MAT215

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Section : 02

Ans to the problem no':- 1

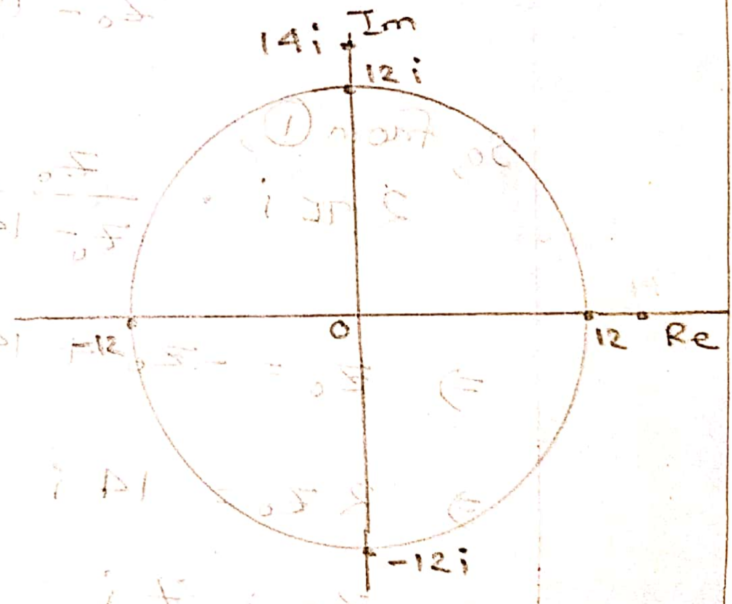
Given integral,

$$\oint_{|z|=12} \frac{z}{(z-z_0)(z-14i)}\,dz = -2\pi i \quad \text{--- (1)}$$

Let's find the L.H.S. firstly.

The contour is given by $|z|=12$.

So, it is a circle centered at $(0,0)$ and its radius is 12.



Let's find the points of similarity.

Here, $z - z_0 = 0$

$\Rightarrow z = z_0$

And, $z - 14i = 0$

$\Rightarrow z = 14i$

Now, $14i$ lies outside the contour. The point of similarity is only at z_0.

Let, $f(z) = \dfrac{z}{(z-14i)}$

$$\therefore \oint_c \frac{z}{(z-z_0)(z-14i)} dz$$

$$= \oint_c \frac{f(z)}{z-z_0} dz$$

$$= 2\pi i \cdot f(z_0)$$

$$= 2\pi i \cdot \frac{z_0}{z_0 - 14i}$$

So, from (1),

$$2\pi i \cdot \frac{z_0}{z_0 - 14i} = -2\pi i$$

$$\Rightarrow z_0 = -z_0 + 14i$$

$$\Rightarrow 2z_0 = 14i$$

$$\therefore z_0 = 7i$$

(Ans)

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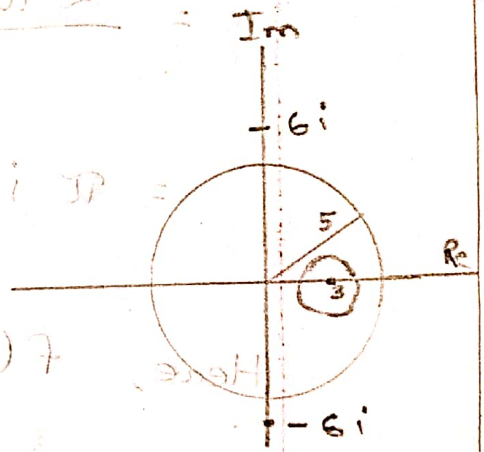
Ans to the problem no:- 2

Given integral,

$$\oint_{|z|=5} \frac{1}{(z-3)^3 (z^2+36)^2} dz$$

Here, the contour is described by a circle centered at (0,0) and radius 5.



Lets find the point of similarity.

