Polynomial Interpolation Examples

Monday, February 6, 2023 2:39 AM

Given the nodes $\{x_0 = 0,\ x_1 = \frac{\pi}{2},\ x_2 = \pi\}$ and the function $f(x) = \cos(x)$, find the interpolating polynomial $P_n(x)$ using (i) — the Vandermonde matrix method.

(ii) — the Lagrange Polynomial method.

Vandermonde Matrix

Remember that

$P_2(x) = a_0 + a_1 x + a_2 x^2$

(i)
$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & \pi/2 & \frac{\pi^2}{4} \\ 1 & \pi & \pi^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \cos 0 \\ \cos \pi/2 \\ \cos \pi \end{bmatrix}$$

$$\therefore \quad a_0 + 0a_1 + 0a_2 = \cos 0 \quad \text{— (i)}$$
$$a_0 + \frac{\pi}{2}a_1 + \frac{\pi^2}{4}a_2 = \cos\frac{\pi}{2} \quad \text{— (ii)}$$
$$a_0 + \pi a_1 + \pi^2 a_2 = \cos \pi \quad \text{— (iii)}$$

Solve the above system of linear equations however you like. Options

- Substitution Method
- Gaussian Elimination
- Inverse Matrix
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(i) — $a_0 = \cos 0 \quad \therefore \boxed{a_0 = 1}$

(ii) — $1 + \frac{\pi}{2}a_1 + \frac{\pi^2}{4}a_2 = \cos\frac{\pi}{2} \rightarrow \frac{\pi}{2}a_1 + \frac{\pi^2}{4}a_2 = -1 \rightarrow \cancel{\pi a_1} - \frac{\pi^2}{2}a_2 = 2$

(iii) — $1 + \pi a_1 + \pi^2 a_2 = \cos\pi \qquad \pi a_1 + \pi^2 a_2 = -2 \qquad \cancel{\pi a_1} + \pi^2 a_2 = -2$

$$-\frac{\pi^2}{2}a_2 + \pi^2 a_2 = 0 \quad \therefore \boxed{a_2 = 0}$$

Since $a_2 = 0$, (iii) — $1 + \pi a_1 + \pi^2 a_2 = \cos\pi$

$$\hookrightarrow a_1 = \frac{\cos\pi - 1}{\pi} \rightarrow \boxed{a_1 = \frac{-2}{\pi}}$$

$$\therefore P_2(x) = a_0 + a_1 x + a_2 x^2 \rightarrow \boxed{1 - \frac{2}{\pi}x = P_2(x)}$$

Notice that technically, the degree is $n=2$, but in reality, the degree is 1. But, we will write $n=2$.

(ii)
$$P_2(x) = f(x_0)\cdot l_0(x) + f(x_1)\cdot l_1(x) + f(x_2)\cdot l_2(x)$$

$$\hookrightarrow f(x_0)\left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) + f(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) + f(x_2)\left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$

Substitute all x_i and $f(x_i)$

$$P_2(x) = 1 \cdot \left(\frac{x - \frac{\pi}{2}}{0 - \frac{\pi}{2}}\right)\left(\frac{x - \pi}{0 - \pi}\right) + 0 - 1 \cdot \left(\frac{x - 0}{\pi - 0}\right)\left(\frac{x - \frac{\pi}{2}}{\pi - \frac{\pi}{2}}\right) \rightarrow \frac{x^2 - \pi x - \frac{\pi}{2}x + \frac{\pi^2}{2}}{\frac{\pi^2}{2}} - \frac{x^2 - \frac{\pi}{2}x}{\pi^2 - \frac{\pi^2}{2}}$$

$$\hookrightarrow \frac{\cancel{x^2} - \pi x - \cancel{\frac{\pi}{2}x} + \frac{\pi^2}{2} - \cancel{x^2} + \cancel{\frac{\pi}{2}x}}{\frac{\pi^2}{2}} \rightarrow \frac{\frac{\pi^2}{2} - \pi x}{\frac{\pi^2}{2}} = \boxed{1 - \frac{2}{\pi}x = P_2(x)}$$