{3} R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{array}\right]$$

$R_2 \leftarrow -\frac{1}{12} R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -12 & 6 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 + 12R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 - \frac{x_3}{2} = 0 \quad -(i)$

$x_2 - \frac{x_3}{2} = 0 \quad -(ii)$

$$\Rightarrow \vec{X} = \begin{bmatrix} x_3/2 \\ x_3/2 \\ x_3 \end{bmatrix}$$

For $x_3 = 1$,

$$\vec{v}_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

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**7.** $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

**Sol$^n$:** $AA^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Eigen values of $AA^T$:

$$\begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0$$

$= (3-\lambda)[(1-\lambda)(2-\lambda)] - 1[2-\lambda]$
$+ 2[-2(1-\lambda)]$

$= (3-\lambda)[2-\lambda-2\lambda+\lambda^2] + (\lambda-2)$
$+ (4\lambda - 4)$

$= 6 - 9\lambda + 3\lambda^2 - 2\lambda$
$+ 3\lambda^2 - \lambda^3 + 5\lambda - 6$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 6\lambda = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 6\lambda = 0$

$\Rightarrow$ Eigen values:
$0,\ 3+\sqrt{3},\ 3-\sqrt{3}$

For singular values, we take square root of eigen values.

$\sigma_1 = \sqrt{3+\sqrt{3}}$

$\sigma_2 = \sqrt{3-\sqrt{3}}$

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**8.** $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

**Sol$^n$:** $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$|A^TA - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 0 & 1 & 0 & 1-\lambda \end{vmatrix}$$

$= (1-\lambda)^4 - (-\lambda+1)^2$
$- (-\lambda+1)^2 + 1$

$\Rightarrow \lambda^4 - 4\lambda^3 + 4\lambda^2 = 0$

Eigen values: 0, 0, 2, 2.

For $\lambda = 0$,

$$A^TA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

For eigen vectors: $(A^TA).\vec{x} = 0$

$R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$R_4 \leftarrow R_4 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad x_1 + x_3 = 0 \; ; \; x_1 = -x_3$

$x_2 + x_4 = 0 \; ; \; x_2 = -x_4$

$$\vec{X} = \begin{bmatrix} -x_3 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

Let $x_3 = 1$, $x_4 = 0$

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; \quad \hat{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $x_3 = 0$, $x_4 = 1$

$$v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} ; \quad \hat{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 2$,

$$|A^TA - 2I| = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & +1 & 0 & -1 & 0 \end{array}\right]$$

$R_1 \leftarrow -1 * R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$R_2 \leftarrow -1 * R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$R_4 \leftarrow R_4 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad x_1 - x_3 = 0 \; ; \; x_1 = x_3$

$x_2 - x_4 = 0 \; ; \; x_2 = x_4$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 \\ x_2 \end{bmatrix}$$

Let $x_1 = 1$, $x_2 = 0$

$$V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \; ; \; \hat{V}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $x_1 = 0$, $x_2 = 1$

$$V_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \; ; \; \hat{V}_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

(Taking square root of eigen values & filling the principal diagonal)

We know,

$$AV = U\Sigma$$

LHS: $(A.V)$

$$\underset{2\times 4}{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \underset{4\times 4}{\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}}$$

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

For RHS to be equal to LHS

$$U.\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

$$U. \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{bmatrix}$$

Then $U$ must be identity matrix, i.e,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \boxed{A = U \Sigma V^T}$$