WAH YAN COLLEGE, HONG KONG

MATHEMATICS Extended Part	**Name :** __________________
Module 2 (Algebra and Calculus)	**Class :** __________________
Form 6 Mock Exam	**Class No.:** __________________
Date: 15/1/2024	

8:15 am - 10:45 am (2.5 hours)

This paper must be answered in English

INSTRUCTIONS

1. After the announcement of the start of the examination, you should first write down your Name, Class and Class Number in the spaces provided on Page 1.
2. This paper consists of TWO sections, A and B.
3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book.
4. Supplementary answer sheets will be supplied on request. Write your Name, Class and Class Number on each sheet.
5. Unless otherwise specified, all working must be clearly shown.
6. Unless otherwise specified, numerical answers must be exact.

This exam paper contains 32 pages (including this cover page) and 12 questions.

Total of marks is 100.

Grade Table (for teacher's use only)

Question:	1	2	3	4	5	6	7
Marks	5	4	5	6	7	7	7
Score:							

Question:	8	9	10	11	12		Total
Marks	9	13	13	12	12		100
Score:							

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

Section A (50 marks)

1. (a) Let k be a constant. Express $\lim\limits_{h \to 0} \dfrac{e^{-kh-h^2} - 1}{h}$ in terms of k.

   (b) Let $\mathrm{f}(x) = xe^{-x^2}$. Using the result of (a), find $\mathrm{f}'(x)$ from first principles.

(5 marks)

3. Let n be an integer greater than 1. Define $(x-a)^n = \sum_{k=0}^{n} \mu_k x^k$, where a is an integral constant. It is given that $\frac{\mu_2}{\mu_1} = -\frac{4}{3}$.

   (a) Chungchung claims that n is an odd number and $a > 0$. Do you agree? Explain your answer.

   (b) Let $(bx-6)^{2n} = \sum_{k=0}^{2n} \lambda_k x^{2n-k}$, where b is an integral constant. If $\lambda_0 = \mu_n$ and $\lambda_1 = -4\mu_{n-1}$, find a, b and n .

(5 marks)

4. (a) Prove that $\sin \pi x^2 + \sin 2\pi x - \sin \pi \left(x^2 + 2x\right) = 4 \sin \dfrac{\pi \left(x^2 + 2x\right)}{2} \sin \dfrac{\pi x^2}{2} \sin \pi x$.

(b) Find the least positive value of x such that $\sin \pi x^2 + \sin 2\pi x = \sin \pi \left(x^2 + 2x\right)$.

(6 marks)

5. Consider the curve $\Gamma : y = e^{-2x}$, where $x \geq 0$. Let P be a moving point on Γ with h as its $x-$coordinate, where $h > 0$. Denote the tangent to Γ at P by L and the area of the region bounded by Γ, L and the y-axis by A.

   (a) Prove that $A = \dfrac{e^{-2h}\left(e^{2h} - 2h^2 - 2h - 1\right)}{2}$.

   (b) If $h = \ln(t+1)$, where t is the time, find the rate of change of A when $t = 2$.

(7 marks)

6. (a) Find $\displaystyle\int \frac{x+\sin x}{1+\cos x}\,\mathrm{d}x$.

(b) At any point (x, y) on the curve Γ , the slope of the tangent to Γ is $\dfrac{1+x-\cos x+\sin x}{1+\cos x}$.

Given that Γ passes through the point $\left(\frac{\pi}{2}, 2+\pi\right)$, does Γ pass through the point $\left(\frac{3\pi}{4}, \left(\frac{3\pi}{4}+2\right)\tan\frac{3\pi}{8}+\frac{\pi}{4}\right)$.

(7 marks)

7. (a) Let A, B be two square matrices of the same order. If $AB = BA$, show by mathematical induction that for any positive integer n, $(A+B)^n = \sum_{r=0}^{n} C_r^n A^{n-r} B^r$, where A^0 and B^0 are by definition the identity matrix I.

(b) Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta \in \mathbb{R}$. It is given that $A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$ for all positive integers n. Using the result of (a), prove that

$$\sum_{r=0}^{n} C_r^n \cos(n-2r)\theta = 2^n \cos^n \theta \quad \text{and} \quad \sum_{r=0}^{n} C_r^n \sin(n-2r)\theta = 0\,.$$

(7 marks)

11. Consider the system of linear equations in real variables x, y and z

$$(E):\begin{cases} x-3y+2z=0 \\ (a-1)\,x+y+2z=0 \\ 2x+ay-2z=0 \end{cases},\ \text{where } a\in\mathbb{R}\,.$$

It is given that (E) has infinitely many solutions when $a=\alpha$ or $a=\beta$, where $\alpha<\beta$.

(a) (i) Find α and β.

(ii) Solve (E) when $a=\alpha$.

(4 marks)

(b) Consider the system of linear equations in real variables x, y and z

$$(F):\begin{cases} x-3y+2z=2 \\ (c-1)\,x+y+2z=3b+5 \\ 2x+cy-2z=2b+1 \end{cases},\ \text{where } b,c\in\mathbb{R}\,.$$

(i) Assume that $c\neq\alpha$ and $c\neq\beta$. Express x in terms of b and c.

(ii) Assume that $c=\beta$. Find b so that (F) is consistent. Hence, solve (F).

(iii) Assume that $c=\beta$ and (F) is consistent. Find h and k such that all the solutions (x,y,z) of (F) satisfy $hx+ky-2z=7$.

(8 marks)

12. Let $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all positive integers n and let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a) Prove that $A^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$ for all positive integers n . (2 marks)

(b) (i) Prove that $F_{n+m-1} = F_n F_m + F_{n-1} F_{m-1}$ for all positive integers m and n .

(ii) Prove that $F_{n-1} F_{n+1} - F_n^2 = (-1)^n$ for all positive integers n .

(4 marks)

(c) (i) Let α and β be real roots of $\lambda^2 - \lambda - 1 = 0$, where $\alpha > \beta$. Prove that $A \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \mu_1 \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$ for some $\mu_1 \in \mathbb{R}$ and $A \begin{pmatrix} 1 \\ \beta \end{pmatrix} = \mu_2 \begin{pmatrix} 1 \\ \beta \end{pmatrix}$ for some $\mu_2 \in \mathbb{R}$.

(ii) Hence, using (a), prove that $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$ for all non-negative integers n .

(6 marks)