Here, $(z-3)^3 = 0$

$\Rightarrow z-3=0$

$\Rightarrow z=3$

And, $(z^2+36)^2 = 0$

$\Rightarrow z^2+36=0$

$\Rightarrow z^2=-36$

$\Rightarrow z=\sqrt{-36}$

$\therefore z = \pm 6i$

$\therefore$ The only point of similarity is $z=3$

So, let $f(z) = \frac{1}{(z^2+36)^2}$

$$\therefore \oint_c \frac{1}{(z-3)^3(z^2+36)^2}\,dz$$

$$= \oint_c \frac{f(z)}{(z-3)^{2+1}}\,dz$$

$$= \frac{2\pi i \cdot f''(3)}{2!}$$

$$= \pi i \cdot f''(3) \quad \text{--- (1)}$$

Here, $f(z) = \frac{1}{(z^2+36)^2} = (z^2+36)^{-2}$

$$\therefore f'(z) = -2 \cdot (z^2+36)^{-3} \cdot \frac{d}{dz}(z^2+36)$$

$$= -2 \cdot (z^2+36)^{-3} \cdot 2z$$

$$= -4z \cdot (z^2+36)^{-3}$$

$$\therefore f''(z) = \frac{d}{dz}\left[\frac{-4z}{(z^2+36)^3}\right]$$

$$= -4\frac{d}{dz}\left[\frac{z}{(z^2+36)^3}\right]$$

$$= -4 \cdot \frac{\frac{d}{dz}z \cdot (z^2+36)^3 - z \cdot \frac{d}{dz}(z^2+36)^3}{\left[(z^2+36)^3\right]^2}$$

$$= -4 \cdot \frac{(z^2+36)^3 - z \cdot 3 \cdot (z^2+36)^2 \cdot \frac{d}{dz}(z^2+36)}{(z^2+36)^6}$$

$$= -4 \; \frac{(z^2+36)^3 - 6z^2 (z^2+36)^2}{(z^2+36)^6}$$

$$= \frac{-4(z^2+36)^3}{(z^2+36)^6} + \frac{24 z^2 (z^2+36)^2}{(z^2+36)^6}$$

$$= \frac{-4}{(z^2+36)^3} + \frac{24z^2}{(z^2+36)^4}$$

$$= \frac{-4(z^2+36) + 24z^2}{(z^2+36)^4}$$

$$= \frac{-4z^2 - 144 + 24z^2}{(z^2+36)^4}$$

$$= \frac{20z^2 - 144}{(z^2+36)^4}$$

$$\therefore f''(3) = \frac{20 \cdot (3)^2 - 144}{(3^2+36)^4} = \frac{36}{4100625} = \frac{4}{455625}$$

$\therefore$ From (1), $\pi i \cdot f''(3) = \frac{4}{455625} \pi i$

(Ans)

Ans to the ques no :- 3

Given function, $f(z) = \frac{1}{(z-2)(z-9)}$

Given domain, $2 < |z| < 9$

Decomposing $f(z)$ into partial fraction,

$$\frac{1}{(z-2)(z-9)} \equiv \frac{A}{z-2} + \frac{B}{z-9}$$

$$\Rightarrow 1 = A(z-9) + B(z-2)$$

When, $z = 9$, $B = \frac{1}{7}$

And, $z = 2$, $A = -\frac{1}{7}$

$$\therefore \frac{1}{(z-2)(z-9)} = -\frac{1}{7(z-2)} + \frac{1}{7(z-9)} = f(z)$$

Here,

$$|z| > 2$$

$$\Rightarrow \left|\frac{z}{2}\right| > 1$$

$$\Rightarrow \left|\frac{2}{z}\right| < 1$$

$$|z| < 9$$

$$\Rightarrow \left|\frac{z}{9}\right| < 1$$

Then, $f(z) = -\dfrac{1}{7(z-2)} + \dfrac{1}{7(z-9)}$

$$= -\frac{1}{7 \cdot z\left(1-\frac{2}{z}\right)} + \frac{1}{7 \cdot 9\left(\frac{z}{9}-1\right)}$$

$$= -\frac{1}{7z\left(1-\frac{2}{z}\right)} - \frac{1}{63\left(1-\frac{z}{9}\right)}$$

$$= -\frac{1}{7z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right) - \frac{1}{63}\left(1+\frac{z}{9}+\frac{z^2}{9^2}+\cdots\right)$$

$$= -\frac{1}{7z} - \frac{2}{7z^2} - \frac{4}{7z^3} - \frac{1}{63} - \frac{z}{207} - \frac{z^2}{5103} - \cdots$$

$$= \cdots - \frac{4}{7z^3} - \frac{2}{7z^2} - \frac{1}{7z} - \frac{1}{63} - \frac{z}{207} - \frac{z^2}{5103} - \cdots$$

(Ans)

Ans to the ques no:- 4

Given function, $f(z) = \frac{z}{z-8}$

So, the singular point of similarity $\Rightarrow z = 8$

So, the domain is divided into two parts.

First one is, $|z| < 8$

Second one is, $|z| > 8$

Now, $f(z) = \frac{z}{z-8}$

$\Rightarrow f(z) = \frac{z}{z-10+2}$

$\Rightarrow f(z) = \frac{z}{2+(z-10)}$

$\therefore f(z) = z \cdot \frac{1}{2+(z-10)}$

For, $|z| < 8$,

$f(z) = z \cdot \frac{1}{2+(z-10)}$

$$= \frac{z}{z-10} \cdot \frac{1}{\frac{2}{z-10} + 1}$$

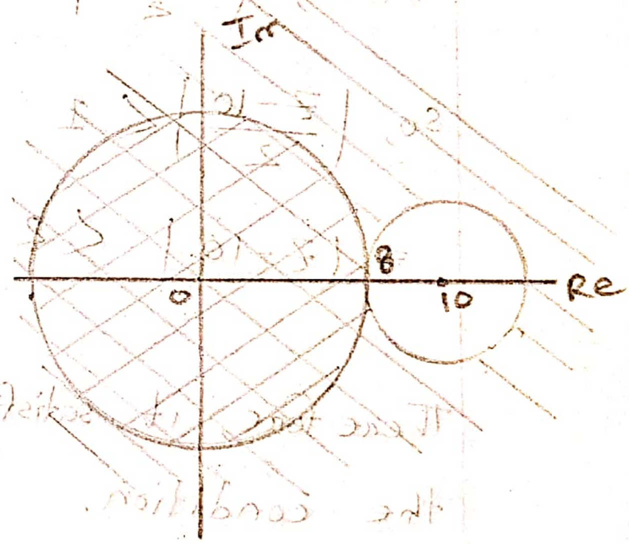
$$= \frac{z}{z-10} \cdot \frac{1}{1 + \left(\frac{2}{z-10}\right)}$$

Here, $\left|\frac{2}{z-10}\right|$ should be less than 1.

So, $\left|\frac{2}{z-10}\right| < 1$

$\Rightarrow 2 < |z-10|$

$\Rightarrow |z-10| > 2$



Therefore, it satisfies the condition.

$$\text{so, } f(z) = \frac{z}{z-10} \cdot \frac{1}{1 + \left(\frac{2}{z-10}\right)}$$

$$= \frac{z}{z-10} \cdot \left(1 - \frac{2}{z-10} + \frac{2^2}{(z-10)^2} - \cdots\cdots\right)$$

$$= \frac{z}{z-10} - \frac{2z}{(z-10)^2} + \frac{4z}{(z-10)^3} - \cdots\cdots$$

And, for, $|z| > 8$

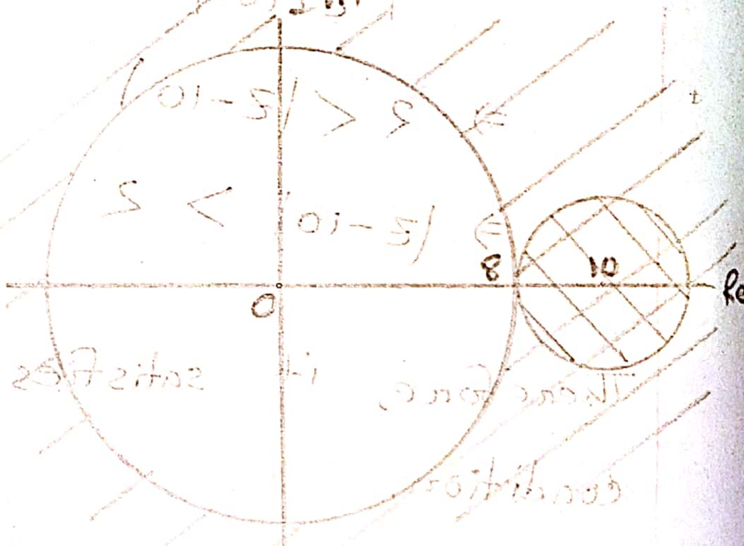
$$f(z) = z \cdot \frac{1}{2 + (z-10)}$$

$$= \frac{z}{2} \cdot \frac{1}{1 + \left(\frac{z-10}{2}\right)}$$

Here, $\left|\frac{z-10}{2}\right|$ should be less than 1.

So, $\left|\frac{z-10}{2}\right| < 1$

$\Rightarrow |z-10| < 2$



Therefore, it satisfies the condition.

So, $f(z) = \frac{z}{2} \cdot \frac{1}{1 + \left(\frac{z-10}{2}\right)}$

$$= \frac{z}{2}\left(1 - \frac{z-10}{2} + \frac{(z-10)^2}{2^2} - \cdots\right)$$

$$= \frac{z}{2} - \frac{z(z-10)}{4} + \frac{z(z-10)^2}{8} - \cdots$$

Ans:-

For $|z| < 8$,

$$f(z) = \frac{z}{z-10} - \frac{2z}{(z-10)^2} + \frac{4z}{(z-10)^3} - \cdots$$

For $|z| > 8$,

$$f(z) = \frac{z}{2} - \frac{z(z-10)}{4} + \frac{z(z-10)^2}{8} - \cdots$$

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